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MINIMAL REDUCIBLE BOUNDS FOR HOM-PROPERTIES OF GRAPHS

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Abstract

Let H be a fixed finite graph and let $\rightarrow H$ be a hom-property, i.e. the set of all graphs admitting a homomorphism into H. We extend the definition of $\rightarrow H$ to include certain infinite graphs H and then describe the minimal reducible bounds for $\rightarrow H$ in the lattice of additive hereditary properties and in the lattice of hereditary properties.

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1. Definitions

In general we follow the notation and terminology of [1]. Denote by $\mathcal{I}$ the set of all finite undirected simple graphs. Any isomorphism-closed subset $\mathcal{P}$ of $\mathcal{I}$ is called a *property* of graphs. A property $\mathcal{P}$ is *hereditary* if whenever a graph G is in $\mathcal{P}$, then all subgraphs of G are also in $\mathcal{P}$. A property $\mathcal{P}$ is *additive* if whenever graphs G and H are in $\mathcal{P}$, then their disjoint union, denoted by $G \cup H$, is in $\mathcal{P}$ too. When partially ordered under set inclusion, the poset of all additive hereditary properties forms a complete distributive lattice, which we will denote by $\mathbb{L}^a$. We use $\mathbb{L}$ to denote the lattice of hereditary properties. A property is called *non-trivial* if it contains at least one non-null graph and it is not equal to $\mathcal{I}$.

Let $\mathcal{P}_1, \mathcal{P}_2, ..., \mathcal{P}_n$ be any properties of graphs. A *vertex* $(\mathcal{P}_1, \mathcal{P}_2, ..., \mathcal{P}_n)$-*partition* of a graph G is a partition $(V_1, V_2, ..., V_n)$ of $V(G)$ such that for each $i = 1, 2, ..., n$, the induced subgraph $G[V_i]$ has the property $\mathcal{P}_i$. Any of the V_i may be empty. The property $\mathcal{P}_1 \circ \mathcal{P}_2 \circ ... \circ \mathcal{P}_n$ is defined as the set of all graphs having a vertex $(\mathcal{P}_1, \mathcal{P}_2, ..., \mathcal{P}_n)$-*partition*. If $\mathcal{P}_1, \mathcal{P}_2, ..., \mathcal{P}_n$ are all (additive) hereditary properties, then $\mathcal{P}_1 \circ \mathcal{P}_2 \circ ... \circ \mathcal{P}_n$ is an (additive) hereditary property too. For convenience, we will write $\mathcal{P}_1 \circ \mathcal{P}_2 \circ ... \circ \mathcal{P}_n$ as $\mathcal{P}_1\mathcal{P}_2...\mathcal{P}_n$, omitting the binary operation symbol.

An additive hereditary property $\mathcal{R}$ is called *reducible in* $\mathbb{L}^a$ if there exist non-trivial properties $\mathcal{P}$ and $\mathcal{Q}$ in $\mathbb{L}^a$ such that $\mathcal{R} = \mathcal{PQ}$. Otherwise $\mathcal{R}$ is called *irreducible*. A reducible property $\mathcal{R} \in \mathbb{L}^a$ is called a *minimal reducible bound* for property $\mathcal{P} \in \mathbb{L}^a$ if $\mathcal{P} \subseteq \mathcal{R}$ and there is no reducible property $\mathcal{R}_1$ satisfying $\mathcal{P} \subseteq \mathcal{R}_1 \subsetneq \mathcal{R}$. From this definition, each reducible property is the unique minimal reducible bound for itself. We use the symbol $\mathbf{B}(\mathcal{P})$ to denote the class of all minimal reducible bounds for property $\mathcal{P}$. We do not know whether a minimal reducible bound exists for every property $\mathcal{P}$, and $\mathbf{B}(\mathcal{P})$ is known for only a few properties $\mathcal{P}$. Similar definitions hold in $\mathbb{L}$.

Given any $\mathcal{P} \in \mathbb{L}^a$ (or in $\mathbb{L}$), we define the class of all $\mathcal{P}$-maximal graphs by $\mathbf{M}(\mathcal{P}) = \{G \in \mathcal{P} : G + e \notin \mathcal{P}$ for any $e \in E(\overline{G})\}$. $\mathbf{M}(\mathcal{P})$ determines $\mathcal{P}$ in the sense that $H \in \mathcal{P}$ iff there exists some $\mathcal{P}$-maximal graph G such that $H \subseteq G$.

A *homomorphism* from a graph G to a graph H is a mapping f of the vertex set $V(G)$ to the vertex set $V(H)$ which preserves edges, i.e. if $\{u, v\} \in E(G)$, then $\{f(u), f(v)\} \in E(H)$. We say that G is *homomorphic to* H if there exists a homomorphism from G to H, and we write $G \to H$. If $G \to H$, then $\chi(G) \leq \chi(H)$. If H is a finite graph, then the *hom-property generated by* H is the set $\to H = \{G \in \mathcal{I} : G \to H\}$. Note that $\to H$ is an additive hereditary property for any $H \in \mathcal{I}$.

In Section 2 we summarise some fundamental properties of hom-properties. In Section 3 we extend the definition of hom-properties to include $\to H$ where H may be an infinite union of finite graphs. We then describe $\mathbf{B}(\to H)$ in the lattice $\mathbb{L}^a$ in Section 4 and consider some applications of these results in Section 5. Section 6 describes $\mathbf{B}(\to H)$ in the lattice $\mathbb{L}$.

2. Fundamental Properties of Hom-Properties

Given a graph G, a *core of* G is any subgraph G' of G such that $G \to G'$, and such that G is not homomorphic to any proper subgraph of G'. Every graph G has a unique core up to isomorphism (see [2]) which is denoted by $C(G)$. If $G = C(G)$, i.e. if G is not homomorphic to any of its proper subgraphs, then we call G a *core*. Since any graph homomorphic to G is also homomorphic to $C(G)$, and any element of $\to C(G)$ is in $\to G$, we have that $\to G =\to C(G)$. Hence, given any hom-property, we can assume it is of the form $\to H$ where H is a core.

The $(\to H)$-maximal graphs are known and described in [4]:

Given any $G \in \mathcal{I}$, with $V(G) = \{v_1, v_2, ..., v_n\}$, its *multiplications* $G^{::}$ are defined as follows:

1. $V(G^{::}) = W_1 \cup W_2 \cup ... \cup W_n$,
2. for each $1 \le i \le n, |W_i| \ge 1$,
3. for any pair $1 \le i < j \le n, W_i \cap W_j = \emptyset$,
4. The only edges of $G^{::}$ are all the edges of the form $\{u, v\}$ where $u \in W_i, v \in W_j$ and $\{v_i, v_j\} \in E(G)$.

