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ON THE COMPUTATIONAL COMPLEXITY OF $(\mathcal{O}, \mathcal{P})$-PARTITION PROBLEMS

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Abstract

We prove that for any additive hereditary property $\mathcal{P} > \mathcal{O}$, it is NP-hard to decide if a given graph G allows a vertex partition $V(G) = A \cup B$ such that $G[A] \in \mathcal{O}$ (i.e., A is independent) and $G[B] \in \mathcal{P}$.

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1. Introduction

We consider finite undirected simple graphs. A *graph property* is any isomorphism closed class of graphs. A graph property is *hereditary* if it is closed under taking subgraphs, and it is *additive* if it is closed under taking disjoint unions. The class $\mathcal{O}$ of all edgeless graphs is the simplest additive hereditary property.

The *join* $G \oplus H$ of two graphs G and H is the graph consisting of the disjoint union of G and H and all the edges between $V(G)$ and $V(H)$.

Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be graph properties. A *vertex* $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-*partition* of a graph G is a partition $(V_1, V_2, \ldots, V_n)$ of $V(G)$ such that

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for each $i = 1, 2, \ldots, n$, the induced subgraph $G[V_i]$ has the property $\mathcal{P}_i$. The composition $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$ is defined as the class of all graphs having a vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition. A graph property $\mathcal{P}$ is *reducible* if $\mathcal{P} = \mathcal{P}_1 \circ \mathcal{P}_2$ for some nonempty properties $\mathcal{P}_1, \mathcal{P}_2$, and it is called *irreducible* otherwise. Every additive hereditary property is uniquely factorizable into irreducible properties [3].

In this paper we address the complexity of recognizing graphs from reducible properties. This question may be viewed as a generalization of graph coloring problems. It is plausible to conjecture that recognition of $\mathcal{P} \circ \mathcal{Q}$ graphs is NP-hard for any two nonempty properties, if at least one of them is nontrivial. In this note we present a general reduction which shows the hardness for $\mathcal{Q} = \mathcal{O}$ and $\mathcal{P} \neq \mathcal{O}$.

2. The NP-Hardness Results

Theorem 1. *If $\mathcal{P} \neq \mathcal{O}$ is an additive hereditary property not divisible by $\mathcal{O}$, then the $(\mathcal{O}, \mathcal{P})$-partition problem is NP-hard.*

Proof. Let k be the maximum size of a complete graph belonging to $\mathcal{P}$, i.e., $K_k \in \mathcal{P}$ and $K_{k+1} \notin \mathcal{P}$. Clearly, $k > 1$. Let S_{k+1} be the graph obtained by taking $k+1$ complete graphs of size k and unifying one vertex from each of them, and let F_k denote the graph obtained from K_k by hanging a copy of K_k on each of its vertices. (Here and later on, to *hang* a copy of K_k onto a vertex x in a graph G means to add a clique of size $k-1$ to G and make all of its $k-1$ vertices adjacent to x.)

We will reduce from **1-in-$(k+1)$-SAT** which is known to be NP-complete even when the input formula has no negations and all variables occur in exactly $k+1$ clauses each (this is the exact cover problem for $(k+1)$-regular $(k+1)$-uniform hypergraphs) [1].

Suppose first that S_{k+1} and F_k are both in $\mathcal{P}$. Given a formula Φ as specified above, we construct a graph G as follows: For each clause $c = (x_{c,1}, x_{c,2}, \ldots, x_{c,k+1})$ we introduce a complete graph on $k+1$ vertices $c_1, c_2, \ldots, c_{k+1}$. For each variable x we regard x as a vertex of G. If x occurs in a clause c, say $x = x_{c,i}$ in c, we construct a so called connector gadget by taking a complete graph on $k+1$ vertices $c_i, x, v_1(c,x), \ldots, v_{k-1}(c,x)$, and forcing the vertices $v_1(c,x), \ldots, v_{k-1}(c,x)$ to "be in P". This forcing is done as follows. Mihók proved in [2] that for any property $\mathcal{P} > \mathcal{O}$ not divisible by $\mathcal{O}$, there exists a graph uniquely partitionable into $\mathcal{O} \circ \mathcal{P}$. Take a copy of such a graph and make one vertex from the $\mathcal{O}$ part of it adjacent to $v_1(c,x),\dots,v_{k-1}(c,x)$. In any $\mathcal{O}\circ\mathcal{P}$ partition of G, this vertex is in the $\mathcal{O}$ part, and since this part is independent, the vertices $v_1(c,x),\dots,v_{k-1}(c,x)$ are all in the $\mathcal{P}$ part. We claim that G constructed in this way allows an $(\mathcal{O},\mathcal{P})$-partition if and only if Φ is satisfiable.

