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SOME ADDITIONS TO THE THEORY OF STAR PARTITIONS OF GRAPHS

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Abstract

This paper contains a number of results in the theory of star partitions of graphs. We illustrate a variety of situations which can arise when the Reconstruction Theorem for graphs is used, considering in particular *galaxy graphs* — these are graphs in which every star set is independent. We discuss a recursive ordering of graphs based on the Reconstruction Theorem, and point out the significance of galaxy graphs in this connection.

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0. Introduction

We take G to be an undirected graph without loops or multiple edges, with vertex set $V(G) = \{1, \dots, n\}$, and with $(0,1)$-adjacency matrix $A(G)$. Let P denote the orthogonal projection of $\mathcal{R}^n$ onto the eigenspace $\mathcal{E}(\mu)$ corresponding to the eigenvalue μ of $A(G)$, and let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be the standard orthonormal basis of $\mathcal{R}^n$. Since $\mathcal{E}(\mu)$ is spanned by the vectors $P\mathbf{e}_j$ $(j = 1, \dots, n)$ there exists $X \subseteq V(G)$ such that the vectors $P\mathbf{e}_j$ $(j \in X)$ form a basis for $\mathcal{E}(\mu)$. Such a subset X of $V(G)$ is called a *star set* for μ in G. A partition of $V(G)$ such that each cell is a star set for a different eigenvalue is called a *star partition*. (The terminology reflects the fact that the vectors $P\mathbf{e}_1, \dots, P\mathbf{e}_n$ form a eutactic star in the sense of Seidel [15]. In the context of star partitions [5, Section 7.1], star sets are called *star cells*.)

An equivalent definition of star set, needed below, is the following: if μ has multiplicity k then a star set for μ in G is a set X of k vertices of G such that μ is not an eigenvalue of $G - X$. Here $G - X$, called the *star complement* for μ corresponding to X, is the subgraph of G induced by $\overline{X}$, the complement of X in $V(G)$.

The paper consists of the following sections: 1. Some consequences of the Reconstruction Theorem; 2. Canonical star complements; 3. Graphs with prescribed star sets; 4. Ordering of graphs.

1. Some Consequences of the Reconstruction Theorem

The Reconstruction Theorem ([5, Theorem 7.4.1]) enables us to reconstruct a graph G from knowledge of an eigenvalue μ, a star complement H $(= G - X)$ for μ, and the H-neighbourhoods of vertices in X. The theorem has a converse which is also valid ([5, Theorem 7.4.4]), and the two can be combined as follows (see also [13, Theorem 3.1]):

Theorem 1.1. *Let G be a graph with adjacency matrix of the form*

$$A(G) = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix}, \tag{1}$$

where the principal submatrix A is determined by the vertex set X. Then X is a star set for μ in G if and only if μ is not an eigenvalue of C and

$$\mu I - A = B^T(\mu I - C)^{-1}B. \tag{2}$$

In this section we discuss various questions involving relations (1) and (2). We may regard (2) as a matrix equation with one real parameter where some of the matrices are fixed, and the remaining matrices need to be determined. We are interested in solutions to this equation in which A and C are symmetric $(0, 1)$-matrices with diagonal entries zero. Here A is to be the adjacency matrix of the graph induced by a star set X, while C is to be the adjacency matrix of the star complement $H = G - X$; we shall also say that the $(0, 1)$-matrix B corresponds to the interconnection graph of these two graphs. Accordingly, we will denote the three graphs by $G(A)$, $G(B)$ and $G(C)$, respectively.

Given a graph $H = G(C)$, it is of interest to determine all possible graphs G for which H is a star complement in G for some eigenvalue μ : such indeed was the main purpose in the series of papers [11, 12, 13]. It is known [5] that, given an eigenvalue $\mu \notin \{-1, 0\}$, there are only finitely many graphs G with a prescribed star complement $H = G - X$. If $\mu = 0$, X may contain arbitrarily many independent vertices with the same open neighbourhood in G, while if $\mu = -1$, X may contain arbitrarily many pairwise adjacent vertices with the same closed neighbourhood in G. If these so-called *duplicate* and *co-duplicate* vertices are excluded then again only finitely many graphs arise for a given H, and these graphs are referred to by Ellingham [8] as *reduced* ($\mu = 0$) and *co-reduced* ($\mu = -1$).

