A NOTE ON STRONG AND CO-STRONG PERFECTNESS OF THE X-JOIN OF GRAPHS

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Abstract

Strongly perfect graphs were introduced by C. Berge and P. Duchet in [1]. In [4], [3] the following was studied: the problem of strong perfectness for the Cartesian product, the tensor product, the symmetrical difference of $n$, $n \geq 2$, graphs and for the generalized Cartesian product of graphs. Co-strong perfectness was first studied by G. Ravindra and D. Basavayya [5]. In this paper we discuss strong perfectness and co-strong perfectness for the generalized composition (the lexicographic product) of graphs named as the X-join of graphs.

Keywords: strongly perfect graphs, co-strongly perfect graphs, the X-join of graphs.

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1. Introduction

Let $G$ be a finite undirected connected simple graph. By $V(G)$ and $E(G)$ we denote its vertex set and edge set, respectively. The notation $H = < V_0 >_G$, $V_0 \subseteq V(G)$ means that $H$ is the subgraph of $G$ induced by $V_0$. A subset $S \subseteq V(G)$ is said to be stable in $G$ if no two distinct vertices of $S$ are adjacent in $G$. A subset $Q \subseteq V(G)$ is a clique of $G$ if $< Q >_G$ is a complete subgraph of $G$. If the stable set $S$ meets every maximal (with respect to the set inclusion) clique $Q$, then we will call it a stable
transversal of $G$. A graph $G$ is called strongly perfect ([1]) if its every induced subgraph (including $G$ itself) has a stable transversal. We call $G$ co-strongly perfect ([5]) if $G$ and the complementary graph $\overline{G}$ to $G$ are strongly perfect. Let $G_1, \ldots, G_n$, $n \geq 2$, be graphs of the same order $m \geq 2$ with the vertex sets $V(G_i) = V = \{y_1, \ldots, y_m\}$ for $i = 1, \ldots, n$ and $X$ be a graph such that $V(X) = \{x_1, \ldots, x_n\}$. The $X$-join ([2]) of the sequence of graphs $G_1, \ldots, G_n$ and the graph $X$ is the graph $X[G_1, \ldots, G_n]$ with the vertex set $V(X) \times V$ and the edge set $\{(x_j, y_r), (x_k, y_q) : j = k \text{ and } \} \in E(G_i)$ or $[x_j, x_k] \in E(X)\}$.

Observe that if $G_1 = G_2 = \ldots = G_n = Y$, then we obtain the composition (the lexicographic product) of graphs $Y$ and $X$ denoted by $X[Y]$.

Let $V_0 \subseteq V(X) \times V$. By the projection $Pr_XV_0$ of the subset $V_0$ on the graph $X$ we mean the set $Pr_XV_0 = \{x \in V(X) : \text{there exists } y \in V(G_i), 1 \leq i \leq n, \text{ such that } (x, y) \in V_0\}$.

2. Results

Put $G = X[G_1, \ldots, G_n]$, for convenience. Let $H$ be a connected induced subgraph of $G$ such that it is not isomorphic to any induced subgraph $H'$ of the graph $X$ or $G_i$, for $i = 1, \ldots, n$.

Let $Pr_XV(H) = \{x_i, \ldots, x_{ip}\}$, $2 \leq p \leq n$.

We partition the set $V(H)$ on $p$ disjoint sets $V_{ij}(H)$ such that $Pr_XV_{ij}(H) = \{x_{ij}\}$ for $j = 1, \ldots, p$. For an arbitrary subset $R \subseteq V(H)$, in a natural way we can write $R = \bigcup_{j=1}^{t} R \cap V_{ij}(H)$, where $1 \leq t \leq p$.

For $G$ and $H$ given above it follows immediately.

Lemma 1. If $Q$ is a maximal clique of $H$, then $Pr_XQ$ is a maximal clique of $< Pr_XV(H) >$.

Lemma 2. A subset $Q \subseteq V(H)$ is a maximal clique of $H$ if and only if

1. $Q \cap V_{ij}(H)$ is a maximal clique of $< V_{ij}(H) >$ for $j = 1, \ldots, p$ or $Q \cap V_{ij}(H) = \emptyset$ and

2. $Pr_XQ$ is a maximal clique of $< Pr_XV(H) >$.

Proof. 1. Assume that $Q$ is a maximal clique of $H$. We can write $Q = \bigcup_{j=1}^{t} Q \cap V_{ij}(H)$ where $1 \leq t \leq p$ with $Q \cap V_{ij}(H) \neq \emptyset$ for each $j = 1, \ldots, t$. Moreover, each of the sets $Q \cap V_{ij}(H)$ must be a clique of $< V_{ij}(H) >$. Suppose there exists $j$, $1 \leq j \leq t$ such that $Q \cap V_{ij}(H)$, is not maximal. In consequence, there exists a vertex $(x_{ij}, y_r) \in V_{ij}(H) \backslash Q \cap V_{ij}(H), 1 \leq j \leq t$.
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(of course \( (x_{ij}, y_r) \notin Q \)) which is adjacent to each vertex from \( Q \cap V_{ij}(H) \). Moreover, by the definition of \( G \) and from the fact that \( Q \cap V_{ij}(H) \subset Q \) it follows that \( (x_{ij}, y_r) \) must be adjacent to each vertex from \( Q \setminus Q \cap V_{ij}(H) \). In consequence, \( (x_{ij}, y_r) \) is adjacent to all vertices from \( Q \) and \( (x_{ij}, y_r) \notin Q \), a contradiction with the assumption that \( Q \) is a maximal clique of \( H \). This shows that the condition in (1) holds.