Thus each vertex v_i of G is replaced by a non-empty set of vertices W_i (also denoted by $v_i^{::}$) and if $u \in W_i, v \in W_j$, then u and v are adjacent in $G^{::}$ iff v_i and v_j are adjacent in G. $W_1, W_2, ..., W_n$ are independent sets called the *multivertices* of $G^{::}$. We also write $G^{::}$ as $G^{::}(W_1, W_2, ..., W_n)$ to emphasize its structure, and $G^{::}(k)$ for $G^{::}(W_1, W_2, ..., W_n)$ if $|W_i| = k$ for each $i = 1, 2, ..., n$. By mapping all the vertices in W_i to v_i for each $i = 1, 2, ..., n$, it is readily seen that $G^{::} \to G$, i.e. $G^{::} \in\to G$ and that $C(G^{::}) = G$ if G is a core.

Kratochvíl, Mihók and Semanišin proved in [4] that every $(\to H)$-maximal graph is a multiplication of a subgraph of H that is itself a core. Thus for every $(\to H)$-maximal graph G, there exists an integer $k \ge 1$ such that G is contained in $H^{::}(k)$.

The following lemma describes properties of hom-properties that will be used often in what follows. We use the notation $H + G$ for the *join* of two graphs H and G, i.e. for the graph obtained from $H \cup G$ by adding all edges joining vertices of H to vertices of G. A graph that is the join of two non-nul graphs is called *decomposable*, while a graph that is not decomposable is called *indecomposable*.

Lemma 1. 1. $\to K_1$ *is the set of all edgeless graphs, also denoted by* $\mathcal{O}$. *We have* $\to K_1 =\to H$ *for any edgeless graph* H, *since* $C(H) = K_1$.

2. $\to K_2$ *is the set of all bipartite graphs and* $\to K_2 =\to H$ *for any graph* H *with chromatic number* 2, *since* $C(H) = K_2$.
3. *For any graphs* H *and* $G, \to (H + G) = (\to H)(\to G)$ (*see* [3]).
4. $\to H$ *is irreducible in* $\mathbb{L}^a$ *iff* H *is indecomposable* (*see* [3]).
5. *For any graphs* H *and* $G, \to H \subseteq\to G$ *iff* $H \to G$ *iff* $H \in\to G$ (*see* [2]).

3. The Hom-Property $\to H$ for Infinite H

Although each hom-property is an additive hereditary property and is thus an element of the complete lattice $\mathbb{L}^a$, the hom-properties do not form a complete sublattice of $\mathbb{L}^a$. For example $\vee\{\to R : R$ is a triangle-free core$\}$ cannot be a hom-property: If $\vee\{\to R : R$ is a triangle-free core$\} = \to H$ for some graph H, then $\to R \subseteq \to H$ for each triangle-free core R. This would imply that $\chi(R) \leq \chi(H)$ for each triangle-free core R, which is not true, since triangle-free graphs of arbitrarily high chromatic number can be constructed.

To enable the supremum and infimum (intersection) of an arbitrary set of hom-properties to again be a hom-property, we extend the definition of hom-properties by including $\to H$, where H is any union of finite graphs. For such a graph H we define $\to H$ by $\to H = \{G \in \mathcal{I} : G \to H\}$, i.e. $\to H$ is the set of all *finite* graphs admitting a homomorphism into H. Since the set of all finite graphs is countable, and since only one copy of each connected component of H is sufficient, we can always assume that H is a countable union of finite cores and that these cores are pairwise non-isomorphic. Unlike in the case where H is finite, H itself need no longer have a core e.g. $K_1 \cup K_2 \cup K_3 \cup ...$ has no core, and H need not have a finite chromatic number.

Extending the definition of hom-properties to allow $\to H$ where H is either finite or a countable union of finite graphs makes the hom-properties a complete sublattice of $\mathbb{L}^a$, i.e. the supremum and infimum of any set of hom-properties is again a hom-property, as the following two results show.

Theorem 2. *Let $\{H_\alpha : \alpha \in A\}$ be a set of graphs, each of which is finite or a countable union of finite graphs. Then $\vee\{\to H_\alpha : \alpha \in A\} = \to (\cup\{H_\alpha : \alpha \in A\})$.*

Proof. In the lattice $\mathbb{L}^a$, $\vee\{\to H_\alpha : \alpha \in A\}$ is the least additive hereditary property which contains each $\to H_\alpha$, $\alpha \in A$. We show that $\to (\cup\{H_\alpha : \alpha \in A\})$ satisfies this.

Clearly, if $G \in \to H_\alpha$ for any $\alpha \in A$, then $G \in \to (\cup\{H_\alpha : \alpha \in A\})$. Therefore $\to H_\alpha \subseteq \to (\cup\{H_\alpha : \alpha \in A\})$ for each $\alpha \in A$.

Now suppose that $\to H_\alpha \subseteq \mathcal{P}$ for each $\alpha \in A$, for some property $\mathcal{P} \in \mathbb{L}^a$. We show that $\to (\cup\{H_\alpha : \alpha \in A\}) \subseteq \mathcal{P}$: Let $G \in \to (\cup\{H_\alpha : \alpha \in A\})$. By definition, G is finite, and hence there is a homomorphism from G to a finite union of H_α's, say $G \in \to H_1 \cup H_2 \cup ... \cup H_n$. Since each connected component of G is homomorphically mapped to exactly one H_i, G has a decomposition $G = G_1 \cup G_2 \cup ... \cup G_n$, such that $G_i \to H_i$, for $i = 1, 2, ..., n$. But then we have $G_i \in\to H_i \in \mathcal{P}$ for $i = 1, 2, ..., n$. As each G_i is in $\mathcal{P}$, by the additivity of $\mathcal{P}$, G is in $\mathcal{P}$ too. ■

Theorem 3. *Let $\{H_\alpha : \alpha \in A\}$ be a set of graphs, each of which is finite or a countable union of finite graphs. Then $\wedge\{\to H_\alpha : \alpha \in A\} =\to (\cup\{R : R$ is a core contained in a multiplication of a finite subgraph of H_α for each $\alpha \in A\})$.*

Proof. Suppose $G \in \cap\{\to H_\alpha : \alpha \in A\}$. Then $G \to C(G)$ and $C(G) \in \cap\{\to H_\alpha : \alpha \in A\}$. Then for each $\alpha \in A, C(G) \in\to H_\alpha$ and so $C(G)$ is contained in a multiplication of a finite subgraph of H_α. So we have $G \in\to C(G) \subseteq\to (\cup\{R : R$ is a core contained in a multiplication of a finite subgraph of H_α for each $\alpha \in A\})$.

Conversely, suppose $G \in\to (\cup\{R : R$ is a core contained in a multiplication of a finite subgraph of H_α for each $\alpha \in A\})$. Then there exists a homomorphism $f : G \to (\cup\{ R : R$ is a core contained in a multiplication of a finite subgraph of H_α for each $\alpha \in A\})$. Consider any connected component K of G : It is mapped by f to one of these cores, say R. By the definition of R, $R \in \cap\{\to H_\alpha : \alpha \in A\}$ and so $K \in\to R \subseteq \cap\{\to H_\alpha : \alpha \in A\}$. But then $\cap\{\to H_\alpha : \alpha \in A\}$ is an additive property containing each connected component of G and we conclude that G itself is in $\cap\{\to H_\alpha : \alpha \in A\}$. ■

4. Minimal Reducible Bounds for $\to H$ in $\mathbb{L}^a$

In this section we describe the set of all minimal reducible bounds for $\to H$ in the lattice $\mathbb{L}^a$, first dealing with the case where H is finite, and then with the infinite case. The following lemma and its corollary are useful for both cases.

