Discussiones Mathematicae
Graph Theory 19 (1999) 175–197

ON-LINE RANKING NUMBER FOR CYCLES AND PATHS

Erik Bruoth and Mirko Horňák

Department of Geometry and Algebra
P.J. Šafárik University, Jesenná 5
041 54 Košice, Slovakia

e-mail: ebruoth@duro.upjs.sk
e-mail: hornak@turing.upjs.sk

Abstract

A k-ranking of a graph G is a colouring $\varphi : V(G) \to \{1, \dots, k\}$ such that any path in G with endvertices x, y fulfilling $\varphi(x) = \varphi(y)$ contains an internal vertex z with $\varphi(z) > \varphi(x)$. On-line ranking number $\chi_r^*(G)$ of a graph G is a minimum k such that G has a k-ranking constructed step by step if vertices of G are coming and coloured one by one in an arbitrary order; when colouring a vertex, only edges between already present vertices are known. Schiermeyer, Tuza and Voigt proved that $\chi_r^*(P_n) < 3\log_2 n$ for $n \geq 2$. Here we show that $\chi_r^*(P_n) \leq 2\lfloor \log_2 n \rfloor + 1$. The same upper bound is obtained for $\chi_r^*(C_n), n \geq 3$.

Keywords: ranking number, on-line vertex colouring, cycle, path.

1991 Mathematics Subject Classification: 05C15.

1 Introduction

In this article we deal with simple finite undirected graphs. For formal reasons we also use the empty graph $K_0 = (\emptyset, \emptyset)$. A *$k$-ranking* of a graph G is a vertex colouring of G which takes as colours integers $1, \dots, k$ in such a way that, whenever a path of G has endvertices of the same colour, it contains an internal vertex with a greater colour. If k is not specified, we speak simply about a *ranking*. Evidently, a ranking is a proper vertex colouring and a k-ranking of a connected graph uses k at most once. Rankings are important in the parallel Cholesky factorization of matrices (Liu [3]) and also in VLSI layout (Leiserson [2]).

Ranking number $\chi_{\mathrm{r}}(G)$ of a graph G is a minimum k such that G has a k-ranking. The problem of finding the ranking number of an arbitrary graph is NP-complete, see Llewelyn et al. [4]. Katchalski et al. [1] proved, among other results on trees, that $\chi_{\mathrm{r}}(P_n) = \lfloor \log_2 n \rfloor + 1$ for $n \geq 1$. They have also an upper bound for the ranking number of a planar graph G, namely $\chi_{\mathrm{r}}(G) \leq 3(\sqrt{6}+2)\sqrt{|V(G)|}$.

In an *on-line* version of the problem vertices of a graph G are coming in an arbitrary order. They are coloured one by one in such a way that only a local information concerning edges between already present vertices is known in a moment when a colour for a vertex is to be chosen. Schiermeyer et al. [5] showed that, for $n \geq 2$, there is an on-line algorithm providing a ranking of n-vertex path, for which the maximum used number is smaller than $3 \log_2 n$, independently from arriving order of vertices. Our main aim is to show that this number is $\leq 2\lfloor \log_2 n \rfloor + 1$.

For a graph G and a set $W \subseteq V(G)$ let $G\langle W \rangle$ be the subgraph of G induced by W. The notation C_n and P_n is used for n-vertex cycle and n-vertex path, respectively.

For integers p, q we denote by $[p, q]$ the set of all integers r with $p \leq r \leq q$, and by $[p, \infty)$ the set of all integers r with $p \leq r$.

The *length* of a finite sequence A (i.e., the number of terms of A), is denoted by $|A|$. For finite sequences $A = (a_1, \dots, a_m)$ and $B = (b_1, \dots, b_n)$ let $AB = (a_1, \dots, a_m, b_1, \dots, b_n)$ be the *concatenation* of A and B (in this order); the concatenation can be generalized to any finite number of finite sequences. The concatenation is, clearly, associative, and we will use $\Pi_{i=1}^{k} A_i$ for the concatenation of finite sequences $A_1, \dots, A_k$ (in this order).

Now, let us describe our on-line version of the ranking problem more precisely. An *input sequence* for a graph G is any sequence of vertices of G containing all vertices of G exactly once. Let $\mathrm{Is}(G)$ be the set of all input sequences for G and let $Y = \Pi_{i=1}^{n}(y_i) \in \mathrm{Is}(G)$. Vertices $y_1, \dots, y_n$ are coloured in this order one by one in the following way: We denote by $G(Y, y_i)$ the graph $G\langle \{y_j : j \in [1, i]\} \rangle$ induced by all vertices that come in Y not later than y_i does, $i \in [1, n]$. We colour y_1 with an arbitrary positive integer. In the moment when y_i, $i \in [2, n]$, is to be coloured, only the graph $G(Y, y_i)$ and a ranking of $G(Y, y_{i-1})$ is known; the colour of y_i has to be chosen in such a way that a ranking of $G(Y, y_i)$ results (without altering "old" colours).

We would like to analyze all possibilities of forming a ranking of a graph G in the above on-line fashion. To that aim, we denote by $\mathcal{Q}$ the set of all quadruples (G, H, φ, x) such that G is a non-empty graph, H is an induced subgraph of G with $|V(H)| = |V(G)| - 1$, φ is a ranking of H

and $\{x\} = V(G) - V(H)$. We say that two quadruples (G, H, φ, x) and (G', H', φ', x') are *equivalent* (and we do not distinguish them in $\mathcal{Q}$) if there is an isomorphism ι between G and G' which maps H onto H' (so that $\iota(x) = x'$) and an automorphism α' of H' such that for any $y \in V(H)$ it holds $\varphi(y) = \varphi'(\alpha'(\iota(y)))$. A *ranking algorithm* is a mapping $\mathcal{A} : \mathcal{Q} \to [1, \infty)$ such that, for any $(G, H, \varphi, x) \in \mathcal{Q}$, $\varphi \cup \{(x, \mathcal{A}(G, H, \varphi, x))\}$ is a ranking of G.

Let $\mathcal{A}$ be a ranking algorithm, let G be a graph and let $Y = \Pi_{i=1}^{n}(y_i) \in \mathrm{Is}(G)$. The algorithm $\mathcal{A}$ provides a ranking $\mathrm{rank}(\mathcal{A}, G, Y, y_i)$ of the graph $G(Y, y_i), i \in [1, n]$, recurrently as follows:

$$\mathrm{rank}(\mathcal{A}, G, Y, y_1) := \{(y_1, \mathcal{A}(K_1, K_0, \emptyset, y_1))\},$$

$$\mathrm{rank}(\mathcal{A}, G, Y, y_i) := \mathrm{rank}(\mathcal{A}, G, Y, y_{i-1})$$

$$\cup \{(y_i, \mathcal{A}(G(Y, y_i), G(Y, y_{i-1}), \mathrm{rank}(\mathcal{A}, G, Y, y_{i-1}), y_i))\},\ i \in [2, n].$$

We denote by $\mathrm{rank}(\mathcal{A}, G, Y)$ the ranking $\mathrm{rank}(\mathcal{A}, G, Y, y_n)$ of the graph $G(Y, y_n) = G$ provided by the algorithm $\mathcal{A}$ if the vertices of G are coming in the input sequence Y. Clearly, the ranking $\mathrm{rank}(\mathcal{A}, G, Y, y_i)$ is a restriction of the ranking $\mathrm{rank}(\mathcal{A}, G, Y)$ to the graph $G(Y, y_i), i \in [1, n]$. By $\max\,(\mathcal{A}, G, Y)$ we will denote the maximum number attributed to a vertex of G by $\mathrm{rank}(\mathcal{A}, G, Y)$ and by $\max(\mathcal{A}, G)$ the maximum of $\max(\mathcal{A}, G, Y)$ over all $Y \in \mathrm{Is}(G)$. The *on-line ranking number* $\chi_{\mathrm{r}}^*(G)$ of the graph G is the minimum of $\max(\mathcal{A}, G)$ over all ranking algorithms $\mathcal{A}$. Evidently, for any graph G and any ranking algorithm $\mathcal{A}$ we have

$$\chi_{\mathrm{r}}(G) \leq \chi_{\mathrm{r}}^*(G) \leq \max(\mathcal{A}, G).$$

Proposition 1. *If G_1 is an induced subgraph of G_2 and $\mathcal{A}$ is a ranking algorithm, then* $\max(\mathcal{A}, G_1) \leq \max(\mathcal{A}, G_2)$.

Proof. Consider an input sequence $Y_1 = \Pi_{i=1}^{n}(y_i) \in \mathrm{Is}(G_1)$ such that $\max(\mathcal{A}, G_1, Y_1) = \max(\mathcal{A}, G_1)$ and an arbitrary input sequence Y_2 of the graph $G_2\langle V(G_2) - V(G_1)\rangle$. Then $Y_1Y_2 \in \mathrm{Is}(G_2)$, and we have $\mathrm{rank}(\mathcal{A}, G_2, Y_1Y_2, y_n) = \mathrm{rank}(\mathcal{A}, G_1, Y_1)$, so that $\max(\mathcal{A}, G_2) \geq \max(\mathcal{A}, G_2, Y_1Y_2) \geq \max(\mathcal{A}, G_1, Y_1) = \max(\mathcal{A}, G_1)$. ∎

Corollary 2. *If G_1 is an induced subgraph of G_2, then $\chi_{\mathrm{r}}^*(G_1) \leq \chi_{\mathrm{r}}^*(G_2)$.* ∎

2 Reduction

A natural *greedy* algorithm $\mathcal{G}$ (called also First Fit Algorithm) is determined by the requirement that, for any $(G, H, \varphi, x) \in \mathcal{Q}$, $\mathcal{G}(G, H, \varphi, x)$ is the minimum positive integer k such that $\varphi \cup \{(x, k)\}$ is a ranking of G. In other words, we can describe $\mathcal{G}$ as follows: A colour $l \in [1, \infty)$ is *forbidden* for x if the colouring $\psi = \varphi \cup \{(x, l)\}$ produces a (u, v)-path P in G with $\psi(u) = \psi(v) = \max\{\psi(y) : y \in V(P)\}$ (clearly, $x \in V(P)$). The greedy algorithm colours x with the smallest colour that is not forbidden for x. Evidently, the colour $\max\{\varphi(y) : y \in V(H)\} + 1$ is not forbidden for x. That is why, we know that for any graph G and any input sequence $Y \in \mathrm{Is}(G)$ the ranking $\mathrm{rank}(\mathcal{G}, G, Y)$ of G uses every integer from the interval $[1, \max(\mathcal{G}, G, Y)]$ at least once.

Now we are going to analyze how $\mathcal{G}$ works for cycles and paths. For that purpose suppose that $G = C_n$, $n \in [3, \infty)$, or $G = P_n$, $n \in [1, \infty)$, with $V(G) = \{x_i : i \in [1, n]\}$ and $E(G) \supseteq \{x_i x_{i+1} : i \in [1, n-1]\}$ (there is an equality in this inclusion if $G = P_n$, and, if $G = C_n$, there is an additional edge $x_n x_1$). Sometimes it will be necessary to use for indices arithmetics modulo n, i.e., $x_{i-n} = x_i = x_{i+n}$ for any $i \in [1, n]$.

As an example, consider the input sequence $Y = (x_6, x_7, x_3, x_5, x_2, x_4, x_1) \in \mathrm{Is}(C_7) = \mathrm{Is}(P_7)$. We have $\mathrm{rank}(\mathcal{G}, C_7, Y) = \{(x_6, 1), (x_7, 2), (x_3, 1), (x_5, 3), (x_2, 2), (x_4, 4), (x_1, 5)\}$ and $\mathrm{rank}(\mathcal{G}, P_7, Y)$ differs from $\mathrm{rank}(\mathcal{G}, C_7, Y)$ only by attributing 1 to x_1.

