

**A NOTE ON THE RAMSEY NUMBER AND THE  
PLANAR RAMSEY NUMBER FOR  $C_4$   
AND COMPLETE GRAPHS**

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**Abstract**

We give a lower bound for the Ramsey number and the planar Ramsey number for  $C_4$  and complete graphs. We prove that the Ramsey number for  $C_4$  and  $K_7$  is 21 or 22. Moreover we prove that the planar Ramsey number for  $C_4$  and  $K_6$  is equal to 17.

**Keywords:** planar graph, Ramsey number.

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1 INTRODUCTION

Let  $F, G, H$  be simple graphs with at least two vertices. The Ramsey number  $R(G, H)$  is the smallest integer  $n$  such that in arbitrary two-colouring (say red and blue) of  $K_n$  a red copy of  $G$  or a blue copy of  $H$  is contained (as subgraphs).

Let the *planar Ramsey number*  $PR(G, H)$  be the smallest integer  $n$  such that any planar graph on  $n$  vertices contains a copy of  $G$  or its complement contains a copy of  $H$ .

So we have an immediate inequality between planar and ordinary Ramsey number, i.e.,  $PR(G, H) \leq R(G, H)$ .

Walker in [9] and Steinberg and Tovey in [8] studied the planar Ramsey number but only in the case when both graphs are complete.

In this paper we will only consider the case when  $G$  is a cycle  $C_4$  of order 4 and  $H$  is a complete graph  $K_t$  of order  $t$ . In that case one can say that the Ramsey number is the smallest integer  $n$  such that any graph on  $n$  vertices contains a copy of  $C_4$  or an independent set of cardinality  $t$ . The

problem for the case when  $G$ , i.e., the first graph of the pair, is a cycle has been studied by J.A. Bondy, P. Erdős in [3] and by P. Erdős, R.J. Faudree, C.C. Rousseau, R.H. Schelp in [6]. We give a lower bound for the Ramsey number and the planar Ramsey number for  $C_4$  and complete graphs. We prove that the Ramsey number for  $C_4$  and  $K_7$  is 21 or 22.

Moreover in Theorem 6 we prove that  $PR(C_4, K_6) = 17$ .

A graph  $F$  is said to be a  $(G, K_t)$ -Ramsey-free graph if it does not contain any copy of  $G$  and any independent set of cardinality  $t$ . For graphs  $G, H$  the symbol  $G \cup H$  denotes a disjoint union of graphs,  $tG$  a disjoint union of  $t$  copies of the graph  $G$ ,  $\overline{G}$  a complement of  $G$ ,  $G - S$  a subgraph of  $G$  induced by a subset  $V(G) - S$  of the vertices of  $G$  where  $S \subset V(G)$ , and  $G \supset H$  express the fact that a graph  $H$  is a subgraph of  $G$ . Then  $deg_G(x)$  denotes the degree of the vertex  $x$  in the graph  $G$ , and  $\delta(G)$  is the minimum vertex degree over all vertices of  $G$ . Moreover  $N(x)$  is the set of vertices adjacent to  $x$ , and  $N[x]$  is the closed neighbourhood, i.e.,  $N[x] = N(x) \cup \{x\}$ .

The following theorems summarises the results for ordinary and planar Ramsey numbers known so far referring to the cases when the first graph is a cycle of order 4 and the second one is a complete graph.

**Theorem 1** [4], [5], [7]. (i)  $R(C_4, K_3) = 7$ ;  
(ii)  $R(C_4, K_4) = 10$ ;  
(iii)  $R(C_4, K_5) = 14$ ;  
(iv)  $R(C_4, K_6) = 18$ .

