ON THE COMPUTATIONAL COMPLEXITY OF \((\mathcal{O}, \mathcal{P})\)-PARTITION PROBLEMS

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Abstract

We prove that for any additive hereditary property \(\mathcal{P} > \mathcal{O}\), it is
NP-hard to decide if a given graph \(G\) allows a vertex partition \(V(G) = A \cup B\)
such that \(G[A] \in \mathcal{O}\) (i.e., \(A\) is independent) and \(G[B] \in \mathcal{P}\).

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1. Introduction

We consider finite undirected simple graphs. A graph property is any isomorphism closed class of graphs. A graph property is hereditary if it is closed under taking subgraphs, and it is additive if it is closed under taking disjoint unions. The class \(\mathcal{O}\) of all edgeless graphs is the simplest additive hereditary property.

The join \(G \oplus H\) of two graphs \(G\) and \(H\) is the graph consisting of the disjoint union of \(G\) and \(H\) and all the edges between \(V(G)\) and \(V(H)\).

Let \(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n\) be graph properties. A vertex \((\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)\)-partition of a graph \(G\) is a partition \((V_1, V_2, \ldots, V_n)\) of \(V(G)\) such that

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for each $i = 1, 2, \ldots, n$, the induced subgraph $G[V_i]$ has the property $P_i$. The composition $P_1 \circ P_2 \circ \ldots \circ P_n$ is defined as the class of all graphs having a vertex $(P_1, P_2, \ldots, P_n)$-partition. A graph property $P$ is reducible if $P = P_1 \circ P_2$ for some nonempty properties $P_1, P_2$, and it is called irreducible otherwise. Every additive hereditary property is uniquely factorizable into irreducible properties [3].

In this paper we address the complexity of recognizing graphs from reducible properties. This question may be viewed as a generalization of graph coloring problems. It is plausible to conjecture that recognition of $P \circ Q$ graphs is NP-hard for any two nonempty properties, if at least one of them is nontrivial. In this note we present a general reduction which shows the hardness for $Q = O$ and $P \neq O$.

2. The NP-Hardness Results

**Theorem 1.** If $P \neq O$ is an additive hereditary property not divisible by $O$, then the $(O, P)$-partition problem is NP-hard.

**Proof.** Let $k$ be the maximum size of a complete graph belonging to $P$, i.e., $K_k \in P$ and $K_{k+1} \notin P$. Clearly, $k > 1$. Let $S_{k+1}$ be the graph obtained by taking $k + 1$ complete graphs of size $k$ and unifying one vertex from each of them, and let $F_k$ denote the graph obtained from $K_k$ by hanging a copy of $K_k$ on each of its vertices. (Here and later on, to hang a copy of $K_k$ onto a vertex $x$ in a graph $G$ means to add a clique of size $k - 1$ to $G$ and make all of its $k - 1$ vertices adjacent to $x$.)

We will reduce from 1-in-$(k + 1)$-SAT which is known to be NP-complete even when the input formula has no negations and all variables occur in exactly $k + 1$ clauses each (this is the exact cover problem for $(k + 1)$-regular $(k + 1)$-uniform hypergraphs) [1].

Suppose first that $S_{k+1}$ and $F_k$ are both in $P$. Given a formula $\Phi$ as specified above, we construct a graph $G$ as follows: For each clause $c = (x_{c,1}, x_{c,2}, \ldots, x_{c,k+1})$ we introduce a complete graph on $k + 1$ vertices $c_1, c_2, \ldots, c_{k+1}$. For each variable $x$ we regard $x$ as a vertex of $G$. If $x$ occurs in a clause $c$, say $x = x_{c,i}$ in $c$, we construct a so called connector gadget by taking a complete graph on $k + 1$ vertices $c_i, x, v_1(c, x), \ldots, v_{k-1}(c, x)$, and forcing the vertices $v_1(c, x), \ldots, v_{k-1}(c, x)$ to “be in $P$”. This forcing is done as follows. Mihók proved in [2] that for any property $P > O$ not divisible by $O$, there exists a graph uniquely partitionable into $O \circ P$. Take a copy of such a graph and make one vertex from the $O$ part of it adjacent
to \(v_1(c, x), \ldots, v_{k-1}(c, x)\). In any \(O \circ P\) partition of \(G\), this vertex is in the \(O\) part, and since this part is independent, the vertices \(v_1(c, x), \ldots, v_{k-1}(c, x)\) are all in the \(P\) part. We claim that \(G\) constructed in this way allows an \((O, P)\)-partition if and only if \(\Phi\) is satisfiable.