For an arbitrary graph H we are here interested in answers to the following questions:

- what are the possible values for μ in (2)?
- for each such value of μ, how many vertices may be added? (i.e., what is the maximum possible value of $|X|$?)

Motivated by these questions, we make the following definitions:

Definition 1.1. μ is an exterior eigenvalue of a graph H if there exists a graph G in which H is a star complement for the eigenvalue μ. The exterior multiplicity of μ is the maximum multiplicity of μ as an eigenvalue of any such graph G (reduced if $\mu = 0$, co-reduced if $\mu = -1$). The exterior spectrum of H is the set of all exterior eigenvalues together with their multiplicities.

Note that every graph H possesses exterior eigenvalues. This can be seen by taking G to be a connected graph obtained from H by adding a single vertex with a suitable H-neighbourhood. Then (see [3, Theorem 0.6, p. 19]) the *index* of G (i.e., the largest eigenvalue μ of G) is not an eigenvalue of H, and since its multiplicity in G is 1, H is a star complement for μ.

As noted earlier, the multiplicity of μ as an eigenvalue of a graph G is equal to $|X|$, where X is a star set for μ in G. Any graph with H as a star complement for μ is an induced subgraph of such a graph G for which X is maximal. There may, however, be a number of different graphs G with the same star complement H but different maximal star sets X for μ, and these sets X may be of different sizes (see [12, Section 3]). The multiplicity of μ as an exterior eigenvalue of H is the largest of these various sizes. Equivalently, it is the largest size of a maximal clique in the *extendability graph* $\Gamma(H, \mu)$ [12].

We consider now some bounds for the multiplicity k of an exterior eigenvalue μ of a graph H, in terms of t, the number of vertices of H. In general, k has a quadratic bound in t, but if μ is non-integral there are linear bounds:

Theorem 1.2. *Let k be the exterior multiplicity of an eigenvalue μ of a t-vertex graph ($t > 1$, $\mu \notin \{-1, 0\}$). Then*

$$k \le \frac{1}{2}(t-1)(t+4), \tag{3}$$

$$k \le t-1 \quad (\mu \notin \mathcal{Z}). \tag{4}$$

Proof. A proof of (3) may be found in [13, Theorem 2.1].

For (4), let G be an extension of H with μ as an eigenvalue of multiplicity k. Note that, since μ is non-integral, its algebraic degree is at least 2, and because any conjugate of μ is also an eigenvalue of G of multiplicity k, we have $k \le t$. Suppose that $k = t$. Then, in any star partition of G, there are just two star cells, each containing t vertices. The index of G must therefore have multiplicity t, and since the index of any connected graph has multiplicity 1, G must have t components. It follows (e.g. by consideration of the characteristic polynomial of G) that each component of G is isomorphic to P_2, and this contradicts the fact that μ is non-integral. Thus $k \le t - 1$. ∎

As noted in [13], the bound in (3) is asymptotically best possible. The bound in (4) is best possible, being attained by any of the infinite family of Paley graphs.

Remark 1.1. We may establish in a similar way the following bounds:

(i) If μ (or one of its conjugates) is the index of G, then $k \le \alpha$, where α is the number of components of H;

(ii) If $\mu \notin \mathcal{Z}$, and the index of G is not equal to μ or any of its conjugates, then $k \le \frac{t-1}{l-1}$, where l is the algebraic degree of μ.

(For (ii), note that in a star partition, the star cells for eigenvalues $\neq \mu$ include $l-1$ cells for algebraic conjugates of μ and one for the index.)

We have seen that every graph is isomorphic to a star *complement* in some graph. We now show that an analogous result holds when 'star complement' is replaced by 'star *set*'. We write '$u \sim v$' to mean that vertices u, v are adjacent.

Theorem 1.3. *Every graph is isomorphic to the subgraph induced by a star set in some graph.*

Proof. Let G be a graph with vertex set $\{u_1, \ldots, u_n\}$, and consider the strong product G' of G and K_2. This graph G' has vertices (u_i, v_1), (u_i, v_2), $(1 \leq i \leq n)$, where v_1 and v_2 are the vertices of K_2, and (u, v) is adjacent to (u', v') if and only if $u = u', v \sim v'$ or $u \sim u', v = v'$ or $u \sim u', v \sim v'$. Let G_1 be the subgraph of G' induced by vertices of the form (u_i, v_1) and G_2 that induced by vertices of the form (u_i, v_2): then G_1 and G_2 are both isomorphic to G. If G has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (repeated according to multiplicity) then G' has eigenvalues $\lambda_i \cdot 1 + \lambda_i + 1$ and $\lambda_i \cdot (-1) + \lambda_i - 1$ $(i = 1, \ldots, n)$ (see [5, p. 70]), i.e., $2\lambda_i + 1$ $(i = 1, \ldots, n)$ and -1 (n times).