An important role in our analysis is played by the following reduction process: We suppose that $G = C_n$, $n \in [5, \infty)$, or $G = P_n$, $n \in [2, \infty)$, $Y \in \mathrm{Is}(G)$ and $\varphi = \mathrm{rank}(\mathcal{G}, G, Y)$. A vertex $x_i \in V(G)$ is said to be a *survivor* of G (with respect to the input sequence Y) if $\varphi(x_i) \geq 2$; if $\varphi(x_i) = 1$, it is a *non-survivor*. We transform G into a non-empty graph $R(G, Y)$ homeomorphic to G as follows: We delete from G all non-survivors and we join by a new edge any two survivors having a non-survivor as a common neighbour (i.e., we delete all non-survivors of degree 1 and we "smooth out" all non-survivors of degree 2). We can do this because it is easy to see that the number of survivors is always positive and, in the case $G = C_n$, it is ≥ 3. The input sequence Y induces in a natural way an input sequence $R(Y, G)$ for the graph $R(G, Y)$ – we simply delete from Y all non-survivors.

If $Y \in \mathrm{Is}(C_7)$ is as above, then $R(C_7, Y) = C_5$, $R(Y, C_7) = (x_7, x_5, x_2, x_4, x_1)$ and $R(P_7, Y) = P_4$, $R(Y, P_7) = (x_7, x_5, x_2, x_4)$.

Lemma 3. *Let $G = C_n$, $n \in [5, \infty)$, or $G = P_n$, $n \in [2, \infty)$, let $Y \in \mathrm{Is}(G)$, $\varphi = \mathrm{rank}(\mathcal{G}, G, Y)$, $\dot{G} = R(G, Y)$, $\dot{Y} = R(Y, G)$ and $\dot{\varphi} = \mathrm{rank}(\mathcal{G}, \dot{G}, \dot{Y})$. Then, for any survivor x_i of G with respect to Y, it holds $\dot{\varphi}(x_i) = \varphi(x_i) - 1$.*

Proof. Consider a sequence $Y' \in \mathrm{Is}(G)$ in which all non-survivors (with respect to Y) come first (in an arbitrary order) and then all survivors (with respect to Y) come in the order induced by that of Y. It is easy to see that $\varphi = \mathrm{rank}(\mathcal{G}, G, Y')$.

Let $Y' = \Pi_{i=1}^{n}(y_i)$ and let y_s be the first survivor with respect to Y' (and Y as well). We are going to show by induction on i that $\dot{\varphi}(y_i) = \varphi(y_i) - 1$ for any $i \in [s, n]$. Obviously, $\dot{\varphi}(y_s) = 1 = 2 - 1 = \varphi(y_s) - 1$.

Now suppose that $i \in [s + 1, n]$ and that $\dot{\varphi}(y_j) = \varphi(y_j) - 1$ for every $j \in [s, i - 1]$. Note that survivors y_j, y_k with $j, k \in [s, i], j \neq k$, are joined by a path P in $G(Y', y_i)$ if and only if they are joined in $\dot{G}(\dot{Y}, y_i)$ by the path $\dot{P}$ such that $V(\dot{P}) = V(P) - \{y_l : l \in [1, s - 1]\}$. Hence, by the induction hypothesis and the fact that $\varphi(y_l) = 1$ for any $l \in [1, s - 1]$, a colour $a \in [2, \infty)$ is forbidden for y_i in $G(Y, y_i)$ by a path P if and only if the colour $a - 1$ is forbidden for y_i in $\dot{G}(\dot{Y}, y_i)$ by the corresponding path $\dot{P}$. Since $\varphi(y_i) \geq 2$, we obtain $\dot{\varphi}(y_i) = \varphi(y_i) - 1$, as necessary. ■

We define a *section* of our graph G as follows: A section of P_n is any sequence $\Pi_{i=j}^{k}(x_i)$ of vertices of P_n with $j, k \in [1, n]$ and $j \leq k$. A section of C_n is any sequence $\Pi_{i=j}^{k}(x_i)$ of vertices of C_n with $j, k \in [1-n, 2n]$ and $j \leq k \leq j-1+n$. From the definition we see that a section $\Pi_{i=j}^{k}(x_i)$ consists of $k + 1 - j \leq n$ distinct vertices of G and that $x_i x_{i+1}$ is an edge of G for every $i \in [j, k - 1]$. An *endsection* of P_n is any section of P_n containing an endvertex of P_n. The *type* of a section $\Pi_{i=j}^{k}(x_i)$ (with respect to the ranking $\varphi = \mathrm{rank}(\mathcal{G}, G, Y)$) is the sequence formed from $\Pi_{i=j}^{k}(\varphi(x_i))$ by replacing any term $\varphi(x_i)$ fulfilling $\varphi(x_i) \geq 3$ with 3+. The ranking $\varphi = \mathrm{rank}(\mathcal{G}, G, Y)$ determines two types of vertices in G: a vertex $x \in V(G)$ is *high* (with respect to φ), if $\varphi(x) \geq 3$, otherwise it is *low*. A section of G containing only high [low] vertices, which is maximal (non-extendable with respect to this property), is called a *high* [*low*] section of G. The *defect* of a section S of G is the difference $\mathrm{def}(S)$ between the number of low vertices in S and the number of high vertices in S. The *defect* of a graph G is the difference $\mathrm{def}(G)$ between the number of low vertices in $V(G)$ and the number of high vertices in $V(G)$, i.e., the defect of (any) section S of G with $|S| = |V(G)|$.

Lemma 4. *Let $G = C_n$, $n \in [3, \infty)$, or $G = P_n$, $n \in [1, \infty)$, let $Y \in \mathrm{Is}(G)$, $\varphi = \mathrm{rank}(\mathcal{G}, G, Y)$ and $q \in [1, n]$.*

1. *If* $\Pi_{i=q}^{q+3}(x_i)$ *is a section of* G*, then there are* $j, k \in [q, q+3]$ *such that* $\varphi(x_j) = 1$ *and* $\varphi(x_k) \geq 3$*.*
2. *If* $\Pi_{i=q}^{q+2}(x_i)$ *is such a section of* G *that* $\varphi(x_{q+1}) = 2$*, then* $\min\{\varphi(x_q), \varphi(x_{q+2})\} = 1$*.*
3. *If* $G = P_n$ *and* $\varphi(x_1) \geq 2$*, then* $n \geq 2$ *and* $\varphi(x_2) = 1$*.*
4. *If* $G = P_n$ *and* $\varphi(x_1) \geq 3$*, then* $n \geq 3$*,* $\varphi(x_2) = 1$ *and* $\varphi(x_3) = 2$*.*
5. *If* $G = P_n$ *and* $\varphi(x_n) \geq 2$*, then* $n \geq 2$ *and* $\varphi(x_{n-1}) = 1$*.*
6. *If* $G = P_n$ *and* $\varphi(x_n) \geq 3$*, then* $n \geq 3$*,* $\varphi(x_{n-1}) = 1$ *and* $\varphi(x_{n-2}) = 2$*.*
7. *If* $\Pi_{i=q}^{q+2}(x_i)$ *is a section of* G *of type* $(3+, 3+, 3+)$*, then* $\Pi_{i=q-2}^{q+4}(x_i)$ *also is a section of* G *and it is of type* $(2, 1, 3+, 3+, 3+, 1, 2)$*.*
8. *If* $\Pi_{i=q}^{q+3}(x_i)$ *is a section of* G *of type* $(3+, 3+, 1, 3+)$*, then* $\Pi_{i=q-2}^{q+5}(x_i)$ *also is a section of* G *and it is of type* $(2, 1, 3+, 3+, 1, 3+, 1, 2)$ *or* $(2, 1, 3+, 3+, 1, 3+, 2, 1)$*.*
9. *If* $\Pi_{i=q}^{q+3}(x_i)$ *is a section of* G *of type* $(3+, 1, 3+, 3+)$*, then* $\Pi_{i=q-2}^{q+5}(x_i)$ *also is a section of* G *and it is of type* $(1, 2, 3+, 1, 3+, 3+, 1, 2)$ *or* $(2, 1, 3+, 1, 3+, 3+, 1, 2)$*.*
10. *If* $G = P_n$*,* $n \geq 3$*,* $\varphi(x_1) = 1$ *and* $\varphi(x_3) \geq 3$*, then* $\varphi(x_2) = 2$*.*
11. *If* $G = P_n$*,* $n \geq 3$*,* $\varphi(x_n) = 1$ *and* $\varphi(x_{n-2}) \geq 3$*, then* $\varphi(x_{n-1}) = 2$*.*
12. *If* $G = P_n$ *and* $\Pi_{i=q}^{q+1}(x_i)$ *is a section of* G *of type* $(3+, 3+)$*, then* $n \geq 6$ *and* $q \in [3, n-3]$*.*
13. *If* $G = P_n$ *and* $\Pi_{i=q}^{q+2}(x_i)$ *is a section of* G *of type* $(3+, 1, 3+)$*, then* $n \geq 7$ *and* $q \in [3, n-4]$*.*
14. *If* $G = P_n$ *and* $\Pi_{i=q}^{q+2}(x_i)$ *is a section of* G *of type* $(3+, 3+, 2)$*, then* $n \geq 7$ *and* $q \in [3, n-4]$*.*
15. *If* $G = P_n$ *and* $\Pi_{i=q}^{q+2}(x_i)$ *is a section of* G *of type* $(2, 3+, 3+)$*, then* $n \geq 7$ *and* $q \in [3, n-4]$*.*

Proof. 1. The existence of k follows immediately from the definition of a ranking. As concerns the existence of j, we may suppose that $\min\{\varphi(x_q), \varphi(x_{q+3})\} \geq 2$ – otherwise we are done. Let x_j be that vertex from among x_{q+1}, x_{q+2}, which comes sooner in Y. Then, clearly, $\varphi(x_j) = 1$.

2. Suppose that $\varphi(x_q) \geq 3$ and $\varphi(x_{q+2}) \geq 3$. We have $\varphi(x_{q+1}) \neq 1$, hence the colour 1 is forbidden for x_{q+1} because of an (x_s, x_t)-path with $\varphi(x_s) = \varphi(x_t) = a$ containing x_{q+1} as an internal vertex. Clearly, $\min\{\varphi(x_s), \varphi(x_t)\} \geq 3$ implies $a \geq 3$. Then, however, the colour 2 is forbidden for x_{q+1}, too, a contradiction.

3. The inequality $n \geq 2$ is immediate. Also, we cannot have $\varphi(x_2) \geq 2$, because then $\varphi(x_1) = 1$.

4. Since φ uses each colour from $[1, \max(\mathcal{G}, G, Y)]$ at least once, we have $n \geq 3$. From 3 we know that $\varphi(x_2) = 1$. The assumption $\varphi(x_3) \geq 3$ then would lead to $\varphi(x_1) = 2$.

5,6. The situation is symmetric with that of 3 and 4.

7. Since, clearly, $n \geq 5$ (1 and 2 are used at least once), the reduction process applies and yields $\dot{G} = R(G, Y), \dot{Y} = R(Y, G)$, $\dot{\varphi} = \mathrm{rank}(\mathcal{G}, \dot{G}, \dot{Y})$.

Suppose first that $G = P_n$. From 4 and 6 it follows that $\Pi_{i=q-1}^{q+3}(x_i)$ is a section of G and from 1 we obtain $\varphi(x_{q-1}) = \varphi(x_{q+3}) = 1$. From Lemma 3 we know that $\dot{\varphi}(x_i) = \varphi(x_i) - 1 \geq 2$ for $i = q, q+1, q+2$; then, from 3 and 5 (applied to the ranking $\dot{\varphi}$ of $\dot{G}$) we see that x_q and x_{q+2} are not endvertices of $\dot{G}$, which (since x_{q-1} and x_{q+3} as non-survivors are not in $\dot{G}$) means that $x_{q-2}, x_{q+4} \in V(\dot{G})$ and $S = \Pi_{i=q-2}^{q+4}(x_i)$ is a section of G. Then, from 1 applied to $\dot{\varphi}$, we have $\dot{\varphi}(x_{q-2}) = \dot{\varphi}(x_{q+4}) = 1$, and, by Lemma 3 again, S is a section of G of type $(2, 1, 3+, 3+, 3+, 1, 2)$.