Suppose first that \(G\) does allow a partition \(V(G) = A \cup B\) such that \(G[A] \in O\) and \(G[B] \in P\). We set \(x = \text{true}\) iff \(x \in B\). Since \(K_{k+1} \not\in P\), at least one vertex of each clause gadget is in \(A\), and since \(A\) is independent, such vertex is unique. Say this is a vertex \(c_i\) in a clause \(c\). Since \(A\) is independent, the corresponding variable vertex \(x_{c,i}\) is in \(B\) and this variable is \text{true} in the clause. For any other variable \(x_{c,j}\) in \(c, c_j \in B\). Since the connector gadget is another copy of \(K_{k+1}\) and \(k - 1\) of its vertices are forced to be in \(B\), the only vertex which can be in \(A\) is the corresponding variable vertex \(x_{c,j}\), hence every \(x_{c,j}, j \neq i\) is \text{false}. Thus \(\Phi\) is 1-in-(\(k+1\))-satisfied.

Suppose on the other hand that \(\Phi\) is 1-in-(\(k+1\)) satisfied by a truth valuation \(\phi\). We set

\[
A = \{x | \phi(x) = \text{false}\} \cup \bigcup_c \{c_i | \phi(x_{c,i}) = \text{true}\}
\]

\[
B = \{x | \phi(x) = \text{true}\} \cup \bigcup_c \{c_i | \phi(x_{c,i}) = \text{false}\}
\]

and we add the vertices whose membership is forced to the particular classes (\(A\) representing \(O\) and \(B\) representing \(P\)). Obviously, \(A\) is an independent set. The components of \(G[B]\) in the forcing uniquely partitionable graphs which hang on \(v_i(c, x)\)'s are in \(P\) by construction, the remaining components of \(G[B]\) are copies of \(F_k\) (around the clause gadgets) and \(S_{k+1}\) (around the variable vertices which were valued \text{true}). Thus \(G \in O \circ P\) and we are done.

The situation is slightly more complex if \(F_k \not\in P\) or \(S_{k+1} \not\in P\). Here we first need to introduce some notation.

Let \(H\) be a rooted graph and let \(s = (s_1, s_2, \ldots, s_n)\) be a finite sequence of positive integers. We denote by \(H[s]\) the graph obtained from \(H\) by hanging \(n\) complete graphs \(K_{s_i}, i = 1, 2, \ldots, n\) on the root of \(H\).

For a sequence of \(k\) positive integers \(s = (s_1, s_2, \ldots, s_k)\), we denote by \(F_k(s)\) the graph obtained by hanging complete graphs \(K_{s_i}, i = 1, 2, \ldots, k\) on the vertices of \(K_k\), one on each. Thus

\[
F_k = F_k(k, k, \ldots, k) - k \text{ terms in the parentheses}
\]

\[
S_{k+1} = K_1[k, k, \ldots, k] - k + 1 \text{ terms in the parentheses}
\]
If $F_k(k, k, \ldots, k) \in \mathcal{P}$, we set $m = k$ and $t^+ = (k, k, \ldots, k)$. If $F_k(k, k, \ldots, k) \notin \mathcal{P}$, we let $m$ be the least number such that $F_k(m, m, \ldots, m) \notin \mathcal{P}$ and we let $h$ be the smallest index such that for $t_1 = t_2 = \ldots = t_h = m - 1, t_{h+1} = \ldots = t_k = m$ and $t = (t_1, t_2, \ldots, t_k)$, $F_k(t) \in \mathcal{P}$. We then set $t^+ = (t_1^+, t_2^+, \ldots, t_k^+)$ so that $t_1^+ = t_2^+ = \ldots = t_{h-1}^+ = m - 1$, $t_h^+ = \ldots = t_k^+ = m$. Thus $F_k(t) \in \mathcal{P}$ and $F_k(t^+) \notin \mathcal{P}$. We denote by $H$ the graph obtained from $F_k(t^+)$ by deleting one of the hanging cliques of size $m$, rooted in the vertex whose clique was deleted (e.g., in the vertex corresponding to $t_k^+$).

