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AN INEQUALITY CHAIN OF DOMINATION PARAMETERS FOR TREES

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Abstract

We prove that the smallest cardinality of a maximal packing in any tree is at most the cardinality of an R-annihilated set. As a corollary to this result we point out that a set of parameters of trees involving packing, perfect neighbourhood, R-annihilated, irredundant and dominating sets is totally ordered. The class of trees for which all these parameters are equal is described and we give an example of a tree in which most of them are distinct.

Keywords: domination, irredundance, packing, perfect neighbourhoods, annihilation.

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1. Introduction

This paper is concerned with the relative values of certain graph parameters for trees. These parameters involve several types of vertex subsets X of a simple graph G, including dominating, irredundant, packing, perfect neighbourhood and R-annihilated subsets. Our first task is the definition of such sets and to observe that each may be characterized in terms of a certain partition of the vertex set V of G induced by X.

We denote by $N(X)$ ($N[X]$) the open (closed) neighbourhood of the set X. As usual $N(\{x\})$ and $N[\{x\}]$ will be abbreviated to $N(x)$ and $N[x]$. For $A, B \subseteq V$, we say that A *dominates* B, written $A \succ B$, (or B *is dominated by* A) if $B \subseteq N[A]$.

The *private neighbourhood* $pn(x, X)$ of x in X is defined by

$$pn(x, X) = N[x] - N[X - \{x\}].$$

An element u of $pn(x, X)$ is called a *private neighbour* of x relative to X and is one of two types. Either u is an isolate of $G[X]$, in which case $u = x$, or $u \in V - X$ and is adjacent to precisely one vertex of X. The latter type is called an *external private neighbour* (epn) of X.

The concept of private neighbourhood enables us to define from X, a partition $\mathcal{P}(X) = Z_X \cup Y_X \cup E_X \cup F_X \cup C_X \cup R_X$ (disjoint union) of V, where:

$$\begin{aligned}
Z_X &= \{x \in X \mid x \text{ is isolated in } G[X]\}, \\
Y_X &= X - Z_X, \\
E_X &= \{v \in V - X \mid v \text{ is an epn of some } y \in Y_X\}, \\
F_X &= \{v \in V - X \mid v \text{ is an epn of some } z \in Z_X\}, \\
C_X &= \{v \in V - X \mid |N(v) \cap X| \geq 2\}, \\
\text{and} \quad R_X &= V - N[X].
\end{aligned}$$

When the basic subset X is clear from the context, we will omit the subscripts X.

We now give the definition of four of the above mentioned types of vertex subsets X and the characterization of each in terms of $\mathcal{P}(X)$.

X is *dominating* if $N[X] = V$ (i.e., if $R = \emptyset$); X is *irredundant* if for all $x \in X$, $pn(x, X) \neq \emptyset$ (i.e. if $E \cap N(y) \neq \emptyset$ for each $y \in Y$); X is a *packing* if for all distinct $x_1, x_2 \in X$, $N[x_1] \cap N[x_2] = \emptyset$ (i.e., $C \cup Y = \emptyset$), and X is a *perfect neighbourhood set* (abbr. PN-set) if $\phi(X) = \bigcup_{x \in X} pn(x, X) \succ V$ (i.e., if $Z \cup E \cup F \succ V$). A vertex v is called an X-*perfect vertex* if $v \in Z \cup E \cup F$.

In order to motivate the definition of the next principal property, we first state a condition given in [3] for an irredundant set to be maximal. We need one additional concept about private neighbourhoods. For $x \in X$ and

$u \in V - X$, u *annihilates* x (or x is *annihilated by* u) if $\emptyset \neq pn(x, X) \subseteq N[u]$. Observe that if u annihilates x, then $pn\,(x, X \cup \{u\}) = \emptyset$, *i.e.,* (informally) addition of u to X destroys (or annihilates) the private neighbourhood of x. Let

$$A_X = \{u \in V - X \mid u \text{ annihilates some } x \in X\}\,.$$

We write A for A_X, if the basic subset X is clear. For $U \subseteq V - X$, define X to be U-*annihilated* if $U \subseteq A$. We can now state a condition for an irredundant set X to be maximal in terms of the partition $\mathcal{P}(X)$.

Theorem 1. [3] *The set X is maximal irredundant if and only if X is irredundant and $N[R]$-annihilated.*

We observe that the class of $N[R]$-annihilated sets (such sets have also been called *external redundant sets* ([3, 4])) is contained in the larger class of R-annihilated sets (abbreviated *Ra-sets*), which is a class of sets of principal interest in this work. We will also consider sets which are both R-annihilated and irredundant, that is, *Rai-sets.*

Notice that for each $z \in Z$ and $r \in R$, $z \in pn(z, X) - N[r]$ and so z is not annihilated by r. Thus any vertex of X which is annihilated by $r \in R$, is necessarily in Y.

