ON THE COST CHROMATIC NUMBER OF OUTERPLANAR, PLANAR, AND LINE GRAPHS

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Abstract

We consider vertex colorings of graphs in which each color has an associated cost which is incurred each time the color is assigned to a vertex. The cost of the coloring is the sum of the costs incurred at each vertex. The cost chromatic number of a graph with respect to a cost set is the minimum number of colors necessary to produce a minimum cost coloring of the graph. We show that the cost chromatic number of maximal outerplanar and maximal planar graphs can be arbitrarily large and construct several infinite classes of counterexamples to a conjecture of Harary and Plantholt on the cost chromatic number of line graphs.

Keywords: cost coloring, outerplanar, planar, line graphs.

1991 Mathematics Subject Classification: Primary 05C15, Secondary 05C10.

1. Introduction

In this paper we consider the problem of coloring a graph as efficiently as possible when each color has an associated positive rational cost. Supowit [11] first posed this problem. Independently various subsets of E. Kubicka, Erdős, Thomassen, Schwenk, Alavi, Malde, G. Kubicka, and Kountanis published papers [1], [5], [6], [7], [8], and [12] on a restricted version of this problem where the costs are the first \( n \) positive integers. Recently Nicoloso et.

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al. [10] investigated this restricted cost coloring problem for interval graphs and discussed its application to VLSI routing. In this paper we continue the investigation of the general cost coloring problem begun by Mitchem and Morriss [9]. We give now the formal basic definitions.

For any palette $P = \{p_1, \ldots, p_n\}$ of colors we have an associated set $C_p = \{c_1, \ldots, c_n\}$ of costs where $c_i$ is a positive rational number associated with color $p_i$. Each time that $p_i$ is used on a vertex the cost $c_i$ is accrued. Our goal is to color graph $G$ using colors from $P$ so that a minimum total cost is attained. For convenience we assume that $c_1 < c_2 < \ldots < c_n$. The cost-chromatic number of $G$, denoted by $\chi_{C_p}(G)$, with respect to cost set $C_p$, is the minimum number of colors necessary to produce a minimum cost coloring of $G$. A minimum cost coloring of $G$ with $C_p$ which uses only $\chi_{C_p}(G)$ colors is called a best coloring. In the special case of cost coloring studied in [1], [5], [6], [7], [8], [10], and [12] the set $C_p$ is $\mathbb{N}$, the set of positive integers, and the minimum cost of coloring $G$ is called the chromatic sum. Other cost coloring papers are [3], [13], [14], and [15].

We will simplify notation by no longer distinguishing between the color $p_i$ and the cost $c_i$. Thus we will have a cost set $C = \{c_1, \ldots, c_n\}$ of positive rational numbers which we will also call colors, and the cost of using color $c_i$ on any vertex is $c_i$. Given $C, \chi_c(G)$ denotes the minimum number of colors from $C$ required to obtain a minimum cost coloring of graph $G$ using $C$.

2. Outerplanar and Planar Graphs

In [1] it is shown that for any integer $n \geq 2$ there is a tree $T$ such that $\chi_{\mathbb{N}}(T) = n$. Mitchem and Morriss extended that result.

**Theorem 1** [9]. For any cost set $C$ with $n \geq 2$ positive rational numbers there exists a tree $T$ such that $\chi_C(T) = n$.

In this section we prove similar results for maximal outerplanar and maximal planar graphs. At first consideration these two new results may appear to be obvious. However, the following two theorems show that cost chromatic number behaves very differently from chromatic number, and Theorem 1 does not immediately imply its outerplanar and planar analogues.

**Theorem 2** [9]. Let $C$ be any cost set of $n$ colors and $G$ be a graph such that $\chi_C(G) = n > \chi(G)$. Then there is a graph $G'$ with $V(G) = V(G')$ and $E(G)$ a subset of $E(G')$ such that $\chi_C(G') = \chi(G)$.
Theorem 3 [9]. Let $C$ be any cost set of $n \geq 2$ colors and $T$ be any tree with $\chi_C(T) = n$. Then there exists a tree $T'$ which contains $T$ such that $\chi_C(T') = 2$.

Before stating and proving our two main theorems of this section we make the following observations. The first fact is immediate, and the easy proof of the second appears in [9].

Fact 4. In any minimum cost coloring of $G$ with cost set $C$, any vertex $v$ of color $c_i$, $i \geq 2$, has neighbors of all colors $c_1, \ldots, c_{i-1}$. Thus if $v$ is colored $c_i$, then $\deg(v) \geq i-1$.

