Discussiones Mathematicae
Graph Theory 17 (1997) 229–241

ON THE COST CHROMATIC NUMBER OF OUTERPLANAR, PLANAR, AND LINE GRAPHS*

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Abstract

We consider vertex colorings of graphs in which each color has an associated cost which is incurred each time the color is assigned to a vertex. The cost of the coloring is the sum of the costs incurred at each vertex. The cost chromatic number of a graph with respect to a cost set is the minimum number of colors necessary to produce a minimum cost coloring of the graph. We show that the cost chromatic number of maximal outerplanar and maximal planar graphs can be arbitrarily large and construct several infinite classes of counterexamples to a conjecture of Harary and Plantholt on the cost chromatic number of line graphs.

Keywords: cost coloring, outerplanar, planar, line graphs.

1991 Mathematics Subject Classification: Primary 05C15, Secondary 05C10.

1. Introduction

In this paper we consider the problem of coloring a graph as efficiently as possible when each color has an associated positive rational cost. Supowit [11] first posed this problem. Independently various subsets of E. Kubicka, Erdös, Thomassen, Schwenk, Alavi, Malde, G. Kubicka, and Kountanis published papers [1], [5], [6], [7], [8], and [12] on a restricted version of this problem where the costs are the first n positive integers. Recently Nicoloso et.

*Supported in part by the National Science Foundation under Grant DMS 92-06991.

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al. [10] investigated this restricted cost coloring problem for interval graphs and discussed its application to VLSI routing. In this paper we continue the investigation of the general cost coloring problem begun by Mitchem and Morriss [9]. We give now the formal basic definitions.

For any palette $P = \{p_1, \ldots, p_n\}$ of colors we have an associated set $C_p = \{c_1, \ldots, c_n\}$ of costs where c_i is a positive rational number associated with color p_i. Each time that p_i is used on a vertex the cost c_i is accrued. Our goal is to color graph G using colors from P so that a minimum total cost is attained. For convenience we assume that $c_1 < c_2 < \ldots < c_n$. The *cost-chromatic number* of G, denoted by $\chi_{Cp}(G)$, with respect to cost set C_p, is the minimum number of colors necessary to produce a minimum cost coloring of G. A minimum cost coloring of G with C_p which uses only $\chi_{Cp}(G)$ colors is called a *best coloring.* In the special case of cost coloring studied in [1], [5], [6], [7], [8], [10], and [12] the set C_p is $\mathcal{N}$, the set of positive integers, and the minimum cost of coloring G is called the chromatic sum. Other cost coloring papers are [3], [13], [14], and [15].

We will simplify notation by no longer distinguishing between the color p_i and the cost c_i. Thus we will have a cost set $C = \{c_1, \ldots, c_n\}$ of positive rational numbers which we will also call colors, and the cost of using color c_i on any vertex is c_i. Given $C, \chi_c(G)$ denotes the minimum number of colors from C required to obtain a minimum cost coloring of graph G using C.

2. Outerplanar and Planar Graphs

In [1] it is shown that for any integer $n \geq 2$ there is a tree T such that $\chi_{\mathcal{N}}(T) = n$. Mitchem and Morriss extended that result.

Theorem 1 [9]**.** *For any cost set C with $n \geq 2$ positive rational numbers there exists a tree T such that $\chi_C(T) = n$.*

In this section we prove similar results for maximal outerplanar and maximal planar graphs. At first consideration these two new results may appear to be obvious. However, the following two theorems show that cost chromatic number behaves very differently from chromatic number, and Theorem 1 does not immediately imply its outerplanar and planar analogues.

Theorem 2 [9]**.** *Let C be any cost set of n colors and G be a graph such that $\chi_C(G) = n > \chi(G)$. Then there is a graph G' with $V(G) = V(G')$ and $E(G)$ a subset of $E(G')$ such that $\chi_C(G') = \chi(G)$.*

Theorem 3 [9]**.** *Let C be any cost set of $n \geq 2$ colors and T be any tree with $\chi_C(T) = n$. Then there exists a tree T' which contains T such that $\chi_C(T') = 2$.*

Before stating and proving our two main theorems of this section we make the following observations. The first fact is immediate, and the easy proof of the second appears in [9].