Lemma 4. *Let H be a finite core or a countable union of finite cores. If $\mathcal{P}$ and $\mathcal{Q}$ are non-trivial properties in $\mathbb{L}$ with $\mathcal{O} \subseteq \mathcal{P}$ and $\mathcal{O} \subseteq \mathcal{Q}$ such that $\to H \subseteq \mathcal{PQ}$ then there exists a partition (V_1, V_2) of $V(H)$ with $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$ such that $\to H \subseteq (\to H[V_1])(\to H[V_2]) \subseteq \mathcal{PQ}$ and $\to H[V_1] \subseteq \mathcal{P}$ and $\to H[V_2] \subseteq \mathcal{Q}$.*

Proof. First suppose that H is finite and let $V(H) = \{v_1, v_2, ..., v_n\}$. We will show that there exists a partition (V_1, V_2) of $V(H)$ with $V_1 \neq \emptyset$ and

$V_2 \neq \emptyset$ such that $H[V_1]^{::}(k) \in \mathcal{P}$ for all $k \geq 1$ and $H[V_2]^{::}(k) \in \mathcal{Q}$ for all $k \geq 1$. Then all maximal elements of $\rightarrow H[V_1]$ are in $\mathcal{P}$ and so $\rightarrow H[V_1] \subseteq \mathcal{P}$, and similarly $\rightarrow H[V_2] \subseteq \mathcal{Q}$.

Fix $k \geq 1$. Since $H^{::}(2k-1) \in\rightarrow H \subseteq \mathcal{PQ}$, $H^{::}(2k-1)$ has a ($\mathcal{P}$,$\mathcal{Q}$)-partition. For each $i = 1, 2, ..., n$, $v_i^{::}(2k-1)$ has at least k vertices in the $\mathcal{P}$ part or at least k vertices in the $\mathcal{Q}$ part. By deleting $k-1$ vertices from each $v_i^{::}(2k-1)$, we can ensure that the remaining $v_i^{::}(k)$ is completely in the $\mathcal{P}$ part or completely in the $\mathcal{Q}$ part. We can also ensure that neither the $\mathcal{P}$ nor the $\mathcal{Q}$ part is empty: One of the $v_i^{::}(k)$ can be moved to the empty part if necessary.

We now have disjoint sets $I_1$and I_2 such that $I_1 \cup I_2 = \{1, 2, ..., n\}$ and $(\{v : v \in v_i^{::}(k), i \in I_1\}, \{v : v \in v_i^{::}(k), i \in I_2\})$ forms a $(\mathcal{P}, \mathcal{Q})$ partition of $H^{::}(k)$.

Since $\mathcal{P}$ and $\mathcal{Q}$ are hereditary properties, each such pair (I_1, I_2) induces a $(\mathcal{P}, \mathcal{Q})$-partition of $H^{::}(r)$ for each $r \leq k$,with each $v_i^{::}(r)$ entirely in the $\mathcal{P}$ part or entirely in the $\mathcal{Q}$ part. Since there are only finitely many partitions (I_1, I_2) of $\{1, 2, ..., n\}$, there exists a pair (I_1^*, I_2^*) which serves for infinitely many values of k, and hence for every value of k. Let $V_1 = \{v_i \in V(H) : i \in I_1^*\}$ and $V_2 = \{v_i \in V(H) : i \in I_2^*\}$. Then $H[V_1]^{::}(k) \in \mathcal{P}$ for all $k \geq 1$ and $H[V_2]^{::}(k) \in \mathcal{Q}$ for all $k \geq 1$.

Suppose now that H is a countable union of finite graphs, $H = H_1 \cup H_2 \cup ....$ Denote by G_n the graph $H_1 \cup H_2 \cup .... \cup H_n$, $n \geq 1$, and let $\mathcal{G}$ be the set of all G_n i.e. $\mathcal{G} = \{\ G_n : n \geq 1\}$.

For each $n \geq 1, \rightarrow G_n \subseteq \mathcal{PQ}$ and so by the finite case above, there exists a partition (W_1^n, W_2^n) of $V(G_n)$ with neither part empty such that $\rightarrow G_n[W_1^n] \subseteq \mathcal{P}$ and $\rightarrow G_n[W_2^n] \subseteq \mathcal{Q}$. Restricted to $V(H_1)$, each (W_1^n, W_2^n) induces a partition of $V(H_1)$ such that $\rightarrow H_1[W_1^n] \subseteq \mathcal{P}$ and $\rightarrow H_1[W_2^n] \subseteq \mathcal{Q}$. Since $V(H_1)$ has only finitely many partitions, there exists a partition of $V(H_1)$ with these properties induced by infinitely many (W_1^n, W_2^n). Call this partition (V_1^1, V_2^1) and note that $\rightarrow H_1[V_1^1] \subseteq \mathcal{P}$ and $\rightarrow H_1[V_2^1] \subseteq \mathcal{Q}$.

Now delete from $\mathcal{G}$ all those G_n whose corresponding (W_1^n, W_2^n) do not induce (V_1^1, V_2^1) and call the resulting set $\mathcal{G}'$. Suppose that $i \geq 2$ is the least integer such that G_i is in $\mathcal{G}'$. For each $n \geq i$ for which $G_n \in \mathcal{G}'$, the partition (W_1^n, W_2^n) of $V(G_n)$ restricted to $V(G_i)$ induces a partition of $V(G_i)$. Since $V(G_i)$ has only finitely many partitions, there exists a partition of $V(G_i)$ induced by infinitely many (W_1^n, W_2^n). This partition of $V(G_i)$ induces (V_1^1, V_2^1) in $V(H_1)$. Label the partitions induced by this partition of $V(G_i)$ in $V(H_2)$, $V(H_3), ..., V(H_i)$ by $(V_1^2, V_2^2)(V_1^3, V_2^3), ..., (V_1^i, V_2^i)$, respectively. For each $k = 1, 2, ..., i$ we have $\rightarrow H_k[V_1^k] \subseteq \mathcal{P}$ and $\rightarrow H_k[V_2^k] \subseteq \mathcal{Q}$.

We now repeat the procedure: delete from $\mathcal{G}'$ all those G_n whose corresponding (W_1^n, W_2^n) do not induce (V_1^1, V_2^1), $(V_1^2, V_2^2), ..., (V_1^i, V_2^i)$ and call the resulting set $\mathcal{G}''$. If $j \geq i+1$ is the least integer such that $G_j \in \mathcal{G}''$, choose a partition of $V(G_j)$ that is induced by infinitely many of the (W_1^n, W_2^n) which satisfy $G_n \in \mathcal{G}''$, etc.