Suppose first that -1 is not an eigenvalue of G: then $2\lambda_i + 1 \neq -1$, so the multiplicity of -1 as an eigenvalue of G' is n. In this case $V(G_1)$ (or $V(G_2)$) is a star set for -1 in G'. Now suppose that -1 is an eigenvalue of G of multiplicity m; then the multiplicity of -1 as an eigenvalue of G' is $m + n$. Let X be a star set for -1 in G_1. Then $X \cup V(G_2)$ contains $m + n$ vertices, and its complement in G' does not have -1 as an eigenvalue since it coincides with the star complement of X in G_1. $X \cup V(G_2)$ is therefore a star set for -1 in G', and it follows that $V(G_2)$ is a star set for -1 in the graph $G' - X$. This establishes the result in general. ■

Remark 1.2. We may consider, instead of the strong product, the *NEPS* of G and K_2 with respect to the basis $\{(1,0), (1,1)\}$. (See [5, p. 66].) This leads to a proof in which the rôle of -1 in the above proof is played by 0.

In the remainder of this section we provide some examples to illustrate the variety of situations captured by the Reconstruction Theorem. For the sake of clarity, we shall assume that matrices A, B, C (as in (1)) correspond to labelled graphs, in contrast to the unlabelled graphs $G(A), G(B), G(C)$. We have seen that if B, C and μ are given then there is at most one corresponding graph G. We now consider three other types of partial information, and ask whether G is uniquely determined.

(i) B and C given

Note first that if B and C are matrices corresponding to some non-integral eigenvalue μ of a graph G, then any conjugate of μ will yield the same matrix A, and therefore a graph isomorphic to G. This is a simple consequence of the fact that any star set for μ is also a star set for any conjugate of μ (see [5, p. 187]). Suppose next that matrices B and C are given, along with two values of μ which are not conjugates of each other; is it possible that the matrices A obtained from (2) give rise to non-isomorphic graphs G? This can indeed happen, as the following example shows.

Example 1. Let $X = \{1,2\}$ and $\overline{X} = \{3,4,5,6,7\}$. If we take $\mu = 1$ in equation (2), we get a graph in which vertices 1 and 2 are non-adjacent, as in Figure 1(a). In contrast, when $\mu = \frac{-1\pm\sqrt{5}}{2}$, a graph is obtained in which vertices 1 and 2 are adjacent, as shown in Figure 1(b).

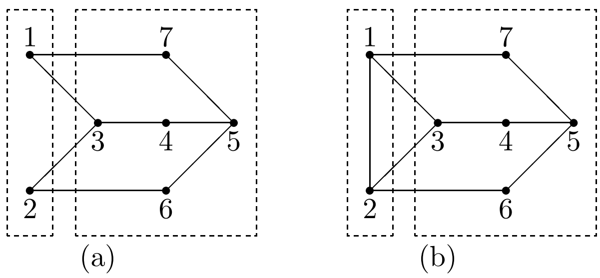

Figure 1. The graphs of Example 1

(ii) $G(C)$ and μ given

We show that in this case $G(A)$ and $G(B)$ need not be uniquely determined: it is even possible to choose $G(C)$ and μ in such a way that *different* choices of $G(A)$ and $G(B)$ yield the *same* graph G.

Example 2. Let $G = L(G')$, where G' is a connected bipartite graph containing two isomorphic spanning trees T_1, T_2, which are differently embedded in G'. We can label the edges of T_1 and T_2 in such a way that $L(T_1)$ and $L(T_2)$ have the same adjacency matrix C; the corresponding matrices B are different, however. From [7], -2 is an eigenvalue of G of multiplicity $m-n+1$, where m and n are the numbers of edges and vertices, respectively, of G'. Since C does not have -2 as an eigenvalue, each of $L(T_1)$ and $L(T_2)$ is a star complement in G. The graph G obtained by using the Reconstruction Theorem is the same in the two cases. An example is illustrated in Figure 2, in which the edges of trees T_1 and T_2 appear as bold lines.