Fact 4. In any minimum cost coloring of G with cost set C, any vertex v of color $c_i, i \geq 2$, has neighbors of all colors $c_1, \ldots, c_{i-1}$. Thus if v is colored c_i, then $\deg(v) \geq i - 1$.

Fact 5. Let $C = \{c_1, \ldots, c_n\}$ be any cost set of positive rational numbers. Then there exists a cost set $C' = \{c'_1, \ldots, c'_n\}$ of positive integers such that for any graph G a best coloring K of G with C corresponds to a best coloring K' of G with C' in the following sense: if K colors vertex v of G with c_i, then K' colors v with c'_i. Furthermore, c'_1 may be taken to be equal to 1.

Theorem 6. *For any cost set C of $n \geq 2$ positive rational numbers, there exists an outerplanar block B such that $\chi_C(B) = n$.*

Proof. By Fact 5, without loss of generality, we may take C to be a set of positive integers. The construction of block $B = B_n$ will depend on the parity of n. Let $B_2 = K_2$ and $B_3 = K_3$. For $t = 4, 5, \ldots, n$ we construct B_t from B_{t-2}.

Consider an outerplanar embedding of B_{t-2} where $v_1, v_2, \ldots, v_r, v_1$ are its consecutive vertices on the outer border. For $i = 1, 2, \ldots, r$ and subscripts taken modulo r, insert a path from v_i to v_{i+1} with $2(c_n+1)r$ internal vertices and construct a chord from each of these new vertices to v_i. Then B_t is an outerplanar block.

We state and prove a lemma from which Theorem 6 follows.

Lemma. *Let $n \geq j \geq i \geq 2$ and S be the subset consisting of the j largest elements of C. Then $\chi_S(B_i) = i$ and any best coloring uses the i smallest elements of S.*

Proof. We use induction on i noting that the lemma is clearly true for $i = 2$ and 3. Consider $B_i, n \geq i \geq 4$. Let $n \geq j \geq i$ and $S = \{c_{n-j+1}, c_{n-j+2}, \ldots, c_n\}$. By the induction hypothesis any best coloring of B_{i-2} with $S' = S - \{c_{n-j+1}, c_{n-j+2}\}$ uses the $i-2$ smallest colors of S'. Give such a coloring to B_{i-2}, and alternate colors c_{n-j+1} and c_{n-j+2} on each path added to B_{i-2} to form B_i. This clearly gives a coloring of B_i with i colors from S.

In order to complete the proof of the lemma it suffices to show that any coloring using other than i colors from S is more costly than the current coloring. On the contrary first assume that there is a coloring of B_i using less than i colors from S which is no more costly than the current coloring. From the inductive hypothesis it follows that in such a coloring either c_{n-j+1} or c_{n-j+2} is used on a vertex of B_{i-2}, and so it is not used on any interior vertex of at least one of the paths added to B_{i-2} to form B_i. Therefore at least half of the interior vertices of this path are colored with a color larger than or equal to $c_{n-j+3} \geq 1 + c_{n-j+2}$. Thus the cost of coloring that path has increased by at least $(c_n + 1)r$, where r is the number of vertices in B_{i-2}. This increase is larger than any possible savings that can accrue to B_{i-2}. A similar argument shows that every coloring of B_i using more than i colors from S is more costly than the current coloring of B_i. Hence our presumed improved coloring does not exist and the lemma is proved. This proves Theorem 6. ■

Theorem 7. *For any cost set C with $n \geq 4$ colors, there exists a maximal planar graph G with $\chi_C(G) = n$.*

Proof. By Fact 5, without loss of generality, we can assume that the elements of C are positive integers. Let G be the maximal planar block formed by joining a single new vertex v to each vertex of the maximal outerplanar block B_{n-1} described in Theorem 6.