The parameters considered in this paper are $\gamma(G)$, $i(G)$, $\theta(G)$, $\theta_i(G)$, $ra(G)$, $rai(G)$, $er(G)$ and $ir(G)$, which are the smallest cardinalities of dominating sets, independent dominating sets, PN-sets, independent PN-sets, Ra-sets, Rai-sets, external redundant and maximal irredundant sets, respectively; $\rho_L(G)$ ($\rho(G)$) which is the smallest (largest) cardinality of a maximal packing and $\gamma_2(G)$ which is the smallest cardinality of X such that each vertex of V is within distance two of X, i.e., such that X 2-*dominates* G.

We abbreviate $\gamma(G)$ to γ etc. when the graph G involved is clear. Further, for example, a dominating set (maximal irredundant set) of minimum cardinality $\gamma(G)$ ($ir(G)$) will be called a γ-*set* (an *ir-set*).

The following inequalities are immediately implied by the definitions, Theorem 1 and the well-known inequalities $ir \leq \gamma \leq i$.

Proposition 2. *For any graph G,*

$$\gamma_2 \leq \left\{ \begin{array}{c} ra \leq \left\{ \begin{array}{c} rai \\ er \end{array} \right\} \leq ir \\ \theta \leq \theta_i \leq \rho_L \leq \rho \end{array} \right\} \leq \gamma \leq i.$$

In Section 2 we prove our principal result, namely that for any tree T, $\rho_L(T) \leq ra(T)$ and hence establish a longer total order for trees than those given in Proposition 2 for general graphs. The trees for which all parameters in the total order are equal, are presented in Section 3 and finally an example with most of the parameters unequal, is given in Section 4.

This research evolved from attempts to prove the conjecture of Fricke, Haynes, Hedetniemi and Henning [8] that $\theta \leq ir$ for arbitrary graphs. This was shown to be false by Favaron and Puech [7]. However, Cockayne, Hedetniemi, Hedetniemi and Mynhardt [5] and Cockayne and Mynhardt [6] established the inequality for trees and claw-free graphs, respectively. Favaron and Puech [7, 12] observed that the proof for trees actually establishes the stronger result $\theta_i \leq rai$ (see Proposition 2) and found other classes of graphs (including claw-free graphs and chordal graphs) for which this latter inequality holds.

In [2] the present authors observed that some other known results concerning the parameter ir may be strengthened to theorems concerning R-annihilation. In the same paper other classes of graphs (defined by degree conditions) for which $\theta_i \leq rai$, were found. It was further proved that for some of these classes, the even stronger inequality (see Proposition 2) $\rho_L \leq rai$ holds. This was the motivation for the main theorem in the present paper.

An area of research that has received considerable attention is the study of classes of graphs for which some of the above-mentioned parameters are equal (or not equal). For any two graph theoretical parameters λ and μ, we define G to be a (λ, μ)-*graph* if $\lambda(G) = \mu(G)$ and a (λ, μ)-*tree* if, in addition, G is a tree. In general, if λ and μ are domination parameters, the class of (λ, μ)-graphs is very difficult to characterise. For trees, however, some success has been achieved. For example, (γ, i)-trees were characterised by the present authors in [1]. More relevant to the present paper, Meir and Moon [11] showed that $\rho = \gamma$ for all trees, Hartnell [9] characterised (ρ_L, ρ)-trees while Topp and Volkmann [13] characterised (γ_2, ρ)-trees. (They actually proved a more general result but we only mention the relevant part here.) In Section 3 we show that the two classes of trees described by Hartnell and Topp and Volkmann are the same and that it is in fact precisely the class of (γ_2, i)-trees. We also give a different description of these trees.

References to further work on domination, irredundance and packing may be found in the comprehensive bibliography of the book by Haynes, Hedetniemi and Slater [10].

2. The Main Result

In order to prove the principal theorem, further notation and structures are now defined. Let X be an Ra-set of any graph G. For $x \in X$, B_x denotes the set of epns of x and $B = \bigcup_{x \in X} B_x$ $(= E \cup F)$.

We now define a partition of $X \cup B \cup R$ into exactly $|X|$ non-empty subsets. With each $x \in X$ we associate a subset R_x of R sequentially. Suppose $X = \{x_1, \ldots, x_p\}$. Let

$$R_{x_1} = \{r \in R \,|\, r \text{ annihilates } x_1\}$$

and for $j = 2, \ldots, p$, let

$$R_{x_j} = \{r \in R - \bigcup_{k=1}^{j-1} R_{x_k} \,|\, r \text{ annihilates } x_j\}.$$

Observe that the R_{x_j}'s are disjoint and that the R-annihilation property implies that $R_x = \emptyset$ for $x \in Z$ and $\bigcup_{x \in X} R_x = R$. It now follows that

$$X \cup B \cup R = \bigcup_{x \in X} (\{x\} \cup B_x \cup R_x) \qquad \text{(disjoint union)},$$

which defines the required partition.