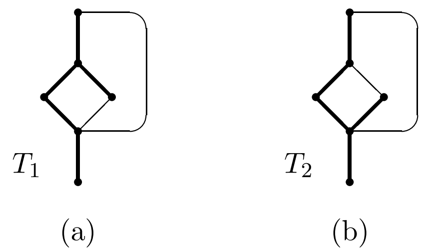

Figure 2. The graphs of Example 2

(iii) A, C **and** μ **given**

We first show that different matrices B may result in graphs G which are isomorphic.

Example 3. In Figure 3 we have a graph G with two different embeddings of $G(A)$ and $G(C)$: in (a) the embeddings are determined by $X = \{4,5,8\}$ and $\overline{X} = \{1,2,3,6,7,9,10\}$), while in (b) they are determined by $X = \{2,4,8\}$ and $\overline{X} = \{1,3,5,6,7,9,10\}$). In each case, the edges of star complements are given by bold lines, and μ is equal to 1 or -2. The corresponding graphs $G(B)$ are non-isomorphic.

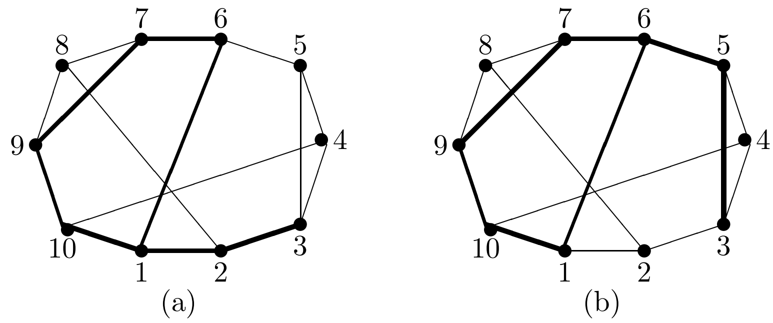

Figure 3. The graphs of Example 3

We next show that the graphs G obtained may be cospectral but non-isomorphic. To this end, take G_1 and G_2 to be respectively $L(K_8)$, the line graph of the complete graph on 8 vertices, and the Chang graph Ch_2 (see [5, p. 5]). (G_2 can be obtained from G_1 by switching with respect to the subgraph isomorphic to $C_3 \cup C_5$, which is for both graphs a star complement for the eigenvalue -2.) Since switching does not change the star set, the matrices A and C are the same. It may be checked that the corresponding graphs $G(B)$ are, like the graphs $L(K_8)$ and Ch_2 we started with, cospectral but non-isomorphic.

2. Canonical Star Complements

In general a graph has non-isomorphic star complements for the same eigenvalue. For this and other reasons (in particular in addressing the graph isomorphism problem), it is useful to seek a star complement which is canonical in some way. Moreover it is desirable for such a star complement to be determined by solving optimization problems; for example, one could first find the star complements with maximal (or minimal) index. Further, one could order star complements lexicographically, thereafter by graph angles and other invariants.

Optimization problems in our case are of the following type: find extremal values of a function defined on a finite set (the set of all star complements of a graph for a given eigenvalue). Such problems are known as problems of combinatorial optimization. The (computational) complexity of such a problem, selected to determine the canonical star complement, is crucial; it would of course be convenient to have a polynomial algorithm to solve the problem. The complexity of finding extremal graphs for an eigenvalue in a set of graphs is not much studied in the literature. A brute force method is not promising in the general case; although an eigenvalue can be determined in polynomial time, the number of graphs grows exponentially with the number of vertices. Hence the cardinality of the set of star complements is important.

One means of reducing the complexity of relevant optimization problems is to consider line star complements instead of star complements.

The edge set of a graph G is denoted by $E(G)$. Let μ be an eigenvalue of the line graph $L(G)$, and let Y be a subset of $E(G)$. In accordance with the definition of line star partitions in [5, Section 7.8.3] we say that Y is a *line star set* for μ in G if it is a star set for μ in $L(G)$. In this situation $G \setminus Y$ is the corresponding *line star complement* for μ in G.

The following definitions appear in [6, Section 1].

Definition 2.1. Let G be a graph whose line graph has least eigenvalue -2. A foundation for G is a line star complement for -2 in G.

Definition 2.2. An orchid is a unicyclic graph whose cycle has odd length and an orchid garden is a graph whose components are orchids.