Assume that a best coloring of G with C uses $t < n$ colors. Then that coloring assigns some color $c_j, 1 \leq j \leq t$, to v leaving colors in $S = C - \{c_j\}$ available for B_{n-1}. Now by Theorem 6 any best coloring of $G - v = B_{n-1}$ with S uses all $n-1$ colors. Using this coloring of $G - v$ and color c_j on v yields a better coloring of G than the presumed best one we started with, a contradiction. ■

3. Upper Bounds for the Cost Chromatic Number

Given the results from Section 2, it is clear that chromatic number and cost chromatic number behave quite differently. Since the cost chromatic number of trees, maximal outerplanar graphs, and maximal planar graphs can be arbitrarily large, in this section we consider upper bounds for the cost chromatic number. We state some theorems from [9] and prove additional bounds.

Theorem 8 [9]. *If G is any graph except an odd cycle or a complete graph and C is any cost set of n colors, $n \geq \chi(G)$, then $\chi_C(G) \leq \Delta(G)$.*

Theorem 9. *Let r be the number of vertices in a longest path of graph G. Then for any cost set C of $n \geq \chi(G)$ colors, $\chi_C(G) \leq r$.*

Proof. Given any best coloring of G let v_t be a vertex colored with the maximum color c_t. By Fact 4, v_t is adjacent to a vertex v_{t-1} of color c_{t-1}, v_{t-1} is adjacent to a vertex v_{t-2} of color c_{t-2}, and so on. Hence $\chi_C(G) = t \leq r$. ■

For trees the bounds given in Theorems 8 and 9 can be substantially reduced.

Theorem 10 [9]. *For any cost set C with $n \geq 2$ colors and for any tree T whose longest path has r vertices, $\chi_C(T) \leq \lfloor r/2 \rfloor + 1$. Furthermore for any C there exists a tree T for which this bound is attained.*

Theorem 11 [9]. *For any tree T and cost set C with at least 2 colors $\chi_C(T) \leq \lceil \Delta(T)/2 \rceil + 1$.*

As noted above the bound in Theorem 10 is the best possible. We show now that the bound in Theorem 11 is the best possible in a weaker sense.

Theorem 12. *For any integer $n \geq 0$, there exists a tree T_n and cost set C such that $\chi_C(T_n) = n + 1 = \lceil \Delta(T_n)/2 \rceil + 1$.*

Proof. Let $C = \{1, 1.1, 1.11, 1.111, \ldots\}$. For each $n \geq 0$, we construct tree T_n with $n+1$ levels, $\Delta(T_n) = 2n$, and $\chi_C(T_n) = n+1$. Along with the construction we give a coloring of T_n with C using $n+1$ colors. We show that the given coloring is the unique best coloring and any other coloring of T_n with C increases the cost by at least $(1/10)^{n+1}$.

Let T_0 consist of a single vertex colored 1. Let T_1 consist of a root colored 1.1 joined to the roots of 2 colored copies of T_0. Tree T_2 is formed by joining a root colored 1.11 with the roots of two colored copies of T_0 and 2 colored copies of T_1. In general, T_n is defined recursively. It consists of a root colored $1 + (1/10) + \ldots + (1/10)^n$ joined with the roots of 2 colored copies of each of $T_{n-1}, T_{n-2}, \ldots, T_0$. Clearly we have an $n+1$ coloring of T_n with C, and T_n has $n+1$ levels and maximum degree $2n$.

We use induction to show that the current coloring is a unique best coloring and any other coloring of T_n costs at least $(1/10)^{n+1}$ more than the given coloring. This is certainly true for $n = 0$ or 1. Consider tree T_n, $n > 1$.

Let the given coloring be denoted by K, and assume that there is a different coloring K' at least as good as K.