Next, for $x \in X$ let $D_x = \{x\} \cup B_x \cup R_x$, for each component Q of $G[X]$ let $D_Q = \bigcup_{x \in Q} D_x$ and observe that $Q \neq Q'$ implies that $D_Q \cap D_{Q'} = \emptyset$. From this point onwards G is a tree T. Contract each set D_Q to a single vertex d_Q. This forms a tree $f(T)$ with vertex set $C \cup \left(\bigcup_{\text{components of } G[X]} \{d_Q\} \right)$. Each leaf of $f(T)$ is a vertex d_Q, since each $c \in C$ is adjacent in T to at least two sets D_Q (by the tree property).

Having defined the above structures, we now prove the following important preliminary result.

Theorem 3. *For each Ra-set X of a tree T, there exists a maximal packing P of T such that $P \cap C_X = \emptyset$.*

Proof. We use induction on n, the order of T, and observe that the statement holds for any tree T and Ra-set X for which $C_X = \emptyset$, which includes all Ra-sets of K_1 and K_2.

Now suppose that the conclusion holds for all trees of order less than n and let X be an Ra-set of an n-vertex tree T with $C = C_X \neq \emptyset$. The contracted tree $f(T)$ has a leaf l such that at most one vertex at distance two from l is not a leaf (e.g., let l be an endvertex of a longest path of $f(T)$). Let Q be the component of T such that $d_Q = l$. In T the set D_Q is linked to $V - D_Q$ by exactly one edge ut with $u \in D_Q$ and $t = V - D_Q$. There are now two cases to consider.

Case 1. $t \notin C$.
Consider $T' = T[V']$ where $V' = V - D_Q$. The set $X \cap V'$ is an R-annihilated set of the tree T' and the set $C' \subseteq V'$ of vertices adjacent in T' to at least two vertices of $X \cap V'$, is equal to C. By the induction hypothesis, there exists a maximal packing P' of T' with $P' \cap C' = P' \cap C = \emptyset$. Let w be a vertex of P' whose distance $d(w,t)$ is minimum and observe that the maximality of P' implies that $0 \leq d(w,t) \leq 2$. The required packing P is now formed in one of the following ways. Let $N_2[w]$ denote the set of all vertices at distance at most two from w.

(i) If $D_Q - N_2[w] = \emptyset$, then $P = P'$.

Otherwise:

(ii) If $d(w,t) = 2$, then let P'' be a maximal packing of $T[D_Q]$ which contains u and set $P = P' \cup P''$.

(iii) If $d(w,t) = 1$, then let P'' be a maximal packing of $T[D_Q]$ which does not contain u (e.g., which contains a neighbour of u in D_Q) and set $P = P' \cup P''$.

(iv) If $d(w,t) = 0$ (i.e., $w = t$), then let P'' be a maximal packing of $T\left[D_Q - N_2[w]\right]$ and set $P = P' \cup P''$.

In each of these four situations, P is a maximal packing of T with $P \cap C = \emptyset$.

Case 2. $t \in C$.
Let $k+1 (\geq 2)$ be the degree of t in $f(T)$ and let d_{Q_j}, $j = 1, \ldots, k$ be leaves of $f(T)$ adjacent to t, where d_{Q_j} is the contraction of D_{Q_j} in T. The k subscripts j are chosen so that $|D_{Q_j}| > 1$ for $1 \leq j \leq s$ and $|D_{Q_j}| = 1$ for $s+1 \leq j \leq k$, where s is possibly equal to 0 or to k. For $1 \leq j \leq k$, let u_j be the (unique) neighbour of t in Q_j and if $s > 0$, then for $1 \leq j \leq s$, let v_j be a vertex of D_{Q_j} at distance two from t.

The set $V' = V - \left(\bigcup_{j=1}^{k} D_{Q_j} \cup \{t\} \right)$ induces a subtree T' of T. The set $X' = X \cap V'$ is an Ra-set of T' and the set $C' \subseteq V'$ of vertices adjacent in

T' to at least two vertices $X \cap V'$, is equal to $C - \{t\}$. By the induction hypothesis, T' has a maximal packing P' such that $P' \cap C' = P' \cap C = \emptyset$.

Let $w \in P'$ such that $d(w,t)$ is minimum. By the maximality of P', $1 \le d(w,t) \le 3$. There are now two subcases to consider.

Subcase (i). $2 \le d(w,t) \le 3$.
Let P_1 be a maximal packing of D_{Q_1} containing u_1 and when $s \ge 2$, for $2 \le j \le s$ let P_j be a maximal packing of D_{Q_j} containing v_j. Then $P'' = \bigcup_{j=1}^{s} P_j$ is a maximal packing of $T\left[\bigcup_{j=1}^{k} D_{Q_j} \cup \{t\}\right]$ which does not contain t. Set $P = P' \cup P''$.

Subcase (ii). $d(w,t) = 1$.
If $s = 0$, then let $P = P'$. If $s > 0$, then let P_j be a maximal packing of $T\left[D_{Q_j}\right]$ containing v_j for $1 \le j \le s$. Then $P'' = \bigcup_{j=1}^{s} P_j$ is a maximal packing of $T\left[\bigcup_{j=1}^{k} D_{Q_j} \cup \{t\}\right]$ not dominating t. Set $P = P' \cup P''$.