Following this procedure, we obtain for each $n \geq 1$ a partition (V_1^n, V_2^n) of $V(H_n)$ which satisfies $\rightarrow H_n[V_1^n] \subseteq \mathcal{P}$ and $\rightarrow H_n[V_2^n] \subseteq \mathcal{Q}$. With $V_1 = \bigcup_{n\geq 1} V_1^n$ and $V_2 = \bigcup_{n\geq 1} V_2^n$, we have a partition of $V(H)$. If either V_1 or V_2 is empty, move an arbitrary vertex into this set. By the construction of V_1 and V_2, $\rightarrow H[V_1] \subseteq \mathcal{P}$ and $\rightarrow H[V_2] \subseteq \mathcal{Q}$. ■

Corollary 5. *Let H be a finite core or a countable union of finite cores. If $\mathcal{P}$ and $\mathcal{Q}$ are non-trivial properties in $\mathbb{L}^a$ such that $\rightarrow H \subseteq \mathcal{P}\mathcal{Q}$ then there exists a partition (V_1, V_2) of $V(H)$ with $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$ such that $\rightarrow H \subseteq (\rightarrow H[V_1])(\rightarrow H[V_2]) \subseteq \mathcal{P}\mathcal{Q}$ and $\rightarrow H[V_1] \subseteq \mathcal{P}$ and $\rightarrow H[V_2] \subseteq \mathcal{Q}$.*

We can now describe the minimal reducible bounds for the hom-properties in $\mathbb{L}^a$.

4.1. Finite H

Let H be a finite core such that $\rightarrow H$ is irreducible in $\mathbb{L}^a$ (i.e. H is indecomposable). Let $\mathbf{H}$ be the set of all hom-properties $\rightarrow C_1+C_2 = (\rightarrow C_1)(\rightarrow C_2)$ formed as follows:

For each partition (V_1, V_2) of $V(H)$ with $V_1 \neq \emptyset$, $V_2 \neq \emptyset$, let $C_1 = C(H[V_1])$ and $C_2 = C(H[V_2])$.

Lemma 6. $\rightarrow H \subseteq \rightarrow C_1 + C_2$ *for each* $\rightarrow C_1 + C_2 \in \mathbf{H}$.

Proof. This will follow if we can show that there is a homomorphism from H to $C_1 + C_2$. By the definition of C_1 and C_2, there exist homomorphisms $f_1 : V_1 \rightarrow V(C_1)$ and $f_2 : V_2 \rightarrow V(C_2)$. Define $f : V(H) \rightarrow V(C_1 + C_2)$ by $f(x) = f_i(x)$ if $x \in V_i$, $i = 1, 2$. ■

Since H is a finite graph, the set $\mathbf{H}$ is finite and thus minimal elements (under inclusion of properties) exist. These minimal elements of $\mathbf{H}$ are precisely all the minimal reducible bounds of $\rightarrow H$, i.e. they form $\mathbf{B}(\rightarrow H)$.

Theorem 7. $\mathbf{B}(\rightarrow H) = \mathrm{Min}_{\subseteq}\mathbf{H}$.

Proof. We must show that if there are non-trivial properties $\mathcal{P}$ and $\mathcal{Q}$ in $\mathbb{L}^a$ such that $\to H \subset \mathcal{P}\mathcal{Q}$, then there exists a $\to C_1 + C_2 \in \mathbf{H}$ such that $\to H \subset\to C_1 + C_2 \subseteq \mathcal{P}\mathcal{Q}$. This follows immediately by Corollary 5: there exists a $(\mathcal{P}, \mathcal{Q})$ partition (V_1, V_2) of $V(H)$ with $V_1 \neq \emptyset$, $V_2 \neq \emptyset$ such that $\to H \subseteq\to H[V_1] \to H[V_2] \subseteq \mathcal{P}\mathcal{Q}$, and so $\to H \subseteq (\to C(H[V_1]))(\to C(H[V_2])) \subseteq \mathcal{P}\mathcal{Q}$. ■

All the minimal reducible bounds in $\mathbb{L}^a$ for a hom-property $\to H$, where H is finite, can thus be found by forming the finite set $\mathbf{H}$ (by considering all partitions (V_1, V_2) of $V(H)$ with $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$, and then forming the hom-properties $\to (C(H[V_1]) + C(H[V_2]))$ and then determining which of these reducible properties are minimal under inclusion.

4.2. Infinite H

We now consider minimal reducible bounds in $\mathbb{L}^a$ for an irreducible $\to H$, where H is an infinite union of finite cores. By Corollary 5, if a minimal reducible bounds exists for such a $\to H$, it is of the same form as in the finite case, i.e. it has the form $(\to H[V_1])(\to H[V_2])$ for some partition (V_1, V_2) of $V(H)$ with $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$. We can again form the set $\mathbf{H}$ for an infinite graph H, $\mathbf{H} = \{(\to H[V_1])(\to H[V_2]) : (V_1, V_2)$ is a partition of $V(H)$ and $V_1 \neq \emptyset$, $V_2 \neq \emptyset\}$ and clearly $\to H \subseteq (\to H[V_1])(\to H[V_2])$ for each $(\to H[V_1])(\to H[V_2])$ in $\mathbf{H}$. However $\mathbf{H}$ will now be an infinite set and the existence of minimal elements is no longer trivial. In the following theorem we show that $\mathbf{H}$ has minimal elements and that every element of $\mathbf{H}$ contains a minimal element. These minimal elements thus form $\mathbf{B}(\to H)$, the set of all minimal reducible bounds for $\to H$.

Theorem 8. *Let H be an countable union of finite cores. Then the set* $\mathbf{H}$ *contains minimal elements, and each element of* $\mathbf{H}$ *contains a minimal element of* $\mathbf{H}$.

Proof. We will first use Zorn's lemma to show that $\mathbf{H} = \{(\to H[V_1])(\to H[V_2]) : (V_1, V_2)$ is a partition of $V(H)$, $V_1 \neq \emptyset, V_2 \neq \emptyset\}$ has minimal elements. This will follow if we can show that every chain in $\mathbf{H}$ has a lower bound in $\mathbf{H}$.

Suppose to the contrary that $\mathcal{C} = \{(\to H[V_1^\alpha])(\to H[V_2^\alpha]) : \alpha \in A\}$ is an infinite chain in $\mathbf{H}$ that does not have a lower bound in $\mathbf{H}$. Then given any element of the chain, there exists an infinite chain of elements of $\mathcal{C}$ below it.