Then either $K'(r) \geq K(r)$ or $K'(r) < K(r)$ where r is the root of T_n. The former cannot occur because, by the induction hypothesis, K is the unique best coloring of each of the designated subtrees of T_n. Hence K' must color r with a smaller color than K. So $K'(r) = 1 + (1/10) + \ldots + (1/10)^i < 1 + (1/10) + \ldots + (1/10)^n = K(r)$ for some $i, 0 \leq i < n$. Let r_1, r_2 be adjacent to r and roots of two copies of T_i. Now r_1, r_2 cannot be colored with $1 + (1/10) + \ldots + (1/10)^i$, and hence by inductive assumption their corresponding T_i must both be colored by K' in a way that costs at least $(1/10)^{i+1}$ more than the cost of coloring T_i by K. Thus $K'(T_n)$ costs at least $2(1/10)^{i+1} - (1 + (1/10) + \ldots + (1/10)^n - (1 + (1/10) + \ldots + (1/10)^i)) > (1/10)^{n+1}$ more than $K(T_n)$. ■

Before stating and proving an upperbound for outerplanar graphs analogous to Theorem 11 we give some notation and a definition. For any vertex $v, N(v)$ denotes the set of neighbors of v. Given a coloring of G with cost set C, we say that vertex v in G has *missing color* c_j if neither v nor any vertex in $N(v)$ has color c_j.

Theorem 13. *Let G be an outerplanar block and C be any cost set with at least 3 colors, then $\chi_C(G) \leq \lceil \Delta(G)/2 \rceil + 3$.*

Proof. Let $t = \lceil \Delta(G)/2 \rceil + 3$, and assume that in a best cost coloring G has a vertex v_0 colored c_{t+1}. In order to obtain a contradiction we show the existence of a path $Q : v_0, v_1, \ldots, v_r$ for some positive integer r such that

(i) The subgraph of G induced by $V(Q)$ is the path Q.
(ii) Vertex v_1 is the only vertex of its color adjacent to v_0.
(iii) For $i = 2, 3, \ldots, r, v_i$ is the only vertex of its color in $N(v_{i-1}) - \{v_{i-2}\}$.
(iv) Vertex v_r has a missing color c_j for some $j \leq t$.

By Fact 4, v_0 is adjacent to vertices of every color $c_i, 1 \leq i \leq t$. Since $t > \Delta(G)/2$, there is some color c_s which is used on only one neighbor of v_0. Let v_1 be the neighbor of v_0 colored c_s. If v_1 has a missing color $c_j, j \leq t$, then $Q : v_0, v_1$ has the properties i-iv. So we assume that v_1 has a neighbor of each color $c_j \neq c_s, j \leq t$.

Claim A. *There exists a vertex v_2 in $N(v_1) - \{v_0\}$ such that v_2 is not adjacent to v_0, and v_2 is the only vertex of its color $c_x, x \leq t$, in $N(v_1) - \{v_0\}$.*

In order to verify Claim A, note that since G is outerplanar, v_0 is adjacent to at most two other neighbors of v_1. Thus if A is not true, then $\deg(v_1) \geq 2(t-3)+3 > \Delta(G)$, a contradiction. Thus v_2 as given in A exists.

If v_2 has a missing color $c_j, 1 \leq j \leq t$, then $Q : v_0, v_1, v_2$ has properties (i) – (iv). Otherwise Q has properties (i) – (iii), and we continue building Q.

Claim B. *Suppose thus far that path* $Q : v_0, v_1, \ldots v_r$ *has been found with properties* (i) – (iii)*, and property* (iv) *does not hold. Then there is a vertex* v_{r+1} *in* $N(v_r)$ *such that* v_{r+1} *is not in* Q, v_{r+1} *is not adjacent to any vertex of* Q *except* v_r*, and* v_{r+1} *is the only vertex of its color* $c_y, y \leq t$*, in* $N(v_r) - \{v_{r-1}\}$*.*

In order to verify B, we note that because G is outerplanar at most 2 vertices of $N(v_r) - \{v_{r-1}\}$ are adjacent to vertices of $Q - \{v_r\}$. Since v_r has no missing color, if Claim B is false, then $\deg(v_r) \geq 2(t-4)+3 > \Delta(G)$, a contradiction. Thus Claim B is true.

If v_{r+1} has a missing color, then the Q we require is $v_0, v_1, \ldots, v_{r+1}$. If v_{r+1} has no missing color we apply B again with r replaced by $r+1$. Since graph G is finite we eventually find path Q with properties (i) – (iv).