In each situation in the subcases, the constructed set P is a maximal packing of T such that $P \cap C = \emptyset$. The induction proof is complete. ■

Theorem 4. *For any tree,* $\rho_L \le ra$.

Proof. Let X be an ra-set of a tree T and P a maximal packing of T whose existence is guaranteed by Theorem 3, i.e., such that $P \cap C = \emptyset$. The R-annihilation property implies that for each $x \in X$ with $R_x \ne \emptyset$, every vertex of R_x dominates B_x. We deduce that for each $x \in X$, $T[D_x]$ has diameter at most two and so $|P \cap D_x| \le 1$. Therefore

$$\rho_L \le |P| = |P \cap C| + \left| P \cap \left(\bigcup_{x \in X} D_x \right) \right| \le |X| = ra.$$

■

Corollary 5. *For any tree,*

$$\gamma_2 \le \theta \le \theta_i \le \rho_L \le ra \le \left\{ \begin{array}{c} rai \\ er \end{array} \right\} \le ir \le \gamma = \rho \le i.$$

Proof. Immediate from Proposition 2, Theorem 4 and the result of Meir and Moon [11] that $\rho = \gamma$ for trees. ■

3. The Class of (γ_2, i)-Trees

A vertex of a tree T is said to be *remote* if it is adjacent to a leaf, and to be a *branch vertex* if it has degree at least three. The set of branch vertices of T is denoted by $B(T)$ and the set of leaves by $L(T)$. A path P in T is said to be a $v-L$ *path* if P is a path from v to a leaf of T. With every branch vertex v of T of degree d and $N(v) = \{x_1, x_2, \ldots, x_d\}$ we associate d integers $l_1, l_2, \ldots l_d$, where l_j is the length of a shortest $v-L$ path containing x_j. Without loss of generality we assume that $l_1 \leq l_2 \leq \ldots \leq l_d$. We define the following types of branch vertices:

type 1: $l_1 = 1$ and $l_j \in \{1, 4\}$ for each $j = 2, \ldots, d$,

type 2: $l_1 = 2$ and $l_j = 3$ for each $j = 2, \ldots, d$.

We characterise (γ_2, i)-trees in terms of branch vertices of types 1 and 2. We begin with the following characterisation of (ρ_L, ρ)-trees by Hartnell [9].

Theorem 6 [9]. *A tree with at least three vertices is a (ρ_L, ρ)-tree if and only if no two remote vertices are adjacent and every non-remote vertex is adjacent to exactly one remote vertex.*

A special case of a more general result by Topp and Volkmann [13] provides a characterisation of (γ_2, ρ)-trees:

Theorem 7 [13]. *A tree T satisfies $\gamma_2(T) = \rho(T) = n$ if and only if either*

(1) *T has diameter at most two (in which case $\gamma_2(T) = \rho(T) = 1$)*

or

(2) *there exists a decomposition (partition of the vertex set) of T into n subgraphs $T_1, \ldots, T_n$ in such a way that:*
   (a) *T_j is a tree of diameter two ($j = 1, \ldots, n$), and*
   (b) *if T_0 is the subgraph of T induced by the edges which do not belong to $T_1, \ldots, T_n$, then for each $j \in \{1, \ldots, n\}$ there exists $u_j \in V(T_j) - V(T_0)$ such that $d_T(u_j, V(T_0)) = 2$.*

Since any (γ_2, ρ)-tree is a (ρ_L, ρ)-tree, if $\mathcal{T}_2'$ ($\mathcal{T}_3$, respectively) is the class of trees described in Theorem 6 (Theorem 7), then $\mathcal{T}_3 - \{P_1, P_2\} \subseteq \mathcal{T}_2'$. We show that $\mathcal{T}_2 = \mathcal{T}_2' \cup \{P_1, P_2\}$ and $\mathcal{T}_3$ are equal to the following class of trees, for which membership is decided only by properties of branch vertices. Let $T \in \mathcal{T}_1$ if and only if $T \in \{P_1, P_2, P_3, P_6\}$ or $B(T) \neq \emptyset$ and each branch vertex of T is of type 1 or type 2. (See Figure 1, in which u_j is of type 1 and v_j of type 2 for $j = 1, 2, 3$.)

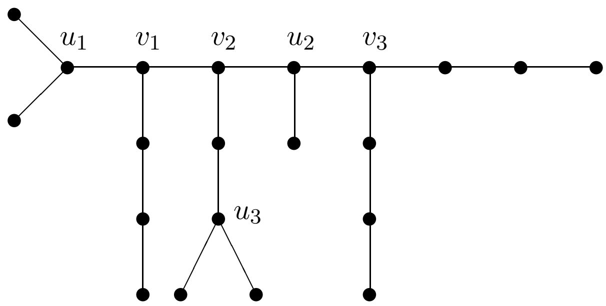

Figure 1. A tree in $\mathcal{T}_1$

Theorem 8. *Let T be a tree. The following conditions are equivalent*:

(a) $T \in \mathcal{T}_1$.

