MINIMAL REDUCIBLE BOUNDS FOR HOM-PROPERTIES OF GRAPHS

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Abstract

Let $H$ be a fixed finite graph and let $\rightarrow H$ be a hom-property, i.e. the set of all graphs admitting a homomorphism into $H$. We extend the definition of $\rightarrow H$ to include certain infinite graphs $H$ and then describe the minimal reducible bounds for $\rightarrow H$ in the lattice of additive hereditary properties and in the lattice of hereditary properties.

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1. Definitions

In general we follow the notation and terminology of [1]. Denote by $\mathcal{I}$ the set of all finite undirected simple graphs. Any isomorphism-closed subset $\mathcal{P}$ of $\mathcal{I}$ is called a property of graphs. A property $\mathcal{P}$ is hereditary if whenever a graph $G$ is in $\mathcal{P}$, then all subgraphs of $G$ are also in $\mathcal{P}$. A property $\mathcal{P}$ is additive if whenever graphs $G$ and $H$ are in $\mathcal{P}$, then their disjoint union, denoted by $G \cup H$, is in $\mathcal{P}$ too. When partially ordered under set inclusion, the poset of all additive hereditary properties forms a complete distributive lattice, which we will denote by $\mathbb{L}^a$. We use $\mathbb{L}$ to denote the lattice of hereditary properties. A property is called non-trivial if it contains at least one non-null graph and it is not equal to $\mathcal{I}$.

Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be any properties of graphs. A vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition of a graph $G$ is a partition $(V_1, V_2, \ldots, V_n)$ of $V(G)$ such that for
each $i = 1, 2, \ldots, n$, the induced subgraph $G[V_i]$ has the property $P_i$. Any of
the $V_i$ may be empty. The property $P_1 \circ P_2 \circ \ldots \circ P_n$ is defined as the set of
all graphs having a vertex $(P_1, P_2, \ldots, P_n)$-partition. If $P_1, P_2, \ldots, P_n$ are
all (additive) hereditary properties, then $P_1 \circ P_2 \circ \ldots \circ P_n$ is an (additive) hereditary property too. For convenience, we will write $P_1 \circ P_2 \circ \ldots \circ P_n$ as $\mathcal{P} \circ \mathcal{P}_2 \ldots \mathcal{P}_n$, omitting the binary operation symbol.

An additive hereditary property $\mathcal{R}$ is called reducible in $\mathbb{L}^a$ if there exist non-trivial properties $\mathcal{P}$ and $\mathcal{Q}$ in $\mathbb{L}^a$ such that $\mathcal{R} = \mathcal{PQ}$. Otherwise $\mathcal{R}$ is called irreducible. A reducible property $\mathcal{R} \in \mathbb{L}^a$ is called a minimal reducible bound for property $\mathcal{P} \in \mathbb{L}^a$ if $\mathcal{P} \subseteq \mathcal{R}$ and there is no reducible property $\mathcal{R}_1$ satisfying $\mathcal{P} \subseteq \mathcal{R}_1 \subseteq \mathcal{R}$. From this definition, each reducible property is the unique minimal reducible bound for itself. We use the symbol $\mathbf{B}(\mathcal{P})$ to denote the class of all minimal reducible bounds for property $\mathcal{P}$. We do not know whether a minimal reducible bound exists for every property $\mathcal{P}$, and $\mathbf{B}(\mathcal{P})$ is known for only a few properties $\mathcal{P}$. Similar definitions hold in $\mathbb{L}$.

Given any $\mathcal{P} \in \mathbb{L}^a$ (or in $\mathbb{L}$), we define the class of all $\mathcal{P}$-maximal graphs by $\mathbf{M}(\mathcal{P}) = \{G \in \mathcal{P} : G + e \not\in \mathcal{P}$ for any $e \in E(G)\}$. $\mathbf{M}(\mathcal{P})$ determines $\mathcal{P}$ in the sense that $H \in \mathcal{P}$ iff there exists some $\mathcal{P}$-maximal graph $G$ such that $H \subseteq G$.

A homomorphism from a graph $G$ to a graph $H$ is a mapping $f$ of the vertex set $V(G)$ to the vertex set $V(H)$ which preserves edges, i.e. if $\{u, v\} \in E(G)$, then $\{f(u), f(v)\} \in E(H)$. We say that $G$ is homomorphic to $H$ if there exists a homomorphism from $G$ to $H$, and we write $G \rightarrow H$. If $G \rightarrow H$, then $\chi(G) \leq \chi(H)$. If $H$ is a finite graph, then the hom-property generated by $H$ is the set $\rightarrow H = \{G \in \mathcal{I} : G \rightarrow H\}$. Note that $\rightarrow H$ is an additive hereditary property for any $H \in \mathcal{I}$.

In Section 2 we summarise some fundamental properties of hom-properties. In Section 3 we extend the definition of hom-properties to include $\rightarrow H$ where $H$ may be an infinite union of finite graphs. We then describe $\mathbf{B}(\rightarrow H)$ in the lattice $\mathbb{L}^a$ in Section 4 and consider some applications of these results in Section 5. Section 6 describes $\mathbf{B}(\rightarrow H)$ in the lattice $\mathbb{L}$.

2. Fundamental Properties of Hom-Properties

Given a graph $G$, a core of $G$ is any subgraph $G'$ of $G$ such that $G \rightarrow G'$, and such that $G$ is not homomorphic to any proper subgraph of $G'$. Every graph $G$ has a unique core up to isomorphism (see [2]) which is denoted by $C(G)$. If $G = C(G)$, i.e. if $G$ is not homomorphic to any of its proper subgraphs, then we call $G$ a core. Since any graph homomorphic to $G$ is
also homomorphic to $C(G)$, and any element of $\rightarrow C(G)$ is in $\rightarrow G$, we have that $\rightarrow G = \rightarrow C(G)$. Hence, given any hom-property, we can assume it is of the form $\rightarrow H$ where $H$ is a core.

The $(\rightarrow H)$-maximal graphs are known and described in [4]:

Given any $G \in \mathcal{I}$, with $V(G) = \{v_1, v_2, ..., v_n\}$, its multiplications $G^i$ are defined as follows:

1. $V(G^i) = W_1 \cup W_2 \cup ... \cup W_n$,
2. for each $1 \leq i \leq n$, $|W_i| \geq 1$,
3. for any pair $1 \leq i < j \leq n$, $W_i \cap W_j = \emptyset$,
4. The only edges of $G^i$ are all the edges of the form $\{u, v\}$ where $u \in W_i, v \in W_j$ and $\{v_i, v_j\} \in E(G)$.

Thus each vertex $v_i$ of $G$ is replaced by a non-empty set of vertices $W_i$ (also denoted by $w_i^i$) and if $u \in W_i, v \in W_j$, then $u$ and $v$ are adjacent in $G^i$ iff $v_i$ and $v_j$ are adjacent in $G$. $W_1, W_2, ..., W_n$ are independent sets called the multivertices of $G^i$. We also write $G^i$ as $G^i(W_1, W_2, ..., W_n)$ to emphasize its structure, and $G^i(k)$ for $G^i(W_1, W_2, ..., W_n)$ if $|W_i| = k$ for each $i = 1, 2, ..., n$. By mapping all the vertices in $W_i$ to $v_i$ for each $i = 1, 2, ..., n$, it is readily seen that $G^i \rightarrow G$, i.e. $G^i \in \rightarrow G$ and that $C(G^i) = G$ if $G$ is a core.