Fact 5. Let $C = \{c_1, \ldots, c_n\}$ be any cost set of positive rational numbers. Then there exists a cost set $C' = \{c'_1, \ldots, c'_n\}$ of positive integers such that for any graph $G$ a best coloring $K$ of $G$ with $C$ corresponds to a best coloring $K'$ of $G$ with $C'$ in the following sense: if $K$ colors vertex $v$ of $G$ with $c_i$, then $K'$ colors $v$ with $c'_i$. Furthermore, $c'_1$ may be taken to be equal to 1.

Theorem 6. For any cost set $C$ of $n \geq 2$ positive rational numbers, there exists an outerplanar block $B$ such that $\chi_C(B) = n$.

Proof. By Fact 5, without loss of generality, we may take $C$ to be a set of positive integers. The construction of block $B = B_n$ will depend on the parity of $n$. Let $B_2 = K_2$ and $B_3 = K_3$. For $t = 4, 5, \ldots, n$ we construct $B_t$ from $B_{t-2}$.

Consider an outerplanar embedding of $B_{t-2}$ where $v_1, v_2, \ldots, v_r, v_1$ are its consecutive vertices on the outer border. For $i = 1, 2, \ldots, r$ and subscripts taken modulo $r$, insert a path from $v_i$ to $v_{i+1}$ with $2(c_n+1)r$ internal vertices and construct a chord from each of these new vertices to $v_i$. Then $B_t$ is an outerplanar block.

We state and prove a lemma from which Theorem 6 follows.

Lemma. Let $n \geq j \geq i \geq 2$ and $S$ be the subset consisting of the $j$ largest elements of $C$. Then $\chi_S(B_i) = i$ and any best coloring uses the $i$ smallest elements of $S$.

Proof. We use induction on $i$ noting that the lemma is clearly true for $i = 2$ and 3. Consider $B_i, n \geq i \geq 4$. Let $n \geq j \geq i$ and $S = \{c_{n-j+1}, c_{n-j+2}, \ldots, c_n\}$. By the induction hypothesis any best coloring of $B_{i-2}$ with $S' = S - \{c_{n-j+1}, c_{n-j+2}\}$ uses the $i - 2$ smallest colors of $S'$. Give such a coloring to $B_{i-2}$, and alternate colors $c_{n-j+1}$ and $c_{n-j+2}$ on
each path added to $B_{i-2}$ to form $B_i$. This clearly gives a coloring of $B_i$ with $i$ colors from $S$.

In order to complete the proof of the lemma it suffices to show that any coloring using other than $i$ colors from $S$ is more costly than the current coloring. On the contrary first assume that there is a coloring of $B_i$ using less than $i$ colors from $S$ which is no more costly than the current coloring. From the inductive hypothesis it follows that in such a coloring either $c_{n-j+1}$ or $c_{n-j+2}$ is used on a vertex of $B_{i-2}$, and so it is not used on any interior vertex of at least one of the paths added to $B_{i-2}$ to form $B_i$. Therefore at least half of the interior vertices of this path are colored with a color larger than or equal to $c_{n-j+3} \geq 1 + c_{n-j+2}$. Thus the cost of coloring that path has increased by at least $(c_n + 1)r$, where $r$ is the number of vertices in $B_{i-2}$. This increase is larger than any possible savings that can accrue to $B_{i-2}$. A similar argument shows that every coloring of $B_i$ using more than $i$ colors from $S$ is more costly than the current coloring of $B_i$. Hence our presumed improved coloring does not exist and the lemma is proved. This proves Theorem 6.

**Theorem 7.** For any cost set $C$ with $n \geq 4$ colors, there exists a maximal planar graph $G$ with $\chi_C(G) = n$.

**Proof.** By Fact 5, without loss of generality, we can assume that the elements of $C$ are positive integers. Let $G$ be the maximal planar block formed by joining a single new vertex $v$ to each vertex of the maximal outerplanar block $B_{n-1}$ described in Theorem 6.

Assume that a best coloring of $G$ with $C$ uses $t < n$ colors. Then that coloring assigns some color $c_j, 1 \leq j \leq t$, to $v$ leaving colors in $S = C - \{c_j\}$ available for $B_{n-1}$. Now by Theorem 6 any best coloring of $G - v = B_{n-1}$ with $S$ uses all $n - 1$ colors. Using this coloring of $G - v$ and color $c_j$ on $v$ yields a better coloring of $G$ than the presumed best one we started with, a contradiction.