(b) T *is a* (ρ_L, ρ)*-tree.*

(c) T *is a* (γ_2, ρ)*-tree.*

(d) T *is a* (γ_2, i)*-tree.*

Proof. Since the theorem obviously holds for P_1 and P_2 we only consider trees with at least three vertices.

(a) $\Longrightarrow$ (b). Let $T \in \mathcal{T}_1$. If T is a path (i.e., $T \in \{P_3, P_6\}$) the result is easy to check, so suppose $B(T) \neq \emptyset$ and let u_1, u_2 be two remote vertices of T. Suppose contrary to the statement of Theorem 6 that u_1 and u_2 are adjacent, with v_j $(j = 1, 2)$ a leaf adjacent to u_j. If $\{u_1, u_2\} \cap B(T) = \emptyset$, then $T \cong P_4$, a contradiction. Hence we may assume that $u_1 \in B(T)$. But $d(u_1, v_1) = 1$ and $d(u_1, v_2) = 2$ so that $l_1(u_1) = 1$ and $l_j(u_1) = 2$ for some j and therefore u_1 is not a type 1 or a type 2 branch vertex. Thus T contains no adjacent remote vertices. Next, let u be a non-remote vertex. If u is a leaf then obviously u is adjacent to exactly one remote vertex, so assume that $\deg u \geq 2$. If $u \in B(T)$, then u is of type 2 (since u is not remote) and thus u is adjacent to exactly one remote vertex as required. Hence suppose $\deg u = 2$. If u is adjacent to two remote vertices u_1 and u_2, and $\{u_1, u_2\} \cap B(T) = \emptyset$, then $T \cong P_5$, a contradiction. On the other hand, if (say) $u_1 \in B(T)$, then $l_1(u_1) = 1$ and (if v_2 is a leaf adjacent to u_2) $d(u_1, v_2) = 3$, i.e., $l_j(u_1) = 3$ for some j with $2 \leq j \leq \deg u_1$, a contradiction. Hence u is adjacent to at most one remote vertex. If u is not adjacent to any remote vertex, let $N(u) = \{x_1, x_2\}$ and let u_1 (say) be a branch vertex at minimum distance from u such that x_1 lies on the $u - u_1$ path in T, while u_2 is a remote vertex at minimum distance from u with x_2 on the $u - u_2$ path. Note that $d(u, u_2) \geq 2$ and either $u_1 = x_1$ with u_1 of type 2 (since u_1 is not a remote vertex by hypothesis), or $d(u, u_1) \geq 2$. In the former case the shortest $u_1 - L$ path in T through u contains u_2 and has length at least four, contradicting u_1 being of type 2. In the latter case, if w is the neighbour of u_1 on the $u_1 - u$ path, then the shortest $u_1 - L$ path through w contains both u and u_2 and hence has length at least five, which is also impossible. Therefore u is adjacent to exactly one remote vertex and we have proved that $\mathcal{T}_1 \subseteq \mathcal{T}_2$.

(b) $\Longrightarrow$ (a). Let $T \in \mathcal{T}_2'$ and note that if T is a path, then $T \cong P_3$ or P_6 and the result holds. We may thus assume that $B(T) \neq \emptyset$. Let u be any vertex in $B(T)$ and suppose firstly that u is remote. Let $N(u) = \{x_1, \ldots, x_d\}$ with $d = \deg u \geq 3$. Without loss of generality we may assume that $x_1, \ldots, x_t$ are leaves and $x_{t+1}, \ldots, x_d$ are non-leaves, for some $1 \leq t \leq d$. If $t = d$, then u is of type 1 and we are done, so suppose $t < d$. Then for any s with $t < s \leq d$, $\deg x_s \geq 2$ and by hypothesis, x_s is not remote. For any $w \in N(x_s) - \{u\}$, w is not remote (since the neighbour u of x_s is remote) and there exists $v \in N(w) - \{x_s\}$ such that v is remote. But then the length of the shortest $u - L$ path through x_s is equal to four and it follows that u is of type 1. Now suppose that u is not remote. Then u is adjacent to exactly one remote vertex, say x_1, so that $l_1(u) = 2$. For any $j = \{2, \ldots, d\}$, x_j is not remote and $N(x_j) - \{u\}$ contains a remote vertex w. Hence the length of a shortest $u - L$ path through x_j is equal to three and so u is of type 2. We have thus shown that $\mathcal{T}_1 = \mathcal{T}_2$.

(b) $\Longrightarrow$ (c). Consider any $T \in \mathcal{T}_2'$ and denote the remote vertices of T by M. Then $\bigcup_{m \in M} N[m]$ is a partition of $V(T)$ since no two remote vertices of T are adjacent and each non-remote vertex is adjacent to exactly one remote vertex. Let T_m be the subtree of T induced by $N[m]$. Clearly, $\operatorname{diam}(T_m) = 2$. Further, any edge joining a vertex of T_m to a vertex of T_s, $m \neq s$, joins a non-remote vertex of T to another non-remote vertex. Let T_0 be the subgraph of T induced by these edges and let v be a leaf of T adjacent to m. Then clearly $d_T(v, V(T_0)) = 2$. We have thus shown that conditions 2(a) and (b) of Theorem 7 are satisfied. Finally, P_1 and P_2 satisfy condition 1 of Theorem 7.