If $G = C_n$, then, by 1, $\Pi_{i=q-1}^{q+3}(x_i)$ is a section of G of type $(1, 3+, 3+, 3+, 1)$, hence $n \geq 6$ (φ as a ranking is a proper vertex colouring of G). If $n \geq 7$, then, as in the case $G = P_n$, we conclude that S is a section of G of type $(2, 1, 3+, 3+, 3+, 1, 2)$. If $n = 6$, $\Pi_{i=q-2}^{q+3}(x_i)$ would be a section of G of type $(2, 1, 3+, 3+, 3+, 1)$. Then, however, $\dot{G} = C_4$ and $\dot{\varphi} = \mathrm{rank}(\mathcal{G}, C_4, \dot{Y})$ uses 1 exactly once in contradiction with the following fact (which can be easily checked out):

(*) For any input sequence $\bar{Y} \in \mathrm{Is}(C_4)$ the ranking $\mathrm{rank}(\mathcal{G}, C_4, \bar{Y})$ uses 1 exactly twice.

8. As in 7, we use the reduction process leading to $\dot{G}, \dot{Y}$ and $\dot{\varphi}$. In the case $G = P_n$, we obtain from 4 and 6 that $\Pi_{i=q-1}^{q+4}(x_i)$ is a section of G. Clearly, because of 7, we have $\varphi(x_{q-1}) \leq 2$. Then, the assumption $q = 2$ would mean $\varphi(x_q) \leq 2$, a contradiction. Thus, $q \geq 3$. Suppose that $\varphi(x_{q-1}) = 2$. If x_q comes in Y before x_{q+1}, then $\varphi(x_q) = 1$, and, if x_{q+1} comes in Y before x_q, then $\varphi(x_{q+1}) \leq 2$, in both cases a contradiction. Thus, $\varphi(x_{q-1}) = 1$; we cannot have $\varphi(x_{q-2}) \geq 3$, because in such a case, by Lemma 3, $(x_{q-2}, x_q, x_{q+1}, x_{q+3})$ would be a section of $\dot{G}$ contradicting 1 (applied to $\dot{\varphi}$). The mentioned contradiction yields $\varphi(x_{q-2}) = 2$. If $\varphi(x_{q+4}) \geq 3$, considering the section $(x_q, x_{q+1}, x_{q+3}, x_{q+4})$ of $\dot{G}$ supplies an analogous contradiction. So, there are two possibilities for $\varphi(x_{q+4})$: If $\varphi(x_{q+4}) = 1$, then $n \geq q + 5$, as $n = q + 4$ would imply $\varphi(x_{q+3}) = 2$, a contradiction; then, by 1 applied to $\dot{\varphi}$,we get $\dot{\varphi}(x_{q+5}) = 1$ and $\varphi(x_{q+5}) = 2$.

The assumption $\varphi(x_{q+4}) = 2$ excludes $n = q + 4$, by 5. Then, by 2, $\varphi(x_{q+5}) \geq 3$ is impossible and $\varphi(x_{q+5}) = 1$, as necessary.

Now, consider the case $G = C_n$. Since φ must use 2, we have $n \geq 5$. However, $n = 5$ is impossible, because then $\dot{\varphi}$ would contradict (*). Thus, $n \geq 6$ and, just as in the case $G = P_n$, we can show that $\varphi(x_{q-1}) = 1$ and $\varphi(x_{q-2}) = 2$. That is why, $n = 6$ is impossible – use again (*) for $\dot{\varphi}$. We cannot have $\varphi(x_{q+4}) \geq 3$ from the same reason as applied for $G = P_n$. Then the assumption $n = 7$ would lead to $\varphi(x_{q+4}) = 1$ (φ is proper) and a contradiction involving once more (*) for $\dot{\varphi}$. Finally, for $n \geq 8$, the reasoning for $G = P_n$ can be repeated, and we are done.

9. Use the symmetry with the situation of 8.

10,11. The proof is immediate.

12. From 4 we see that $q \geq 2$. If $\varphi(x_{q-1}) \geq 2$, from 3 we obtain $q \geq 3$. If $\varphi(x_{q-1}) = 1$, then $q \geq 3$, since $q = 2$ would lead to $\varphi(x_q) = 2$. Thus, $q \geq 3$ in any case, and, because of the symmetry of the type $(3+, 3+)$, we have $n \geq q + 3$, too.

13. The proof is analogous to that of 12.

14. By 5 we have $n \geq q + 3$, so that 1 yields $\varphi(x_{q+3}) = 1$. Now, $n = q + 3$ is impossible – this would mean that $\varphi(x_{q+1}) = 1$. To show that $q \geq 3$, proceed as in 12.

15. Symmetry with 14. ■

For a ranking algorithm $\mathcal{A}$, we will denote by $f_i(\mathcal{A}, G, Y), i \in [1, \infty)$, the number of vertices that are coloured with i by $\text{rank}(\mathcal{A}, G, Y)$.

Lemma 5. *Let $G = C_n$, $n \in [3, \infty)$, or $G = P_n$, $n \in [1, \infty)$, and let $Y \in \text{Is}(G)$. Then the sequence $\{f_i(\mathcal{G}, G, Y)\}_{i=1}^{\infty}$ is non-increasing.*

Proof. We proceed by induction on n. First, it is straightforward to see that $f_1(\mathcal{G}, P_1, Y) = 1$ for (the unique) $Y \in \text{Is}(P_1)$, $f_i(\mathcal{G}, C_3, Y) = 1, i = 1, 2, 3$, for any $Y \in \text{Is}(C_3)$, and $f_1(\mathcal{G}, C_4, Y) = 2$ (in fact, this is (*)), $f_i(\mathcal{G}, C_4, Y) = 1, i = 2, 3$, for any $Y \in \text{Is}(C_4)$.

Now, suppose that $n \geq 5$ (if $G = C_n$) or $n \geq 2$ (if $G = P_n$) and that $\{f_i(\mathcal{G}, G', Y')\}_{i=1}^{\infty}$ is a non-increasing sequence for any graph G' homeomorphic to G with $|V(G')| < n$ and any input sequence $Y' \in \text{Is}(G')$. Let $\varphi = \text{rank}(\mathcal{G}, G, Y)$, $\dot{G} = R(G, Y), \dot{Y} = R(Y, G), \dot{\varphi} = \text{rank}(\mathcal{G}, \dot{G}, \dot{Y})$. From Lemma 3 we know that, for any $i \in [2, \infty)$, we have $f_{i-1}(\mathcal{G}, \dot{G}, \dot{Y}) = f_i(\mathcal{G}, G, Y)$ and, since $|V(\dot{G})| < n$ (there are non-survivors of G with respect to Y, because φ uses 1 at least once), from the induction hypothesis we obtain $f_i(\mathcal{G}, G, Y) = f_{i-1}(\mathcal{G}, \dot{G}, \dot{Y}) \geq f_i(\mathcal{G}, \dot{G}, \dot{Y}) = f_{i+1}(\mathcal{G}, G, Y)$.

Put $V_i = \{x \in V(G) : \varphi(x) = i\}, i = 1, 2$, and consider a mapping $\alpha : V_2 \to V_1$ defined in such a way that $x\alpha(x)$ is an edge of G for any $x \in V_2$. From Lemmas 4.2, 4.3 and 4.5 it follows that α is well defined. Moreover, the definition of a ranking implies that α is an injection; thus, $f_1(\mathcal{G}, G, Y) = |V_1| \geq |V_2| = f_2(\mathcal{G}, G, Y)$, which represents the last wanted inequality. ■

Suppose that $G \in \{C_n, P_n\}$, $n \in [4, \infty)$ and let $\tilde{G}$ be the cycle defined as follows: $\tilde{G} = G$ if $G = C_n$, $\tilde{G} = G + x_n x_1$ if $G = P_n$. The ranking φ of G is then also a vertex colouring of $\tilde{G}$, which, if $G = P_n$, in general is *not* a ranking of $\tilde{G}$ (it may be even not proper). When working with $\tilde{G}$, types of vertices will be always related to this colouring "inherited" from the ranking φ of the "underlying" graph G. With respect to this colouring we define also high and low sections of $\tilde{G}$.

By Lemma 4.1, rotating around $\tilde{G}$ we meet alternately high and low sections; their possible lengths are between 1 and 3 if $G = C_n$, and between 1 and 6 if $G = P_n$ (and in this case, due to Lemmas 4.4 and 4.6, only one section, namely low, obtained by joining two low endsections of P_n, can be of length greater than 3). Let s be the number of high (and low as well) sections of $\tilde{G}$. We will denote those sections $S_i, i \in [1, 2s]$, in such a way that S_1 is that high section of maximum length which contains a vertex x_t with minimum index t. Consider a (high) section S_{2i-1}, $i \in [1, s]$. Starting from it and rotating around $\tilde{G}$ in the sense of the orientation of $\tilde{G}$ given by the growing order of sections indices (modulo 2s) we take all sections until we arrive at the first high section not shorter than S_{2i-1} (maybe S_{2i-1} itself). The section which arises by the concatenation of those sections (in their natural "rotating" order) is called the *closure* of S_{2i-1} and is denoted by $\mathrm{cl}(S_{2i-1})$. Thus, $\mathrm{cl}(S_{2i-1}) = \Pi_{k=2i-1}^{2j} S_k$, where $j \in [i, s]$ is (uniquely) chosen to fulfill the conditions $|S_{2k-1}| < |S_{2i-1}|$ for each $k \in [i+1, j]$ and $|S_{2j+1}| \geq |S_{2i-1}|$ (note that $j \leq s$ because S_1 is the longest high section).

In our example we have $S_1 = (x_4, x_5), \mathrm{cl}(S_1) = S_1 S_2 = (x_4, x_5, x_6, x_7)$, $S_3 = (x_1), \mathrm{cl}(S_3) = S_3 S_4 = (x_1, x_2, x_3)$ (for $G = C_7$) and $S_1 = (x_4, x_5)$, $\mathrm{cl}(S_1) = S_1 S_2 = (x_4, x_5, x_6, x_7, x_1, x_2, x_3)$ (for $G = P_7$).

Lemma 6. *The closure of any high section of $\tilde{G}$ has a nonnegative defect.*

Proof. Let S_{2i-1} be a high section of $\tilde{G}$ and suppose that $\mathrm{cl}(S_{2i-1}) = \Pi_{k=2i-1}^{2j} S_k$.

1. If $|S_{2i-1}| = 1$, then $\mathrm{cl}(S_{2i-1}) = S_{2i-1} S_{2i}$ and $\mathrm{def}(\mathrm{cl}(S_{2i-1})) = |S_{2i}| - 1 \geq 0$.

2. Assume that $|S_{2i-1}| = 2$. Evidently, we have $\mathrm{def}(\mathrm{cl}(S_{2i-1})) = \mathrm{def}(S_{2i-1} S_{2i}) + \sum_{k=i+1}^{j} \mathrm{def}(S_{2k-1} S_{2k})$. Since $2 = |S_{2i-1}| > |S_{2k-1}| = 1$

for each $k \in [i+1, j]$, the sum consists of nonnegative summands $|S_{2k}| - 1$. Thus, we are done if $\mathrm{def}(S_{2i-1}S_{2i}) \geq 0$.