Let G be a connected graph which is neither a tree nor an orchid. Then the least eigenvalue of $L(G)$ is -2 [7]; moreover a foundation of G is a spanning tree of G if G is bipartite and a spanning orchid garden in G if G is non-bipartite [5, Theorem 7.8.13].

In the context of foundations, we can expect the complexity of the optimization process to be reduced in so far as the family of graphs to be considered is restricted to trees or orchid gardens. For example, among the spanning trees of the complete bipartite graph $K_{m,n}$ the corresponding double star has maximal index [1]. (In this connection it would be of interest to identify families of bipartite graphs in which the number of spanning trees grows polynomially with the number of vertices.) In the following example we identify possible canonical foundations for K_6.

Example 4. The graph $L(K_6)$ has spectrum $8,\ 2^5,\ -2^9$ and the star complements for -2 have the form $L(F)$ where F is one of the graphs illustrated in Figure 4, where they are shown in order of increasing index (cf. [4]). The foundation with maximal index has the form of a star with one edge added, but that with minimal index among connected graphs F is not a 6-cycle because C_6 has -2 as an eigenvalue.

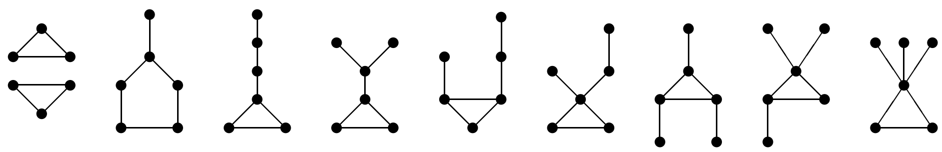

Figure 4. The foundations for K_6

In seeking a canonical foundation we could choose instead to maximize or minimize spectral moments (ordered lexicographically, cf. [5, Section 8.2]). For trees, the first three spectral moments are fixed, and for any graph the fourth moment S_4 is equal to $2m+4p+8q$, where m is the number of edges, p is the number of paths of length 2 and q is the number of quadrilaterals. For a tree with n vertices, we have $m = n-1$ and $q = 0$, and so the corresponding foundations F are first distinguished by p, which is the number of edges in $L(F)$. Now a canonical star complement for -2 in $L(G)$ (G bipartite) would have an extremal number of edges. In the case of a non-bipartite graph G, if we apply the same criterion, the foundations are ordered first by the number t of triangles (since $S_3 = 6t$) and then by p as above. For the graph in Example 4, we see that we obtain the same candidates for a canonical foundation as before.

In the case that G is bipartite, we could first restrict ourselves to foundations realized as minimal spanning trees in a weighted graph associated with G (the corresponding optimization problem having polynomial complexity).

Explicitly, let the adjacency matrix A of G have spectral decomposition

$$A = \mu_1 P_1 + \mu_2 P_2 + \cdots + \mu_m P_m \quad (\mu_1 < \mu_2 < \cdots < \mu_m)$$

and let $W_i(G)$ be the weighted graph obtained from G by weighting each edge uv with the (u, v)-entry of P_i $(i = 1, \ldots, m)$. We can first find minimal spanning trees in $W_1(G)$ and among them those which are of least weight in $W_2(G), \ldots, W_m(G)$ in turn. If more than one foundation remains, we apply one of the criteria described above. In the case that G is non-bipartite we can carry out an analogous procedure to find a minimal spanning orchid. Indeed a spanning unicyclic subgraph which contains the vertex v is called a *v-tree*, and a minimal v-tree can be found by first finding a minimal spanning tree T of $G - v$ and then adding two edges of least weight joining v to T. It is customary in the traveling salesperson problem literature to take $v = 1$ and to find a minimal 1-tree (cf. [10]). The procedure described in this paragraph affords no refinement in cases where for each i all weights in $W_i(G)$ are the same, in particular for strongly regular graphs.

3. Graphs With Prescribed Star Sets

It is easy to find examples of graphs for which the star cells are independent sets (i.e., they induce graphs without edges), and also ones for which they induce complete graphs. Motivated by considerations of ordering of graphs (which we come to in Section 4), we now ask whether there exist graphs in which *all* star sets (for all eigenvalues) are independent, or induce complete graphs (cf. [14, p. 96]). The answer is obvious for null graphs (graphs without edges) and for complete graphs, and also for graphs with all their eigenvalues simple. In order to discuss some non-trivial examples, we next prove a lemma which provides a necessary and sufficient condition for all star sets to be independent.