(c) $\Longrightarrow$ (b). This is obvious and it follows that $\mathcal{T}_2 = \mathcal{T}_3$.

(c) $\Longrightarrow$ (d). Since $\rho = \gamma$ for all trees (Meir and Moon [11]) we only need to show that $i = \gamma$ for any tree $T \in \mathcal{T}_1 = \mathcal{T}_3$. Let $\{T_m \,|\, m \in M\}$ be the decomposition of T as defined in the previous paragraph; by Theorem 7, $\rho(T) = |M|$. But M is independent and dominating, hence $i(T) \leq |M|$ and the desired result follows.

(d) $\Longrightarrow$ (c). Obvious. ■

Corollary 9. *Let λ be any of the parameters γ_2, θ, θ_i, ρ_L and let μ be any of the parameters ρ, γ, i. Then T is a (λ, μ)-tree if and only if $T \in \mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}_3$.*

Let β denote the independence number, i.e., the cardinality of a maximum independent set, of a graph. (The notation β_0 or α is also sometimes used.) We conclude this section by showing that the class of (γ_2, β)-trees is not particularly interesting. We obtain this as a corollary to the following result which holds for general graphs.

Proposition 10. *Let λ be any of the parameters γ_2, θ, θ_i and ρ_L. Then G is a connected (λ, β)-graph if and only if G is complete.*

Proof. If G is complete, then $\lambda(G) = \beta(G) = 1$. Conversely, suppose $\rho_L(G) = \beta(G)$ but G is not complete. Let X be any ρ_L-set of G. We claim that $\bigcup_{x \in X} N[x]$ is a partition of $V(G)$. Indeed, by the packing property, $N[x_1] \cap N[x_2] = \emptyset$ for distinct $x_1, x_2 \in X$; moreover, if $y \in V(G) - N[X]$, then since $X \cup \{y\}$ is independent we have $\beta(G) \geq \rho_L(G) + 1$, a contradiction. Now suppose that there exist two non-adjacent neighbours u and v of some vertex $x \in X$. Since $(X - \{x\}) \cup \{u, v\}$ is independent, we have $\beta(G) \geq \rho_L(G) + 1$, a contradiction. Therefore $G[N[x]]$ is complete for each $x \in X$. Since G is not complete by assumption, it follows that $|X| \geq 2$. Moreover, since G is connected, there exists a neighbour w of some $x \in X$ such that $X' = \{x' \in X : N(w) \cap N[x'] \neq \emptyset\}$ satisfies $|X'| \geq 2$. (Note that $x \in X'$.) But then $(X - X') \cup \{w\}$ is a maximal packing of cardinality less than ρ_L, a contradiction. The result now follows from Proposition 2. ■

Corollary 11. *Let λ be any of the parameters γ_2, θ, θ_i and ρ_L. Then T is a (λ, β)-tree if and only if $T \in \{K_1, K_2\}$.*

4. A Tree with Distinct Parameter Values

Consider the tree T in Figure 2. Let $A = \{a_j : 1 \leq j \leq 13\}$,$B = \{b_j : 1 \leq j \leq 12\}$,$C = \{c_j : 1 \leq j \leq 23\}$, $D = \{d_j : 1 \leq j \leq 10\}$, $E = \{e_j : 1 \leq j \leq 10\}$,

$F = \{b_1, w\} \cup D \cup E$ and $B^* = B - \{b_1\}$. We illustrate that the two total orders given in Corollary 5 can be strict by showing that $\gamma_2(T) = 16$, $\theta(T) = 17$, $\theta_i(T) = 18$, $\rho_L(T) = 19$, $ra(T) = 23$, $rai(T) = er(T) = 24$, $ir(T) = 25$, $\gamma(T) = 26$ and $i(T) = 27$. (It is also possible to obtain trees T_1 and T_2 in which all these parameters are distinct and $er(T_1) < rai(T_1)$, while $rai(T_2) < er(T_2)$. This shows that er and rai are incomparable even for trees. However, we do not exhibit examples of such trees here.)