If $\mathrm{def}(S_{2i-1}S_{2i}) = |S_{2i}| - |S_{2i-1}| < 0$, then, necessarily, $|S_{2i}| = 1$. From Lemmas 4.2, 4.3 and 4.5 we then see that S_{2i} is of type (1). Suppose that $S_{2i-1}S_{2i} = \Pi_{k=q}^{q+2}(x_k)$, $q \in [1, n]$, and consider the section $S = \Pi_{k=q}^{q+3}(x_k)$ of $\tilde{G}$ of type $(3+, 3+, 1, 3+)$. If S is also a section of G, then, by Lemma 4.8, S_{2i+1} is of length 1 (so that $j \geq i+1$) and $\mathrm{def}(S_{2i+1}S_{2i+2}) \geq 1$, which implies $\mathrm{def}(\mathrm{cl}(S_{2i-1})) \geq -1 + 1 + \sum_{k=i+2}^{j}(|S_{2k}| - 1) \geq 0$. If S is not a section of G, then $G = P_n$ and $n \in [q, q+2]$. However, $n = q$ is impossible by Lemma 4.4, $n = q+1$ by Lemma 4.5 and $n = q+2$ by Lemma 4.11.

3. Now, let $|S_{2i-1}| = 3$. First we show that, for any $l \in [i, j]$, we have $d_l = \mathrm{def}(\Pi_{k=2i-1}^{2l} S_k) \geq -1$, and, if $d_k = -1$ for every $k \in [i, l]$, then either S_{2l} is of type (1,2) or $S_{2l-1}S_{2l}$ is of type (3+,1). We proceed by induction on l. If $l = i$ and $S_{2i-1} = \Pi_{k=q}^{q+2}(x_k)$ with $q \in [1, n]$, we know that S_{2i-1} is a section of G (otherwise $G = P_n$ and $n \in [q, q+1]$, which contradicts Lemma 4.3 or Lemma 4.5). Thus, we can use Lemma 4.7, from which it follows that $d_i \geq -1$ and $d_i = -1$ only if S_{2i} is of type (1,2).

Suppose that $j > i$ and that our statement is true for some $l \in [i, j-1]$ (so that $|S_{2l+1}| \leq 2$). Since $d_{l+1} = d_l + |S_{2l+2}| - |S_{2l+1}| \geq d_l + 1 - 2 = d_l - 1$, to prove the statement for $l+1$ it is sufficient to analyze the case $d_l = -1$. (If $d_l \geq 0$, then $d_{l+1} \geq -1$ and it is not true that $d_k = -1$ for any $k \in [i, l+1]$.) By the induction hypothesis, we have two possibilities:

a) $S_{2l} = \Pi_{k=q}^{q+1}(x_k)$, where $q \in [1, n]$, is of type (1,2). If $|S_{2l+1}| = 2$, then $\Pi_{k=q}^{q+5}(x_k)$ is the section of the graph G ($G = P_n$ and $n \in [q, q+4]$ would be in contradiction with one of Lemmas 4.3, 4.5 and 4.11) and S_{2l+2} is neither of type (1,1) nor of type (2,2) (this would mean $G = P_n$ and $n = q+4$). Next, by Lemma 4.1, S_{2l+2} cannot be of type (2) or (2,1), and, by Lemma 4.8, of type (1); thus, either $d_{l+1} = d_l = -1$ and S_{2l+2} is of type (1,2) (as necessary) or $d_{l+1} \geq 0$ (and there is nothing more to prove). Let $|S_{2l+1}| = 1$. The only interesting case (in which $d_{l+1} = -1$) is that with $|S_{2l+2}| = 1$. Then, because of Lemma 4.2 or 4.5, S_{2l+2} is not of type (2), and, consequently, $S_{2l+1}S_{2l+2}$ is of type $(3+, 1)$, as needed.

b) $S_{2l-1}S_{2l} = (x_q, x_{q+1})$, where $q \in [1, n]$, is of type (3+,1). If $|S_{2l+1}| = 2$, then $\Pi_{k=q}^{q+3}(x_k)$ is the section of the graph G ($G = P_n$ and $n \in [q, q+2]$ would be in contradiction with one of Lemmas 4.3, 4.6 and 4.10). Then, by Lemma 4.9, $\varphi(x_{q+4}) = 1$ and $\varphi(x_{q+5}) = 2$, so that either $d_{l+1} = -1$ and S_{2l+2} is of type (1,2) or $d_{l+1} = 0$; in both cases we are done. Suppose $|S_{2l+1}| = 1$. It is sufficient to deal with the case $d_{l+1} = -1$, in which

$|S_{2l+2}| = 1$. If $S_{2l+1}S_{2l+2}$ is of type (3+,1), we are done. On the other hand, by Lemmas 4.2 and 4.5, S_{2l+2} cannot be of type (2) and our statement is completely proved.

Now, it is clear that we cannot have $d_k = -1$ for each $k \in [i, j]$, because $|S_{2j+1}| = 3$ and, by Lemma 4.7, the type of S_{2j} ends up with (2,1). Thus, there exists (uniquely determined) $l \in [i, j]$ fulfilling $d_l \geq 0$ and $d_k = -1$ for any $k \in [i, l-1]$. If $l = j$, then $\mathrm{def}(\mathrm{cl}(S_{2i-1})) = d_l \geq 0$. Suppose therefore $l < j$. If $|S_{2k-1}| = 1$ for any $k \in [l+1, j]$, then $\mathrm{def}(\mathrm{cl}(S_{2i-1})) = d_l + \sum_{k=l+1}^{j}(|S_{2k}|-1) \geq 0$. If $|S_{2m-1}| = 2$ for some $m \in [l+1, j]$ and $|S_{2k-1}| = 1$ for any $k \in [l+1, m-1]$, delete from the sequence $\Pi_{k=m}^{j}(2k-1)$ all terms $2k-1$ with $|S_{2k-1}| = 1$ and denote by $\Pi_{k=1}^{q}(p_k)$ the resulting sequence. Then it is easy to see directly from the definitions that $\Pi_{k=2m-1}^{2j} S_k = \Pi_{k=1}^{q}\mathrm{cl}(S_{p_k})$ and, as S_{p_k} is a high section of length 2, by 2 we have $\mathrm{def}(\mathrm{cl}(S_{p_k})) \geq 0$ for each $k \in [1, q]$. That is why, $\mathrm{def}(\mathrm{cl}(S_{2i-1})) = d_l + \sum_{k=l+1}^{m-1}(|S_{2k}|-1) + \sum_{k=1}^{q}\mathrm{def}(\mathrm{cl}(S_{p_k})) \geq 0$. ■

Theorem 7. *Let* $G = C_n$, $n \in [3, \infty)$, *or* $G = P_n$, $n \in [1, \infty)$, *and let* $Y \in \mathrm{Is}(G)$. *Then* $\sum_{i=1}^{2} f_i(\mathcal{G}, G, Y) \geq \lceil n/2 \rceil$ *and* $f_1(\mathcal{G}, G, Y) \geq \lceil \lceil n/2 \rceil /2 \rceil$.

Proof. The assertion is immediate if $n \leq 3$. If $n \in [4, \infty)$, consider the graph $\tilde{G}$ and its high and low sections $S_i, i \in [1, 2s]$, as defined before Lemma 6. Let $\Pi_{i=1}^{m}(l_i)$ be the increasing sequence of indices of all longest high sections of $\tilde{G}$. Then, obviously, the section $\Pi_{i=1}^{m}\mathrm{cl}(S_{l_i})$ contains all vertices of $V(\tilde{G}) = V(G)$, and so, by Lemma 6, $\sum_{i=1}^{2} f_i(\mathcal{G}, G, Y) - \sum_{i=3}^{\infty} f_i(\mathcal{G}, G, Y) = \mathrm{def}(G) = \mathrm{def}(\Pi_{i=1}^{m}\mathrm{cl}(S_{l_i})) = \sum_{i=1}^{m}\mathrm{def}(\mathrm{cl}(S_{l_i})) \geq 0$. Thus, we have $n = \sum_{i=1}^{2} f_i(\mathcal{G}, G, Y) + \sum_{i=3}^{\infty} f_i(\mathcal{G}, G, Y) \leq 2\sum_{i=1}^{2} f_i(\mathcal{G}, G, Y)$ and the first inequality follows. The remaining one comes from Lemma 5, since $2f_1(\mathcal{G}, G, Y) \geq \sum_{i=1}^{2} f_i(\mathcal{G}, G, Y) \geq \lceil n/2 \rceil$. ■

Proposition 8. *If* $k \in [1, \infty)$ *and* $l \in [3, \infty)$, *there exist* $q \in [1, \infty)$ *and* $r \in [3, \infty)$ *such that* $\max(\mathcal{G}, P_q) = k$ *and* $\max(\mathcal{G}, C_r) = l$.

Proof. Suppose that there is no $q \in [1, \infty)$ such that $\max(\mathcal{G}, P_q) = k$. Since, evidently, $\max(\mathcal{G}, P_n) = n, n = 1, 2$, we have $k \geq 3$. The sequence $\{\chi_{\mathrm{r}}(P_n)\}_{n=1}^{\infty} = \{\lfloor \log_2 n \rfloor + 1\}_{n=1}^{\infty}$ is unbounded and $\max(\mathcal{G}, P_n) \geq \chi_{\mathrm{r}}^*(P_n) \geq \chi_{\mathrm{r}}(P_n)$, hence there exists $q \in [1, \infty)$ such that $\max(\mathcal{G}, P_q) \geq k+1$; without loss of generality, we may suppose that q is minimum with this property, i.e., $\max(\mathcal{G}, P_n) \leq k-1$ for any $n \in [1, q-1]$. Consider such an input sequence $Y \in \mathrm{Is}(P_q)$ that $\max(\mathcal{G}, P_q, Y) = \max(\mathcal{G}, P_q)$. Clearly, $q \geq k+1 \geq 4$, so we may use our reduction process yielding $\dot{G} = R(P_q, Y)$, $\dot{Y} = R(Y, P_q)$. We have $|V(\dot{G})| < q$, which implies $\max(\mathcal{G}, \dot{G}) \leq k-1$. On the other hand, by Lemma 3, the maximum number used by $\dot{\varphi}$ is by 1 smaller than that used by φ, i.e., $\max(\mathcal{G}, \dot{G}, \dot{Y}) = \max(\mathcal{G}, P_q, Y) - 1 = \max(\mathcal{G}, P_q) - 1 \geq (k+1) - 1 = k$, hence $\max(\mathcal{G}, \dot{G}) \geq \max(\mathcal{G}, \dot{G}, \dot{Y}) \geq k$, a contradiction.

For cycles we proceed analogously using the fact that $\max(\mathcal{G}, C_3) = 3$ and that the reduction process applies if the number of vertices of C_n is at least 5. Note that also the sequence $\{\chi_{\mathrm{r}}(C_n)\}_{n=1}^{\infty}$ is unbounded, because P_{n-1} is an induced subgraph of C_n, and so (as can be easily seen) $\chi_{\mathrm{r}}(P_{n-1}) \leq \chi_{\mathrm{r}}(C_n)$ for any $n \in [3, \infty)$. ■

From Proposition 8 we conclude that the numbers

$$f(k) := \min\{n \in [1, \infty) : \max(\mathcal{G}, P_n) = k\}, \quad k \in [1, \infty),$$

$$g(k) := \min\{n \in [3, \infty) : \max(\mathcal{G}, C_n) = k\}, \quad k \in [3, \infty)$$

($f(k)$ was introduced in [5]) are correctly defined. It is easily seen that $f(k) = k$ for $k = 1, 2, 3$ and $g(3) = 3$. Clearly, from Lemma 3 it follows that $f(k) \neq f(l)$ and $g(k) \neq g(l)$ for $k \neq l$. However, we can say more:

Proposition 9. *The sequences* $\{f(k)\}_{k=1}^{\infty}$ *and* $\{g(k)\}_{k=3}^{\infty}$ *are increasing.*

Proof. In the case of paths use simply Proposition 1 and the fact that P_m is an induced subgraph of P_n if $m < n$.