Recolor $V(Q)$ by moving the color of v_i to $v_{i-1}, i = 1, \ldots, r$ and using c_j on v_r. This results in a proper coloring of G whose cost is reduced by $c_{t+1} - c_j > 0$. This is a contradiction and completes the proof of the theorem. ■

4. Line Graphs and a Conjecture of Harary and Plantholt

Earlier we showed that the cost chromatic number of trees, maximal outerplanar blocks, and maximal planar graphs can be arbitrarily larger than their chromatic number. By contrast, in this section we show that for any line graph the cost chromatic number is at most one more than its chromatic number. In fact Harary and Plantholt conjectured that for cost set $\mathcal{N}, \chi_{\mathcal{N}}(G) = \chi(G)$ for any line graph G. We give an infinite class of counterexamples to this conjecture.

Let $G = L(H)$ be the line graph of H. Since vertex coloring G is equivalent to edge coloring H, we will only consider coloring edges of H. As earlier, C is any set of positive rational numbers. The edge chromatic number and the cost edge chromatic number of H are respectively denoted by $\chi'(H)$ and $\chi'_C(H)$. With this notation the Harary-Plantholt Conjecture, as reported by West [17], is that $\chi'(H) = \chi'_N(H)$ for every graph H.

By Vizing's Theorem [16] $\chi'(H) = \Delta(H)$ or $\Delta(H)+1$ for every graph H. Graph H is called Class 1 if $\chi'(H) = \Delta(H)$ and Class 2 otherwise. We show that for any graph H and any cost set $C, \chi'_C(H) \leq \Delta(H)+1$. The proof is very similar to the proof of Vizing's Theorem given by Fiorini and Wilson [2]. We include the proof here to emphasize the cost coloring ideas.

Theorem 14. *For any graph H and any cost set C of at least $\Delta(H)+1$ positive rational numbers, $\chi'_C(H) \leq \Delta(H)+1$.*

Proof. On the contrary assume that there is a graph H and cost set C with $\chi'_C(H) = k \geq \Delta(H)+2$. We show that we can find a smaller cost coloring of H.

By our assumption, in any best edge coloring of H there exists an edge vw_1 of H which is colored c_k. Since $k \geq \Delta(H)+2$, there are colors which we will denote by b_0 and b_1, both smaller than c_k, such that b_0 is not used at v and b_1 is not used at w_1. If $b_0 = b_1$, then by recoloring vw_1 with b_0 we obtain a smaller cost edge coloring. Thus we conclude $b_0 \neq b_1$.

For any colors b_i and b_j let $H[b_i, b_j]$ denote the subgraph of H induced by all edges colored b_i or b_j. If v and w_1 are not in the same component of $H[b_0, b_1]$, then let P be the path in $H[b_0, b_1]$ containing v. Interchange colors on P and recolor vw_1 with b_1. This results in a proper edge coloring of H and reduces the cost by $c_k - b_0$ or $c_k - b_1$.

Thus v and w_1 are in the same component of $H[b_0, b_1]$. Let vw_2 be the edge colored b_1 adjacent to v. Now since $k \geq \Delta(H)+2$, there is some color $b_2 < c_k$ and different from b_0 and b_1, which is not used on any edge incident with w_2. If b_2 is not used on an edge incident with v we recolor vw_2 with b_2 and vw_1 with b_1. This yeilds a smaller cost edge coloring of H.

Therefore we let vw_3 be the edge colored b_2 adjacent to v. Now $H[b_0, b_2]$ has a component containing v, w_2 and w_3. Otherwise, interchange the colors b_0 and b_2 on the component of $H[b_0, b_2]$ containing v, recolor vw_2 with b_2 and vw_1 with b_1. This yields a proper edge coloring of H and reduces the cost by either $c_k - b_0$ or $c_k - b_2$.