3. **Upper Bounds for the Cost Chromatic Number**

Given the results from Section 2, it is clear that chromatic number and cost chromatic number behave quite differently. Since the cost chromatic number of trees, maximal outerplanar graphs, and maximal planar graphs can be arbitrarily large, in this section we consider upper bounds for the cost chromatic number. We state some theorems from [9] and prove additional bounds.
Theorem 8 [9]. If $G$ is any graph except an odd cycle or a complete graph and $C$ is any cost set of $n$ colors, $n \geq \chi(G)$, then $\chi_C(G) \leq \Delta(G)$.

Theorem 9. Let $r$ be the number of vertices in a longest path of graph $G$. Then for any cost set $C$ of $n \geq \chi(G)$ colors, $\chi_C(G) \leq r$.

Proof. Given any best coloring of $G$ let $v_t$ be a vertex colored with the maximum color $c_t$. By Fact 4, $v_t$ is adjacent to a vertex $v_{t-1}$ of color $c_{t-1}$, $v_{t-1}$ is adjacent to a vertex $v_{t-2}$ of color $c_{t-2}$, and so on. Hence $\chi_C(G) = t \leq r$.

For trees the bounds given in Theorems 8 and 9 can be substantially reduced.

Theorem 10 [9]. For any cost set $C$ with $n \geq 2$ colors and for any tree $T$ whose longest path has $r$ vertices, $\chi_C(T) \leq \lfloor r/2 \rfloor + 1$. Furthermore for any $C$ there exists a tree $T$ for which this bound is attained.

Theorem 11 [9]. For any tree $T$ and cost set $C$ with at least 2 colors $\chi_C(T) \leq \lceil \Delta(T)/2 \rceil + 1$.

As noted above the bound in Theorem 10 is the best possible. We show now that the bound in Theorem 11 is the best possible in a weaker sense.

Theorem 12. For any integer $n \geq 0$, there exists a tree $T_n$ and cost set $C$ such that $\chi_C(T_n) = n + 1 = \lceil \Delta(T_n)/2 \rceil + 1$.

Proof. Let $C = \{1, 1.1, 1.11, 1.111, \ldots \}$. For each $n \geq 0$, we construct tree $T_n$ with $n + 1$ levels, $\Delta(T_n) = 2n$, and $\chi_C(T_n) = n + 1$. Along with the construction we give a coloring of $T_n$ with $C$ using $n + 1$ colors. We show that the given coloring is the unique best coloring and any other coloring of $T_n$ with $C$ increases the cost by at least $(1/10)^{n+1}$.

Let $T_0$ consist of a single vertex colored 1. Let $T_1$ consist of a root colored 1.1 joined to the roots of 2 colored copies of $T_0$. Tree $T_2$ is formed by joining a root colored 1.11 with the roots of two colored copies of $T_0$ and 2 colored copies of $T_1$. In general, $T_n$ is defined recursively. It consists of a root colored $1 + (1/10) + \ldots + (1/10)^n$ joined with the roots of 2 colored copies of each of $T_{n-1}, T_{n-2}, \ldots, T_0$. Clearly we have an $n + 1$ coloring of $T_n$ with $C$, and $T_n$ has $n + 1$ levels and maximum degree $2n$.

We use induction to show that the current coloring is a unique best coloring and any other coloring of $T_n$ costs at least $(1/10)^{n+1}$ more than the given coloring. This is certainly true for $n = 0$ or 1. Consider tree $T_n$, $n > 1$.  

Let the given coloring be denoted by $K$, and assume that there is a different coloring $K'$ at least as good as $K$.

Then either $K'(r) \geq K(r)$ or $K'(r) < K(r)$ where $r$ is the root of $T_n$. The former cannot occur because, by the induction hypothesis, $K$ is the unique best coloring of each of the designated subtrees of $T_n$. Hence $K'$ must color $r$ with a smaller color than $K$. So $K'(r) = 1 + (1/10) + \ldots + (1/10)^i < 1 + (1/10) + \ldots + (1/10)^n = K(r)$ for some $i, 0 \leq i < n$. Let $r_1, r_2$ be adjacent to $r$ and roots of two copies of $T_i$. Now $r_1, r_2$ cannot be colored with $1 + (1/10) + \ldots + (1/10)^i$, and hence by inductive assumption their corresponding $T_i$ must both be colored by $K'$ in a way that costs at least $(1/10)^{i+1}$ more than the cost of coloring $T_i$ by $K$. Thus $K'(T_n)$ costs at least $2(1/10)^{i+1} - (1 + (1/10) + \ldots + (1/10)^n - (1 + (1/10) + \ldots + (1/10)^i)) > (1/10)^{n+1}$ more than $K(T_n)$. \hfill \qed

Before stating and proving an upperbound for outerplanar graphs analogous to Theorem 11 we give some notation and a definition. For any vertex $v$, $N(v)$ denotes the set of neighbors of $v$. Given a coloring of $G$ with cost set $C$, we say that vertex $v$ in $G$ has missing color $c_j$ if neither $v$ nor any vertex in $N(v)$ has color $c_j$.