For cycles suppose that $\{h(k)\}_{k=3}^{\infty}$ is the increasing sequence created by rearranging $\{g(k)\}_{k=3}^{\infty}$, that $\{h(k)\} \neq \{g(k)\}$ and that k is the minimum index with $h(k) \neq g(k)$. Since $g(3) = h(3) = 3$, we have $k \geq 4$ and $h(k) = g(l) < g(k)$ with $k < l$. For $n = g(l)$ take an input sequence $Y \in \mathrm{Is}(C_n)$ fulfilling $\max(\mathcal{G}, C_n, Y) = l$. As $l \geq 5$, $\dot{G} = R(C_n, Y)$ and $\dot{Y} = R(Y, C_n)$ are well defined. Then, by Lemma 3, $\max(\mathcal{G}, \dot{G}, \dot{Y}) = \max(\mathcal{G}, C_n, Y) - 1 = l - 1 \geq k$ and, since $|V(\dot{G})| < |V(C_n)| = g(l)$, we have $g(l-1) \leq |V(\dot{G})| < g(l) < g(k)$ and $l - 1 > k$. Now, $g(l-1) > g(k-1)$ is in contradiction with $h(k) = g(l)$ and $g(l-1) < g(k-1)$ contradicts the minimality of k. ■

Corollary 10. *For any* $k, n \in [1, \infty)$ *it holds* $\max(\mathcal{G}, P_n) = k$ *if and only if* $n \in [f(k), f(k+1) - 1]$.

Proof. A consequence of Propositions 1 and 9. ■

For cycles the situation is unclear, but we conjecture that, analogously, for any $k, n \in [3, \infty)$, $\max(\mathcal{G}, C_n) = k$ if and only if $n \in [g(k), g(k+1) - 1]$.

Theorem 7 has an important consequence:

Theorem 11. *Let $k \in [1, \infty), l \in [3, \infty), q \in [2, \infty)$ and $r \in [7, \infty)$.*

1. *If $f(k) \geq q$, then $f(k + 2i) \geq q \cdot 2^i$ for any $i \in [0, \infty)$.*
2. *If $g(k) \geq r$, then $g(k + 2i) \geq r \cdot 2^i$ for any $i \in [0, \infty)$.*

Proof. 1. We proceed by induction on i. For $i = 0$ there is nothing to prove, so we suppose that $i \in [1, \infty)$ and $f(k + 2i - 2) \geq q \cdot 2^{i-1}$. With respect to Proposition 9 it is sufficient to show that $\max(\mathcal{G}, P_n, Y) \leq k + 2i - 1$ for any $n \in [q \cdot 2^{i-1} + 2, q \cdot 2^i - 1]$ and any $Y \in \mathrm{Is}(P_n)$. Since $n \geq q \cdot 2^{i-1} + 2 \geq q + 2 \geq 4$, the reduction process applied to P_n and Y yields $\dot{G} = R(P_n, Y)$ and $\dot{Y} = R(Y, P_n)$. The ranking $\mathrm{rank}(\mathcal{G}, P_n, Y)$ is a proper vertex colouring of P_n, hence $f_1(\mathcal{G}, P_n, Y) \leq \lceil n/2 \rceil$, $|V(\dot{G})| = n - f_1(\mathcal{G}, P_n, Y) \geq n - \lceil n/2 \rceil = \lfloor n/2 \rfloor \geq 2$, so that the reduction process applied to $\dot{G}$ and $\dot{Y}$ leads to $\ddot{G} = R(\dot{G}, \dot{Y})$ and $\ddot{Y} = R(\dot{Y}, \dot{G})$. By a repeated use of Lemma 3 we see that $|V(\ddot{G})| = n - \sum_{i=1}^{2} f_i(\mathcal{G}, P_n, Y)$, hence, by Theorem 7, $|V(\ddot{G})| \leq n - \lceil n/2 \rceil = \lfloor n/2 \rfloor \leq q \cdot 2^{i-1} - 1$, and, by the induction hypothesis, $\max(\mathcal{G}, \ddot{G}, \ddot{Y}) \leq \max(\mathcal{G}, \ddot{G}) \leq k + 2i - 3$. Using Lemma 3 twice then $\max(\mathcal{G}, P_n, Y) = \max(\mathcal{G}, \dot{G}, \dot{Y}) + 1 = \max(\mathcal{G}, \ddot{G}, \ddot{Y}) + 2 \leq k + 2i - 1$, as needed.

2. We proceed as in 1 and use the fact that $f_1(\mathcal{G}, C_n, Y) \leq \lfloor n/2 \rfloor$, so that $|V(R(C_n, Y))| \geq n - \lfloor n/2 \rfloor = \lceil n/2 \rceil \geq 5$ for any $n \in [r \cdot 2^{i-1} + 2, r \cdot 2^i - 1], i \in [1, \infty)$ and any $Y \in \mathrm{Is}(C_n)$, which enables us to use the reduction process twice, as above. ∎

3 INSERTION

Now we are going to show that, in some extent, our reduction process can be inverted. Let $\mathcal{A}_{m,n}, n \in [1, \infty), m \in [0, n]$, be the set of all non-empty increasing sequences of integers from $[m, n]$.

We will analyze in detail the case $G = P_n$. For $A = \Pi_{i=1}^{l}(a_i) \in \mathcal{A}_{0,n}$ we denote by $I(P_n, A)$ the path with $n + l$ vertices constructed as follows: Add to $V(P_n) = \{x_i : i \in [1, n]\}$ l new vertices (called *newcomers*) z_i, $i \in [1, l]$. If $i \in [1, l]$ is such that $a_i \in [1, n - 1]$, the newcomer z_i is inserted between vertices x_{a_i} and x_{a_i+1} (i.e., the edge $x_{a_i} x_{a_i+1}$ is deleted and edges $x_{a_i} z_i$ and $z_i x_{a_i+1}$ are added). If $a_1 = 0$, the newcomer z_1 is a new endvertex – the edge $z_1 x_1$ is added. Similarly, if $a_l = n$, the newcomer z_l is a new endvertex – the edge $x_n z_l$ is added. Note that the set of newcomers is an independent set of vertices of $I(P_n, A)$. An input sequence $Y \in \mathrm{Is}(P_n)$ for the path P_n yields in a natural way an input sequence $I(P_n, A, Y) = [\Pi_{i=1}^{l}(z_i)]Y$ for the path $I(P_n, A)$ – newcomers are coming first (z_i comes as i-th, $i \in [1, l]$) and then vertices of P_n arrive in the order given by Y. Consider the ranking $\varphi = \operatorname{rank}(\mathcal{G}, P_n, Y)$. An internal vertex x_i of P_n, $i \in [2, n-1]$, is *Y-good*, if it comes in Y as the last from among x_{i-1}, x_i, x_{i+1}, and $\varphi(x_{i-1}) = \varphi(x_{i+1})$. A sequence $A \in \mathcal{A}_{0,n}$ is *Y-proper*, if any vertex of P_n, that is not Y-good, has in $I(P_n, A)$ at least one newcomer as a neighbour.

For example, if Y is the input sequence $(x_3, x_2, x_5, x_6, x_4, x_1) \in \operatorname{Is}(P_6)$, there is only one Y-good vertex in P_6, namely x_4 – we have $\operatorname{rank}(\mathcal{G}, P_6, Y) = \{(x_3, 1), (x_2, 2), (x_5, 1), (x_6, 2), (x_4, 3), (x_1, 1)\}$ (x_2 is not Y-good, because it comes in Y before x_1). Thus, the sequence $A = (1, 2, 5) \in \mathcal{A}_{0,6}$ is Y-proper – vertices x_i, $i \in [1, 6] - \{5\}$, that are not Y-good, are "dominated" by newcomers of the graph $I(P_6, A) = P_9$ (its vertices are successively $x_1, z_1, x_2, z_2, x_3, x_4, x_5, z_3, x_6$). The input sequence $I(P_6, A, Y)$ is $(z_1, z_2, z_3, x_3, x_2, x_5, x_6, x_4, x_1)$.

Lemma 12. *Let $n \in [1, \infty), Y \in \operatorname{Is}(P_n)$, let a sequence $A \in \mathcal{A}_{0,n}$ be Y-proper and let $\varphi = \operatorname{rank}(\mathcal{G}, P_n, Y)$, $\hat{G} = I(P_n, A)$, $\hat{Y} = I(P_n, A, Y), \hat{\varphi} = \operatorname{rank}(\mathcal{G}, \hat{G}, \hat{Y})$. Then $\hat{\varphi}(z_i) = 1$ for any newcomer z_i, $i \in [1, |A|]$, and $\hat{\varphi}(x_i) = \varphi(x_i) + 1$ for any $i \in [1, n]$.*

Proof. Newcomers of the graph $\hat{G}$ are attributed 1 by $\hat{\varphi}$ because they form an independent set of vertices in $\hat{G}$ and they are coming at the beginning of $\hat{Y}$, before all remaining vertices of $\hat{G}$.

Let us prove by induction on i that $\hat{\varphi}(y_i) = \varphi(y_i) + 1$ for every $i \in [1, n]$. The vertex y_1, clearly, is not Y-good, hence it has at least one newcomer as a neighbour and $\hat{\varphi}(y_1) = 2 = \varphi(y_1) + 1$.

Suppose that $i \in [2, n]$ and that $\hat{\varphi}(y_j) = \varphi(y_j) + 1$ for any $j \in [1, i-1]$. Vertices y_j, y_k with $j, k \in [1, i], j \neq k$, are joined by a path $\hat{P}$ in $\hat{G}(\hat{Y}, y_i)$ if and only if they are joined in $G(Y, y_i)$ by the path P with $V(P) = V(\hat{P}) - \{z_l : l \in [1, |A|]\}$. Since $\hat{\varphi}(z_l) = 1$ for any $l \in [1, |A|]$, using the induction hypothesis we see that a colour $a \in [2, \infty)$ is forbidden for y_i in $\hat{G}(\hat{Y}, y_i)$ because of a path $\hat{P}$ if and only if the colour $a-1$ is forbidden for y_i in $G(Y, y_i)$ because of the corresponding path P. Moreover, the colour 1 is forbidden for y_i in $\hat{G}(\hat{Y}, y_i)$, too – either a neighbour of y_i is a newcomer (and so is coloured with 1 in $\hat{G}(\hat{Y}, y_i)$) or both neighbours of y_i are coloured in $\hat{G}(\hat{Y}, y_i)$ and they received the same colour. This means that $\varphi(y_i) = \hat{\varphi}(y_i) - 1$ and we are done. ∎

In our illustrative example with $n = 6$ we have $\hat{\varphi} = \operatorname{rank}(\mathcal{G}, P_9, I(P_6, A, Y)) = \{(z_1, 1), (z_2, 1), (z_3, 1), (x_3, 2), (x_2, 3), (x_5, 2), (x_6, 3), (x_4, 4), (x_1, 2)\}$.

Put $e_l := 3 \cdot 2^{l-1} - 1$ and $o_l := 2^{l+1} - 1$, $l \in [1, \infty)$.

Theorem 13. *For any* $l \in [1, \infty)$ *there exists*

1. *an input sequence* $Y_{2l} \in \mathrm{Is}(P_{e_l})$ *such that* $\max(\mathcal{G}, P_{e_l}, Y_{2l}) = 2l$ *and the set of* Y_{2l}*-good vertices of the path* P_{e_l} *is* $\{x_{3i} : i \in [1, 2^{l-1} - 1]\}$*;*
2. *an input sequence* $Y_{2l+1} \in \mathrm{Is}(P_{o_l})$ *such that* $\max(\mathcal{G}, P_{o_l}, Y_{2l+1}) = 2l + 1$ *and the set of* Y_{2l+1}*-good vertices of the path* P_{o_l} *is* $\{x_{4i} : i \in [1, 2^{l-1} - 1]\}$.