Condition (2) follows from Lemma 1.

II. Suppose that conditions (1) and (2) hold. We can write \( Q = \bigcup_{t=1}^{p} Q \cap V_{ij}(H), 1 \leq t \leq p \). Note that \( |Q| > 1 \), by the assumption about \( H \). Firstly, we shall show that \( Q \) is a clique of \( H \). Let \( (x_{ij}, y_r), (x_{ik}, y_s) \) be two distinct vertices from \( Q \). If \( j = k \), then they belong to \( Q \cap V_{ij}(H) \) and are adjacent by (1). If \( j \neq k \), then \( x_{ij}, x_{ik} \in Pr_X Q \) and by (2) they are adjacent in \( X \). Thus, by the definition of \( G \) the vertices \( (x_{ij}, y_r), (x_{ik}, y_s) \) are adjacent in \( G \). This proves that \( Q \) is a clique of \( H \).

Assume that \( Q \) is not maximal. This means that there exists \( (x_{il}, y_r) \not\in Q \) but it is adjacent to each vertex from \( Q \). Moreover, by the definition of \( G \), the vertex \( x_{il} \) is adjacent to all vertices from \( Pr_X Q \). This implies that \( x_{il} \in Pr_X Q \) by (2). In consequence, it must be that \( (x_{il}, y_r) \in V_{il}(H) \setminus Q \cap V_{il}(H) \) (evidently \( (x_{il}, y_r) \notin Q \cap V_{il}(H) \)). Since \( Q \cap V_{il}(H) \subset Q \) and \( (x_{il}, y_r) \) is adjacent to each vertex from \( Q \), then it is adjacent to each vertex from \( Q \cap V_{il}(H) \). Hence by (1) it must be that \( (x_{il}, y_r) \in Q \cap V_{il}(H) \), a contradiction. Hence, \( Q \) is a maximal clique of \( H \) and this completes the proof of the lemma.

Using the same method as in the proof of Lemma 2 we prove.

**Lemma 3.** A subset \( S \subset V(H) \) is a maximal stable set of \( H \) if and only if

1. \( S \cap V_{ij}(H) \) is a maximal stable set of \(< V_{ij}(H) > \) for \( j = 1, \ldots, s \) or \( S \cap V_{ij}(H) = \emptyset \) and

2. \( Pr_X S \) is a maximal stable set of \(< Pr_X V(H) > \).

Lemma 4 follows directly from the definition of the graph \( X[G_1, \ldots, G_n] \).

**Lemma 4.** \( X[G_1, \ldots, G_n] = X[G_n, \ldots, G_1] \).

**Theorem 1.** \( X[G_1, \ldots, G_n] \) is strongly perfect if and only if the graphs \( X, G_1, \ldots, G_n \) are strongly perfect.

**Proof.** I. Let \( X[G_1, \ldots, G_n] \) be strongly perfect. Then \( X, G_1, \ldots, G_n \) are strongly perfect since they are isomorphic to some induced subgraphs of \( G \).
II. Suppose that the graphs $X, G_1, \ldots, G_n$ are strongly perfect. We shall show that $G$ is strongly perfect. Let $H$ be a connected induced subgraph of $G$. We shall prove that $H$ has a stable transversal.

If $H$ is an induced subgraph of $X$ or $G_i, 1 \leq i \leq n$, then $H$ has a stable transversal, by the assumption that $X, G_1, \ldots, G_n$ are strongly perfect.

Let $H$ be not induced subgraph of $X, G_i, i = 1, \ldots, n$. Assume that $H$ does not have a stable transversal, i.e., for every maximal stable set $S \subseteq V(H)$ there exists a maximal clique $Q \subseteq V(H)$ such that $S \cap Q = \emptyset$. Moreover, by the definition of $G$ and Lemmas 2, 3 we have that for every maximal stable set $Pr_X S$ of $\langle Pr_X V(H) \rangle$ there exists a maximal clique $Pr_X Q$ of $\langle Pr_X V(H) \rangle$ such that $Pr_X S \cap Pr_X Q = \emptyset$. This is a contradiction, since $\langle Pr_X V(H) \rangle$ has a stable transversal.

This proves that $X[G_1, \ldots, G_n]$ is strongly perfect and the proof is complete.

For $G_1 = G_2 = \ldots = G_n = Y$ we obtain

**Corollary 1.** The composition $X[Y]$ of graphs $X$ and $Y$ is strongly perfect if and only if both $X$ and $Y$ are strongly perfect.

Using Lemma 4 and Theorem 1 we obtain

**Corollary 2.** $X[G_1, \ldots, G_n]$ is strongly perfect if and only if the graphs $X, G_1, \ldots, G_n$ are strongly perfect.

In consequence, it follows immediately

**Theorem 2.** $X[G_1, \ldots, G_n]$ is co-strongly perfect if and only if the graphs $X, G_1, \ldots, G_n$ are co-strongly perfect.

**References**


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