Kratochvil, Mihók and Semanišin proved in [4] that every $(\rightarrow H)$-maximal graph is a multiplication of a subgraph of $H$ that is itself a core. Thus for every $(\rightarrow H)$-maximal graph $G$, there exists an integer $k \geq 1$ such that $G$ is contained in $H^i(k)$.

The following lemma describes properties of hom-properties that will be used often in what follows. We use the notation $H + G$ for the join of two graphs $H$ and $G$, i.e. for the graph obtained from $H \cup G$ by adding all edges joining vertices of $H$ to vertices of $G$. A graph that is the join of two non-null graphs is called decomposable, while a graph that is not decomposable is called indecomposable.

**Lemma 1.**

1. $\rightarrow K_1$ is the set of all edgeless graphs, also denoted by $\mathcal{O}$.

   We have $\rightarrow K_1 = \rightarrow H$ for any edgeless graph $H$, since $C(H) = K_1$.

2. $\rightarrow K_2$ is the set of all bipartite graphs and $\rightarrow K_2 = \rightarrow H$ for any graph $H$ with chromatic number 2, since $C(H) = K_2$.

3. For any graphs $H$ and $G$, $\rightarrow (H + G) = (\rightarrow H)(\rightarrow G)$ (see [3]).

4. $\rightarrow H$ is irreducible in $\mathbb{L}^\alpha$ iff $H$ is indecomposable (see [3]).

5. For any graphs $H$ and $G$, $\rightarrow H \subseteq \rightarrow G$ iff $H \rightarrow G$ iff $H \in \rightarrow G$ (see [2]).
3. The Hom-Property $\rightarrow H$ for Infinite $H$

Although each hom-property is an additive hereditary property and is thus an element of the complete lattice $\mathbb{L}^{a}$, the hom-properties do not form a complete sublattice of $\mathbb{L}^{a}$. For example $\forall \{ \rightarrow R : R \text{ is a triangle-free core} \} \neq \rightarrow H$ for some graph $H$, then $\rightarrow R \subseteq \rightarrow H$ for each triangle-free core $R$. This would imply that $\chi(R) \leq \chi(H)$ for each triangle-free core $R$, which is not true, since triangle-free graphs of arbitrarily high chromatic number can be constructed.

To enable the supremum and infimum (intersection) of an arbitrary set of hom-properties to again be a hom-property, we extend the definition of hom-properties by including $\rightarrow H$, where $H$ is any union of finite graphs. For such a graph $H$ we define $\rightarrow H$ by $\rightarrow H = \{ G \in \mathcal{I} : G \rightarrow H \}$, i.e. $\rightarrow H$ is the set of all finite graphs admitting a homomorphism into $H$. Since the set of all finite graphs is countable, and since only one copy of each connected component of $H$ is sufficient, we can always assume that $H$ is a countable union of finite cores and that these cores are pairwise non-isomorphic. Unlike in the case where $H$ is finite, $H$ itself need no longer have a core e.g. $K_1 \cup K_2 \cup K_3 \cup \ldots$ has no core, and $H$ need not have a finite chromatic number.

Extending the definition of hom-properties to allow $\rightarrow H$ where $H$ is either finite or a countable union of finite graphs makes the hom-properties a complete sublattice of $\mathbb{L}^{a}$, i.e. the supremum and infimum of any set of hom-properties is again a hom-property, as the following two results show.

**Theorem 2.** Let $\{ H_\alpha : \alpha \in A \}$ be a set of graphs, each of which is finite or a countable union of finite graphs. Then $\forall \{ \rightarrow H_\alpha : \alpha \in A \} = \rightarrow (\cup \{ H_\alpha : \alpha \in A \})$.

**Proof.** In the lattice $\mathbb{L}^{a}$, $\forall \{ \rightarrow H_\alpha : \alpha \in A \}$ is the least additive hereditary property which contains each $\rightarrow H_\alpha$, $\alpha \in A$. We show that $\rightarrow (\cup \{ H_\alpha : \alpha \in A \})$ satisfies this.

Clearly, if $G \in \rightarrow H_\alpha$ for any $\alpha \in A$, then $G \in \rightarrow (\cup \{ H_\alpha : \alpha \in A \})$. Therefore $\rightarrow H_\alpha \subseteq \rightarrow (\cup \{ H_\alpha : \alpha \in A \})$ for each $\alpha \in A$.

Now suppose that $\rightarrow H_\alpha \subseteq \mathcal{P}$ for each $\alpha \in A$, for some property $\mathcal{P} \in \mathbb{L}^{a}$. We show that $\rightarrow (\cup \{ H_\alpha : \alpha \in A \}) \subseteq \mathcal{P}$: Let $G \in \rightarrow (\cup \{ H_\alpha : \alpha \in A \})$. By definition, $G$ is finite, and hence there is a homomorphism from $G$ to a finite union of $H_\alpha$’s, say $G \in \rightarrow H_1 \cup H_2 \cup \ldots \cup H_n$. Since each connected component
of $G$ is homomorphically mapped to exactly one $H_i$, $G$ has a decomposition $G = G_1 \cup G_2 \cup ... \cup G_n$, such that $G_i \rightarrow H_i$, for $i = 1, 2, ..., n$. But then we have $G_i \in \mathcal{H} \in \mathcal{P}$ for $i = 1, 2, ..., n$. As each $G_i$ is in $\mathcal{P}$, by the additivity of $\mathcal{P}$, $G$ is in $\mathcal{P}$ too.

**Theorem 3.** Let $\{H_\alpha : \alpha \in A\}$ be a set of graphs, each of which is finite or a countable union of finite graphs. Then $\{\rightarrow H_\alpha : \alpha \in A\} = \{\cup \{R : R$ is a core contained in a multiplication of a finite subgraph of $H_\alpha$ for each $\alpha \in A\}\}$.

**Proof.** Suppose $G \in \cap \{\rightarrow H_\alpha : \alpha \in A\}$. Then $G \rightarrow C(G)$ and $C(G) \in \cap \{\rightarrow H_\alpha : \alpha \in A\}$. Then for each $\alpha \in A, C(G) \in \rightarrow H_\alpha$ and so $C(G)$ is contained in a multiplication of a finite subgraph of $H_\alpha$. So we have $G \in \rightarrow C(G) \subseteq \{\cup \{R : R is a core contained in a multiplication of a finite subgraph of $H_\alpha$ for each $\alpha \in A\}\}$.

Conversely, suppose $G \in \cap \{\rightarrow \cup \{R : R$ is a core contained in a multiplication of a finite subgraph of $H_\alpha$ for each $\alpha \in A\}\}$. Then there exists a homomorphism $f : G \rightarrow \cup \{R : R is a core contained in a multiplication of a finite subgraph of $H_\alpha$ for each $\alpha \in A\}$). Consider any connected component $K$ of $G$. It is mapped by $f$ to one of these cores, say $R$. By the definition of $R$, $R \in \cap \{\rightarrow H_\alpha : \alpha \in A\}$ and so $K \in \rightarrow R \subseteq \cap \{\rightarrow H_\alpha : \alpha \in A\}$. But then $\cap \{\rightarrow H_\alpha : \alpha \in A\}$ is an additive property containing each connected component of $G$ and we conclude that $G$ itself is in $\cap \{\rightarrow H_\alpha : \alpha \in A\}$.