Suppose $H = H_1 \cup H_2 \cup ....$ For each $\alpha \in A$, the partition (V_1^α, V_2^α) of $V(H)$ induces a partition of $V(H_1)$. Since $V(H_1)$ has only finitely many partitions, there exists a partition $(V_{1,1}, V_{2,1})$ of $V(H_1)$ that is induced infinitely many times and that satisfies: given any $\alpha \in A$, there exists $\alpha' \in A$ such that $(\to H[V_1^{\alpha'}])(\to H[V_2^{\alpha'}]) \subset (\to H[V_1^\alpha])(\to H[V_2^\alpha])$ and $(V_1^{\alpha'}, V_2^{\alpha'})$ induces $(V_{1,1}, V_{2,1})$ in $V(H_1)$. (If for each induced partition of $V(H_1)$ occuring infinitely many times, there exists an α such that every $\alpha' \in A$ satisfying $(\to H[V_1^{\alpha'}])(\to H[V_2^{\alpha'}]) \subset (\to H[V_1^\alpha])(\to H[V_2^\alpha])$ induces some different partition of $V(H_1)$, then, since these α are finite, we can choose the one among them corresponding to the least element of $\mathcal{C}$. This element of $\mathcal{C}$ contains only finitely many other elements of $\mathcal{C}$ below it, contradicting our hypothesis.) We have $H_1[V_{1,1}] \in\to H[V_1^{\alpha'}]$ and $H_2[V_{1,2}] \in\to H[V_2^{\alpha'}]$.

Now form A' from A by deleting all those α for which (V_1^α, V_2^α) does not induce $(V_{1,1}, V_{2,1})$. For any $\alpha \in A$, there exists α' in A' such that $(\to H[V_1^{\alpha'}])(\to H[V_2^{\alpha'}]) \subset (\to H[V_1^\alpha])(\to H[V_2^\alpha])$ and $H_1[V_{1,1}] \in\to H[V_1^{\alpha'}]$ and $H_1[V_{2,1}] \in\to H[V_2^{\alpha'}]$. We now have a new infinite chain, $\mathcal{C}' = \{(\to H[V_1^\alpha])(\to H[V_2^\alpha]) : \alpha \in A'\}$, and we repeat the procedure using H_2 and $\mathcal{C}'$, to form $\mathcal{C}''$, etc. For each H_i we obtain a partition $(V_{1,i}, V_{2,i})$ of $V(H_i)$ and after completing the procedure i times, we have a chain of $(\to H[V_1^\alpha])(\to H[V_2^\alpha])$ such that for all α in the new index set, the partition (V_1^α, V_2^α) of $V(H)$ induces the partition $(V_{1,j}, V_{2,j})$ of $V(H_j)$ for all $j = 1, 2, ..., i$. Also, for any $\alpha \in A$, there exists α' in the new index set such that $(\to H[V_1^{\alpha'}])(\to H[V_2^{\alpha'}]) \subset (\to H[V_1^\alpha])(\to H[V_2^\alpha])$ and $H_j[V_{1,j}] \in\to H[V_1^{\alpha'}]$ and $H_j[V_{2,j}] \in\to H[V_2^{\alpha'}]$ for all $j = 1, 2, ..., i$.

Now let $V_1 = \bigcup_{i\geq 1} V_{1,i}$ and let $V_2 = \bigcup_{i\geq 1} V_{2,i}$. There are now two possibilities: either both V_1 and V_2 are non-empty, or one of them (say V_2) is empty while the other (V_1) equals $V(H)$.

Suppose first that both V_1 and V_2 are non-empty. Then $(\to H[V_1])(\to H[V_2])$ is itself in $\mathbf{H}$. We will show that $(\to H[V_1])(\to H[V_2])$ is a lower bound for the chain $\mathcal{C}$.

Let $\alpha \in A$ and let $G \in (\to H[V_1])(\to H[V_2])$. Then there exists a partition (A, B) of $V(G)$ such that $G[A] \to H[V_1]$ and $G[B] \to H[V_2]$. Since both $G[A]$ and $G[B]$ are finite, there exists an integer n such that $G[A] \to \cup\{H_i[V_{1,i}] : i = 1, 2, ..., n\}$ and $G[B] \to \cup\{H_i[V_{2,i}] : i = 1, 2, ..., n\}$. Now by the remark at the end of the previous paragraph, after n steps of the procedure,there exists an α' in the modified index set of the chain with $(\to H[V_1^{\alpha'}])(\to H[V_2^{\alpha'}]) \subset (\to H[V_1^\alpha])(\to H[V_2^\alpha])$ and such that $H_i[V_{1,i}] \in\to H[V_1^{\alpha'}]$ and $H_i[V_{2,i}] \in\to H[V_2^{\alpha'}]$ for $i = 1, 2, ..., n$. Hence $G[A] \in\to H[V_1^{\alpha'}]$

and $G[B] \in \rightarrow H[V_2^{\alpha'}]$, so $G \in (\rightarrow H[V_1^{\alpha'}])(\rightarrow H[V_2^{\alpha'}]) \subset (\rightarrow H[V_1^{\alpha}])(\rightarrow H[V_2^{\alpha}])$, i.e. $(\rightarrow H[V_1])(\rightarrow H[V_2]) \subseteq (\rightarrow H[V_1^{\alpha}])(\rightarrow H[V_2^{\alpha}])$.

Now suppose that V_2 is empty and that $V_1 = V(H)$. We claim that in this case, any element of $\mathbf{H}$ of the form $(\rightarrow H[W_1])(\rightarrow H[W_2])$ where W_2 is independent, is a lower bound for the chain $\mathcal{C}$. To prove this, fix such an element of $\mathbf{H}$. Suppose it is $(\rightarrow H[W_1])(\rightarrow H[W_2])$, with W_2 independent. Let $\alpha \in A$ and let $G \in (\rightarrow H[W_1])(\rightarrow H[W_2])$. We must show that $G \in (\rightarrow H[V_1^{\alpha}])(\rightarrow H[V_2^{\alpha}]$: Since G is finite, there exists an integer n such that $G \in (\rightarrow (H_1 \cup H_2 \cup ... \cup H_n)[W_1])(\rightarrow (H_1 \cup H_2 \cup ... \cup H_n)[W_2])$. Now there exists an $\alpha' \in A$ such that $(\rightarrow H[V_1^{\alpha'}])(\rightarrow H[V_2^{\alpha'}]) \subset (\rightarrow H[V_1^{\alpha}])(\rightarrow H[V_2^{\alpha}])$ and $(V_1^{\alpha'}, V_2^{\alpha'})$ induces $(V_{1,i}, V_{2,i}) = (V(H_i), \emptyset)$ for each $i = 1, 2, ..., n$. Then $(H_1 \cup H_2 \cup ... \cup H_n)[W_1] \rightarrow H[V_1^{\alpha'}]$ (the inclusion map) and $(H_1 \cup H_2 \cup ... \cup H_n)[W_2] \rightarrow H[V_2^{\alpha'}]$ (since W_2 is independent and $V_2^{\alpha'}$ is non-empty.) Hence $G \in (\rightarrow H[V_1^{\alpha'}])(\rightarrow H[V_2^{\alpha'}]) \subset (\rightarrow H[V_1^{\alpha}])(\rightarrow H[V_2^{\alpha}])$.