Suppose first that G does allow a partition $V(G) = A \cup B$ such that $G[A] \in \mathcal{O}$ and $G[B] \in \mathcal{P}$. We set $x = \mathsf{true}$ iff $x \in B$. Since $K_{k+1} \notin \mathcal{P}$, at least one vertex of each clause gadget is in A, and since A is independent, such vertex is unique. Say this is a vertex c_i in a clause c. Since A is independent, the corresponding variable vertex $x_{c,i}$ is in B and this variable is true in the clause. For any other variable $x_{c,j}$ in c, $c_j \in B$. Since the connector gadget is another copy of K_{k+1} and $k-1$ of its vertices are forced to be in B, the only vertex which can be in A is the corresponding variable vertex $x_{c,j}$, hence every $x_{c,j}, j \neq i$ is false. Thus Φ is 1-in-$(k+1)$-satisfied.

Suppose on the other hand that Φ is 1-in-$(k+1)$ satisfied by a truth valuation ϕ. We set

$$A = \{x|\phi(x) = \mathsf{false}\} \cup \bigcup_c \{c_i|\phi(x_{c,i}) = \mathsf{true}\}$$

$$B = \{x|\phi(x) = \mathsf{true}\} \cup \bigcup_c \{c_i|\phi(x_{c,i}) = \mathsf{false}\}$$

and we add the vertices whose membership is forced to the particular classes (A representing $\mathcal{O}$ and B representing $\mathcal{P}$). Obviously, A is an independent set. The components of $G[B]$ in the forcing uniquely partitionable graphs which hang on $v_i(c,x)$'s are in $\mathcal{P}$ by construction, the remaining components of $G[B]$ are copies of F_k (around the clause gadgets) and S_{k+1} (around the variable vertices which were valued true). Thus $G \in \mathcal{O}\circ\mathcal{P}$ and we are done.

The situation is slightly more complex if $F_k \notin \mathcal{P}$ or $S_{k+1} \notin \mathcal{P}$. Here we first need to introduce some notation.

Let H be a rooted graph and let $s = (s_1, s_2, \dots, s_n)$ be a finite sequence of positive integers. We denote by $H[s]$ the graph obtained from H by hanging n complete graphs $K_{s_i}, i = 1, 2, \dots, n$ on the root of H.

For a sequence of k positive integers $s = (s_1, s_2, \dots, s_k)$, we denote by $F_k(s)$ the graph obtained by hanging complete graphs $K_{s_i}, i = 1, 2, \dots, k$ on the vertices of K_k, one on each. Thus

$$F_k = F_k(k, k, \dots, k) - k \text{ terms in the parentheses}$$

$$S_{k+1} = K_1[k, k, \dots, k] - k+1 \text{ terms in the parentheses.}$$

If $F_k(k,k,\ldots,k) \in \mathcal{P}$, we set $m = k$ and $t^+ = (k,k,\ldots,k)$. If $F_k(k,k,\ldots,k) \notin \mathcal{P}$, we let m be the least number such that $F_k(m,m,\ldots,m) \notin \mathcal{P}$ and we let h be the smallest index such that for $t_1 = t_2 = \ldots = t_h = m-1, t_{h+1} = \ldots = t_k = m$ and $t = (t_1,t_2,\ldots,t_k)$, $F_k(t) \in \mathcal{P}$. We then set $t^+ = (t_1^+,t_2^+,\ldots,t_k^+)$ so that $t_1^+ = t_2^+ = \ldots = t_{h-1}^+ = m-1$, $t_h^+ = \ldots = t_k^+ = m$. Thus $F_k(t) \in \mathcal{P}$ and $F_k(t^+) \notin \mathcal{P}$. We denote by H the graph obtained from $F_k(t^+)$ by deleting one of the hanging cliques of size m, rooted in the vertex whose clique was deleted (e.g., in the vertex corresponding to t_k^+).