Note first that if P is the projection matrix corresponding to an eigenvalue μ, then a vertex v belongs to some star set for μ if and only if $P\mathbf{e}_v \neq \mathbf{0}$ (see e.g. [5, p. 166]). Thus $\{v \in V(G) : P\mathbf{e}_v = \mathbf{0}\}$ consists of all vertices which do not lie in any star set for μ.

Lemma 3.1. *Let μ be an eigenvalue of G, and P the corresponding projection matrix. Let $V_0 = \{v \in V(G) : P\mathbf{e}_v = \mathbf{0}\}$, and write $\langle P\mathbf{e}_v \rangle$ for the subspace spanned by $P\mathbf{e}_v$. Then the following is a necessary and sufficient condition for all star sets for μ to be independent:*

(5) $\langle P\mathbf{e}_u \rangle = \langle P\mathbf{e}_v \rangle$ *for all adjacent vertices u, v of $G - V_0$.*

Proof. Suppose that all star sets for μ are independent. Let u, v be adjacent vertices of $G - V_0$. If $\langle P\mathbf{e}_u \rangle \neq \langle P\mathbf{e}_v \rangle$ then (see [12, Theorem 7.4.6]) there is a star cell containing u and v, and this is a contradiction.

Conversely, suppose (5) holds, and let u, $v \in X$, where X is a star set for μ. Then u, v cannot belong to V_0. Suppose that u and v are adjacent. Then, by (5), $\langle P\mathbf{e}_u \rangle = \langle P\mathbf{e}_v \rangle$, and this contradicts the linear independence of $P\mathbf{e}_u$, $P\mathbf{e}_v$. Thus u and v are not adjacent. It follows that the star set X has no edges. ∎

Graphs such that all star sets, for all eigenvalues, are independent will be referred to as *galaxy* graphs. Lemma 3.1 shows that a graph G is a galaxy graph if and only if (5) holds for each eigenvalue of G.

Condition (5) tells us the following fact about the structure of a galaxy graph G. For a given eigenvalue μ, all vertices u in the same connected component of $G - V_0$ have their *star arms* (the vectors $P\mathbf{e}_u$) in the same direction. Vertices in different components may have their star arms in the same or different directions. We may observe also that each component of $G - V_0$ itself has μ as an eigenvalue. This is because the condition $P\mathbf{e}_u = \mathbf{0}$ implies that each eigenvector of G corresponding to μ has its u-th entry equal to 0, and it follows that the eigenvector equation continues to hold at each vertex of G even when the vertices of V_0 have been deleted.

We now give some examples of galaxy graphs.

Example 5. *Trees with* 0 *as the only multiple eigenvalue.* Let T be a tree on n vertices which has 0 as an eigenvalue of multiplicity $s > 1$, all other eigenvalues being simple. We show that T is a galaxy graph. Note first that the multiplicity $m(F)$ of the eigenvalue 0 in a forest F is given by $m(F) = n - 2k$, where n is the number of vertices and k is the cardinality of a largest matching (cf. [3, p. 233]). Suppose that T has a star set X for 0 which is not independent — i.e. the graph induced by X contains at least one edge. Then $m(T - X) = 2\{k(T) - k(T - X)\} > 0$, a contradiction. Hence all star sets (for all eigenvalues of T) are independent, i.e., T is a galaxy graph.

In particular, all stars and all double stars (i.e. graphs consisting of an edge plus pendant edges at its ends) are galaxy graphs. For stars, this result is immediate, since it is well known that the star $K_{1,n-1}$ has simple eigenvalues $\pm\sqrt{n-1}$, and 0 as an eigenvalue of multiplicity $n - 2$. We may note that the set V_0 in Lemma 3.1 is the center of the tree. For double stars, we have similarly that any double star has 4 non-zero simple eigenvalues and

0 as an eigenvalue of multiplicity $n-4$, as may be seen by using divisors. The only possible star complement for 0 is a graph isomorphic to P_4, and the set V_0 consists of the two central vertices. (To verify this last statement, note that deletion of a central vertex increases the multiplicity of 0 by 1, whereas the deletion of any vertex in a star set decreases the multiplicity of 0 by 1.)