**Theorem 13.** Let $G$ be an outerplanar block and $C$ be any cost set with at least 3 colors, then $\chi_C(G) \leq \lceil \Delta(G)/2 \rceil + 3$.

**Proof.** Let $t = \lceil \Delta(G)/2 \rceil + 3$, and assume that in a best cost coloring $G$ has a vertex $v_0$ colored $c_{t+1}$. In order to obtain a contradiction we show the existence of a path $Q : v_0, v_1, \ldots, v_r$ for some positive integer $r$ such that

(i) The subgraph of $G$ induced by $V(Q)$ is the path $Q$.
(ii) Vertex $v_1$ is the only vertex of its color adjacent to $v_0$.
(iii) For $i = 2, 3, \ldots, r$, $v_i$ is the only vertex of its color in $N(v_{i-1}) - \{v_{i-2}\}$.
(iv) Vertex $v_i$ has a missing color $c_j$ for some $j \leq t$.

By Fact 4, $v_0$ is adjacent to vertices of every color $c_i, 1 \leq i \leq t$. Since $t > \Delta(G)/2$, there is some color $c_s$ which is used on only one neighbor of $v_0$. Let $v_1$ be the neighbor of $v_0$ colored $c_s$. If $v_1$ has a missing color $c_j, j \leq t$, then $Q : v_0, v_1$ has the properties i-iv. So we assume that $v_1$ has a neighbor of each color $c_j \neq c_s, j \leq t$.

**Claim A.** There exists a vertex $v_2$ in $N(v_1) - \{v_0\}$ such that $v_2$ is not adjacent to $v_0$, and $v_2$ is the only vertex of its color $c_x, x \leq t$, in $N(v_1) - \{v_0\}$.
In order to verify Claim A, note that since $G$ is outerplanar, $v_0$ is adjacent to at most two other neighbors of $v_1$. Thus if $A$ is not true, then $\deg(v_1) \geq 2(t - 3) + 3 > \Delta(G)$, a contradiction. Thus $v_2$ as given in $A$ exists.

If $v_2$ has a missing color $c_j$, $1 \leq j \leq t$, then $Q : v_0, v_1, v_2$ has properties (i) – (iv). Otherwise $Q$ has properties (i) – (iii), and we continue building $Q$.

Claim B. Suppose thus far that path $Q : v_0, v_1, \ldots, v_r$ has been found with properties (i) – (iii), and property (iv) does not hold. Then there is a vertex $v_{r+1}$ in $N(v_r)$ such that $v_{r+1}$ is not in $Q$, $v_{r+1}$ is not adjacent to any vertex of $Q$ except $v_r$, and $v_{r+1}$ is the only vertex of its color $c_y, y \leq t$, in $N(v_r) - \{v_{r-1}\}$.

In order to verify B, we note that because $G$ is outerplanar at most 2 vertices of $N(v_r) - \{v_{r-1}\}$ are adjacent to vertices of $Q - \{v_r\}$. Since $v_r$ has no missing color, if Claim B is false, then $\deg(v_r) \geq 2(t - 4) + 3 > \Delta(G)$, a contradiction. Thus Claim B is true.

If $v_{r+1}$ has a missing color, then the $Q$ we require is $v_0, v_1, \ldots, v_{r+1}$. If $v_{r+1}$ has no missing color we apply B again with $r$ replaced by $r + 1$. Since graph $G$ is finite we eventually find path $Q$ with properties (i) – (iv).

Recolor $V(Q)$ by moving the color of $v_i$ to $v_{i-1}, i = 1, \ldots, r$ and using $c_j$ on $v_r$. This results in a proper coloring of $G$ whose cost is reduced by $c_{t+1} - c_j > 0$. This is a contradiction and completes the proof of the theorem.