4. **Minimal Reducible Bounds for $\rightarrow H$ in $\mathbb{L}^a$**

In this section we describe the set of all minimal reducible bounds for $\rightarrow H$ in the lattice $\mathbb{L}^a$, first dealing with the case where $H$ is finite, and then with the infinite case. The following lemma and its corollary are useful for both cases.

**Lemma 4.** Let $H$ be a finite core or a countable union of finite cores. If $\mathcal{P}$ and $\mathcal{Q}$ are non-trivial properties in $\mathbb{L}$ with $\mathcal{O} \subseteq \mathcal{P}$ and $\mathcal{O} \subseteq \mathcal{Q}$ such that $\rightarrow H \subseteq \mathcal{P} \mathcal{Q}$ then there exists a partition $(V_1, V_2)$ of $V(H)$ with $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$ such that $\rightarrow H \subseteq (\rightarrow H[V_1])(\rightarrow H[V_2]) \subseteq \mathcal{P} \mathcal{Q}$ and $\rightarrow H[V_1] \subseteq \mathcal{P}$ and $\rightarrow H[V_2] \subseteq \mathcal{Q}$.

**Proof.** First suppose that $H$ is finite and let $V(H) = \{v_1, v_2, ..., v_n\}$. We will show that there exists a partition $(V_1, V_2)$ of $V(H)$ with $V_1 \neq \emptyset$ and
\( V_2 \neq \emptyset \) such that \( H[V_1]^i(k) \in \mathcal{P} \) for all \( k \geq 1 \) and \( H[V_2]^i(k) \in \mathcal{Q} \) for all \( k \geq 1 \). Then all maximal elements of \( \rightarrow H[V_1] \) are in \( \mathcal{P} \) and so \( \rightarrow H[V_1] \subseteq \mathcal{P} \), and similarly \( \rightarrow H[V_2] \subseteq \mathcal{Q} \).

Fix \( k \geq 1 \). Since \( H^i(2k - 1) \in \rightarrow H \subseteq \mathcal{P}\mathcal{Q} \), \( H^i(2k - 1) \) has a \((\mathcal{P},\mathcal{Q})\)-partition. For each \( i = 1, 2, ..., n \), \( v_i^i(2k - 1) \) has at least \( k \) vertices in the \( \mathcal{P} \) part or at least \( k \) vertices in the \( \mathcal{Q} \) part. By deleting \( k - 1 \) vertices from each \( v_i^i(2k - 1) \), we can ensure that the remaining \( v_i^i(k) \) is completely in the \( \mathcal{P} \) part or completely in the \( \mathcal{Q} \) part. We can also ensure that neither the \( \mathcal{P} \) nor the \( \mathcal{Q} \) part is empty: One of the \( v_i^i(k) \) can be moved to the empty part if necessary.

We now have disjoint sets \( I_1 \) and \( I_2 \) such that \( I_1 \cup I_2 = \{1, 2, ..., n\} \) and \( \langle \{v : v \in v_i^i(k), i \in I_1\}, \{v : v \in v_i^i(k), i \in I_2\} \rangle \) forms a \((\mathcal{P},\mathcal{Q})\)-partition of \( H^i(k) \).

Since \( \mathcal{P} \) and \( \mathcal{Q} \) are hereditary properties, each such pair \( (I_1, I_2) \) induces a \((\mathcal{P},\mathcal{Q})\)-partition of \( H^i(r) \) for each \( r \leq k \), with each \( v_i^i(r) \) entirely in the \( \mathcal{P} \) part or entirely in the \( \mathcal{Q} \) part. Since there are only finitely many partitions \( (I_1, I_2) \) of \( \{1, 2, ..., n\} \), there exists a pair \( (I_1^*, I_2^*) \) which serves for infinitely many values of \( k \), and hence for every value of \( k \). Let \( V_1 = \{v_i \in V(H) : i \in I_1^*\} \) and \( V_2 = \{v_i \in V(H) : i \in I_2^*\} \). Then \( H[V_1]^i(k) \in \mathcal{P} \) for all \( k \geq 1 \) and \( H[V_2]^i(k) \in \mathcal{Q} \) for all \( k \geq 1 \).

Suppose now that \( H \) is a countable union of finite graphs, \( H = H_1 \cup H_2 \cup ... \). Denote by \( G_n \) the graph \( H_1 \cup H_2 \cup ... \cup H_n \), \( n \geq 1 \), and let \( \mathcal{G} \) be the set of all \( G_n \) i.e. \( \mathcal{G} = \{ G_n : n \geq 1 \} \).

For each \( n \geq 1 \), \( \rightarrow G_n \subseteq \mathcal{P}\mathcal{Q} \) and so by the finite case above, there exists a partition \( (W_1^n, W_2^n) \) of \( V(G_n) \) with neither part empty such that \( \rightarrow G_n[W_1^n] \subseteq \mathcal{P} \) and \( \rightarrow G_n[W_2^n] \subseteq \mathcal{Q} \). Restricted to \( V(H_1) \), each \( (W_1^n, W_2^n) \) induces a partition of \( V(H_1) \) such that \( \rightarrow H_1[W_1^n] \subseteq \mathcal{P} \) and \( \rightarrow H_1[W_2^n] \subseteq \mathcal{Q} \). Since \( V(H_1) \) has only finitely many partitions, there exists a partition of \( V(H_1) \) with these properties induced by infinitely many \( (W_1^n, W_2^n) \). Call this partition \( (V_1^1, V_2^1) \) and note that \( \rightarrow H_1[V_1^1] \subseteq \mathcal{P} \) and \( \rightarrow H_1[V_2^1] \subseteq \mathcal{Q} \).

Now delete from \( \mathcal{G} \) all those \( G_n \) whose corresponding \( (W_1^n, W_2^n) \) do not induce \( (V_1^1, V_2^1) \) and call the resulting set \( \mathcal{G}' \). Suppose that \( i \geq 2 \) is the least integer such that \( G_i \) is in \( \mathcal{G}' \). For each \( n \geq i \) for which \( G_n \in \mathcal{G}' \), the partition \( (W_1^n, W_2^n) \) of \( V(G_n) \) restricted to \( V(G_i) \) induces a partition of \( V(G_i) \). Since \( V(G_i) \) has only finitely many partitions, there exists a partition of \( V(G_i) \) induced by infinitely many \( (W_1^n, W_2^n) \). This partition of \( V(G_i) \) induces \( (V_1^i, V_2^i) \) in \( V(H_1) \). Label the partitions induced by this partition of \( V(G_i) \) in \( V(H_2), V(H_3), ..., V(H_t) \) by \( (V_1^2, V_2^2)(V_1^3, V_2^3), ..., (V_1^t, V_2^t) \), respectively. For each \( k = 1, 2, ..., i \) we have \( \rightarrow H_k[V_1^k] \subseteq \mathcal{P} \) and \( \rightarrow H_k[V_2^k] \subseteq \mathcal{Q} \).
We now repeat the procedure: delete from $G_0$ all those $G_n$ whose corresponding $(W_1^n, W_2^n)$ do not induce $(V_1^1, V_2^1), (V_1^2, V_2^2), \ldots, (V_1^i, V_2^i)$ and call the resulting set $G''$. If $j \geq i+1$ is the least integer such that $G_j \in G''$, choose a partition of $V(G_j)$ that is induced by infinitely many of the $(W_1^n, W_2^n)$ which satisfy $G_n \in G''$, etc.