We continue similarly finding edges $vw_1, vw_2, \ldots, vw_j$ and colors $b_1, b_2, \ldots, b_j$ all smaller than c_k where each $b_i, i = 1, 2, \ldots, j$ is missing at w_i and edge vw_i is colored b_{i-1}. Since the graph is finite we eventually find

(i) Edge vw_j and color b_j such that b_j is also missing at v or

(ii) Edge vw_j and color b_j, such that $b_j = b_i$ for some $i < j-1$.

If (i) occurs, then we obtain a smaller cost coloring by recoloring vw_j with b_j, vw_{j-1} with $b_{j-1}, \ldots, vw_2$ with b_2 and vw_1 with b_1.

If (ii) occurs, then vertices v, w_i, and w_{i+1} are in the same component H' of $H[b_o, b_i]$, for otherwise, as previously an interchange of colors leads to a smaller cost coloring. Thus $H' \neq H''$ where H'' is the component of $H[b_0, b_i]$ which contains w_j. Interchange colors on H''. Then recolor vw_j with b_0, vw_{j-1} with $b_{j-2}, \ldots, vw_2$ with b_2, and vw_1 with b_1. Hence we have a smaller cost coloring of G which completes the proof of the theorem. ■

It follows from Theorem 14 that if H is Class 2, then $\chi'_C(H) = \chi'(H)$ for any cost set C. Now we exhibit an infinite set of Class 1 graphs for which $\chi'_{\mathcal{N}}(H) > \chi'(H)$. In order to do that we consider a theorem of Izbicki [4], which we give without proof, and an immediate corollary.

Theorem 15 (Izbicki)**.** *Let H be a Class 1 graph in which every vertex has either degree $\Delta(H)$ or degree 1. For any edge coloring of H with $\Delta(H)$ colors let f_i be the number of end edges with color i for $i = 1, 2, \ldots, \Delta(H)$. Then all f_i have the same parity.*

Corollary 16. *Let H be Class 1 with all vertices of degree $\Delta(H)$ or 1 where $\Delta(H)$ is odd. If H has exactly $\Delta(H)$ end edges, then any edge coloring of H with $\Delta(H)$ colors has each color used on exactly one end edge.*

Corollary 17. *Let H_n be the graph formed by adding an end edge to each vertex of the complete graph K_n. If n is odd, then H_n is Class 1, $\Delta(H_n) = n$, and any edge coloring of H_n with n colors has each color used on exactly one end edge.*

Proof. When n is odd it is easy to n-color the edges of K_n. Then each vertex of K_n has a missing color which can be used on its incident end edge. Thus H is Class 1 and Corollary 17 follows immediately from Corollary 16. ■

Theorem 18. *For odd $n \geq 3$, and for any cost set C with $n+1$ colors where $c_{n-1} + c_n > c_1 + c_{n+1}$, we have $\chi'_C(H_n) = n + 1 > \chi'(H_n)$.*

Proof. Corollary 17 implies that all edge colorings of H_n with $C - (c_{n+1})$ have the same cost. Furthermore each such coloring is a partition of $E(H_n)$ into n sets where each set has exactly one end edge. So we may consider an edge coloring of H_n with n colors in which the end edges colored c_{n-1} and c_n are joined by an edge e of K_n colored c_1. Recolor e with c_{n+1} and the end edges with c_1. This reduces the cost by $c_1 + c_{n-1} + c_n - (c_{n+1} + 2c_1) = c_{n-1} + c_n - c_{n+1} - c_1 > 0$. Hence $\chi'_C(H_n) = n + 1$. ■

Corollary 19.