4. Line Graphs and a Conjecture of Harary and Plantholt

Earlier we showed that the cost chromatic number of trees, maximal outerplanar blocks, and maximal planar graphs can be arbitrarily larger than their chromatic number. By contrast, in this section we show that for any line graph the cost chromatic number is at most one more than its chromatic number. In fact Harary and Plantholt conjectured that for cost set $N, \chi_N(G) = \chi(G)$ for every graph $G$. We give an infinite class of counterexamples to this conjecture.

Let $G = L(H)$ be the line graph of $H$. Since vertex coloring $G$ is equivalent to edge coloring $H$, we will only consider coloring edges of $H$. As earlier, $C$ is any set of positive rational numbers. The edge chromatic number and the cost edge chromatic number of $H$ are respectively denoted by $\chi'(H)$ and $\chi'_C(H)$. With this notation the Harary-Plantholt Conjecture, as reported by West [17], is that $\chi'(H) = \chi'_N(H)$ for every graph $H$. 
By Vizing’s Theorem [16] \( \chi'(H) = \Delta(H) \) or \( \Delta(H) + 1 \) for every graph \( H \). Graph \( H \) is called Class 1 if \( \chi'(H) = \Delta(H) \) and Class 2 otherwise. We show that for any graph \( H \) and any cost set \( C \), \( \chi'_C(H) \leq \Delta(H) + 1 \). The proof is very similar to the proof of Vizing’s Theorem given by Fiorini and Wilson [2]. We include the proof here to emphasize the cost coloring ideas.

**Theorem 14.** For any graph \( H \) and any cost set \( C \) of at least \( \Delta(H) + 1 \) positive rational numbers, \( \chi'_C(H) \leq \Delta(H) + 1 \).

**Proof.** On the contrary assume that there is a graph \( H \) and cost set \( C \) with \( \chi'_C(H) = k \geq \Delta(H) + 2 \). We show that we can find a smaller cost coloring of \( H \).

By our assumption, in any best edge coloring of \( H \) there exists an edge \( vw_1 \) of \( H \) which is colored \( c_k \). Since \( k \geq \Delta(H) + 2 \), there are colors which we will denote by \( b_0 \) and \( b_1 \), both smaller than \( c_k \), such that \( b_0 \) is not used at \( v \) and \( b_1 \) is not used at \( w_1 \). If \( b_0 = b_1 \), then by recoloring \( vw_1 \) with \( b_0 \) we obtain a smaller cost edge coloring. Thus we conclude \( b_0 \neq b_1 \).

For any colors \( b_i \) and \( b_j \) let \( H[b_i, b_j] \) denote the subgraph of \( H \) induced by all edges colored \( b_i \) or \( b_j \). If \( v \) and \( w_1 \) are not in the same component of \( H[b_0, b_1] \), then let \( P \) be the path in \( H[b_0, b_1] \) containing \( v \). Interchange colors on \( P \) and recolor \( vw_1 \) with \( b_1 \). This results in a proper edge coloring of \( H \) and reduces the cost by \( c_k - b_0 \) or \( c_k - b_1 \).

Thus \( v \) and \( w_1 \) are in the same component of \( H[b_0, b_1] \). Let \( vw_2 \) be the edge colored \( b_1 \) adjacent to \( v \). Now since \( k \geq \Delta(H) + 2 \), there is some color \( b_2 \) which is not used at \( v \) and \( b_1 \), which is not used on any edge incident with \( w_2 \). If \( b_2 \) is not used on an edge incident with \( v \) we recolor \( vw_2 \) with \( b_2 \) and \( vw_1 \) with \( b_1 \). This yields a smaller cost edge coloring of \( H \).

Therefore we let \( vw_3 \) be the edge colored \( b_2 \) adjacent to \( v \). Now \( H[b_0, b_2] \) has a component containing \( v, w_2 \) and \( w_3 \). Otherwise, interchange the colors \( b_0 \) and \( b_2 \) on the component of \( H[b_0, b_2] \) containing \( v \), recolor \( vw_2 \) with \( b_2 \) and \( vw_1 \) with \( b_1 \). This yields a proper edge coloring of \( H \) and reduces the cost by either \( c_k - b_0 \) or \( c_k - b_2 \).

We continue similarly finding edges \( vw_1, vw_2, \ldots, vw_j \) and colors \( b_1, b_2, \ldots, b_j \) all smaller than \( c_k \) where each \( b_i, i = 1, 2, \ldots, j \) is missing at \( w_i \) and edge \( vw_i \) is colored \( b_{i-1} \). Since the graph is finite we eventually find

(i) Edge \( vw_j \) and color \( b_j \) such that \( b_j \) is also missing at \( v \) or
(ii) Edge \( vw_j \) and color \( b_j \), such that \( b_j = b_i \) for some \( i < j - 1 \).