Following this procedure, we obtain for each $n \geq 1$ a partition $(V_1^n, V_2^n)$ of $V(H_n)$ which satisfies $\rightarrow H_n[V_1^n] \subseteq P$ and $\rightarrow H_n[V_2^n] \subseteq Q$. With $V_1 = \bigcup V_1^n$ and $V_2 = \bigcup V_2^n$, we have a partition of $V(H)$. If either $V_1$ or $V_2$ is empty, move an arbitrary vertex into this set. By the construction of $V_1$ and $V_2$, $\rightarrow H[V_1] \subseteq P$ and $\rightarrow H[V_2] \subseteq Q$.

**Corollary 5.** Let $H$ be a finite core or a countable union of finite cores. If $P$ and $Q$ are non-trivial properties in $\mathbb{L}^a$ such that $\rightarrow H \subseteq P \cap Q$ then there exists a partition $(V_1, V_2)$ of $V(H)$ with $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$ such that $\rightarrow H \subseteq (\rightarrow H[V_1])(\rightarrow H[V_2]) \subseteq P \cup Q$ and $\rightarrow H[V_1] \subseteq P$ and $\rightarrow H[V_2] \subseteq Q$.

We can now describe the minimal reducible bounds for the hom-properties in $\mathbb{L}^a$.

### 4.1. Finite $H$

Let $H$ be a finite core such that $\rightarrow H$ is irreducible in $\mathbb{L}^a$ (i.e. $H$ is indecomposable). Let $H$ be the set of all hom-properties $\rightarrow C_1 + C_2 = (\rightarrow C_1)(\rightarrow C_2)$ formed as follows:

For each partition $(V_1, V_2)$ of $V(H)$ with $V_1 \neq \emptyset$, $V_2 \neq \emptyset$, let $C_1 = C(H[V_1])$ and $C_2 = C(H[V_2])$.

**Lemma 6.** $\rightarrow H \subseteq \rightarrow C_1 + C_2$ for each $\rightarrow C_1 + C_2 \in H$.

**Proof.** This will follow if we can show that there is a homomorphism from $H$ to $C_1 + C_2$. By the definition of $C_1$ and $C_2$, there exist homomorphisms $f_1 : V_1 \rightarrow V(C_1)$ and $f_2 : V_2 \rightarrow V(C_2)$. Define $f : V(H) \rightarrow V(C_1 + C_2)$ by $f(x) = f_i(x)$ if $x \in V_i$, $i = 1, 2$.

Since $H$ is a finite graph, the set $H$ is finite and thus minimal elements (under inclusion of properties) exist. These minimal elements of $H$ are precisely all the minimal reducible bounds of $\rightarrow H$, i.e. they form $B(\rightarrow H)$.

**Theorem 7.** $B(\rightarrow H) = \text{Min}_{\leq} H$. 

Proof: We must show that if there are non-trivial properties $P$ and $Q$ in $L^a$ such that $\rightarrow H \subseteq C_1 + C_2 \subseteq \mathcal{P}Q$, then there exists a $\rightarrow C_1 + C_2 \subseteq \mathcal{P}Q$. This follows immediately by Corollary 5: there exists a $(P, Q)$ partition $(V_1, V_2)$ of $V(H)$ with $V_1 \neq \emptyset$, $V_2 \neq \emptyset$ such that $\rightarrow H \subseteq H[V_1] \rightarrow H[V_2] \subseteq \mathcal{P}Q$, and so $\rightarrow H \subseteq (\rightarrow C(H[V_1])) (\rightarrow C(H[V_2])) \subseteq \mathcal{P}Q$.

All the minimal reducible bounds in $L^a$ for a hom-property $\rightarrow H$, where $H$ is finite, can thus be found by forming the finite set $H$ (by considering all partitions $(V_1, V_2)$ of $V(H)$ with $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$, and then forming the hom-properties $\rightarrow (C(H[V_1]) + C(H[V_2]))$ and then determining which of these reducible properties are minimal under inclusion.

### 4.2. Infinite $H$

We now consider minimal reducible bounds in $L^a$ for an irreducible $\rightarrow H$, where $H$ is an infinite union of finite cores. By Corollary 5, if a minimal reducible bounds exists for such a $\rightarrow H$, it is of the same form as in the finite case, i.e. it has the form $(\rightarrow H[V_1])(\rightarrow H[V_2])$ for some partition $(V_1, V_2)$ of $V(H)$ with $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$. We can again form the set $H$ for an infinite graph $H$, $H = \{(\rightarrow H[V_1])(\rightarrow H[V_2]) : (V_1, V_2) \text{ is a partition of } V(H) \text{ and } V_1 \neq \emptyset, V_2 \neq \emptyset\}$ and clearly $\rightarrow H \subseteq (\rightarrow H[V_1])(\rightarrow H[V_2])$ for each $\rightarrow H[V_1]) \rightarrow H[V_2])$ in $H$. However $H$ will now be an infinite set and the existence of minimal elements is no longer trivial. In the following theorem we show that $H$ has minimal elements and that every element of $H$ contains a minimal element. These minimal elements thus form $B(\rightarrow H)$, the set of all minimal reducible bounds for $\rightarrow H$.

**Theorem 8.** Let $H$ be an countable union of finite cores. Then the set $H$ contains minimal elements, and each element of $H$ contains a minimal element of $H$.

**Proof:** We will first use Zorn’s lemma to show that $H = \{(\rightarrow H[V_1])(\rightarrow H[V_2]) : (V_1, V_2) \text{ is a partition of } V(H), V_1 \neq \emptyset, V_2 = \emptyset\}$ has minimal elements. This will follow if we can show that every chain in $H$ has a lower bound in $H$.

Suppose to the contrary that $C = \{(\rightarrow H[V_1]) \rightarrow H[V_2]) : \alpha \in A\}$ is an infinite chain in $H$ that does not have a lower bound in $H$. Then given any element of the chain, there exists an infinite chain of elements of $C$ below it.
Suppose $H = H_1 \cup H_2 \cup \ldots$. For each $\alpha \in A$, the partition $(V_{1,\alpha}, V_{2,\alpha})$ of $V(H)$ induces a partition of $V(H_1)$. Since $V(H_1)$ has only finitely many partitions, there exists a partition $(V_{1,1}, V_{2,1})$ of $V(H_1)$ that is induced infinitely many times and that satisfies: given any $\alpha \in A$, there exists $\alpha' \in A$ such that $(\rightarrow H[V_{1,\alpha}])((\rightarrow H[V_{2,\alpha}]) \subset (\rightarrow H[V_{1,\alpha}])((\rightarrow H[V_{2,\alpha}]) and $(V_{1,\alpha}, V_{2,\alpha})$ induces $(V_{1,1}, V_{2,1})$ in $V(H_1)$. (If for each induced partition of $V(H_1)$ occurring infinitely many times, there exists an $\alpha$ such that every $\alpha' \in A$ satisfying $(\rightarrow H[V_{1,\alpha}])((\rightarrow H[V_{2,\alpha}]) \subset (\rightarrow H[V_{1,\alpha}])((\rightarrow H[V_{2,\alpha}]) induces some different partition of $V(H_1)$, then, since these $\alpha$ are finite, we can choose the one among them corresponding to the least element of $C$. This element of $C$ contains only finitely many other elements of $C$ below it, contradicting our hypothesis.) We have $H_1[V_{1,1}] \rightarrow H[V_{1,\alpha}]$ and $H_2[V_{2,1}] \rightarrow H[V_{2,\alpha}]$.