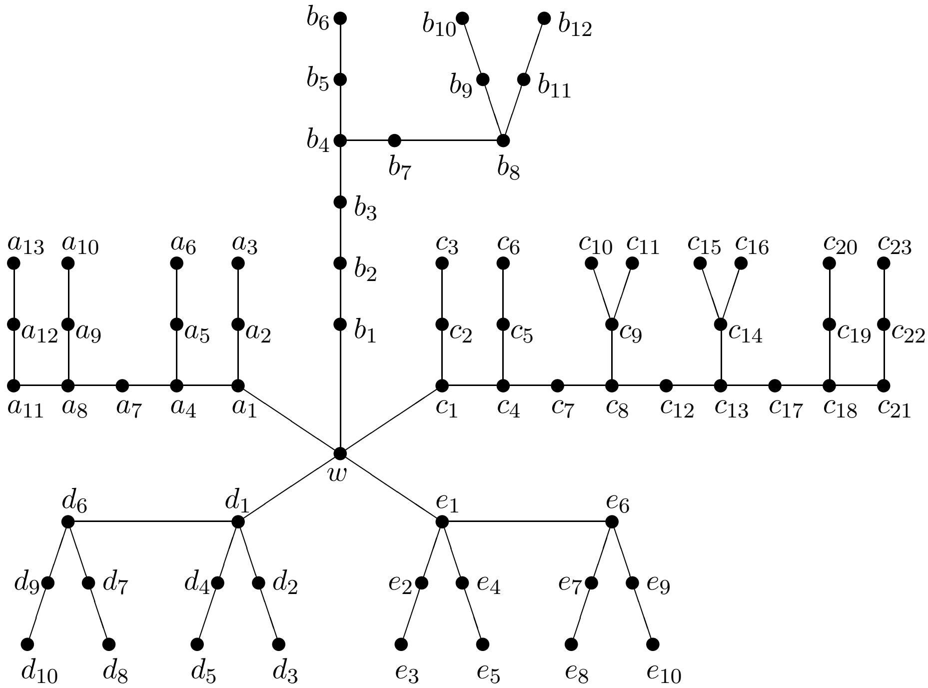

Figure 2. A tree with distinct parameter values

Recall that any maximal packing of a graph G is an independent PN-set, while any PN-set 2-dominates G. Also, any maximal irredundant set is external redundant and any external redundant set of G is R-annihilated. (All these relationships are direct consequences of the definitions.)

Let X be any 2-dominating set of T. In order to 2-dominate the leaves of $T[A]$ and $T[C]$, $|A \cap X| \geq 4$ and $|C \cap X| \geq 6$. To 2-dominate b_6, b_{10} and b_{12}, $|B^* \cap X| \geq 2$ and if $|B^* \cap X| = 2$, then $B^* \cap X = \{b_4, b_8\}$. Since the leaves of $T[F]$ are 2-dominated, $|F \cap X| \geq 4$, and if $|F \cap X| = 4$, then $F \cap X = \{d_1, d_6, e_1, e_6\}$. It follows that $|X| \geq 4 + 6 + 2 + 4 = 16$. Let

$$X_0 = \{a_2, a_5, a_9, a_{12}, c_2, c_5, c_9, c_{14}, c_{19}, c_{22}\}$$

and

$$X_1 = \{b_4, b_8, d_1, d_6, e_1, e_6\}.$$

Then $X_2 = X_0 \cup X_1$ is a 2-dominating set of T and hence $\gamma_2(T) = 16$.

Suppose furthermore that X is a PN-set. If $|X \cap (B^* \cup F)| = 6$, then, as shown above, $X \cap (B^* \cup F) = X_1$. But then w is not X-perfect and b_1 is not dominated by an X-perfect vertex. Thus $|X \cap (B^* \cup F)| \geq 7$ and $|X| \geq 17$. Since $X_2 \cup \{b_1\}$ is a PN-set, it follows that $\theta(T) = 17$.

Moreover, if X is an *independent* PN-set, then in order to 2-dominate the leaves of $T[F]$, we need at least six vertices in $F \cap X$, and if $|F \cap X| = 6$, then no vertex in B^* is 2-dominated by any vertex of $F \cap X$. It follows that $|X| \geq 4+6+2+6 = 18$. On the other hand, $X_0 \cup \{b_4, b_8, e_1, e_8, e_{10}, d_3, d_5, d_6\}$ is an independent PN-set of T and hence $\theta_i(T) = 18$.

Suppose that in addition, X is a maximal packing of T. In particular, since X is independent, $|F \cap X| \geq 6$ and if $|F \cap X| = 6$, then by the above analysis, b_2, b_6, b_{10} and b_{12} are 2-dominated by $X \cap B^*$. Therefore $|B^* \cap X| \geq 2$ and if $|B^* \cap X| = 2$, then $B^* \cap X = \{b_4, b_8\}$. But then $b_7 \in N[b_4] \cap N[b_8]$, a contradiction. Therefore $|X| \geq 19$. Since $X = X_0 \cup \{b_4, b_{10}, b_{12}, d_1, d_8, d_{10}, e_3, e_5, e_6\}$ is a maximal packing, it follows that $\rho_L(T) = 19$.