We can conclude by Zorn's lemma that the set $\mathbf{H}$ has minimal elements. By fixing an element of $\mathbf{H}$ and considering only chains of elements of $\mathbf{H}$ each of which is contained in that fixed element, the same argument as above shows that each element of $\mathbf{H}$ contains at least one of these minimal elements of $\mathbf{H}$. Hence, as in the case where H is finite, the minimal elements of $\mathbf{H}$ form $\mathbf{B}(\rightarrow H)$ when H is an infinite union of finite graphs. ∎

5. Some Applications

In the following applications, we allow the graph H to be either finite or a countable union of finite graphs and we show the existence of minimal reducible bounds of certain types in $\mathbb{L}^a$ for $\rightarrow H$. In this section we assume throughout that $\rightarrow H$ is irreducible, while if H is finite it is assumed to be a core.

Proposition 9. *If H is a graph with chromatic number 3, then $\mathcal{O}^3$ is the unique minimal reducible bound for $\rightarrow H$.*

Proof. Since $\chi(H) = 3$, there exists a partition (V_1, V_2) of $V(H)$ such that $H[V_1]$ is an independent set of vertices and $H[V_2]$ has chromatic number 2, i.e.$\rightarrow C(H[V_1]) \rightarrow C(H[V_2]) = \rightarrow K_1 + K_2 = \rightarrow K_3 = \mathcal{O}^3$.

If $\rightarrow H \subset \rightarrow C_1 \rightarrow C_2$ for any other $\rightarrow C_1 \rightarrow C_2 \in \mathbf{H}$, then either C_1 or C_2 must contain an edge (since $\chi(C_1) + \chi(C_2) \geq 3$) and hence $K_1 + K_2 \in \rightarrow C_1 \rightarrow C_2$, i.e. $\rightarrow H \subset \rightarrow K_1 + K_2 = \mathcal{O}^3 \subseteq \rightarrow C_1 \rightarrow C_2$. ∎

Proposition 10. *If H is a graph with chromatic number 4, then all minimal reducible bounds of $\to H$ are of the form $\mathcal{O}(\to X)$ for some graph $X \subset H$.*

Proof. Since $\chi(H) = 4$, there exists a partition (V_1, V_2) of $V(H)$ such that $\chi(H[V_1]) = 2$ and $\chi(H[V_2]) = 2$, i.e. $\to C(H[V_1]) \to C(H[V_2]) =\to K_2 + K_2 =\to K_1 + K_3 = \mathcal{O}(\to K_3)$.

Consider all partitions (V_1, V_2) of $V(H)$. If $H[V_1]$ or $H[V_2]$ is independent, we get a reducible bound for $\to H$ of the form $\mathcal{O}(\to H[V_1])$ or $\mathcal{O}(\to H[V_2])$. If neither $H[V_1]$ nor $H[V_2]$ is independent, then $K_2 \to H[V_1]$ and $K_2 \to H[V_2]$, so $\to K_2 + K_2 = \mathcal{O}(\to K_3) \subseteq\to H[V_1] \to H[V_2]$.

We can now conclude that all the minimal elements of $\mathbf{H}$ are of the form $\mathcal{O}(\to X)$ for some graph $X \subset H$. ■

Proposition 11. *If H is a graph with chromatic number 5, then $\to H$ has a minimal reducible bound of the form $\mathcal{O}(\to X)$ for some graph $X \subset H$.*

Proof. Since $\chi(H) = 5$, there exists a bound of the form $\mathcal{O}(\to X) = (\to K_1)(\to X)$ for $\to H$ with $X \subset H$ and $\chi(X) = 4$. Suppose that $\to X_1 \to X_2$ is any other element of $\mathbf{H}$ satisfying $\to H \subseteq\to X_1 \to X_2 \subseteq \mathcal{O}(\to X)$. Since $\chi(H) = \chi(K_1) + \chi(X) = 5$, we must have $\chi(X_1) + \chi(X_2) = 5$ and this is only possible if one of X_1 or X_2 has chromatic number at most 2.

Say $\chi(X_1) \leq 2$. Then we can assume that $X_1 = K_1$ or $X_1 = K_2$. In the first case, $\to X_1 \to X_2 =\to K_1 \to X_2 = \mathcal{O}(\to X_2)$, while in the second, $\to X_1 \to X_2 =(\to K_1)(\to K_1 \to X_2)$. By Corollary 5, there exists a bound for $\to H$ of the form $\mathcal{O}(\to Y)$ with $Y \subset H$ satisfying $\to H \subseteq \mathcal{O}(\to Y) \subseteq (\to K_1)(\to K_1 \to X_2)$. In either case there exists a bound for $\to H$ of the form $\mathcal{O}(\to Y)$ with $Y \subset H$ satisfying $\to H \subseteq \mathcal{O}(\to Y) \subseteq\to X_1 \to X_2$, so we conclude that $\mathbf{H}$ has a minimal element of the form $\mathcal{O}(\to Y)$ for some $Y \subset H$. ■

Proposition 12. *If H is a graph with chromatic number either infinite or finite and greater than or equal to 6, and if K_4 is not a subgraph of H, then $\to H$ has a minimal reducible bound of the form $\mathcal{O}(\to X)$ for some $X \subset H$.*

Proof. There exists a bound for $\to H$ of the form $\mathcal{O}(\to X)$ where $X \subset H$, and $\chi(X) \geq 5$, which is minimal of this type.

Suppose $\to H \subset (\to X_1)(\to X_2) \subseteq \mathcal{O}(\to X)$ where $(\to X_1)(\to X_2) \in \mathbf{H}$ is not of the form $\mathcal{O}(\to Y)$ for any graph Y. If the chromatic number of either X_1 or X_2 is one, say $\chi(X_1) = 1$, then $(\to X_1)(\to X_2) = \mathcal{O}(\to X_2)$, contradicting our assumption on the form of $(\to X_1)(\to X_2)$. If one of X_1 or X_2 has chromatic number 2, say $\chi(X_1) = 2$, then $(\to X_1)(\to X_2) = \mathcal{O}$

$(\mathcal{O}(\to X_2))$ and by Corollary 5 there exists an element of $\mathbf{H}$ of the form $\mathcal{O}(\to Y)$ between $\to H$ and $\mathcal{O}(\mathcal{O}(\to X_2))$, contradicting the minimality of $\mathcal{O}(\to X)$.

Thus $\chi(X_1) \geq 3$ and $\chi(X_2) \geq 3$ so that both X_1 and X_2 contain an odd cycle, say S_1 and S_2 respectively. But then $S_1 + S_2 \in (\to X_1)(\to X_2) \subseteq \mathcal{O}(\to X)$, so $V(S_1 + S_2)$ has an $(\mathcal{O}, (\to X))$-partition, say (V_1, V_2). Thus $(S_1 + S_2)[V_1]$ is an independent subgraph of either S_1 or S_2, and (since $\chi(S_1) = 3$ and $\chi(S_2) = 3$), $(S_1 + S_2)[V_2]$ must contain K_4 as a subgraph, a contradiction since $(S_1 + S_2)[V_2] \in\to X$, and any K_4 in $(S_1 + S_2)[V_2]$ would force a K_4 in $X \subset H$.