Now form $A'$ from $A$ by deleting all those $\alpha$ for which $(V_{1,\alpha}, V_{2,\alpha})$ does not induce $(V_{1,1}, V_{2,1})$. For any $\alpha \in A$, there exists $\alpha'$ in $A'$ such that $(\rightarrow H[V_{1,\alpha}])((\rightarrow H[V_{2,\alpha}]) \subset (\rightarrow H[V_{1,\alpha}])((\rightarrow H[V_{2,\alpha}]) and $H_1[V_{1,1}] \rightarrow H[V_{1,\alpha}]$ and $H_1[V_{2,1}] \rightarrow H[V_{2,\alpha}]$. We now have a new infinite chain, $C' = \{(\rightarrow H[V_{1,\alpha}])((\rightarrow H[V_{2,\alpha}]) : \alpha \in A', and we repeat the procedure using $H_2$ and $C'$, to form $C''$, etc. For each $H_i$ we obtain a partition $(V_{1,i}, V_{2,i})$ of $V(H_i)$ and after completing the procedure $i$ times, we have a chain of $(\rightarrow H[V_{1,\alpha}])((\rightarrow H[V_{2,\alpha}]) such that for all $\alpha$ in the new index set, the partition $(V_{1,\alpha}, V_{2,\alpha})$ of $V(H)$ induces the partition $(V_{1,j}, V_{2,j})$ of $V(H_j)$ for all $j = 1, 2, \ldots, i$. Also, for any $\alpha \in A$, there exists $\alpha'$ in the new index set such that $(\rightarrow H[V_{1,\alpha}])((\rightarrow H[V_{2,\alpha}]) \subset (\rightarrow H[V_{1,\alpha}])((\rightarrow H[V_{2,\alpha}]) and $H_j[V_{1,j}] \rightarrow H[V_{1,\alpha}]$ and $H_j[V_{2,j}] \rightarrow H[V_{2,\alpha}]$ for all $j = 1, 2, \ldots, i$.

Now let $V_1 = \bigcup_{i \geq 1} V_{1,i}$ and let $V_2 = \bigcup_{i \geq 1} V_{2,i}$. There are now two possibilities: either both $V_1$ and $V_2$ are non-empty, or one of them (say $V_1$) is empty while the other ($V_2$) equals $V(H)$.

Suppose first that both $V_1$ and $V_2$ are non-empty. Then $\rightarrow H[V_1]((\rightarrow H[V_2]) is itself in $H$. We will show that $\rightarrow H[V_1]((\rightarrow H[V_2]) is a lower bound for the chain $C$.

Let $\alpha \in A$ and let $G \in (\rightarrow H[V_1])(\rightarrow H[V_2])$. Then there exists a partition $(A, B)$ of $V(G)$ such that $G[A] \rightarrow H[V_1]$ and $G[B] \rightarrow H[V_2]$. Since both $G[A]$ and $G[B]$ are finite, there exists an integer $n$ such that $G[A] \rightarrow \cup \{H_{i[V_1,i]} : i = 1, 2, \ldots, n \}$ and $G[B] \rightarrow \cup \{H_{i[V_2,i]} : i = 1, 2, \ldots, n \}$. Now by the remark at the end of the previous paragraph, after $n$ steps of the procedure,there exists an $\alpha'$ in the modified index set of the chain with $(\rightarrow H[V_{1,\alpha'}])(\rightarrow H[V_{2,\alpha'}]) \subset (\rightarrow H[V_{1,\alpha'}])(\rightarrow H[V_{2,\alpha'}]) and such that $H_{i[V_1,i]} \rightarrow H[V_{1,\alpha'}]$ and $H_{i[V_2,i]} \rightarrow H[V_{2,\alpha'}]$ for $i = 1, 2, \ldots, n$. Hence $G[A] \rightarrow H[V_{1,\alpha'}]$
and \( G[B] \in \rightarrow H[V_2'] \), so \( G \in (\rightarrow H[V_1'])(\rightarrow H[V_2']) \subseteq (\rightarrow H[V_1'])(\rightarrow H[V_2']) \), i.e. \( (\rightarrow H[V_1])(\rightarrow H[V_2]) \subseteq (\rightarrow H[V_1'])(\rightarrow H[V_2']) \).

Now suppose that \( V_2 \) is empty and that \( V_1 = V(H) \). We claim that in this case, any element of \( \mathcal{H} \) of the form \( (\rightarrow H[W_1])(\rightarrow H[W_2]) \) where \( W_2 \) is independent, is a lower bound for the chain \( \mathcal{C} \). To prove this, fix such an element of \( \mathcal{H} \). Suppose it is \( (\rightarrow H[W_1])(\rightarrow H[W_2]) \), with \( W_2 \) independent. Let \( \alpha \in A \) and let \( G \in (\rightarrow H[W_1])(\rightarrow H[W_2]) \). We must show that \( G \in (\rightarrow H[V_1])(\rightarrow H[V_2']): \) Since \( G \) is finite, there exists an integer \( n \) such that \( G \in (\rightarrow (H_1 \cup H_2 \cup ... \cup H_n)[W_1])(\rightarrow (H_1 \cup H_2 \cup ... \cup H_n)[W_2]) \). Now there exists an \( \alpha' \in A \) such that \( (\rightarrow H[V_1'])(\rightarrow H[V_2']) \subseteq (\rightarrow H[V_1'])(\rightarrow H[V_2']) \) and \( (V_1', V_2') \) induces \( (V_{1,i}, V_{2,i}) = (V(H_i), \emptyset) \) for each \( i = 1, 2, ..., n \). Then \( (H_1 \cup H_2 \cup ... \cup H_n)[W_1] \rightarrow H[V_1'] \) (the inclusion map) and \( (H_1 \cup H_2 \cup ... \cup H_n)[W_2] \rightarrow H[V_2'] \) (since \( W_2 \) is independent and \( V_2' \) is non-empty.) Hence \( G \in (\rightarrow H[V_1'])(\rightarrow H[V_2']) \subseteq (\rightarrow H[V_1'])(\rightarrow H[V_2']) \).

We can conclude by Zorn’s lemma that the set \( \mathcal{H} \) has minimal elements. By fixing an element of \( \mathcal{H} \) and considering only chains of elements of \( \mathcal{H} \) each of which is contained in that fixed element, the same argument as above shows that each element of \( \mathcal{H} \) contains at least one of these minimal elements of \( \mathcal{H} \). Hence, as in the case where \( H \) is finite, the minimal elements of \( \mathcal{H} \) form \( \mathcal{B}(\rightarrow H) \) when \( H \) is an infinite union of finite graphs.

5. Some Applications

In the following applications, we allow the graph \( H \) to be either finite or a countable union of finite graphs and we show the existence of minimal reducible bounds of certain types in \( \mathbb{L}^n \) for \( \rightarrow H \). In this section we assume throughout that \( \rightarrow H \) is irreducible, while if \( H \) is finite it is assumed to be a core.