For a sequence $s = (s_1, s_2, \ldots, s_n)$ we denote

$$\psi(s) = (s_2, s_3, \ldots, s_n)$$

i.e., the sequence obtained by deleting the first element, and we denote

$$\phi_k(s) = (s_1 - 1, s_1 - 1, \ldots, s_1 - 1, s_2, s_3, \ldots, s_n)$$

i.e., the sequence obtained by replacing the first element $s_1$ by $k$ occurrences of $s_1 - 1$. Note that $|\psi(s)| = |s| - 1$ and $|\phi_k(s)| = |s| + k - 1$.

If $F_k \notin \mathcal{P}$, we have $H[m] \notin \mathcal{P}$. If $F_k \in \mathcal{P}$, we must have $S_{k+1} \notin \mathcal{P}$ and hence $H[k, k, \ldots, k] \notin \mathcal{P}$ (but $H = H[1] \in \mathcal{P}$). In either case, Lemma 1 says that there is a sequence $s = (s_1, \ldots, s_n)$ such that

$$2 \leq s_1 \leq s_2 \leq \ldots \leq s_n \leq m,$$

$$H[s] \notin \mathcal{P} \text{ but } H[\phi_k(s)] \in \mathcal{P}.$$ 

We denote by $\tilde{H}$ the graph $H[\psi(s)] = H[s_2, s_3, \ldots, s_n]$ and we use this $\tilde{H}$ for the construction of the graph $G$.

Given a formula $\Phi$ as in the first part of the paper, we again plug into $G$ $(k + 1)$-cliques for the clauses of $\Phi$. Each variable $x$ will be replaced by a copy of $\tilde{H}$ with the root in the vertex $x$ and with all vertices except for $x$ being forced to “be in $P$”. If variable $x$ occurs as the $i$-th variable of a clause $c$, the connector of $x$ and $c$ will be a copy of $K_{s_1,i}$ containing $c_i$, $x$ and $s_1 - 2$ extra vertices which will be also forced to “be in $P$”.

Now the proof is straightforward. Suppose first that $G \in \mathcal{O} \circ \mathcal{P}$, say $V(G) = A \cup B$ such that $A$ is independent and $G[B] \in \mathcal{P}$. Again we set $x = \text{true}$ iff $x \in B$. Since $K_{k+1} \notin \mathcal{P}$, at least one vertex of each clause gadget is in $A$, and since $A$ is independent, such a vertex is unique. Say this be a vertex $c_i$ in a clause $c$. Since $A$ is independent, the corresponding variable vertex $x_{c,i}$ is in $B$ and this variable is true in the clause. For any
other variable \(x_{c,j}\) in \(c, c_j \in B\). Since the connector gadget \(K_{s_1}\) together with the vertices of the variable gadget which are forced to be in \(B\) forms \(H[s] \not\in \mathcal{P}\), it must be \(x_{c,j} \in A\) for every \(j \neq i\). Thus \(\Phi\) is \(\Phi\) is 1-in-(\(k+1\)) satisfied.

Suppose, on the other hand, that \(\Phi\) is 1-in-(\(k+1\)) satisfied by a truth valuation \(\phi\). We set

\[
A = \{x | \phi(x) = \text{false}\} \cup \bigcup_c \{c_i | \phi(x_{c,i}) = \text{true}\}
\]

\[
B = \{x | \phi(x) = \text{true}\} \cup \bigcup_c \{c_i | \phi(x_{c,i}) = \text{false}\}
\]

and we add the vertices whose membership is forced to the particular classes \((A\) representing \(O\) and \(B\) representing \(P\)). Obviously, \(A\) is an independent set. The components of \(G[B]\) in the forcing uniquely partitionable graphs which hang on \(v_i(c,x)\)'s are in \(P\) by construction. The remaining components of \(G[B]\) are copies \(F_k(s_1-1, \ldots, s_1-1) \subset F_k(m-1, m-1, \ldots, m-1) \subset F_k(t) \in \mathcal{P}\) (around the clause gadgets) and \(\tilde{H}[s_1-1, \ldots, s_1-1] = H[\phi_k(s)] \in \mathcal{P}\) (around the variable vertices which are valued \text{true}). Thus \(G \in O \circ P\) and we are done.