We conclude that $\mathbf{H}$ has a minimal element of the form $\mathcal{O}(\to Y)$ for some $Y \subset H$. ∎

Proposition 13. *If H is a graph with finite chromatic number satisfying $\chi(H) = n \geq 6$, and $K_{n-1} \subset H$, then $\to H$ has a minimal reducible bound of the form $\mathcal{O}(\to X)$ for some $X \subset H$.*

Proof. There exists an element $\mathcal{O}(\to X) \in \mathbf{H}$ with $\chi(X) = n-1$. Suppose now that $\to H \subset (\to H[V_1])(\to H[V_2]) \subseteq \mathcal{O}(\to X)$, with $(\to H[V_1])(\to H[V_2]) \in \mathbf{H}$. Then $\chi(H[V_1]) + \chi(H[V_2]) = n$. Since $K_{n-1} \subset H$, there exists $K_i \subseteq H[V_1]$ and $K_j \subseteq H[V_2]$ with $i + j = n - 1$.

If $i \geq \chi(H[V_1])$, then $C(H[V_1]) = K_i$, so $(\to H[V_1])(\to H[V_2]) = (\to K_1)(\to K_{i-1} \to H[V_2])$ and by Corollary 5, there exists a bound for $\to H$ of the form $\mathcal{O}(\to Y)$ for some $Y \subset H$, contained in $(\to H[V_1])(\to H[V_2])$. However if $i < \chi(H[V_1])$, then $j \geq \chi(H[V_2])$ and $\mathrm{C}(H[V_2]) = K_j$, and once again $(\to H[V_1])(\to H[V_2])$ contains a bound for $\to H$ of the form $\mathcal{O}(\to Y)$ for some $Y \subset H$.

We conclude that $\mathbf{H}$ has a minimal element of the form $\mathcal{O}(\to Y)$ for some $Y \subset H$. ∎

Proposition 14. *If H is a triangle-free graph with finite chromatic number satisfying $\chi(H) \geq 6$, then $\to H$ has a minimal reducible bound not of the form $\mathcal{O}\mathcal{P}$ for any $\mathcal{P} \in \mathbb{L}^a$.*

Proof. Since $\chi(H) \geq 6$, there exists $(\to X_1)(\to X_2) \in \mathbf{H}$ such that $\chi(X_1) \geq 3, \chi(X_2) \geq 3$, $\chi(X_1) + \chi(X_2) = \chi(H)$. Suppose $(\to X_1)(\to X_2) = \mathcal{O}(\to X)$ for some $X \subset H$. $X_1$and X_2 each contain an odd cycle, say S_1, and S_2 respectively. We then have that $S_1 + S_2 \in \mathcal{O}(\to X)$ so $V(S_1 + S_2)$ has an $(\mathcal{O}, \to X)$-partition, say (V_1, V_2). Since $(S_1 + S_2)[V_1]$ is an independent subset of either S_1 or S_2, $(S_1 + S_2)[V_2]$ must contain a triangle, forcing H to contain a triangle, contradicting our hypothesis. So $(\to X_1)(\to X_2)$ is not of the form $\mathcal{OP}$ for any $\mathcal{P} \in \mathbb{L}^a$.

Suppose now that $\to H \subset \mathcal{O}(\to X) \subset (\to X_1)(\to X_2)$ for some $X \subset H$. Since $\chi(H) = \chi(X_1)+\chi(X_2)$, it must be true that $\chi(X) = \chi(H)-1$. Let G be any finite subgraph of X with $\chi(G) = \chi(X)$. The graph $G+\{v\}$ is in $\mathcal{O}(\to X)$ and therefore in $(\to X_1)(\to X_2)$, and so $V(G + \{v\})$ has a $(\to X_1, \to X_2)$-partition (V_1, V_2). Suppose that $v \in V_1$. If $\{w \in V(G) : w \in V_1\}$ is not an independent set of vertices, then $(G + v)[V_1]$ contains a triangle, and so X_1 contains a triangle, which is not possible. If $\{w \in V(G) : w \in V_1\}$ is an independent set of vertices, then $\chi((G + v)[V_2]) \geq \chi(H) - 2$. But $(G+v)[V_2] \in\to X_2$ and $\chi(X_2) \leq \chi(H)-3$, again a contradiction. Hence no bound of the form $\mathcal{OP}$ with $\mathcal{P} \in \mathbb{L}^a$ can occur between $\to H$ and $(\to X_1)$ $(\to X_2)$.

We conclude that $\mathbf{H}$ has a minimal element not of the form $\mathcal{O}(\to Y)$ for any $Y \subset H$. ■

The previous result is not true if we allow $\chi(H)$ to be infinite since the set of all triangle-free graphs, $\mathcal{I}_1$, has the unique minimal reducible bound $\mathcal{OI}_1$ (see [1], [6]). $\mathcal{I}_1$ is the hom-property $\to \cup\{R : R \text{ is a triangle free core}\}$, with infinite chromatic number.

Corollaries 12 and 14 show that if H has a finite chromatic number greater than or equal to 6, and H is triangle-free, then $\to H$ has a minimal reducible bound of the form $\mathcal{OP}$ for some $\mathcal{P} \in \mathbb{L}^a$ and a minimal reducible bound not of this form.

6. Minimal Reducible Bounds for $\to H$ in $\mathbb{L}$

We now describe the minimal reducible bounds of a hom-property $\to H$ in the lattice of hereditary properties, $\mathbb{L}$. Again, we will describe the case for a finite H first, and then draw conclusions about an infinite H. The following lemma and its corollary are useful in both the finite and infinite cases.

Lemma 15. *Let H be a finite graph or a countable union of finite graphs. If $\to H \subseteq \mathcal{PQ}$, where $\mathcal{P}$ and $\mathcal{Q}$ are non-trivial properties in $\mathbb{L}$ such that $\mathcal{O} \not\subseteq \mathcal{Q}$, then $\to H \subseteq \mathcal{P}$.*

Proof. Suppose first that H is finite, and suppose that the cardinality of the largest edgeless graph in $\mathcal{Q}$ is k. For any $m > k$, $H^{::}(m) \in \mathcal{PQ}$ and by the restriction on $\mathcal{Q}$, $H^{::}(m-k)$ must be in $\mathcal{P}$. This is true for any $m > k$ so that $H^{::}(r) \in \mathcal{P}$ for all $r \geq 1$, i.e. $\to H \subseteq \mathcal{P}$.