(i) *For odd* $n \geq 5, \chi'_{\mathcal{N}}(H_n) > \chi'(H_n)$, *and*

(ii) *For each positive integer* m *there exists* n *sufficiently large such that there exists a* $(\Delta + 1)$*-coloring of* H_n *with* $\mathcal{N}$ *which has cost at least* m *smaller than any* Δ*-coloring.*

Proof. Let $C = \mathcal{N}$. Part i follows immediatelly from Theorem 18 because for $n \geq 5, (n-1)+n > (n+1)+1$. Part ii follows from the proof of Theorem 18 because for sufficiently large $n, (n-1)+n-(n+1)-1 > m$. ■

The counterexamples H_n, given above to the Harary-Plantholt Conjecture, have minimum degree 1. On the other hand any n-regular Class 1 graph H has n mutually disjoint perfect matchings. Thus $\chi'(H) = \chi'_C(H)$ for any cost set C. In Theorem 20, we use H_n to construct additional counterexamples $H_{n,m}$ with maximum degree n and minimum degree m, $m = 2, 3, \ldots, n-2$. Then in Theorem 21, we construct counterexamples to Harary-Plantholt which have maximum degree n and minimum degree $n-1$.

Theorem 20. *For any odd integer* $n \geq 5$ *and* $m = 2, 3, \ldots, n-2$, *there exists a Class* 1 *graph* $H_{n,m}$ *with* $\Delta(H_{n,m}) = n$ *and* $\delta(H_{n,m}) = m$ *such that* $\chi'_{\mathcal{N}}(H_{n,m}) = n+1$.

Proof. Let $u_1, \ldots, u_n$ be the endvertices of H_n. We form $H_{n,m}$ by adding $m-1$ vertices $w_1, w_2, \ldots, w_{m-1}$ to H_n and joining each w_i to each u_j. For $m = 2, 3, \ldots, n-2$, $H_{n,m}$ is Class 1. To see this, n-color the edges of H_n so that the edge e_j incident with u_j has color j. For each $i = 1, 2, \ldots, m-1$, color $w_iu_1, w_iu_2, \ldots, w_iu_n$ respectively with colors $n-i+1, n-i+2, \ldots, n, 1, 2, \ldots, n-i$. This is an n-edge coloring of $H_{n,m}$, and thus $H_{n,m}$ is Class 1. Furthermore with this coloring color 1 is missing from both u_{n-1} and u_n. Also note that all n-edge colorings of $H_{n,m}$ have the same cost. ■

Without loss of generality we may assume that the edge e of K_n which is incident with e_n and e_{n-1} is colored 1. We obtain a lower cost edge coloring by using color $n+1$ on e and color 1 on both e_n and e_{n-1}.

Theorem 21. *For each* $r \geq 2$, *there exists a graph* G'' *such that* $\Delta(G'') = 2r+1, \delta(G'') = 2r, \chi'_{\mathcal{N}}(G'') = 2r+2$, *and* $\chi'(G'') = 2r+1$.

Proof. As we construct G'' we also give a minimum cost $(2r+1)$-coloring of its edges. We then show that the cost can be reduced by using color $2r+2$.

It is well known that the edge set of any complete graph of odd order can be partitioned into mutually disjoint Hamiltonian cycles. For $r \geq 2$, we choose r of these cycles from K_{4r+1}. Let $B_1 : w_1, w_2, \ldots, w_{4r+1}, w_1$ be any Hamiltonian cycle of K_{4r+1}. Let $B_2, B_3, \ldots, B_r$ be the other cycles where B_r is the unique Hamiltonian cycle in the partition which contains edge w_2w_4.

We now $(2r+1)$-color the edges of the resulting $2r$-regular graph G. Color the edges $w_1w_2, w_2w_3, \ldots, w_{4r+1}w_1$ of B_1 respectively with $3, 1, 3, 1, 2, 3, 2, 3, \ldots, 2$. In B_r color edge w_2w_4 with 2 and use colors $2r, 2r+1$ on the other edges of B_r. Thus we have a proper edge-coloring of $B_1 \cup B_r$ which has two edges colored 1 and $2r$ edges of each color $2, 3, 2r, 2r+1$.

Each remaining cycle $B_i, i = 2, 3, \ldots, r-1$, of G is colored so that one edge has color 1 and the other edges alternate colors $2i, 2i+1$. In coloring B_i, an edge which is not adjacent to an edge already colored 1 can be selected for color 1. In order to see this, we note that each of the less than r edges already colored 1 is adjacent with at most 4 edges of B_i. Since B_i has $4r+1$ edges, one of them can be colored 1.