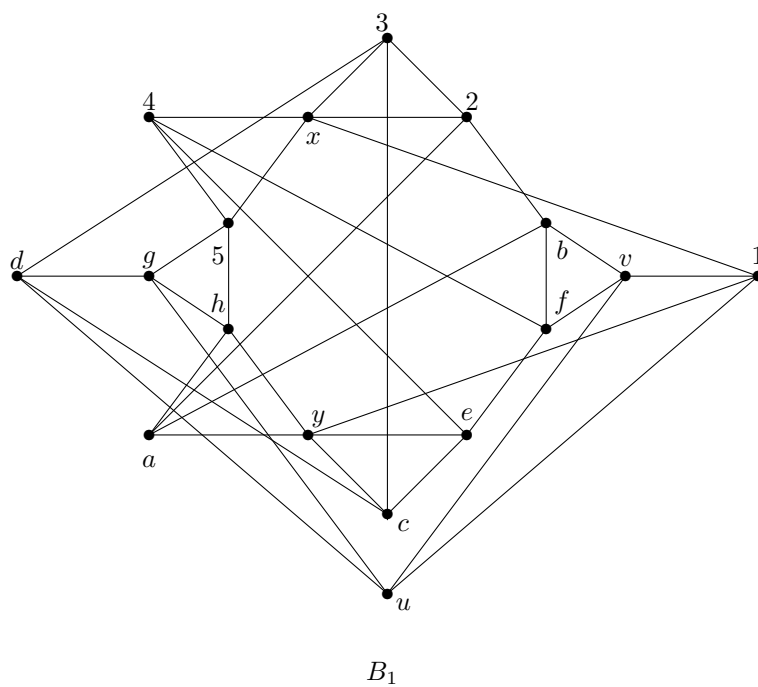
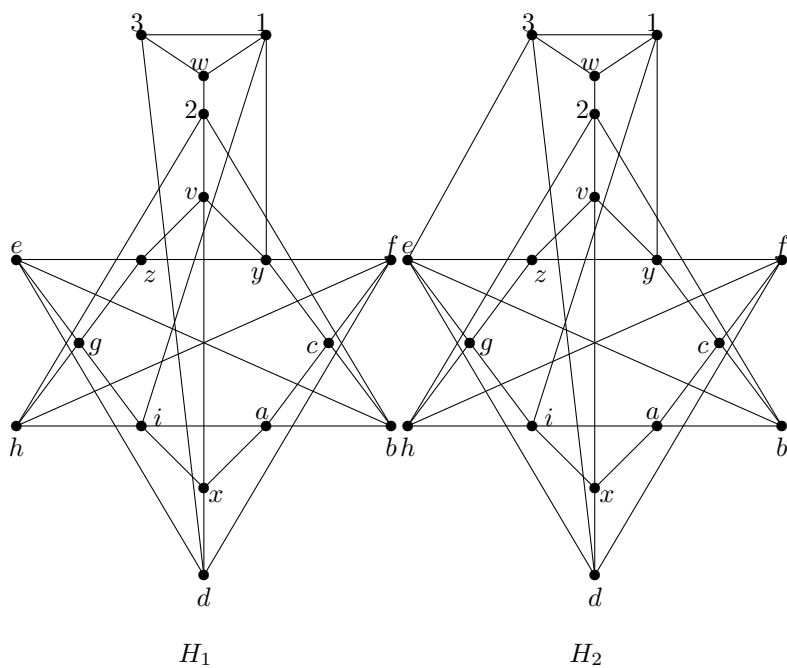
**Theorem 2** [1]. (i)  $PR(C_4, K_3) = 7$ ;  
(ii)  $PR(C_4, K_4) = 10$ ;  
(iii)  $PR(C_4, K_5) = 13$ .

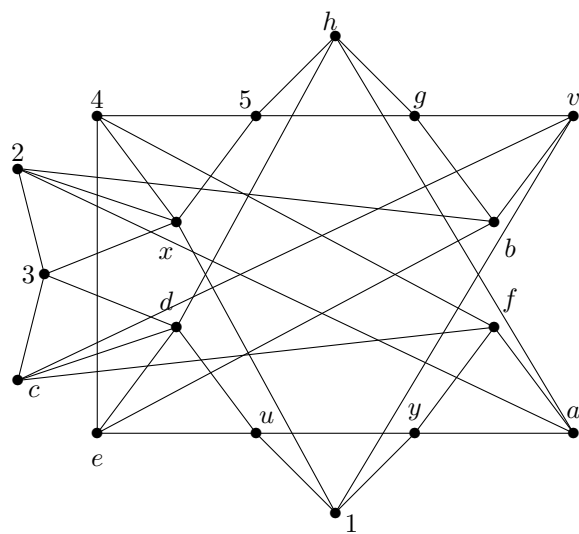
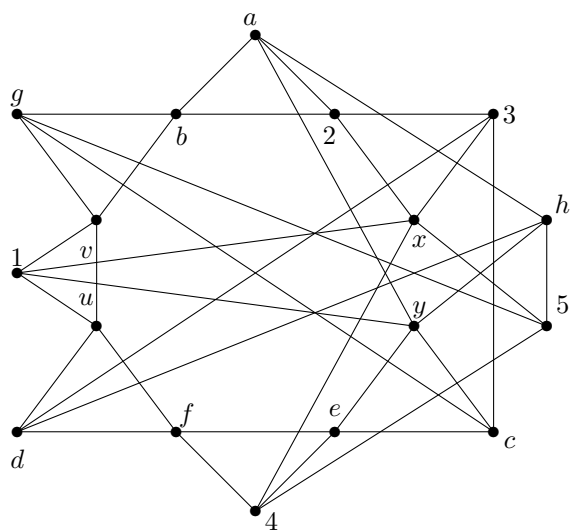
## 2 MAIN RESULTS