Proof. Evidently, for $l = 1$ any input sequence $Y_2 \in \mathrm{Is}(P_2)$ has all the properties required by 1 (no vertex of P_2 is Y_2-good). We are going to show that for any $l \in [1, \infty)$ the existence of Y_{2l} implies that of Y_{2l+1} and the existence of Y_{2l+1} implies that of Y_{2l+2}. So, suppose that there is an input sequence $Y_{2l} \in \mathrm{Is}(P_{e_l})$ with properties given by 1. The sequence $A_{2l} := \Pi_{i=1}^{2^{l-1}}(3i - 2) \in \mathcal{A}_{0,e_l}$ is Y_{2l}-proper – note that vertices of P_{e_l}, that are not Y_{2l}-good, are in pairs x_{3i-2}, x_{3i-1}, and an "old" edge $x_{3i-2}x_{3i-1}$ is subdivided by the newcomer z_i, $i \in [1, 2^{l-1}]$. The graph $I(P_{e_l}, A_{2l})$ is a path with $e_l + 2^{l-1} = o_l$ vertices and, if we define $Y_{2l+1} := I(P_{e_l}, A_{2l}, Y_{2l})$, then, by Lemma 12, $\max(\mathcal{G}, P_{o_l}, Y_{2l+1}) = \max(\mathcal{G}, P_{e_l}, Y_{2l}) + 1 = 2l + 1$. Moreover, any Y_{2l}-good vertex $x_{3i}, i \in [1, 2^{l-1} - 1]$, is Y_{2l+1}-good. There are no other Y_{2l+1}-good vertices, because, by Lemma 12, any vertex of the path P_{e_l}, that is Y_{2l+1}-good and not Y_{2l}-good, must have two newcomers as neighbours (and the distance between any two newcomers in $I(P_{o_l}, A_{2l})$ is at least 3). Now, if we rename vertices of $I(P_{e_l}, A_{2l}) = P_{o_l}$ in our ordinary way (i.e., they will be x_i, $i \in [1, o_l]$), then x_{3i} becomes x_{4i}, $i \in [1, 2^{l-1} - 1]$, and the set of Y_{2l+1}-good vertices of P_{o_l} is $\{x_{4i} : i \in [1, 2^{l-1} - 1]\}$.

The sequence $A_{2l+1} := \Pi_{i=1}^{2^{l-1}}(4i - 3, 4i - 2) \in \mathcal{A}_{0,o_l}$ is Y_{2l+1}-proper, because vertices of P_{o_l}, that are not Y_{2l}-good, occur in triples $x_{4i-3}, x_{4i-2}, x_{4i-1}$, which are "dominated" by newcomers z_{2i-1} and z_{2i}, $i \in [1, 2^{l-1}]$. The graph $I(P_{o_l}, A_{2l+1})$ is a path with $o_l + 2 \cdot 2^{l-1} = e_{l+1}$ vertices and, for $Y_{2l+2} := I(P_{o_l}, A_{2l+1}, Y_{2l+1})$, we have, by Lemma 12, $\max(\mathcal{G}, P_{e_{l+1}}, Y_{2l+2}) = \max(\mathcal{G}, P_{o_l}, Y_{2l+1}) + 1 = 2l + 2$. Any Y_{2l+1}-good vertex x_{4i}, $i \in [1, 2^{l-1}]$, is Y_{2l+2}-good. Moreover, the vertex x_{4i-2}, $i \in [1, 2^{l-1}]$, is Y_{2l+2}-good, too (it has two newcomers as neighbours). There are no other Y_{2l+2}-good vertices, because there are no more pairs of newcomers which are at the distance 2 apart. Thus, after renaming vertices of $I(P_{o_l}, A_{2l+1}) = P_{e_{l+1}}$ in our ordinary way (so that x_{4i} becomes x_{6i}, $i \in [1, 2^{l-1} - 1]$, and x_{4i-2} becomes x_{6i-3}, $i \in [1, 2^{l-1}]$), the set of Y_{2l+2}-good vertices of $P_{e_{l+1}}$ is $\{x_{3i} : i \in [1, 2^l - 1]\}$. ■

Corollary 14. *For any* $l \in [1, \infty)$, $f(2l) \le e_l$ *and* $f(2l + 1) \le o_l$. ■

Evidently, the reduction process can also be (partially) inverted for cycles. In this case the sequence $A = \Pi_{i=1}^{l}(a_i)$, characterizing positions of newcomers, is from the set $\mathcal{A}_{1,n}$ (if the original cycle is C_n), a newcomer z_i subdivides the edge $x_{a_i}x_{a_i+1}$, $i \in [1,l]$, and there is no restriction on index of a Y-good vertex. (Recall that, for paths, endvertices are not Y-good.) Thus, an analogue of Lemma 12 is presented without proof (no new idea is necessary).

Lemma 15. *Let* $n \in [3,\infty)$, $Y \in \mathrm{Is}(C_n)$, *let a sequence* $A \in \mathcal{A}_{1,n}$ *be* Y*-proper and let* $\varphi = \mathrm{rank}(\mathcal{G}, C_n, Y)$, $\hat{G} = I(C_n, A)$, $\hat{Y} = I(C_n, A, Y)$, $\hat{\varphi} = \mathrm{rank}(\mathcal{G}, \hat{G}, \hat{Y})$. *Then* $\hat{\varphi}(z_i) = 1$ *for any newcomer* z_i, $i \in [1, |A|]$ *and* $\hat{\varphi}(x_i) = \varphi(x_i) + 1$ *for any* $i \in [1,n]$. ■

4 Main Results

Now we are able to analyze First Fit Algorithm for cycles and paths in a detailed way.

Proposition 16. $g(4) \le 5$, $g(5) \le 7$, $g(6) \le 10$ *and* $g(7) \le 15$.

Proof. It is easy to check that the sequences $\hat{A}_3 = (1,2)$, $\hat{A}_4 = (1,4)$, $\hat{A}_5 = (2,5,7)$ and $\hat{A}_6 = (1,3,5,7,9)$ are such that $\hat{A}_n$ is $\hat{Y}_n$-proper, $n \in [3,6]$, if the graph $\hat{G}_n$ and the input sequence $\hat{Y}_n$ for $\hat{G}_n$, $n \in [3,7]$, are defined by the following recurrence: $\hat{G}_3 := C_3$, $\hat{Y}_3 := (x_1,x_2,x_3)$ and $\hat{G}_{n+1} := I(\hat{G}_n, \hat{A}_n)$, $\hat{Y}_{n+1} := I(\hat{G}_n, \hat{A}_n, \hat{Y}_n)$, $n \in [3,6]$. Since $\max(\mathcal{G}, \hat{G}_3, \hat{Y}_3) = 3$, $\hat{G}_4 = C_5$, $\hat{G}_5 = C_7$, $\hat{G}_6 = C_{10}$, $\hat{G}_7 = C_{15}$ and, by Lemma 15, $\max(\mathcal{G}, \hat{G}_{n+1}, \hat{Y}_{n+1}) = \max(\mathcal{G}, \hat{G}_n, \hat{Y}_n) + 1$ for $n \in [3,6]$, the proof follows. ■

Proposition 17. *If* $k \in [3,\infty)$, *then*

1. $f(k+1) \ge \min\{n \in [f(k)+1,\infty) : n - \lceil \lceil n/2 \rceil /2 \rceil \ge f(k)\}$;
2. $g(k+1) \ge \min\{n \in [g(k)+1,\infty) : n - \lceil \lceil n/2 \rceil /2 \rceil \ge g(k)\}$.

Proof. 1. Suppose that $f(k+1) = n$; by Proposition 9 then $n \ge f(k)+1$. Take an input sequence $Y \in \mathrm{Is}(P_n)$ such that $\max(\mathcal{G}, P_n, Y) = k+1$ and put $\dot{G} = R(P_n, Y)$, $\dot{Y} = R(Y, P_n)$. For the path $\dot{G}$ we have, by Theorem 7, $|V(\dot{G})| = n - f_1(\mathcal{G}, P_n, Y) \le n - \lceil \lceil n/2 \rceil /2 \rceil$, and, by Lemma 3, $\max(\mathcal{G}, \dot{G}, \dot{Y}) = \max(\mathcal{G}, P_n, Y) - 1 = k$. Since $|V(\dot{G})| < n = f(k+1)$, due to Proposition 9 we obtain $\max(\mathcal{G}, \dot{G}) = \max(\mathcal{G}, \dot{G}, \dot{Y}) = k$. Thus, $|V(\dot{G})| \ge f(k)$ and we see that $n - \lceil \lceil n/2 \rceil /2 \rceil \ge f(k)$.

2. The proof is completely analogous to that of 1. ■

Theorem 18. *$f(4) = g(4) = 5, f(5) = g(5) = 7, f(6) = 11, g(6) = 10, f(7) = 15$ and $14 \leq g(7) \leq 15$.*

Proof. Take $k \in [4, 7]$. The upper bounds for $f(k)$ come from Corollary 14 and those for $g(k)$ from Proposition 16. On the other hand, by Theorem 1 and Lemma 7 of [5], $f(4) \geq 5$ and $g(4) \geq 5$, so that $f(4) = g(4) = 5$. Now, by Proposition 17, $f(5) \geq 7$ and $g(5) \geq 7$, which implies $f(5) = g(5) = 7$. By Proposition 17 again, we get $f(6) \geq 10$ and $g(6) \geq 10$, yielding $g(6) = 10$ and, consequently, $g(7) \geq 14$.

Suppose that there is an input sequence $Y \in \text{Is}(P_{10})$ such that $\max(\mathcal{G}, P_{10}, Y) = 6$ and put $\varphi = \text{rank}(\mathcal{G}, P_{10}, Y)$. Since $f(4) = 5$, from Lemma 3 (used twice) we see that $\sum_{i=1}^{2} f_i(\mathcal{G}, P_{10}, Y) \leq 5$. So, with help of Theorem 7, $\sum_{i=1}^{2} f_i(\mathcal{G}, P_{10}, Y) = \sum_{i=3}^{6} f_i(\mathcal{G}, P_{10}, Y) = 5$, and, by Lemma 5, $f_1(\mathcal{G}, P_{10}, Y) = 3$, $f_2(\mathcal{G}, P_{10}, Y) = 2$. Consider the cycle $\tilde{P}_{10} = C_{10}$ introduced before Lemma 6 and its high and low sections. First we show that there is no high section of $\tilde{P}_{10}$ of length 3. Suppose there is one; by Lemmas 4.4 and 4.6, this section $\Pi_{i=q}^{q+2}(x_i)$ must also be a section of P_{10}. Then, by Lemma 4.7, $\Pi_{i=q-2}^{q+4}(x_i)$ is a section of P_{10} of type (2,1,3+,3+,3+,1,2). The remaining three vertices of P_{10} do not form a section of P_{10}, because two of them are high (otherwise we would obtain a contradiction with one of Lemmas 4.4, 4.6, 4.10 and 4.11). Thus, they form two nonempty endsections of P_{10}. That containing only one vertex cannot be of type (3+) (P_{10} would have an endsection of type $(3+, 2)$ or $(2, 3+)$ in contradiction with Lemmas 4.4 and 4.6), hence that of length 2 is of type (3+,3+), which contradicts again Lemmas 4.4 and 4.6.