**Proposition 9.** If \( H \) is a graph with chromatic number 3, then \( \mathcal{O}^3 \) is the unique minimal reducible bound for \( \rightarrow H \).

**Proof.** Since \( \chi(H) = 3 \), there exists a partition \((V_1, V_2)\) of \( V(H) \) such that \( H[V_1] \) is an independent set of vertices and \( H[V_2] \) has chromatic number 2, i.e. \( \rightarrow C(H[V_1]) \rightarrow C(H[V_2]) \rightarrow K_1 + K_2 = \rightarrow K_3 = \mathcal{O}^3 \).

If \( \rightarrow H \subset \rightarrow C_1 \rightarrow C_2 \) for any other \( \rightarrow C_1 \rightarrow C_2 \in \mathcal{H} \), then either \( C_1 \) or \( C_2 \) must contain an edge (since \( \chi(C_1) + \chi(C_2) \geq 3 \)) and hence \( K_1 + K_2 \in \rightarrow C_1 \rightarrow C_2 \), i.e. \( \rightarrow H \subset \rightarrow K_1 + K_2 = \mathcal{O}^3 \subseteq \rightarrow C_1 \rightarrow C_2 \).
Proposition 10. If \( H \) is a graph with chromatic number 4, then all minimal reducible bounds of \( \rightarrow H \) are of the form \( O(\rightarrow X) \) for some graph \( X \subset H \).

Proof. Since \( \chi(H) = 4 \), there exists a partition \((V_1, V_2)\) of \( V(H) \) such that \( \chi(H[V_1]) = 2 \) and \( \chi(H[V_2]) = 2 \), i.e. \( \rightarrow C(H[V_1]) \rightarrow C(H[V_2]) \rightarrow K_2 + K_2 \rightarrow K_1 + K_3 = O(\rightarrow K_3) \).

Consider all partitions \((V_1, V_2)\) of \( V(H) \). If \( H[V_1] \) or \( H[V_2] \) is independent, we get a reducible bound for \( \rightarrow H \) of the form \( O(\rightarrow H[V_1]) \) or \( O(\rightarrow H[V_2]) \). If neither \( H[V_1] \) nor \( H[V_2] \) is independent, then \( K_2 \rightarrow H[V_1] \) and \( K_2 \rightarrow H[V_2] \), so \( K_2 + K_2 = O(\rightarrow K_3) \subseteq \rightarrow H[V_1] \rightarrow H[V_2] \).

We can now conclude that all the minimal elements of \( H \) are of the form \( O(\rightarrow X) \) for some graph \( X \subset H \).

Proposition 11. If \( H \) is a graph with chromatic number 5, then \( \rightarrow H \) has a minimal reducible bound of the form \( O(\rightarrow X) \) for some graph \( X \subset H \).

Proof. Since \( \chi(H) = 5 \), there exists a bound of the form \( O(\rightarrow X) = (\rightarrow K_1)(\rightarrow X) \) for \( \rightarrow H \) with \( X \subset H \) and \( \chi(X) = 4 \). Suppose that \( \rightarrow X_1 \rightarrow X_2 \) is any other element of \( H \) satisfying \( \rightarrow H \subseteq \rightarrow X_1 \rightarrow X_2 \subseteq O(\rightarrow X) \). Since \( \chi(H) = \chi(K_1) + \chi(X) = 5 \), we must have \( \chi(X_1) + \chi(X_2) = 5 \) and this is only possible if one of \( X_1 \) or \( X_2 \) has chromatic number at most 2.

Say \( \chi(X_1) \leq 2 \). Then we can assume that \( X_1 = K_1 \) or \( X_1 = K_2 \). In the first case, \( \rightarrow X_1 \rightarrow X_2 = \rightarrow K_1 \rightarrow X_2 = O(\rightarrow X_2) \), while in the second, \( \rightarrow X_1 \rightarrow X_2 = (\rightarrow K_1)(\rightarrow K_1 \rightarrow X_2) \). By Corollary 5, there exists a bound for \( \rightarrow H \) of the form \( O(\rightarrow Y) \) with \( Y \subset H \) satisfying \( \rightarrow H \subseteq O(\rightarrow Y) \subseteq (\rightarrow K_1)(\rightarrow K_1 \rightarrow X_2) \). In either case there exists a bound for \( \rightarrow H \) of the form \( O(\rightarrow Y) \) with \( Y \subset H \) satisfying \( \rightarrow H \subseteq O(\rightarrow Y) \subseteq \rightarrow X_1 \rightarrow X_2 \), so we conclude that \( H \) has a minimal element of the form \( O(\rightarrow Y) \) for some \( Y \subset H \).

Proposition 12. If \( H \) is a graph with chromatic number either infinite or finite and greater than or equal to 6, and if \( K_4 \) is not a subgraph of \( H \), then \( \rightarrow H \) has a minimal reducible bound of the form \( O(\rightarrow X) \) for some \( X \subset H \).

Proof. There exists a bound for \( \rightarrow H \) of the form \( O(\rightarrow X) \) where \( X \subset H \), and \( \chi(X) \geq 5 \), which is minimal of this type.

Suppose \( \rightarrow H \subset (\rightarrow X_1)(\rightarrow X_2) \subseteq O(\rightarrow X) \) where \( (\rightarrow X_1)(\rightarrow X_2) \in H \) is not of the form \( O(\rightarrow Y) \) for any graph \( Y \). If the chromatic number of either \( X_1 \) or \( X_2 \) is one, say \( \chi(X_1) = 1 \), then \( (\rightarrow X_1)(\rightarrow X_2) = O(\rightarrow X_2) \), contradicting our assumption on the form of \( (\rightarrow X_1)(\rightarrow X_2) \). If one of \( X_1 \) or \( X_2 \) has chromatic number 2, say \( \chi(X_1) = 2 \), then \( (\rightarrow X_1)(\rightarrow X_2) = O(\rightarrow X_2) \).
(\(O(\rightarrow X_2)\)) and by Corollary 5 there exists an element of \(H\) of the form \(O(\rightarrow Y)\) between \(\rightarrow H\) and \(O(\rightarrow X_2)\), contradicting the minimality of \(O(\rightarrow X)\).

Thus \(\chi(X_1) \geq 3\) and \(\chi(X_2) \geq 3\) so that both \(X_1\) and \(X_2\) contain an odd cycle, say \(S_1\) and \(S_2\) respectively. But then \(S_1 + S_2 \subseteq (\rightarrow X_1)(\rightarrow X_2) \subseteq O(\rightarrow X)\), so \(V(S_1 + S_2)\) has an \((O(\rightarrow X))\)-partition, say \((V_1, V_2)\). Thus \((S_1 + S_2)[V_1]\) is an independent subgraph of either \(S_1\) or \(S_2\), and (since \(\chi(S_1) = 3\) and \(\chi(S_2) = 3\)), \((S_1 + S_2)[V_2]\) must contain \(K_4\) as a subgraph, a contradiction since \((S_1 + S_2)[V_2] \subseteq X\), and any \(K_4\) in \((S_1 + S_2)[V_2]\) would force a \(K_4\) in \(X \subset H\).

We conclude that \(H\) has a minimal element of the form \(O(\rightarrow Y)\) for some \(Y \subset H\).

Proposition 13. If \(H\) is a graph with finite chromatic number satisfying \(\chi(H) = n \geq 6\), and \(K_{n-1} \subset H\), then \(\rightarrow H\) has a minimal reducible bound of the form \(O(\rightarrow X)\) for some \(X \subset H\).