For a sequence $s = (s_1,s_2,\ldots,s_n)$ we denote

$$\psi(s) = (s_2,s_3,\ldots,s_n)$$

i.e., the sequence obtained by deleting the first element, and we denote

$$\phi_k(s) = (s_1-1,s_1-1,\ldots,s_1-1,s_2,s_3,\ldots,s_n)$$

i.e., the sequence obtained by replacing the first element s_1 by k occurrences of s_1-1. Note that $|\psi(s)| = |s|-1$ and $|\phi_k(s)| = |s|+k-1$.

If $F_k \notin \mathcal{P}$, we have $H[m] \notin \mathcal{P}$. If $F_k \in \mathcal{P}$, we must have $S_{k+1} \notin \mathcal{P}$ and hence $H[k,k,\ldots,k] \notin \mathcal{P}$ (but $H = H[1] \in \mathcal{P}$). In either case, Lemma 1 says that there is a sequence $s = (s_1,\ldots,s_n)$ such that

$$2 \le s_1 \le s_2 \le \ldots \le s_n \le m,$$

$$H[s] \notin \mathcal{P} \text{ but } H[\phi_k(s)] \in \mathcal{P}.$$

We denote by $\widetilde{H}$ the graph $H[\psi(s)] = H[s_2,s_3,\ldots,s_n]$ and we use this $\widetilde{H}$ for the construction of the graph G.

Given a formula Φ as in the first part of the paper, we again plug into G $(k+1)$-cliques for the clauses of Φ. Each variable x will be replaced by a copy of $\widetilde{H}$ with the root in the vertex x and with all vertices except for x being forced to "be in P". If variable x occurs as the i-th variable of a clause c, the connector of x and c will be a copy of K_{s_1} containing c_i, x and s_1-2 extra vertices which will be also forced to "be in P".

Now the proof is straightforward. Suppose first that $G \in \mathcal{O} \circ \mathcal{P}$, say $V(G) = A \cup B$ such that A is independent and $G[B] \in \mathcal{P}$. Again we set $x = \mathsf{true}$ iff $x \in B$. Since $K_{k+1} \notin \mathcal{P}$, at least one vertex of each clause gadget is in A, and since A is independent, such a vertex is unique. Say this be a vertex c_i in a clause c. Since A is independent, the corresponding variable vertex $x_{c,i}$ is in B and this variable is true in the clause. For any other variable $x_{c,j}$ in c, $c_j \in B$. Since the connector gadget K_{s_1} together with the vertices of the variable gadget which are forced to be in B forms $H[s] \notin \mathcal{P}$, it must be $x_{c,j} \in A$ for every $j \neq i$. Thus Φ is Φ is 1-in-$(k+1)$ satisfied.

Suppose, on the other hand, that Φ is 1-in-$(k+1)$ satisfied by a truth valuation ϕ. We set

$$A = \{x|\phi(x) = \mathsf{false}\} \cup \bigcup_c \{c_i|\phi(x_{c,i}) = \mathsf{true}\}$$

$$B = \{x|\phi(x) = \mathsf{true}\} \cup \bigcup_c \{c_i|\phi(x_{c,i}) = \mathsf{false}\}$$

and we add the vertices whose membership is forced to the particular classes (A representing $\mathcal{O}$ and B representing $\mathcal{P}$). Obviously, A is an independent set. The components of $G[B]$ in the forcing uniquely partitionable graphs which hang on $v_i(c,x)$'s are in $\mathcal{P}$ by construction. The remaining components of $G[B]$ are copies $F_k(s_1-1,\ldots,s_1-1) \subset F_k(m-1,m-1,\ldots,m-1) \subset F_k(t) \in \mathcal{P}$ (around the clause gadgets) and $\widetilde{H}[s_1-1,\ldots,s_1-1] = H[\phi_k(s)] \in \mathcal{P}$ (around the variable vertices which are valued true). Thus $G \in \mathcal{O} \circ \mathcal{P}$ and we are done. ■