It is interesting to observe that there exist infinite families of mutually cospectral galaxy graphs. This can be seen from [3, p. 161, Figure 6.7], which contains a pair of trees on $3s+8$ vertices ($s \geq 0$), one a double star.

Example 6. *Trees having more than one multiple eigenvalue.* We show first that the subdivision graph $S(K_{1,n})$ ($n \geq 3$) is a galaxy graph. The spectrum of this graph consists of three simple eigenvalues (0 and $\pm\sqrt{n+1}$), and the numbers ± 1, each of multiplicity $n-1$. By an argument similar to that used in Example 5, the central vertex cannot belong to a star set for either 1 or -1, and for each of these eigenvalues V_0 consists of this one vertex. If a star set for 1 or for -1 contained an edge — say that between vertices u and v — then deletion of vertices u and v would reduce the multiplicity of the relevant eigenvalue by 2, whereas in fact it is reduced by just 1. Thus the graph is a galaxy graph.

A more complicated example is that shown in Figure 5 with $a+2b+3c+1$ vertices, where $c = c_1 + c_2$. In general, this has $\pm\sqrt{2}, \pm 1, 0$ as eigenvalues of multiplicities $c-1, b-1, a+c-1$, respectively, in addition to six simple eigenvalues. Arguments similar to those above can be used to show that this is a galaxy graph for all values of a, b, c_1, c_2.

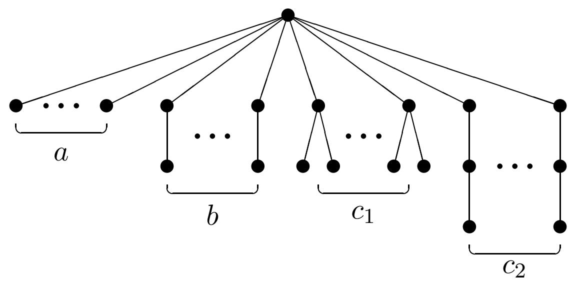

Figure 5. An example of a galaxy graph

Example 7. *Book graphs.* A book graph B_n is a graph obtained as a sum of $K_{1,n}$ and K_2: it consists of a common edge (denoted by $00'$) and n "pages"

(4-cycles whose free edges are denoted by ii', where $i = 1, \ldots, n$). Its simple eigenvalues are $\pm\sqrt{n} \pm 1$, while ± 1 are eigenvalues of multiplicity $n-1$. By virtue of symmetry, in order to show that the graph is a galaxy graph, we have only to check whether three types of edges can be present in a star set. If this were the case for the edge $00'$, then deletion of vertices $0, 0'$, would result in the multiplicity of ± 1 changing to $n-3$, whereas in fact it would be increased to n. Similarly, if vertices i, i' were in the same star set, then their deletion would reduce the multiplicity of ± 1 to $n-3$, whereas in fact it would become $n-2$. Likewise, if we delete vertices $0, i$ (or $0', i'$), then we get a graph in which ± 1 is an eigenvalue of multiplicity not $n-3$ but $n-2$. This can be seen by constructing $n-2$ linearly independent eigenvectors, say in $B_n - \{0', n'\}$, for $\mu = 1$ (resp. for $\mu = -1$) in the following way: the k-th eigenvector x_k ($1 \le k \le n-2$) has all entries equal to 0, except for the entries corresponding to vertices $k, k', (n-1), (n-1)'$ which are given by $+1, -1, -1, +1$ (resp. $-1, -1, +1, +1$). It follows that B_n is a galaxy graph for each n.

It was mentioned at the start of this section that graphs in which all star sets induce *complete* graphs are also of interest. For such graphs an analogue of Lemma 3.1, with 'adjacent' replaced by 'non-adjacent', is easily established.

4. Ordering of Graphs

Let G be a graph with n vertices and m edges, and adjacency matrix $\mathcal{A}$. We shall define recursively a procedure of canonical vertex labelling ($\mathcal{CVL}$), with notation

$$\mathcal{CVL}(G) = \mathcal{CVL}(\mathcal{A}) = \mathcal{A}^*,$$

where $\mathcal{A}^*$ is a unique canonical adjacency matrix of G.