Denote the number of low sections of P_{10} and $\tilde{P}_{10}$ by l and $\tilde{l}$, respectively. Clearly, $\tilde{l} \geq 3$, since for $\tilde{l} = 2$ one of two high sections of $\tilde{P}_{10}$ would be of length 3. By Lemmas 4.2, 4.3 and 4.5, any low section of P_{10} contains a vertex coloured with 1, hence $l \leq 3$. On the other hand, $\tilde{l} \leq l$, and we get $l = \tilde{l} = 3$. Thus, $\tilde{P}_{10}$ has two low sections of type (1,2) or (2,1), one low section of type (1), two high sections of length 2 and one high section of length 1.

A high section of $\tilde{P}_{10}$ of length 2 must be a section of P_{10}, too – otherwise, by Lemmas 4.4 and 4.6, $\Pi_{i=1}^{3}(x_i)$ is of type (3+,1,2) and $\Pi_{i=8}^{10}(x_i)$ is of type (2,1,3+), so that $\Pi_{i=4}^{7}(x_i)$ is of type (3+,3+,1,3+) or (3+,1,3+,3+), which contradicts Lemma 4.8 or Lemma 4.9. Thus, two high sections of P_{10} of length 2 are, by Lemmas 4.8 and 4.9, separated by a low section of P_{10} of length 2; let $\Pi_{i=q}^{q+5}(x_i)$ be the corresponding section of P_{10} with $\min\{\varphi(x_i) : i \in \{q, q+1, q+4, q+5\}\} \geq 3$. Then $q = 1$ is impossible by

Lemma 4.4, $q = 2$ by Lemmas 4.3 and 4.10 and, symmetrically, $q = 4$ by Lemmas 4.5 and 4.11, $q = 5$ by Lemma 4.6. If $q = 3$, one endvertex of P_{10} is high, which contradicts Lemma 4.4 or Lemma 4.6.

So, we conclude that $f(6) = 11$, and then Proposition 17 yields $f(7) = 15$. ■

Corollary 19. *For* $n = 5, 6$, $\chi_r^*(C_n) = \chi_r^*(P_n) = 4$.

Proof. Those on-line ranking numbers must be at least 4, by Theorem 1 of [5]. On the other hand, due to Theorem 18, $\max(\mathcal{G}, C_n) = \max(\mathcal{G}, P_n) = 4$. ■

Note that, by Theorem 1 of [5], it holds $\chi_r^*(C_4) = \chi_r^*(P_4) = 3$. The value of on-line ranking number for simplest cycles and paths (with at most three vertices) is evidently equal to the corresponding number of vertices.

For an input sequence $Y = \Pi_{i=1}^{n}(y_i) \in \mathrm{Is}(C_n)$ and $j \in [0, n-1]$ let Y^{+j} be the input sequence for the graph C_n defined by $Y^{+j} := \Pi_{i=1}^{n}(y_{i+j})$.

Lemma 20. *If* $n \in [3, \infty)$, $j \in [0, n-1]$ *and* $Y \in \mathrm{Is}(C_n)$, *then* $\max(\mathcal{G}, C_n, Y^{+j}) = \max(\mathcal{G}, C_n, Y)$.

Proof. Evidently, $V(C_n(Y^{+j}, x_i)) = \{x_{k+j} : x_k \in V(C_n(Y, x_i))\}$ for any $i \in [1, n]$. If $i \in [1, n]$ and $x_k \in V(C_n(Y, x_i))$, the ranking $\mathrm{rank}(\mathcal{G}, C_n, Y^{+j}, x_{i+j})$ attributes to the vertex x_{k+j} the same colour as the ranking $\mathrm{rank}(\mathcal{G}, C_n, Y, x_i)$ does to the vertex x_k, hence the proof follows. ■

Proposition 21. *If* $n \in [2, \infty)$, *then* $\max(\mathcal{G}, P_n) \le \max(\mathcal{G}, C_{n+1}) \le \max(\mathcal{G}, P_n) + 1$.

Proof. The first inequality comes from Proposition 1, because P_n is an induced subgraph of C_{n+1}.

Take an input sequence $Y = \Pi_{i=1}^{n+1}(y_i) \in \mathrm{Is}(C_{n+1})$ such that $\max(\mathcal{G}, C_{n+1}, Y) = \max(\mathcal{G}, C_{n+1})$. Since $C_{n+1}(Y, y_n)$ is a path with n vertices, with respect to Lemma 20 we may suppose that $V(C_{n+1}(Y, y_n)) = \{x_i : i \in [1, n]\}$. Then, for the input sequence $Y^- = \Pi_{i=1}^{n}(y_i) \in \mathrm{Is}(P_n)$, we have $\mathrm{rank}(\mathcal{G}, P_n, Y^-) = \mathrm{rank}(\mathcal{G}, C_{n+1}, Y, y_n)$. That is why, $\max(\mathcal{G}, P_n, Y^-) \ge \max(\mathcal{G}, C_{n+1}, Y) - 1 = \max(\mathcal{G}, C_{n+1}) - 1$ (the arrival of y_{n+1}, the last vertex of Y, can increase the number of used colours only by 1) and $\max(\mathcal{G}, C_{n+1}) \le \max(\mathcal{G}, P_n, Y^-) + 1 \le \max(\mathcal{G}, P_n) + 1$. ■

Corollary 22. *If* $k \in [3, \infty)$, *then* $g(k) \le f(k) + 1$.

Proof. Suppose that $f(k) = n$. As $n \geq k \geq 3$, Proposition 21 implies $\max(\mathcal{G}, C_{n+1}) \geq \max(\mathcal{G}, P_n) = k$, and so, by Proposition 9, $g(k) \leq n + 1 = f(k) + 1$. ■

Theorem 23. *Let i be a nonnegative integer. Then*

1. $11 \cdot 2^i \leq f(2i + 6) \leq 12 \cdot 2^i - 1$.
2. $15 \cdot 2^i \leq f(2i + 7) \leq 16 \cdot 2^i - 1$.
3. $10 \cdot 2^i \leq g(2i + 6) \leq 12 \cdot 2^i$.
4. $14 \cdot 2^i \leq g(2i + 7) \leq 16 \cdot 2^i$.

Proof. Lower bounds come from Theorems 11 and 18. The upper bounds in 1 and 2 follow from Corollary 14, and then those in 3 and 4 from Corollary 22. ■

Theorem 24. *Let i be a nonnegative integer.*

1. *If* $n \in [12 \cdot 2^i - 1, 15 \cdot 2^i - 1]$, *then* $\max(\mathcal{G}, P_n) = 2i + 6$.
2. *If* $n \in [15 \cdot 2^i, 16 \cdot 2^i - 2]$, *then* $2i + 6 \leq \max(\mathcal{G}, P_n) \leq 2i + 7$.
3. *If* $n \in [16 \cdot 2^i - 1, 22 \cdot 2^i - 1]$, *then* $\max(\mathcal{G}, P_n) = 2i + 7$.
4. *If* $n \in [22 \cdot 2^i, 24 \cdot 2^i - 2]$, *then* $2i + 7 \leq \max(\mathcal{G}, P_n) \leq 2i + 8$.
5. *If* $n \in [12 \cdot 2^i, 14 \cdot 2^i - 1]$, *then* $\max(\mathcal{G}, C_n) = 2i + 6$.
6. *If* $n \in [14 \cdot 2^i, 16 \cdot 2^i - 1]$, *then* $2i + 6 \leq \max(\mathcal{G}, C_n) \leq 2i + 7$.
7. *If* $n \in [16 \cdot 2^i, 20 \cdot 2^i - 1]$, *then* $\max(\mathcal{G}, C_n) = 2i + 7$.
8. *If* $n \in [20 \cdot 2^i, 24 \cdot 2^i - 1]$, *then* $2i + 7 \leq \max(\mathcal{G}, C_n) \leq 2i + 8$.

Proof. Because of Proposition 1, the statements 1–4 follow from Theorems 23.1 and 23.2.

If $n \in [12 \cdot 2^i, \infty)$, then $\max(\mathcal{G}, C_n) \geq 2i + 6$, since otherwise, by Proposition 21, $\max(\mathcal{G}, P_{n-1}) \leq \max(\mathcal{G}, C_n) \leq 2i + 5$, which contradicts Theorem 23.1 (with respect to Proposition 1). Thus, 5 and 6 follow from Theorems 23.3 and 23.4. The remaining two statements are proved analogously. ■

Theorem 25. *Let i be a nonnegative integer.*

1. *If* $n \in [12 \cdot 2^i - 1, 15 \cdot 2^i - 1]$, *then* $\chi_r^*(P_n) \leq 2\lfloor \log_2 n \rfloor$.
2. *If* $n \in [15 \cdot 2^i, 16 \cdot 2^i - 1]$, *then* $\chi_r^*(P_n) \leq 2\lfloor \log_2 n \rfloor + 1$.
3. *If* $n \in [16 \cdot 2^i, 22 \cdot 2^i - 1]$, *then* $\chi_r^*(P_n) \leq 2\lfloor \log_2 n \rfloor - 1$.
4. *If* $n \in [22 \cdot 2^i, 24 \cdot 2^i - 2]$, *then* $\chi_r^*(P_n) \leq 2\lfloor \log_2 n \rfloor$.

5. *If* $n \in [12 \cdot 2^i, 14 \cdot 2^i - 1]$*, then* $\chi_r^*(C_n) \le 2\lfloor \log_2 n \rfloor$.
6. *If* $n \in [14 \cdot 2^i, 16 \cdot 2^i - 1]$*, then* $\chi_r^*(C_n) \le 2\lfloor \log_2 n \rfloor + 1$.
7. *If* $n \in [16 \cdot 2^i, 20 \cdot 2^i - 1]$*, then* $\chi_r^*(C_n) \le 2\lfloor \log_2 n \rfloor - 1$.
8. *If* $n \in [20 \cdot 2^i, 24 \cdot 2^i - 1]$*, then* $\chi_r^*(C_n) \le 2\lfloor \log_2 n \rfloor$.

Proof. If $n \in [12 \cdot 2^i - 1, 15 \cdot 2^i - 1]$, then $\lfloor \log_2 n \rfloor = i + 3$, and, by Theorem 24.1, $\chi_r^*(P_n) \le \max(\mathcal{G}, P_n) = 2i + 6 = 2\lfloor \log_2 n \rfloor$, which represents 1. The remaining assertions follow from Theorem 24, too. ■

Theorem 26. *For any* $n \in [3, \infty)$*,* $\chi_r(C_n) = \lfloor \log_2(n-1) \rfloor + 2$.

Proof. First we show that $\chi_r(C_n) \ge 1 + \chi_r(P_{n-1})$. Suppose, on the contrary, that $\chi_r(C_n) = l \le \chi_r(P_{n-1})$, and consider an l-ranking φ of C_n. If x is the (only) vertex of C_n coloured with l, then $\varphi - \{(x, l)\}$ is an $(l-1)$-ranking of the path $P_{n-1} = C_n - x$, and so $\chi_r(P_{n-1}) \le l - 1$, a contradiction. Thus, according to [1], we have $\chi_r(C_n) \ge 1 + \lfloor \log_2(n-1) \rfloor + 1 = \lfloor \log_2(n-1) \rfloor + 2$.

Now, take $k \in [1, \infty)$, $m \in [1, 2^k - 1]$ and $n = 2^k + m$. From Lemma 2.1 of [1] it is easy to see that $\chi_r(P_{2^k}) = k + 1$ and $\chi_r(P_m) = \lfloor \log_2 m \rfloor + 1 = l(m) \le k$. Let φ_1 be a $(k+1)$-ranking of P_{2^k} with $V(P_{2^k}) = \{x_i : i \in [1, 2^k]\}$ and endvertices x_1, x_{2^k}, and let φ_2 be an $l(m)$-ranking of P_m with $V(P_m) = \{u_i : i \in [1, m]\}$, with endvertices u_1, u_m and with $V(P_{2^k}) \cap V(P_m) = \emptyset$. Without loss of generality, by Proposition 2.1 of [1], we may suppose that $\varphi_1(x_1) = k + 1$. Let C_{2^k+m} be the cycle formed from $P_{2^k} \cup P_m$ by adding the edges $x_1 u_m$ and $x_{2^k} u_1$. The colouring φ of C_{2^k+m} defined by $\varphi(x_i) := \varphi_1(x_i), i \in [1, 2^k]$, $\varphi(u_1) = k+2$ and $\varphi(u_i) = \varphi_2(u_i), i \in [2, m]$, is easily seen to be a $(k+2)$-ranking. Thus, $\chi_r(C_n) \le k + 2 = \lfloor \log_2(n-1) \rfloor + 2$.