Lemma 1. *Let H be a graph such that $H \in \mathcal{P}$ and $H[w] \notin \mathcal{P}$ for some sequence w. Then there exists a sequence s such that*

$$\max s \leq \max w,$$

$$H[s] \notin \mathcal{P},$$

$$H[\phi_k(s)] \in \mathcal{P}.$$

Proof. Let $m = \max w$. Set

$$A = \{s|1 < s_1 \leq \ldots \leq s_n \leq m, H[s] \in \mathcal{P}\},$$

$$A' = \{s|s \notin A, \psi(s) \in A\}.$$

Let $s \in A'$ be a sequence with minimum possible $s_1 (> 1)$.

If $s_1 = 2$ then $\phi_k(s) = (1,1,\ldots,1,s_2,\ldots,s_n)$ and $H[\phi_k(s)] = H[\psi(s)] \in A$ and s has the desired property.

If $s_1 > 2$ then s would be good for us if $H[\phi_k(s)] \in \mathcal{P}$. So we may assume that $H[\phi_k(s)] = H[s_1-1,s_1-1,\ldots,s_1-1,s_2,\ldots,s_n] \notin \mathcal{P}$. Since $(s_2,\ldots,s_n) \in A$, there is a number $j > 0$ such that $(s_1-1,\ldots,s_1-1,s_2,\ldots,s_n) \in A'$ (with j occurrences of s_1-1). But this is a contradiction as $2 \leq s_1 - 1 < s_1$. ■

Theorem 2. *For any property $\mathcal{P} \neq \mathcal{O}$, the $(\mathcal{O} \circ \mathcal{P})$-partition problem is NP-hard.*

Proof. If $\mathcal{P} = \mathcal{O}^n$ for $n \geq 2$, then the $(\mathcal{O}, \mathcal{P})$-partition problem is just the $(n+1)$-colorability of graphs and hence well known NP-complete.

Let $\mathcal{P} = \mathcal{O}^n \circ \mathcal{Q}$, where and $\mathcal{Q}$ is not divisible by $\mathcal{O}$. In view of Theorem 1, we may assume that $n > 0$. We know that $\mathcal{O} \circ \mathcal{Q}$-partition is NP-hard. Suppose G is an input graph for this question. Let G have g vertices and let G' be the join (Zykov sum) of G and n independent sets $I_i, i = 1, 2, \ldots, n$, each of size g. We claim that $G' \in \mathcal{O} \circ \mathcal{P} = \mathcal{O}^{n+1} \circ \mathcal{Q}$ if and only if $G \in \mathcal{O} \circ \mathcal{Q}$.

It is clear that if $G \in \mathcal{O} \circ \mathcal{Q}$ then $G' \in \mathcal{O}^{n+1} \circ \mathcal{Q}$.

On the other hand, suppose that G' allows a partition into a graph from $\mathcal{Q}$ and at most $n+1$ independent sets, say $V(G') = Q \cup \bigcup_{i=1}^{n+1} A_i$, where each A_i is independent and $G'[Q] \in \mathcal{Q}$. Each A_i is either a subset of $V(G) - Q$, or a subset of one of the I_i's. Assume $k+1$ of A_i's being subsets of $V(G) - Q$, and without loss of generality let them be $A_1, \ldots, A_{k+1}$. Then only $n-k$ sets $A_{k+2}, \ldots, A_{n+1}$ lie outside of $V(G)$, and consequently, at least k of the I_i's lie inside Q, say $I_1, \ldots, I_k$. But then $G[(Q \cap V(G)) \cup \bigcup_{i=1}^{k} A_i]$ is isomorphic to a subgraph of $G[V(G) \cap Q] \oplus \sum_{i=1}^{k} I_i \subset G'[Q] \in \mathcal{Q}$ and $G \subset G[(V(G) \cap Q) \cup \bigcup_{i=1}^{k} A_i] \oplus G[A_{k+1}] \in \mathcal{Q} \circ \mathcal{O}$.

Since the construction of G' is linear in the size of G, we have concluded the proof. ∎

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