Let μ be an eigenvalue of G with a star set X and a star complement $\overline{X}$. Then we can write $\mathcal{A}$ in the form

$$\mathcal{A} = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix}, \tag{6}$$

where A and C are the adjacency matrices of the subgraphs induced by X and $\overline{X}$ respectively. We may assume that

$$\mathcal{CVL}\begin{pmatrix} O & B^T \\ B & O \end{pmatrix} = \begin{pmatrix} O & B'^T \\ B' & O \end{pmatrix}.$$

(If the resulting matrix is not in the block-diagonal form as above we can perform a uniquely defined relabelling of vertices of the corresponding bipartite graph so that the adjacency matrix takes this form.)

We can take permutations P and Q such that

$$\begin{pmatrix} P & O \\ O & Q \end{pmatrix}^T \begin{pmatrix} O & B^T \\ B & O \end{pmatrix} \begin{pmatrix} P & O \\ O & Q \end{pmatrix} = \begin{pmatrix} O & P^T B^T Q \\ Q^T B P & O \end{pmatrix} = \begin{pmatrix} O & B'^T \\ B' & O \end{pmatrix}.$$

(We choose P and Q to be the "smallest" such permutations, in some ordering of permutations.) Now we have

$$\begin{pmatrix} P & O \\ O & Q \end{pmatrix}^T \begin{pmatrix} A & B^T \\ B & C \end{pmatrix} \begin{pmatrix} P & O \\ O & Q \end{pmatrix} = \begin{pmatrix} A_1 & B'^T \\ B' & C_1 \end{pmatrix},$$

where $A_1 = P^T AP$, $C_1 = Q^T CQ$.

Again, we may assume that $C^* = \mathcal{CVL}(C) = \mathcal{CVL}(C_1)$ has already been defined, and we now find the "smallest" permutation Q_1 such that $Q_1^T C_1 Q_1 = C^*$.

Finally, we consider the matrix

$$\mathcal{A}' = \begin{pmatrix} I & O \\ O & Q_1^T \end{pmatrix} \begin{pmatrix} A_1 & B'^T \\ B' & C_1 \end{pmatrix} \begin{pmatrix} I & O \\ O & Q_1 \end{pmatrix} = \begin{pmatrix} A_1 & B_1'^T \\ B_1' & C^* \end{pmatrix}, \tag{7}$$

where $B_1 = Q_1^T B'$.

Of course, the bipartition $X, \overline{X}$ of G which gives rise to (6) needs to be uniquely defined, so we select a *canonical* star complement from the collection of all star complements of G. (See Section 2 for a method of doing this.) We classify star complements by the number of edges and spectra. For star complements C', C'' which are cospectral we find $\mathcal{CVL}(C')$ and $\mathcal{CVL}(C'')$, and choose the smaller one (i.e., the smaller of the two binary numbers[2] determined by adjacency matrices of these graphs). In this way we obtain a canonical star complement up to isomorphism, but it is necessary to select a unique one from amongst the isomorphic copies. In order to do this, consider (for each canonical C) the graph

$$\begin{pmatrix} O & B^T \\ B & C \end{pmatrix},$$

[2] By concatenating the rows of the upper triangle of an adjacency matrix (of a graph) we get a binary number. Adjacency matrices will be ordered by this binary number.

and determine a canonical choice by the same procedure by which we have determined a canonical choice of C. Again, there may be several isomorphic possibilities. For each of these, consider the subgraph corresponding to A, and for the third time determine a canonical choice.

For each resulting bipartition $(X, \overline{X})$, we use (6) to determine the matrix $\mathcal{A}'$, and we select from these matrices the one with the smallest (or the largest) corresponding binary number. In this way we define $\mathcal{A}^* = \mathcal{CVL}(G)$.

The recursive procedure described above makes sense if the graph $G(A)$ induced by the star set for the eigenvalue μ has at least one edge. This suggests that we adopt the following rules for choosing μ:

- select the eigenvalues which have the largest multiplicity;
- from these eigenvalues, find the ones whose star sets induce the largest number of edges;
- from amongst these, choose μ to be the largest.

However, the reduction procedure will break down if all star sets are independent sets — i.e., if the graph is a galaxy graph, as discussed in Section 3. Therefore for galaxy graphs G we need to define $\mathcal{CVL}(G)$ independently (for example, by the canonical star basis, cf. [2] or [5, Chap. 8]).

Corresponding to the canonical vertex labelling defined above, we can order graphs lexicographically by their adjacency matrices. Such a graph ordering, similar to that suggested in [14, p. 97], may be useful in studying the graph isomorphism problem.

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