If H is infinite, then since $\rightarrow H' \subseteq \mathcal{PQ}$ for any finite subgraph H' of H, by the finite case we can conclude that $\rightarrow H' \subseteq \mathcal{P}$ for every finite subgraph H' of H. Since any graph in $\rightarrow H$ is contained in some $\rightarrow H'$ where H' is a finite subgraph of H, we can conclude that $\rightarrow H \subseteq \mathcal{P}$. ∎

Corollary 16. *Let H be a finite graph or a countable union of finite graphs. If $\rightarrow H \subseteq \mathcal{PQ}$, where $\mathcal{P}$ and $\mathcal{Q}$ are non-trivial properties in $\mathbb{L}$ such that $\mathcal{O} \nsubseteq \mathcal{Q}$, then $\rightarrow H \subseteq (\rightarrow H)(\{K_1\}) \subseteq \mathcal{PQ}$.*

Proof. The proof is immediate as $\rightarrow H \subseteq \mathcal{P}$ and, since $\mathcal{Q}$ is non-trivial, $K_1 \in \mathcal{Q}$. ∎

We now describe the minimal reducible bounds for hom-properties in $\mathbb{L}$.

6.1. Finite H

Theorem 17. *If H is a finite indecomposable core then the minimal reducible bounds for $\rightarrow H$ in $\mathbb{L}$ are the minimal elements of $\mathbf{H}$ as well as the property $(\rightarrow H)(\{K_1\})$.*

Proof. By Lemma 4 and Corollary 16 we know that if $\rightarrow H \subset \mathcal{PQ}$, where $\mathcal{P}$ and $\mathcal{Q}$ are non-trivial properties in $\mathbb{L}$, then if $\mathcal{O} \subseteq \mathcal{P}$ and $\mathcal{O} \subseteq \mathcal{Q}$, we have a minimal element of $\mathbf{H}$ between $\rightarrow H$ and $\mathcal{PQ}$, while if $\mathcal{O} \nsubseteq \mathcal{Q}$, then $(\rightarrow H)(\{K_1\})$ lies between $\rightarrow H$ and $\mathcal{PQ}$. Note that the case $\mathcal{O} \nsubseteq \mathcal{P}$ and $\mathcal{O} \nsubseteq \mathcal{Q}$ cannot occur since by Lemma 15, if $\mathcal{O} \nsubseteq \mathcal{Q}$, then $\rightarrow H \subseteq \mathcal{P}$, and since H is assumed to have at least one vertex, all multiplications of this vertex must be in $\mathcal{P}$ i.e. $\mathcal{O} \subseteq \mathcal{P}$.

To complete the proof of the theorem, we must show that $(\rightarrow H)(\{K_1\})$ is not contained in any minimal element of $\mathbf{H}$, and that no minimal element of $\mathbf{H}$ is contained in $(\rightarrow H)(\{K_1\})$.

First suppose to the contrary that $\rightarrow H[V_1]+H[V_2]$ is a minimal element of $\mathbf{H}$ satisfying $\rightarrow H[V_1] + H[V_2] \subseteq (\rightarrow H)(\{K_1\})$. By Lemma 15 we then have $\rightarrow H[V_1] + H[V_2] \subseteq\rightarrow H$, and so $H[V_1] + H[V_2] \rightarrow H$. If this homomorphism is a surjection, then H is decomposable, a contradiction, while if this homomorphism is not a surjection, then we can use it to map H into a proper subgraph of itself, a contradiction to the fact that H is a core.

Now suppose that $\rightarrow (H[V_1] + H[V_2])$ is a minimal element of $\mathbf{H}$ and that $(\rightarrow H)(\{K_1\}) \subseteq\rightarrow (H[V_1] + H[V_2])$. Now $H + K_1 \in (\rightarrow H)(\{K_1\}) \subseteq\rightarrow (H[V_1] + H[V_2])$, so we have the inclusions $\rightarrow H \subseteq\rightarrow (H + K_1) = (\rightarrow H)(\mathcal{O}) \subseteq\rightarrow (H[V_1] + H[V_2])$. By Lemma 4 there exists an element

$\to (H[W_1] + H[W_2])$ in $\mathbf{H}$ satisfying $\to H \subseteq\to (H[W_1] + H[W_2]) \subseteq (\to H)(\mathcal{O}) \subseteq\to (H[V_1] + H[V_2])$, and $\to H[W_1] \subseteq\to H$ and $\to H[W_2] = \mathcal{O}$. By the minimality of $\to (H[V_1] + H[V_2])$ in $\mathbf{H}$, the two elements of $\mathbf{H}$ must be equal, and so we have $(\to H)(\mathcal{O}) =\to (H[W_1]+H[W_2])$ i.e. $(\to H)(\mathcal{O}) =\to H[W_1] \to H[W_2]$. By the unique factorisation theorem [3], and the fact that $\to H[W_2] = \mathcal{O}$, we can conclude that $\to H =\to H[W_1]$ and $\mathcal{O} =\to H[W_2]$. But then we have a homomorphism from H to $H[W_1]$, a proper subgraph of H, contradicting the fact that H is a core. ■

6.2. Infinite H

Theorem 18. *If H is an infinite union of finite graphs, then the minimal elements of the set* $\mathbf{H} \cup\{(\to H)(\{K_1\})\}$ *are the minimal reducible bounds for* $\to H$ *in* $\mathbb{L}$.

This result immediately follows from Lemma 4 and Corollary 16. The sharper result from the finite case is no longer true since when H is infinite, it may be possible that $(\to H)(\{K_1\})$ is properly contained in a minimal element of $\mathbf{H}$ e.g. $\mathcal{I}_1$ has the unique minimal reducible bound in $\mathbb{L}^a$ of $\mathcal{I}_1 O$, the unique minimal element of $\mathbf{H}$. In $\mathbb{L}$ however, we have $\mathcal{I}_1 \subsetneq \mathcal{I}_1\{K_1\} \subsetneq \mathcal{I}_1 O$, so that $\mathcal{I}_1$ has unique minimal reducible bound $\mathcal{I}_1\{K_1\}$.

It is not true that $(\to H)(\{K_1\})$ is contained in every minimal element of $\mathbf{H}$, since if $(\to H)(\{K_1\}) \subseteq (\to H[V_1])(\to H[V_2])$ where $(\to H[V_1])(\to H[V_2])$ is minimal in $\mathbf{H}$, then we have $\to H \subseteq (\to H)(\mathcal{O}) \subseteq (\to H[V_1])(\to H[V_2])$. (The second inclusion follows since any graph G in $(\to H)(\mathcal{O})$ is in $\to (H' + K_1)$ for some finite subgraph H' of H, and since $H' + K_1 \in\to H[V_1] \to H[V_2]$, we have that $\to (H'+K_1) \in\to H[V_1] \to H[V_2]$.) By Lemma 4 there should be another element of $\mathbf{H}$ between $\to H$ and $(\to H)(\mathcal{O})$. By the minimality of $\to (H[V_1] + H[V_2])$, we now have that $(\to H)(\mathcal{O}) = (\to H[V_1])(\to H[V_2])$. However (Corollary 14) if H is infinite and triangle-free with finite chromatic number at least six, $\mathbf{H}$ contains at least one minimal element which does not contain the factor $\mathcal{O}$.

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