If (i) occurs, then we obtain a smaller cost coloring by recoloring \( vw_j \) with \( b_j, vw_{j-1} \) with \( b_{j-1} \), \ldots, \( vw_2 \) with \( b_2 \) and \( vw_1 \) with \( b_1 \).
If (ii) occurs, then vertices $v, w_i$, and $w_{i+1}$ are in the same component $H'$ of $H[b_0, b_1]$, for otherwise, as previously an interchange of colors leads to a smaller cost coloring. Thus $H' \neq H''$ where $H''$ is the component of $H[b_0, b_1]$ which contains $w_j$. Interchange colors on $H''$. Then recolor $vw_j$ with $b_0, vw_{j-1}$ with $b_{j-2}, \ldots, vw_2$ with $b_2$, and $vw_1$ with $b_1$. Hence we have a smaller cost coloring of $G$ which completes the proof of the theorem.

It follows from Theorem 14 that if $H$ is Class 2, then $\chi'_{C}(H) = \chi'(H)$ for any cost set $C$. Now we exhibit an infinite set of Class 1 graphs for which $\chi'_{N}(H) > \chi'(H)$. In order to do that we consider a theorem of Izbicki [4], which we give without proof, and an immediate corollary.

**Theorem 15** (Izbicki). Let $H$ be a Class 1 graph in which every vertex has either degree $\Delta(H)$ or degree 1. For any edge coloring of $H$ with $\Delta(H)$ colors let $f_i$ be the number of end edges with color $i$ for $i = 1, 2, \ldots, \Delta(H)$. Then all $f_i$ have the same parity.

**Corollary 16.** Let $H$ be Class 1 with all vertices of degree $\Delta(H)$ or 1 where $\Delta(H)$ is odd. If $H$ has exactly $\Delta(H)$ end edges, then any edge coloring of $H$ with $\Delta(H)$ colors has each color used on exactly one end edge.

**Corollary 17.** Let $H_n$ be the graph formed by adding an end edge to each vertex of the complete graph $K_n$. If $n$ is odd, then $H_n$ is Class 1, $\Delta(H_n) = n$, and any edge coloring of $H_n$ with $n$ colors has each color used on exactly one end edge.

**Proof.** When $n$ is odd it is easy to $n$-color the edges of $K_n$. Then each vertex of $K_n$ has a missing color which can be used on its incident end edge. Thus $H$ is Class 1 and Corollary 17 follows immediately from Corollary 16.

**Theorem 18.** For odd $n \geq 3$, and for any cost set $C$ with $n+1$ colors where $c_{n-1} + c_n > c_1 + c_{n+1}$, we have $\chi'_{C}(H_n) = n + 1 > \chi'(H_n)$.

**Proof.** Corollary 17 implies that all edge colorings of $H_n$ with $C - (c_{n+1})$ have the same cost. Furthermore each such coloring is a partition of $E(H_n)$ into $n$ sets where each set has exactly one end edge. So we may consider an edge coloring of $H_n$ with $n$ colors in which the end edges colored $c_{n-1}$ and $c_n$ are joined by an edge $e$ of $K_n$ colored $c_1$. Recolor $e$ with $c_n + 1$ and the end edges with $c_1$. This reduces the cost by $c_1 + c_{n-1} + c_n - (c_{n+1} + 2c_1) = c_{n-1} + c_n - c_{n+1} - c_1 > 0$. Hence $\chi'_{C}(H_n) = n + 1$. 

\[ \chi'_{C}(H_n) = n + 1. \]
Corollary 19.

(i) For odd \( n \geq 5 \), \( \chi'_N(H_n) > \chi'(H_n) \), and

(ii) For each positive integer \( m \) there exists \( n \) sufficiently large such that there exists a \((\Delta + 1)\)-coloring of \( H_n \) with \( N \) which has cost at least \( m \) smaller than any \( \Delta \)-coloring.

Proof. Let \( C = N \). Part i follows immediately from Theorem 18 because for \( n \geq 5 \), \((n-1)+n > (n+1)+1\). Part ii follows from the proof of Theorem 18 because for sufficiently large \( n \), \((n-1)+n-(n+1)-1 > m\).