We use the following lemma to prove some further results for the Ramsey and the planar Ramsey number of pair of graphs.

**Lemma 3** [2]. *Let  $G$  be a graph of order 17 with independence number less than 6 and without  $C_4$ . Then  $G$  is isomorphic to one of the graphs presented in Figure 1.*

Therefore we have the following simple general observation.



 $B_2$  $B_3$ Figure 1. Graphs of order 17 without  $C_4$  and with  $\alpha(G) < 6$ .

**Proposition 4.** For each integer  $t \geq 6$ ,  $R(C_4, K_{t+1}) \geq 3t + 2\lceil \frac{t}{5} \rceil + 1$ .

**Proof.** Let  $H$  be a graph of order 17 presented in Figure 1. Note that  $H$  does not contain any subgraph  $C_4$  and  $\alpha(H) = 5$ . Therefore  $\lceil \frac{t}{5} \rceil H \cup (t - \lceil \frac{t}{5} \rceil)K_3$ ,  $t \geq 5$ , shows that  $R(C_4, K_{t+1}) \geq 3t + 2\lceil \frac{t}{5} \rceil + 1$ . ■

**Theorem 5.**  $21 \leq R(C_4, K_7) \leq 22$ .

**Proof.** Immediately by Proposition 4 we get  $21 \leq R(C_4, K_7)$ . Suppose for the contrary that  $R(C_4, K_7) > 22$ . Let  $G$  be a  $(C_4, K_7)$ -Ramsey-free graph of order 22. Note that  $\delta(G) < 5$ , else a  $C_4$  should be a subgraph of  $G$ .

Let  $m$  be an arbitrary vertex of  $G$  of the minimum degree  $\delta(G)$ .

Suppose that  $\delta(G) \leq 3$ . Then deleting a 3-degree vertex  $m$  and all its neighbours we get a graph  $F$  of the order at least 18. By Theorem 1(iv) the graph  $F$  contains an independent set  $S$  of cardinality 6. Thus  $S \cup \{m\}$  is an independent set of cardinality 7, a contradiction.

Therefore  $\delta(G) = 4$ . Let  $m_i$ ,  $i = 1, 2, 3, 4$  be the neighbours of  $m$  in  $G$ . Let us consider the graph  $F$  obtained from  $G$  by deleting the vertex  $m$  and all its neighbours. Since  $G$  does not contain any  $C_4$  then by degree condition each  $m_i$ ,  $i = 1, 2, 3, 4$  has at least two neighbours in  $F$ . Evidently the order of  $F$  equals 17 and  $F$  must be isomorphic to one of the  $(C_4, K_6)$ -Ramsey-free graphs presented in Figure 1 (else we get a contradiction as before). Suppose that  $F$  is isomorphic to  $H_1$  or  $H_2$ . Since the vertex  $w$  has degree 3 in  $F$  then it must be adjacent to one of the neighbours of  $m$ , say  $m_1$ . Let us consider the graph  $Y = G - N[w]$ . Note that the vertex  $m$  has degree 3 in  $Y$ . Hence  $Y$  must be one of the  $(C_4, K_6)$ -Ramsey-free graphs  $H_1$  or  $H_2$  presented in Figure 1. Evidently  $m$  is not adjacent to any vertex of the set  $\{d, v, b, h\}$ . Therefore each of the four vertices must be adjacent to a vertex of the set  $\{m_2, m_3, m_4\}$ . It is impossible without creating  $C_4$  because each two vertices of the set  $\{d, v, b, h\}$  are at distance 2. A contradiction. Therefore we can assume that  $F$  is not isomorphic to  $H_i$ ,  $i = 1, 2$ .