Let Y be any R-annihilated set of T. Define

$$\begin{aligned} Y_a &= \{a_1, a_4, a_8, a_{11}\}, \\ Y_b &= \{b_3, b_4, b_8, b_9\}, \\ Y_c &= \{c_1, c_4, c_8, c_{12}, c_{13}, c_{18}, c_{21}\}, \end{aligned}$$

and

$$Y_f = \{d_2, d_4, d_7, d_9, e_2, e_4, e_7, e_9\}.$$

For each $y \in Y$, if $|pn(y, Y)| \geq 2$, then (since T is a tree) no vertex in R_Y annihilates y. Hence if $d_1 \in Y$, then $\{d_2, d_3, d_4, d_5\} \cap Y \neq \emptyset$. If $d_1 \notin Y$, then clearly $\{d_2, d_3\} \cap Y \neq \emptyset$ and $\{d_4, d_5\} \cap Y \neq \emptyset$. A similar argument holds with respect to d_6 and it follows that $|Y \cap D| \geq 4$. Similarly, $|Y \cap E| \geq 4$ and $|Y \cap B| \geq 4$. Moreover, $|Y \cap B| = 4$ if and only if $Y \cap B = Y_b$ or $(Y_b - \{b_9\}) \cup \{b_{11}\}$, and $|Y \cap A| \geq 4$ with $|Y \cap A| = 4$ if and only if $Y \cap A = Y_a$. Let $C_j = \{c_j, c_{j+1}, c_{j+2}\}$ for $j \in \{1, 4, 18, 21\}$ and $C_j = \{c_j, c_{j+1}, c_{j+2}, c_{j+3}\}$ for $j \in \{8, 13\}$. Since Y is 2-dominating, $Y \cap C_j \neq \emptyset$ for each $j \in \{1,4,8,13,18,21\}$ and hence $|Y \cap C| \geq 6$. However, if $|Y \cap C| = 6$, then c_k and c_l with $\{c_k\} = Y \cap C_8$ and $\{c_l\} = Y \cap C_{13}$ are isolated in Y. If $k = 8$, then c_{10} does not annihilate any $y \in Y$, and obviously $k \notin \{10, 11\}$. Hence $k = 9$. Similarly, $l = 14$. But then c_{12} does not annihilate any $y \in Y$. Thus $|Y \cap C| \geq 7$ and as in the case of C_8 and C_{13}, it can be shown that no vertex of $Y \cap C_j$ ($j \in \{1,4,18,21\}$) is isolated in Y. Therefore $|Y \cap C| = 7$ if and only if $Y \cap C = Y_c$. Therefore $|Y| \geq 23$. Since

$$Y_0 = Y_a \cup Y_b \cup Y_c \cup Y_f$$

is an R-annihilated set, it follows that $ra(T) = 23$.

Let Y moreover be irredundant. Then $|Y \cap C| \geq 8$, for otherwise $Y \cap C = Y_c$ and c_{12} is redundant in Y. Thus $|Y| \geq 24$. However,

$$Y_1 = (Y_0 - \{c_8, c_{12}\}) \cup \{c_5, c_9, c_{14}\}$$

is an R-annihilated irredundant set, so that $rai(T) = 24$.

Furthermore, if Y is maximal irredundant, then as above, $|Y \cap C| \geq 8$. If $|Y| \leq 24$, then $|Y \cap B| = 4$ and so $Y \cap B = Y_b$ or $(Y_b - \{b_9\}) \cup \{b_{11}\}$. But then $Y \cup \{w\}$ is irredundant since $b_1 \in pn(w, Y \cup \{w\})$, a contradiction. Hence $|Y \cap B| \geq 5$ and thus $|Y| \geq 25$. On the other hand,

$$Y_2 = Y_1 \cup \{w\}$$

is a maximal irredundant set and so $ir(T) = 25$.

Now suppose that Y is an external redundant (i.e., $N[R]$-annihilated) set but not necessarily irredundant. Then $|Y| \geq ra(T) = 23$. If $|Y| = 23$, then $|Y \cap B| = 4$ with $Y \cap B = Y_b$ or $(Y_b - \{b_9\}) \cup \{b_{11}\}$, $|Y \cap A| = 4$ with $Y \cap A = Y_a$ and $|Y \cap C| = 7$ with $Y \cap C = Y_c$. Moreover, $|Y \cap F| = 8$ and $b_1 \in R_Y$. But then $w \in N[R]$ does not annihilate any $y \in Y$, a contradiction from which it follows that $|Y| \geq 24$. Since $Y_0 \cup \{w\}$ is external redundant, we have that $er(T) = 24$.

Let I be any dominating set of T. Clearly, $|I \cap A| \geq 5$, $|I \cap B| \geq 5$, $|I \cap D| \geq 4$, $|I \cap E| \geq 4$ and $|I \cap C| \geq 8$. Moreover, if $|I \cap C| = 8$, then $\{c_9, c_{14}\} \subseteq I$ and $\{c_8, c_{13}\} \cap I \neq \emptyset$. Thus I is not independent and $|I| \geq 26$. Since

$$I_0 = X_0 \cup Y_f \cup \{a_7, b_1, b_4, b_6, b_9, b_{12}, c_8, c_{17}\}$$

dominates T, it follows that $\gamma(T) = 26$.

As explained above, if I is independent, then $|I| > 26$. On the other hand,

$$I_1 = (I_0 - \{c_8\}) \cup \{c_7, c_{12}\}$$

is an independent dominating set and hence $i(T) = 27$. This completes the discussion of the tree T.

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