The counterexamples \( H_n \), given above to the Harary-Plantholt Conjecture, have minimum degree 1. On the other hand any \( n \)-regular Class 1 graph \( H_n \) has \( n \) mutually disjoint perfect matchings. Thus \( \chi'(H) = \chi'_C(H) \) for any cost set \( C \). In Theorem 20, we use \( H_n \) to construct additional counterexamples \( H_{n,m} \) with maximum degree \( n \) and minimum degree \( m \), \( m = 2, 3, \ldots, n-2 \). Then in Theorem 21, we construct counterexamples to Harary-Plantholt which have maximum degree \( n \) and minimum degree \( n-1 \).

Theorem 20. For any odd integer \( n \geq 5 \) and \( m = 2, 3, \ldots, n-2 \), there exists a Class 1 graph \( H_{n,m} \) with \( \Delta(H_{n,m}) = n \) and \( \delta(H_{n,m}) = m \) such that \( \chi'_N(H_{n,m}) = n+1 \).

Proof. Let \( u_1, \ldots, u_n \) be the endvertices of \( H_n \). We form \( H_{n,m} \) by adding \( m-1 \) vertices \( w_1, w_2, \ldots, w_{m-1} \) to \( H_n \) and joining each \( w_i \) to each \( u_j \). For \( m = 2, 3, \ldots, n-2 \), \( H_{n,m} \) is Class 1. To see this, \( n \)-color the edges of \( H_n \) so that the edge \( e_j \) incident with \( u_j \) has color \( j \). For each \( i = 1, 2, \ldots, m-1 \), color \( w_i u_1, w_i u_2, \ldots, w_i u_n \) respectively with colors \( n-i+1, n-i+2, \ldots, n, 1, 2, \ldots, n-i \). This is an \( n \)-edge coloring of \( H_{n,m} \), and thus \( H_{n,m} \) is Class 1. Furthermore with this coloring color 1 is missing from both \( u_{n-1} \) and \( u_n \). Also note that all \( n \)-edge colorings of \( H_{n,m} \) have the same cost.

Without loss of generality we may assume that the edge \( e \) of \( K_n \) which is incident with \( e_n \) and \( e_{n-1} \) is colored 1. We obtain a lower cost edge coloring by using color \( n+1 \) on \( e \) and color 1 on both \( e_n \) and \( e_{n-1} \).

Theorem 21. For each \( r \geq 2 \), there exists a graph \( G'' \) such that \( \Delta(G'') = 2r+1 \), \( \delta(G'') = 2r \), \( \chi'_N(G'') = 2r+2 \), and \( \chi'(G'') = 2r+1 \).

Proof. As we construct \( G'' \) we also give a minimum cost \((2r+1)\)-coloring of its edges. We then show that the cost can be reduced by using color \( 2r+2 \).
It is well known that the edge set of any complete graph of odd order can be partitioned into mutually disjoint Hamiltonian cycles. For \( r \geq 2 \), we choose \( r \) of these cycles from \( K_{4r+1} \). Let \( B_1 : w_1, w_2, \ldots, w_{4r+1}, w_1 \) be any Hamiltonian cycle of \( K_{4r+1} \). Let \( B_2, B_3, \ldots, B_r \) be the other cycles where \( B_r \) is the unique Hamiltonian cycle in the partition which contains edge \( w_2w_4 \).

We now \((2r + 1)\)-color the edges of the resulting \( 2r \)-regular graph \( G \). Color the edges \( w_1w_2, w_2w_4, \ldots, w_{4r+1}w_1 \) of \( B_1 \) respectively with \( 3, 1, 3, 1, 2, 3, 2, 3, \ldots, 2 \). In \( B_r \) color edge \( w_2w_4 \) with 2 and use colors \( 2r, 2r + 1 \) on the other edges of \( B_r \). Thus we have a proper edge-coloring of \( B_1 \cup B_r \) which has two edges colored 1 and \( 2r \) edges of each color \( 2, 3, 2r, 2r + 1 \).

Each remaining cycle \( B_i, i = 2, 3, \ldots, r - 1 \), of \( G \) is colored so that one edge has color 1 and the other edges alternate colors \( 2i, 2i + 1 \). In coloring \( B_i \), an edge which is not adjacent to an edge already colored 1 can be selected for color 1. In order to see this, we note that each of the less than \( r \) edges already colored 1 is adjacent with at most 4 edges of \( B_i \). Since \( B_i \) has \( 4r + 1 \) edges, one of them can be colored 1.