Suppose that  $F$  is isomorphic to  $B_1$ . Let the vertex  $x$  be adjacent to  $m_1$ . Then  $1m_1 \in E(G)$ , else  $\deg(m_1) < 4$ . Moreover without loss of generality  $m_1m_2 \in E(G)$ . Note that  $\deg(m_1) = 4$ . So we consider the graph  $Y = G - N[m_1]$ . Since  $Y$  cannot be isomorphic to  $H_i$ ,  $i = 1, 2$  then each of the vertices of the set  $\{2, 3, 4, 5\}$  must be adjacent to  $m_3$  or  $m_4$  and we get  $C_4$ , a contradiction. Therefore  $xm_i \notin E(G)$ , for  $i = 1, 2, 3, 4$ . By symmetry,  $ym_i \notin E(G)$ , for  $i = 1, 2, 3, 4$ .

Suppose that  $f$  is adjacent to  $m_1$ . Since  $C_4$  cannot be a subgraph then  $b, v, e$  or 4 is not adjacent to  $m_i$ ,  $i = 2, 3, 4$ . Therefore  $\deg(m_1) > 4$ , else

the graph  $G - N[m_1]$  has a 3-degree vertex, so it should be isomorphic to  $H_i$ ,  $i = 1, 2$  and we get a case above. Then  $m_1$  should be adjacent to 3 and  $h$ , and without loss of generality  $m_1m_2 \in E(G)$ . Note that  $m_2$  can be adjacent to one of the vertices  $d, 1$  or  $u$ . So  $\deg(m_2) < 4$  or a  $C_4$  exists, a contradiction.

Hence  $fm_i \notin E(G)$ , for  $i = 1, 2, 3, 4$ . By symmetry  $bm_i \notin E(G)$ , for  $i = 1, 2, 3, 4$ .

If the vertex 2 is adjacent to  $m_1$  then  $\deg(m_1) > 4$ , else  $G - N[m_1]$  has a 3-degree vertex, and we get a case above. Then  $m_1$  must be adjacent to  $e$  and to one of  $g, u$ . Moreover without loss of generality  $m_1m_2 \in E(G)$ . Note that  $\deg(m_2) < 4$  or a  $C_4$  exists, a contradiction.

Hence  $2m_i \notin E(G)$ , for  $i = 1, 2, 3, 4$ . By symmetry  $em_i \notin E(G)$ , for  $i = 1, 2, 3, 4$ .

Similar arguments give that 5 and  $h$  cannot be adjacent to  $m_i$ ,  $i = 1, 2, 3, 4$ .

Now without loss of generality we can assume that  $m_1, m_2$  and  $u$  create an independent set. Therefore  $\{m_1, m_2, 2, 5, y, f, u\}$  is an independent set.

Suppose that  $F$  is isomorphic to  $B_2$ . Let  $g$  be adjacent to  $m_1$ . Then  $m_1$  must be adjacent to 3 and  $y$ , and without loss of generality  $m_1m_2 \in E(G)$ , else the graph  $G - N[m_1]$  has a 3-degree vertex, so it should be isomorphic to  $H_i$ ,  $i = 1, 2$  and we get a case above. So  $m_2$  must be adjacent to 4 and  $e$ , and it has degree four. Therefore the vertices  $5, b, u, f$  must be adjacent to  $m_3$  or  $m_4$ , else we get a 3-degree vertex in  $G - N[m_2]$ . Without loss of generality we can assume that the vertex  $m_3$  is adjacent to  $b, f$ , and the vertex  $m_4$  is adjacent to  $5, u$ . Note that  $m_4$  has only these two neighbours in  $B_2$ . Hence  $m_4$  must be adjacent to  $m_3$  and  $\deg(m_4) = 4$ . Since  $h$  cannot be adjacent to  $m_i$ ,  $i = 1, 2, 3, 4$  the graph  $G - N[m_4]$  has a 3-degree vertex and we get a case above.

Hence  $gm_i \notin E(G)$ , for  $i = 1, 2, 3, 4$ . By symmetry  $ym_i \notin E(G)$ , for  $i = 1, 2, 3, 4$ .