Thus we have edge-colored G with $2r+1$ colors such that r edges are colored 1 and $2r$ edges are colored $i, i = 2, 3, \ldots, 2r+1$. Now form graph G' by adding one end edge to each vertex of G. Color each end edge with the unique color available from $1, 2, \ldots, 2r+1$. Thus we have edge-colored G' with $2r+1$ colors such that the end edges colored $2r$ and $2r+1$ are both adjacent to edge w_2w_4 of B_r which has color 2.

Hence exactly one end edge of G' has color $i, i = 2, 3, \ldots, 2r+1$, and color 1 is used on $2r+1$ end edges. So in total, G' has $3r+1$ edges colored 1 and $2r+1$ edges colored $i, i = 2, 3, \ldots, 2r+1$. From Theorem 15 it follows that any edge-coloring of G' with $2r+1$ colors must use each color at least $2r+1$ times. Hence among all edge-colorings of G' with $2r+1$ colors, this one has minimum cost.

We now add vertices and edges to G' to form G'' which will have the required properties. Let E_1 be any $2r$ end edges colored $2, 1, 1, \ldots, 1$, listed from left to right. Let E_2 be the other $2r+1$ end edges, which are colored $1, 1, 3, 4, \ldots, 2r, 2r+1$ and are also listed from left to right.

Let $U_1 = \{u_1, \ldots, u_{2r-1}\}$ and $V_1 = \{v_1, \ldots, v_{2r-1}\}$ be disjoint sets of new vertices which we add to G'. Form the edge $u_{2i}u_{2i+1}, i = 1, 2, \ldots, r-1$, and color each of these $r-1$ edges 1. Also join each $u_i, i = 1, \ldots, 2r-1$, to each endvertex of the edges of E_1. The edges from u_1 to E_1 (where E_1 is taken in the above order) are colored consecutively $1, 2, \ldots, 2r$. Similarly the edges from u_2 to E_1 are colored consecutively $3, 4, 5, \ldots, 2r+1, 2$; the edges from u_3 to E_1 are colored consecutively $4, 5, \ldots, 2r+1, 2, 3$; and so forth until finally the edges from u_{2r-1} to E_1 are colored consecutively $2r, 2r+1, 2, 3, \ldots, 2r-1$.

Now join each vertex of V_1 to each endvertex of the edges in E_2. Color the edges from v_1 to E_2 consecutively (in the order given for E_2) $2, 2r+1, 1, 3, 4, \ldots, 2r$. Similarly color the edges from v_2 to E_2 consecutively $2r, 2, 2r+1, 1, 3, 4, \ldots, 2r-1$; the edges from v_3 to E_2 consecutively $2r-1, 2r, 2, 2r+1, 1, 3, 4, \ldots, 2r-2$; and so forth until finally the edges from v_{2r-1} to E_2 are colored consecutively $3, 4, \ldots, 2r, 2, 2r+1, 1$.

The resulting graph G'' satisfies $\delta(G'') = 2r$, and has been properly edge-colored with $2r+1 = \Delta(G'')$ colors. Furthermore our edge-coloring of G'' is a minimum cost Class 1 coloring. In order to see this, recall that our coloring of G' has minimum cost. Also note that all vertices added to G' in forming G'' except u_1 have degree $2r+1$ and hence must use all colors on incident edges. The degree of u_1 is $2r$, and its incident edges use all colors except $2r+1$. Hence among all colorings of G'' with $2r+1$ colors, we have one of minimum cost.

Recall that edge w_2w_4 is colored 2 and is adjacent to end edges e_1 and e_2 of G' colored $2r$ and $2r+1$. Furthermore the endvertices in G' of e_1 and e_2 are not incident to any edge colored 2. Hence we can recolor edge w_2w_4 with $2r+2$ and edges e_1 and e_2 with 2. This reduces the cost of coloring by $4r+3-(2r+6) = 2r-3 > 0$. Thus $\chi'_{\mathcal{N}}(G'') = \chi'(G'')+1$. ■

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Received 15 February 1996
Revised 14 January 1997