Proof. There exists an element \(O(\rightarrow X) \subset H\) with \(\chi(X) = n - 1\). Suppose now that \(\rightarrow H \subset (\rightarrow H[V_1])(\rightarrow H[V_2]) \subseteq O(\rightarrow X)\), with \((\rightarrow H[V_1])(\rightarrow H[V_2]) \subset H\). Then \(\chi(H[V_1]) + \chi(H[V_2]) = n\). Since \(K_{n-1} \subset H\), there exists \(K_i \subseteq H[V_1]\) and \(K_j \subseteq H[V_2]\) with \(i + j = n - 1\).

If \(i \geq \chi(H[V_1])\), then \(C(H[V_1]) = K_i\), so \((\rightarrow H[V_1])(\rightarrow H[V_2]) = (\rightarrow K_i)(\rightarrow K_{i-1} \rightarrow H[V_2])\) and by Corollary 5, there exists a bound for \(\rightarrow H\) of the form \(O(\rightarrow Y)\) for some \(Y \subset H\), contained in \((\rightarrow H[V_1])(\rightarrow H[V_2])\). However if \(i < \chi(H[V_1])\), then \(j \geq \chi(H[V_2])\) and \(C(H[V_2]) = K_j\), and once again \((\rightarrow H[V_1])(\rightarrow H[V_2])\) contains a bound for \(\rightarrow H\) of the form \(O(\rightarrow Y)\) for some \(Y \subset H\).

We conclude that \(H\) has a minimal element of the form \(O(\rightarrow Y)\) for some \(Y \subset H\).

Proposition 14. If \(H\) is a triangle-free graph with finite chromatic number satisfying \(\chi(H) \geq 6\), then \(\rightarrow H\) has a minimal reducible bound not of the form \(O\mathcal{P}\) for any \(\mathcal{P} \in \mathbb{L}_a\).

Proof. Since \(\chi(H) \geq 6\), there exists \((\rightarrow X_1)(\rightarrow X_2) \subseteq H\) such that \(\chi(X_1) \geq 3, \chi(X_2) \geq 3, \chi(X_1) + \chi(X_2) = \chi(H)\). Suppose \((\rightarrow X_1)(\rightarrow X_2) = O(\rightarrow X)\) for some \(X \subset H\). \(X_1\) and \(X_2\) each contain an odd cycle, say \(S_1\) and \(S_2\) respectively. We then have that \(S_1 + S_2 \subseteq O(\rightarrow X)\) so \(V(S_1 + S_2)\) has an \((O, \rightarrow X)\)-partition, say \((V_1, V_2)\). Since \((S_1 + S_2)[V_1]\) is an independent subset of either \(S_1\) or \(S_2\), \((S_1 + S_2)[V_2]\) must contain a triangle, forcing \(H\) to
contain a triangle, contradicting our hypothesis. So \((\rightarrow X_1)(\rightarrow X_2)\) is not of the form \(\mathcal{O}\mathcal{P}\) for any \(\mathcal{P} \in \mathbb{L}^a\).

Suppose now that \(\rightarrow H \subset \mathcal{O}(\rightarrow X) \subset (\rightarrow X_1)(\rightarrow X_2)\) for some \(X \subset H\). Since \(\chi(H) = \chi(X_1) + \chi(X_2)\), it must be true that \(\chi(X) = \chi(H) - 1\). Let \(G\) be any finite subgraph of \(X\) with \(\chi(G) = \chi(X)\). The graph \(G + \{v\}\) is in \(\mathcal{O}(\rightarrow X)\) and therefore in \((\rightarrow X_1)(\rightarrow X_2)\), and so \(V(G + \{v\})\) has a \((\rightarrow X_1, \rightarrow X_2)\)-partition \((V_1, V_2)\). Suppose that \(v \in V_1\). If \(\{w \in V(G) : w \in V_1\}\) is not an independent set of vertices, then \((G + v)[V_1]\) contains a triangle, and so \(X_1\) contains a triangle, which is not possible. If \(\{w \in V(G) : w \in V_1\}\) is an independent set of vertices, then \(\chi((G + v)[V_2]) \geq \chi(H) - 2\). But \((G + v)[V_2] \in \rightarrow X_2\) and \(\chi(X_2) \leq \chi(H) - 3\), again a contradiction. Hence no bound of the form \(\mathcal{O}\mathcal{P}\) with \(\mathcal{P} \in \mathbb{L}^a\) can occur between \(\rightarrow H\) and \((\rightarrow X_1)(\rightarrow X_2)\).

We conclude that \(H\) has a minimal element not of the form \(\mathcal{O}(\rightarrow Y)\) for any \(Y \subset H\).

The previous result is not true if we allow \(\chi(H)\) to be infinite since the set of all triangle-free graphs, \(\mathcal{I}_1\) has the unique minimal reducible bound \(\mathcal{O}\mathcal{I}_1\) (see [1], [6]). \(\mathcal{I}_1\) is the hom-property \(\rightarrow \cup \{R : R\) is a triangle-free core\}, with infinite chromatic number.

Corollaries 12 and 14 show that if \(H\) has a finite chromatic number greater than or equal to 6, and \(H\) is triangle-free, then \(\rightarrow H\) has a minimal reducible bound of the form \(\mathcal{O}\mathcal{P}\) for some \(\mathcal{P} \in \mathbb{L}^a\) and a minimal reducible bound not of this form.

### 6. Minimal Reducible Bounds for \(\rightarrow H\) in \(\mathbb{L}\)

We now describe the minimal reducible bounds of a hom-property \(\rightarrow H\) in the lattice of hereditary properties, \(\mathbb{L}\). Again, we will describe the case for a finite \(H\) first, and then draw conclusions about an infinite \(H\). The following lemma and its corollary are useful in both the finite and infinite cases.

**Lemma 15.** Let \(H\) be a finite graph or a countable union of finite graphs. If \(\rightarrow H \subseteq \mathcal{P}\mathcal{Q}\), where \(\mathcal{P}\) and \(\mathcal{Q}\) are non-trivial properties in \(\mathbb{L}\) such that \(\mathcal{O} \not\subseteq \mathcal{Q}\), then \(\rightarrow H \subseteq \mathcal{P}\).

**Proof.** Suppose first that \(H\) is finite, and suppose that the cardinality of the largest edgeless graph in \(\mathcal{Q}\) is \(k\). For any \(m > k\), \(\mathcal{H}^\dagger(m) \in \mathcal{P}\mathcal{Q}\) and by the restriction on \(\mathcal{Q}\), \(\mathcal{H}^\dagger(m - k)\) must be in \(\mathcal{P}\). This is true for any \(m > k\) so that \(\mathcal{H}^\dagger(r) \in \mathcal{P}\) for all \(r \geq 1\), i.e. \(\rightarrow H \subseteq \mathcal{P}\).
If $H$ is infinite, then since $\rightarrow H' \subseteq \mathcal{P}Q$ for any finite subgraph $H'$ of $H$, by the finite case we can conclude that $\rightarrow H' \subseteq \mathcal{P}$ for every finite subgraph $H'$ of $H$. Since any graph in $\rightarrow H$ is contained in some $\rightarrow H'$ where $H'$ is a finite subgraph of $H$, we can conclude that $\rightarrow H \subseteq \mathcal{P}$.