**Lemma 1.** Let \(H\) be a graph such that \(H \in \mathcal{P}\) and \(H[w] \not\in \mathcal{P}\) for some sequence \(w\). Then there exists a sequence \(s\) such that

\[
\max s \leq \max w,
\]

\[
H[s] \not\in \mathcal{P},
\]

\[
H[\phi_k(s)] \in \mathcal{P}.
\]

**Proof.** Let \(m = \max w\). Set

\[
A = \{s | 1 < s_1 \leq \ldots \leq s_n \leq m, H[s] \in \mathcal{P}\},
\]

\[
A' = \{s | s \not\in A, \psi(s) \in A\}.
\]

Let \(s \in A'\) be a sequence with minimum possible \(s_1(>1)\).

If \(s_1 = 2\) then \(\phi_k(s) = (1, 1, \ldots, 1, s_2, \ldots, s_n)\) and \(H[\phi_k(s)] = H[\psi(s)] \in A\) and \(s\) has the desired property.

If \(s_1 > 2\) then \(s\) would be good for us if \(H[\phi_k(s)] \in \mathcal{P}\). So we may assume that \(H[\phi_k(s)] = H[s_1-1, s_1-1, \ldots, s_1-1, s_2, \ldots, s_n] \not\in \mathcal{P}\). Since \((s_2, \ldots, s_n) \in A\), there is a number \(j > 0\) such that \((s_1-1, \ldots, s_1-1, s_2, \ldots, s_n) \in A'\) (with \(j\) occurrences of \(s_1-1\)). But this is a contradiction as \(2 \leq s_1-1 < s_1\).
Theorem 2. For any property $P \neq \emptyset$, the $(O \circ P)$-partition problem is $NP$-hard.

Proof. If $P = O^n$ for $n \geq 2$, then the $(O, P)$-partition problem is just the $(n+1)$-colorability of graphs and hence well known $NP$-complete.

Let $P = O^n \circ Q$, where and $Q$ is not divisible by $O$. In view of Theorem 1, we may assume that $n > 0$. We know that $O \circ Q$-partition is $NP$-hard.

Suppose $G$ is an input graph for this question. Let $G$ have $g$ vertices and let $G'$ be the join (Zykov sum) of $G$ and $n$ independent sets $I_i, i = 1, 2, \ldots, n$, each of size $g$. We claim that $G' \in O \circ P = O^{n+1} \circ Q$ if and only if $G \in O \circ Q$.

It is clear that if $G \in O \circ Q$ then $G' \in O^{n+1} \circ Q$.

On the other hand, suppose that $G'$ allows a partition into a graph from $Q$ and at most $n+1$ independent sets, say $V(G') = Q \cup \bigcup_{i=1}^{n+1} A_i$, where each $A_i$ is independent and $G'[Q] \in Q$. Each $A_i$ is either a subset of $V(G) - Q$, or a subset of one of the $I_i$’s. Assume $k + 1$ of $A_i$’s being subsets of $V(G) - Q$, and without loss of generality let them be $A_1, \ldots, A_{k+1}$. Then only $n-k$ sets $A_{k+2}, \ldots, A_{n+1}$ lie outside of $V(G)$, and consequently, at least $k$ of the $I_i$’s lie inside $Q$, say $I_1, \ldots, I_k$. But then $G[(Q \cap V(G)) \cup \bigcup_{i=1}^k I_i]$ is isomorphic to a subgraph of $G[V(G) \cap Q] \oplus \sum_{i=1}^k I_i \subset G'[Q] \in Q$ and $G \subset G[V(G) \cap Q] \cup \bigcup_{i=1}^k A_i] \oplus G[A_{k+1}] \in Q \circ O$.

Since the construction of $G'$ is linear in the size of $G$, we have concluded the proof.

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References


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