Let 2 be adjacent to  $m_1$ . Then  $m_1$  should be adjacent to one of the vertices  $u, a, b$ . So the graph  $G - N[m_1]$  contains a 3-degree vertex  $g$  or  $y$ , and we get a case above. Hence  $2m_i \notin E(G)$ , for  $i = 1, 2, 3, 4$ . By symmetry  $cm_i \notin E(G)$ , for  $i = 1, 2, 3, 4$ .

Now without loss of generality we can assume that  $m_1, m_2$  and 4 create an independent set. Therefore  $\{m_1, m_2, 4, c, 2, g, y\}$  is an independent set.

Suppose that  $F$  is isomorphic to  $B_3$ . Let  $d$  be adjacent to  $m_1$ . Then  $m_1$  must be adjacent to one of the vertices  $3, b, g, h$ , and without loss of generality

$m_1m_2 \in E(G)$ . Since  $deg(m_1) = 4$  and  $m_2$  cannot be adjacent to  $3, h, f, u$ , then the graph  $G - N[m_1]$  has a 3-degree vertex, and we get a case above.

Hence  $dm_i \notin E(G)$ , for  $i = 1, 2, 3, 4$ . By symmetry  $gm_i \notin E(G)$ , for  $i = 1, 2, 3, 4$ .

Let  $a$  be adjacent to  $m_1$ . Then  $m_1$  must be adjacent to one of the vertices  $u, f, 4$ . As before  $deg(m_1) = 4$ . Note that one of the vertices  $2, b, h, y$  has 3-degree in  $G - N[m_1]$ , and we get a case above.

Hence  $a$  and  $4$  (by symmetry) cannot be adjacent to  $m_i$ ,  $i = 1, 2, 3, 4$ . Now without loss of generality we can assume that  $m_1, m_2$  and  $1$  create an independent set. Therefore  $\{m_1, m_2, 1, 4, a, d, g\}$  is an independent set.

All cases lead to a contradiction ■

For the planar case we get the following theorem.

**Theorem 6.**  $PR(C_4, K_6) = 17$ .

**Proof.** Since by Lemma 3 each  $(C_4, K_6)$ -Ramsey-free graph of order 17 is not planar and  $R(C_4, K_6) = 18$  we get  $PR(C_4, K_6) \leq 17$ . The graph presented in Figure 2 is  $(C_4, K_6)$ -Ramsey-free planar graph. So  $PR(C_4, K_6) > 16$ . ■

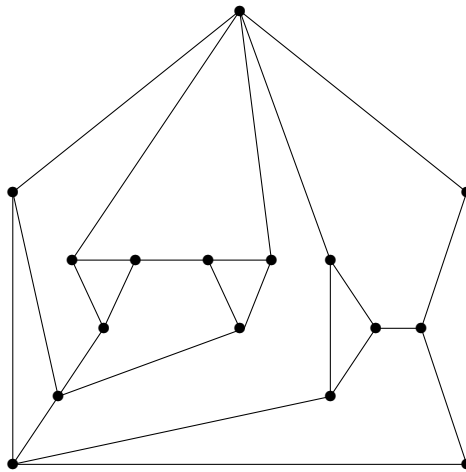


Figure 2. A planar graph of order 16 with independence number less than 6 and without  $C_4$ .

**Proposition 7.** For each integer  $t \geq 5$ ,  $PR(C_4, K_{t+1}) \geq 3t + \lfloor \frac{t}{5} \rfloor + 1$ .

**Proof.** Let  $H$  be a graph of order 16 presented in Figure 2. Note that  $H$  does not contain any subgraph  $C_4$  and  $\alpha(H) = 5$ . Therefore  $\lfloor \frac{t}{5} \rfloor H \cup (t - \lfloor \frac{t}{5} \rfloor 5)K_3$ ,  $t \geq 6$ , shows that  $PR(C_4, K_{t+1}) \geq 3t + \lfloor \frac{t}{5} \rfloor + 1$ . ■

**Added in Proof.** The result cited in Lemma 3 can be also find in: C.J. Jayawardene, C.C. Rousseau, An upper bound for Ramsey number of a quadrilateral versus a complete graph on seven vertices, *Congressus Numerantium* **130** (1998) 175–188.

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