**Corollary 16.** Let $H$ be a finite graph or a countable union of finite graphs. If $\rightarrow H \subseteq \mathcal{P}Q$, where $\mathcal{P}$ and $\mathcal{Q}$ are non-trivial properties in $\mathbb{L}$ such that $\mathcal{O} \nsubseteq \mathcal{Q}$, then $\rightarrow H \subseteq (\rightarrow H)((\{K_1\}) \subseteq \mathcal{P}Q$.

**Proof.** The proof is immediate as $\rightarrow H \subseteq \mathcal{P}$ and, since $\mathcal{Q}$ is non-trivial, $K_1 \in \mathcal{Q}$.

We now describe the minimal reducible bounds for hom-properties in $\mathbb{L}$.

### 6.1. Finite $H$

**Theorem 17.** If $H$ is a finite indecomposable core then the minimal reducible bounds for $\rightarrow H$ in $\mathbb{L}$ are the minimal elements of $H$ as well as the property $(\rightarrow H)((\{K_1\})$.

**Proof.** By Lemma 4 and Corollary 16 we know that if $\rightarrow H \subseteq \mathcal{P}Q$, where $\mathcal{P}$ and $\mathcal{Q}$ are non-trivial properties in $\mathbb{L}$, then if $\mathcal{O} \subseteq \mathcal{P}$ and $\mathcal{O} \subseteq \mathcal{Q}$, we have a minimal element of $H$ between $\rightarrow H$ and $\mathcal{P}Q$, while if $\mathcal{O} \nsubseteq \mathcal{Q}$, then $(\rightarrow H)((\{K_1\})$ lies between $\rightarrow H$ and $\mathcal{P}Q$. Note that the case $\mathcal{O} \nsubseteq \mathcal{P}$ and $\mathcal{O} \nsubseteq \mathcal{Q}$ cannot occur since by Lemma 15, if $\mathcal{O} \nsubseteq \mathcal{Q}$, then $\rightarrow H \subseteq \mathcal{P}$, and since $H$ is assumed to have at least one vertex, all multiplications of this vertex must be in $\mathcal{P}$ i.e. $\mathcal{O} \subseteq \mathcal{P}$.

To complete the proof of the theorem, we must show that $(\rightarrow H)((\{K_1\})$ is not contained in any minimal element of $H$, and that no minimal element of $H$ is contained in $(\rightarrow H)((\{K_1\})$.

First suppose to the contrary that $\rightarrow H[V_1] + H[V_2]$ is a minimal element of $H$ satisfying $\rightarrow H[V_1] + H[V_2] \subseteq (\rightarrow H)((\{K_1\})$. By Lemma 15 we then have $\rightarrow H[V_1] + H[V_2] \subseteq \rightarrow H$, and so $H[V_1] + H[V_2] \rightarrow H$. If this homomorphism is a surjection, then $H$ is decomposable, a contradiction, while if this homomorphism is not a surjection, then we can use it to map $H$ into a proper subgraph of itself, a contradiction to the fact that $H$ is a core.

Now suppose that $\rightarrow (H[V_1] + H[V_2])$ is a minimal element of $H$ and that $(\rightarrow H)((\{K_1\}) \subseteq (H[V_1] + H[V_2])$. Now $H + K_1 \in (\rightarrow H)((\{K_1\}) \subseteq (\rightarrow (H[V_1] + H[V_2])$, so we have the inclusions $\rightarrow H \subseteq \rightarrow (H + K_1) = (\rightarrow H)((\mathcal{O}) \subseteq \rightarrow (H[V_1] + H[V_2])$. By Lemma 4 there exists an element
\[ \rightarrow (H[W_1] + H[W_2]) \text{ in } H \text{ satisfying } \rightarrow H \subseteq \rightarrow (H[W_1] + H[W_2]) \subseteq (\rightarrow H) \]

By the minimality of \( \rightarrow (H[V_1] + H[V_2]) \) in \( H \), the two elements of \( H \) must be equal, and so we have \( (\rightarrow H)(O) \Rightarrow (\rightarrow H[V_1] + H[V_2]) \) i.e. \( (\rightarrow H)(O) \Rightarrow H[W_1] \rightarrow H[W_2] \).

This result immediately follows from Lemma 4 and Corollary 16. The sharper result from the finite case is no longer true since when \( H \) is infinite, it may be possible that \( (\rightarrow H)(\{K_1\}) \) is properly contained in a minimal element of \( H \) e.g. \( \mathcal{I}_1 \) has the unique minimal reducible bound in \( \mathbb{L}^a \) of \( \mathcal{I}_1 \bigcirc \), the unique minimal element of \( H \). In \( \mathbb{L} \) however, we have \( \mathcal{I}_1 \subseteq \mathcal{I}_1 \{K_1\} \subseteq \mathcal{I}_1 \bigcirc \), so that \( \mathcal{I}_1 \) has unique minimal reducible bound \( \mathcal{I}_1 \{K_1\} \).

6.2. Infinite \( H \)

Theorem 18. If \( H \) is an infinite union of finite graphs, then the minimal elements of the set \( H \cup \{(-\rightarrow H)(\{K_1\})\} \) are the minimal reducible bounds for \( \rightarrow H \) in \( \mathbb{L} \).

This result immediately follows from Lemma 4 and Corollary 16. The sharper result from the finite case is no longer true since when \( H \) is infinite, it may be possible that \( (\rightarrow H)(\{K_1\}) \) is properly contained in a minimal element of \( H \) e.g. \( \mathcal{I}_1 \) has the unique minimal reducible bound in \( \mathbb{L}^a \) of \( \mathcal{I}_1 \bigcirc \), the unique minimal element of \( H \). In \( \mathbb{L} \) however, we have \( \mathcal{I}_1 \subseteq \mathcal{I}_1 \{K_1\} \subseteq \mathcal{I}_1 \bigcirc \), so that \( \mathcal{I}_1 \) has unique minimal reducible bound \( \mathcal{I}_1 \{K_1\} \).

It is not true that \( (\rightarrow H)(\{K_1\}) \) is contained in every minimal element of \( H \), since if \( (\rightarrow H)(\{K_1\}) \subseteq (\rightarrow H[V_1])(\rightarrow H[V_2]) \) where \( (\rightarrow H[V_1])(\rightarrow H[V_2]) \) is minimal in \( H \), then we have \( \rightarrow H \subseteq (\rightarrow H)(O) \subseteq (\rightarrow H[V_1])(\rightarrow H[V_2]) \). (The second inclusion follows since any graph \( G \) in \( (\rightarrow H)(O) \) is in \( (\rightarrow H')(\mathcal{I}) \) for some finite subgraph \( H' \) of \( H \), and since \( H' + K_1 \subset H[V_1] \rightarrow H[V_2] \), we have that \( (\rightarrow H' + K_1) \subset H[V_1] \rightarrow H[V_2] \)). By Lemma 4 there should be another element of \( H \) between \( \rightarrow H \) and \( (\rightarrow H)(O) \). By the minimality of \( \rightarrow (H[V_1] + H[V_2]) \), we now have that \( (\rightarrow H)(O) = (\rightarrow H[V_1])(\rightarrow H[V_2]) \). However (Corollary 14) if \( H \) is infinite and triangle-free with finite chromatic number at least six, \( H \) contains at least one minimal element which does not contain the factor \( O \).

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