Thus we have edge-colored \( G \) with \( 2r + 1 \) colors such that \( r \) edges are colored 1 and \( 2r \) edges are colored \( i, i = 2, 3, \ldots, 2r + 1 \). Now form graph \( G' \) by adding one end edge to each vertex of \( G \). Color each end edge with the unique color available from \( 1, 2, \ldots, 2r + 1 \). Thus we have edge-colored \( G' \) with \( 2r + 1 \) colors such that the end edges colored \( 2r \) and \( 2r + 1 \) are both adjacent to edge \( w_2w_4 \) of \( B_r \) which has color 2.

Hence exactly one end edge of \( G' \) has color \( i, i = 2, 3, \ldots, 2r + 1 \), and color 1 is used on \( 2r + 1 \) end edges. So in total, \( G' \) has \( 3r + 1 \) edges colored 1 and \( 2r + 1 \) edges colored \( i, i = 2, 3, \ldots, 2r + 1 \). From Theorem 15 it follows that any edge-coloring of \( G' \) with \( 2r + 1 \) colors must use each color at least \( 2r + 1 \) times. Hence among all edge-colorings of \( G' \) with \( 2r + 1 \) colors, this one has minimum cost.

We now add vertices and edges to \( G' \) to form \( G'' \) which will have the required properties. Let \( E_1 \) be any \( 2r \) end edges colored \( 2, 1, 1, \ldots, 1 \), listed from left to right. Let \( E_2 \) be the other \( 2r + 1 \) end edges, which are colored \( 1, 1, 3, 4, \ldots, 2r, 2r + 1 \) and are also listed from left to right.

Let \( U_1 = \{ u_1, \ldots, u_{2r-1} \} \) and \( V_1 = \{ v_1, \ldots, v_{2r-1} \} \) be disjoint sets of new vertices which we add to \( G' \). Form the edge \( u_2u_{2i+1}, i = 1, 2, \ldots, r - 1 \), and color each of these \( r - 1 \) edges 1. Also join each \( u_i, i = 1, \ldots, 2r - 1 \), to each endvertex of the edges of \( E_1 \). The edges from \( u_1 \) to \( E_1 \) (where \( E_1 \) is taken in the above order) are colored consecutively \( 1, 2, \ldots, 2r \). Similarly the edges from \( u_2 \) to \( E_1 \) are colored consecutively \( 3, 4, 5, \ldots, 2r + 1, 2 \); the edges from \( u_3 \) to \( E_1 \) are colored consecutively \( 4, 5, \ldots, 2r + 1, 2, 3 \); and so forth.
until finally the edges from $u_{2r-1}$ to $E_1$ are colored consecutively $2r, 2r + 1, 2, 3, \ldots, 2r - 1$.

Now join each vertex of $V_1$ to each endvertex of the edges in $E_2$. Color the edges from $v_1$ to $E_2$ consecutively (in the order given for $E_2$) $2, 2r + 1, 1, 3, 4, \ldots, 2r$. Similarly color the edges from $v_2$ to $E_2$ consecutively $2r, 2, 2r + 1, 1, 3, 4, \ldots, 2r - 1$; the edges from $v_3$ to $E_2$ consecutively $2r - 1, 2r, 2, 2r + 1, 1, 3, 4, \ldots, 2r - 2$; and so forth until finally the edges from $v_{2r-1}$ to $E_2$ are colored consecutively $3, 4, \ldots, 2r, 2, 2r + 1, 1$.

The resulting graph $G''$ satisfies $\delta(G'') = 2r$, and has been properly edge-colored with $2r + 1 = \Delta(G'')$ colors. Furthermore our edge-coloring of $G''$ is a minimum cost Class 1 coloring. In order to see this, recall that our coloring of $G'$ has minimum cost. Also note that all vertices added to $G'$ in forming $G''$ except $u_1$ have degree $2r + 1$ and hence must use all colors on incident edges. The degree of $u_1$ is $2r$, and its incident edges use all colors except $2r + 1$. Hence among all colorings of $G''$ with $2r + 1$ colors, we have one of minimum cost.

Recall that edge $w_2w_4$ is colored 2 and is adjacent to end edges $e_1$ and $e_2$ of $G'$ colored $2r$ and $2r + 1$. Furthermore the endvertices in $G'$ of $e_1$ and $e_2$ are not incident to any edge colored 2. Hence we can recolor edge $w_2w_4$ with $2r + 2$ and edges $e_1$ and $e_2$ with 2. This reduces the cost of coloring by $4r + 3 - (2r + 6) = 2r - 3 > 0$. Thus $\chi' G''(G'') = \chi'(G'') + 1$.

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Received 15 February 1996
Revised 14 January 1997