For $k \in [1, \infty)$ let φ' be such a $(k+2)$-ranking of $P_{2^{k+1}}$ that the (unique) appearance of the colour $k + 2$ is at an endvertex of $P_{2^{k+1}}$. Then, φ' is also a $(k+2)$-ranking of the cycle $C_{2^{k+1}}$, which is created from $P_{2^{k+1}}$ by joining its endvertices by a new edge, and, for $n = 2^k + 2^k = 2^{k+1}$, we have $\chi_r(C_n) \le k + 2 = \lfloor \log_2(n-1) \rfloor + 2$.

So, $\chi_r(C_n) \le \lfloor \log_2(n-1) \rfloor + 2$ for any $n \in [2^k + 1, 2^{k+1}]$ and any $k \in [1, \infty)$, and the desired result follows. ■

Theorem 27.

1. *For any* $n \in [1, \infty)$*,* $\lfloor \log_2 n \rfloor + 1 \le \chi_r^*(P_n) \le 2\lfloor \log_2 n \rfloor + 1$.
2. *For any* $n \in [3, \infty)$*,* $\lfloor \log_2(n-1) \rfloor + 2 \le \chi_r^*(C_n) \le 2\lfloor \log_2 n \rfloor + 1$.

Proof. Lower bounds come from the values of $\chi_r(P_n)$ and $\chi_r(C_n)$ due to [1] and Theorem 26.

As concerns upper bounds, for $n \in [12, \infty)$ see Theorem 25; for $n \leq 11$ use Theorem 18 and the fact that $f(i) = i, i = 1, 2, 3$, and $g(3) = 3$. ■

First Fit Algorithm is not necessarily optimal when computing $\chi_r^*(P_n)$, as shows our next statement.

Theorem 28. $\chi_r^*(P_7) = 4 < 5 = \max(\mathcal{G}, P_7)$.

Proof. According to Theorem 1 of [5], we have $\chi_r^*(P_7) \geq 4$. Consider the ranking algorithm $\mathcal{G}'$ functioning just as $\mathcal{G}$ does with the only exception: If $G = P_5$, $H = 2K_2$, $\{x\} = V(G) - V(H)$ and φ is a ranking of H such that both neighbours of x (in G) are coloured with 2, then $\mathcal{G}'(G, H, \varphi, x) = 4$ (and not 3, as required by $\mathcal{G}$). We are going to show that $m' = \max(\mathcal{G}', P_7, Y) \leq 4$ for any $Y \in \mathrm{Is}(P_7)$.

First suppose that $Y = \Pi_{i=1}^7(y_i)$ is such that $\varphi' = \mathrm{rank}(\mathcal{G}', P_7, Y) \neq \mathrm{rank}(\mathcal{G}, P_7, Y) = \varphi$. Then $P_7(Y, y_5) = P_5$ and it is easy to see that any neighbour of (a vertex of) $P_7(Y, y_5)$ is coloured with 3 and any non-neighbour (at most one) of $P_7(Y, y_5)$ is coloured with 1 by φ'; thus, $m' = 4$.

Now, assume that $\varphi' = \varphi$. If $y_7 \in \{x_3, x_4, x_5\}$, then $P_7(Y, y_6) = P_i \cup P_{6-i}, i \in \{2, 3\}$. Clearly, the maximum colour used by $\varphi'_6 = \mathrm{rank}(\mathcal{G}', P_7, Y, y_6)$ is not greater than $\max\{\max(\mathcal{G}, P_i), \max(\mathcal{G}, P_{6-i})\}$; this maximum is equal to 3, by Proposition 1 and $f(3) = 3$, $f(4) = 5$ (Theorem 18), hence $m' \leq 4$.

If $y_7 \in \{x_1, x_2\}$, we may suppose that φ'_6 uses colour 4 – otherwise we are done.

If $y_7 = x_2$, then $P_7(Y, y_6) = P_1 \cup P_5$ and 4 is used by φ'_6 for a vertex of P_5-component of $P_7(Y, y_6)$. If one of x_3, x_4 is coloured with a colour ≥ 3, then, using Lemma 4.3, $\varphi'(x_2) = 2$. On the other hand, $\{\varphi'_6(x_3), \varphi'_6(x_4)\} \neq \{1, 2\}$, because otherwise at least two vertices from among x_5, x_6, x_7 would be coloured with a colour ≥ 3 (3 is used at least once by φ'_6) in contradiction with one of Lemmas 4.2, 4.7, 4.12 and 4.13.

If $y_7 = x_1$, then $P_7(Y, y_6) = P_6$. We may assume that $\varphi'_6(x_2) = 1$ and $\varphi'_6(x_3) = 2$, since if not, we would have $\varphi'(x_1) \leq 2$. Because of Lemmas 4.1, 4.7, 4.8 and 4.9, exactly two vertices from among x_4, x_5, x_6, x_7 are coloured with a colour ≥ 3. From Lemmas 4.2, 4.12, 4.13 and 4.15 it follows that these are x_4 and x_7. If $\varphi'_6(x_4) = 4$, then $\varphi'(x_1) = 3$. Finally, suppose that $\varphi'_6(x_4) = 3$ and $\varphi'_6(x_7) = 4$. Then $\varphi'_6(x_6) = 1$ and $\varphi'_6(x_5) = 2$ (by Lemma 4.6), x_4 comes in Y before x_7 (otherwise $\varphi'_6(x_7) \leq 3$), x_4 comes in Y after each of $x_i, i \in \{2, 3, 5, 6\}$ (otherwise $\varphi'_6(x_4) = 1$), which means that $P_7(Y, y_4) = 2K_2$ and that the vertex $y_5 = x_4$ has in $P_7(Y, y_5)$ both neighbours coloured with 2. This, however, is a contradiction, because in such a case $4 = \varphi'(y_5) = \varphi'_6(y_5)$.

The last possibility, $y_7 \in \{x_6, x_7\}$, leads to a situation which is symmetric with that of $y_7 \in \{x_1, x_2\}$.

Now, to conclude the proof, we use Theorem 18, from which it follows that $\max(\mathcal{G}, P_7) = 5$. ∎

Theorem 29. $\chi_{\mathrm{r}}^*(C_7) = 5$.

Proof. By Theorem 1 of [5], we have $\chi_{\mathrm{r}}^*(C_7) \geq 4$. We are going to show by the way of contradiction, that $\chi_{\mathrm{r}}^*(C_7) \geq 5$; this, together with $\max(\mathcal{G}, C_7) = 5$ (Theorem 18), will mean that $\chi_{\mathrm{r}}^*(C_7) = 5$.

We know from Theorem 26 that $\chi_{\mathrm{r}}(C_7) = 4$. Let φ be a 4-ranking of C_7. It can be immediately seen that φ uses 3 and 4 exactly once, say, for vertices x_i and x_j. Since $\chi_{\mathrm{r}}(P_4) = 3 = \chi_{\mathrm{r}}(P_5)$, no component of $H = C_7 - \{x_i, x_j\}$ can have more than 3 vertices, so that $H = P_2 \cup P_3$. Clearly, φ restricted to P_3-component of H uses 2 just once, for the internal vertex of that P_3. Also, φ restricted to P_2-component of H, uses 2 once. Thus, φ colours two vertices of C_7 with 2 and two vertices with a colour ≥ 3; the mutual distance of vertices in those two pairs is 3.

Now, suppose that there is a ranking algorithm $\mathcal{A}$ such that $\max(\mathcal{A}, C_7) = 4$. Consider an input sequence $Y = \Pi_{i=1}^7(y_i) \in \mathrm{Is}(P_7)$ and the ranking $\varphi = \mathrm{rank}(\mathcal{A}, C_7, Y)$. As $\chi_{\mathrm{r}}(C_7) = 4$, φ is a 4-ranking of C_7. If $C_7(Y, y_2) = P_2$, the ranking $\mathrm{rank}(\mathcal{A}, C_7, Y, y_2)$ must use colours 1 and 2. To see this suppose that a colour $i \in \{3, 4\}$ is used for a vertex y_j of $C_7(Y, y_2)$. Assume, moreover, that $C_7(Y, y_k) = P_2 \cup P_{k-2}, k = 3, 4, 5$ (we cannot avoid this situation). We have $\varphi(y_k) \neq i, k = 3, 4, 5$, hence it may happen that $\varphi(y_k) = 7 - i$ for some $k \in [3, 5]$ and an endvertex y_k of $C_7(Y, y_k)$ – if $\{\varphi(y_3), \varphi(y_4)\} = \{1, 2\}$, y_5 may be an endvertex of $C_7(Y, y_5)$ with the neighbour coloured with 1. Then, however, the distance between y_j and y_k, the vertices coloured with 3 and 4, may be 2 in contradiction with the structure of a 4-ranking of C_7.

If $C_7(Y, y_2) = P_2$, $C_7(Y, y_3) = P_3$ and the neighbour of y_3 in $C_7(Y, y_3)$ is coloured with 1, we have $\varphi(y_3) = i \in \{3, 4\}$. It may happen that $C_7(Y, y_5) = P_3 \cup P_2$. For vertices of P_2-component of $C_7(Y, y_5)$ two from among colours 1,2 and $7 - i$ are used. If 2 is used, it may happen that there are two vertices coloured with 2 by φ, whose distance is 2, a contradiction. On the other hand, the presence of $7 - i$ could yield two vertices of distance 2, coloured with 3 and 4 by φ, a contradiction again. ∎

5 Open Problems

There are several open problems which naturally arise from our analysis.

1. Find nontrivial lower bounds for $\chi_{\rm r}^*(C_n)$ and $\chi_{\rm r}^*(P_n)$.
2. Which is the minimum n such that $\chi_{\rm r}^*(P_n) = 5$?
3. Does there exist $n \in [8, \infty)$ such that $\chi_{\rm r}^*(C_n) < \max(\mathcal{G}, C_n)$? If so, which is the minimum n in such an inequality?
4. Determine $g(7)$. (We conjecture that $g(7) = 15$.)
5. Prove or disprove that the sequence $\{\max(\mathcal{G}, C_n)\}_{n=3}^{\infty}$ is non-decreasing.

Acknowledgements

The authors are indebted to an anonymous referee whose hints helped to shorten proofs of Lemmas 3 and 12.

A support of the Slovak VEGA grant 1/4377/97 for the work of the second author is gratefully acknowledged.

References

[1] M. Katchalski, W. McCuaig and S. Seager, *Ordered colourings*, Discrete Math. **142** (1995) 141–154.

[2] C.E. Leiserson, *Area-efficient graph layouts (for VLSI)*, in: Proc. 21st Annu. IEEE Symp. on Foundations of Computer Science (1980) 270–281.

[3] J.W.H. Liu, *The role of elimination trees in sparse factorization*, SIAM J. Matrix Analysis and Appl. **11** (1990) 134–172.

[4] D.C. Llewelyn, C. Tovey and M. Trick, *Local optimization on graphs*, Discrete Appl. Math. **23** (1989) 157–178.

[5] I. Schiermeyer, Zs. Tuza and M. Voigt, *On-line rankings of graphs*, submitted.

Received 22 February 1999
Revised 29 October 1999