NEW CLASSES OF CRITICAL KERNEL-IMPERFECT DIGRAPHS

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Abstract

A kernel of a digraph $D$ is a subset $N \subseteq V(D)$ which is both independent and absorbing. When every induced subdigraph of $D$ has a kernel, the digraph $D$ is said to be kernel-perfect. We say that $D$ is a critical kernel-imperfect digraph if $D$ does not have a kernel but every proper induced subdigraph of $D$ does have at least one. Although many classes of critical kernel-imperfect digraphs have been constructed, all of them are digraphs such that the block-cutpoint tree of its asymmetrical part is a path. The aim of the paper is to construct critical kernel-imperfect digraphs of a special structure, a general method is developed which permits to build critical kernel-imperfect digraphs whose asymmetrical part has a prescribed block-cutpoint tree. Specially, any directed cactus (an asymmetrical digraph all of whose blocks are directed cycles) whose blocks are directed cycles of length at least 5 is the asymmetrical part of some critical kernel-imperfect digraph.

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1. Introduction

For general concepts we refer the reader to [1]. If $D$ is a digraph, $V(D)$ and $F(D)$ denote the set of vertices and arcs of $D$ respectively. An arc $u_1u_2 \in F(D)$ is asymmetrical (resp. symmetrical) if $u_2u_1 \notin F(D)$ (resp. $u_2u_1 \in F(D)$). The asymmetrical part of $D$ (resp. symmetrical part of $D$) which is denoted by Asym $D$ (resp. sym $D$) is the spanning subdigraph of $D$ whose arcs are the asymmetrical (resp. symmetrical) arcs of $D$. If $D_1$ and $D_2$ are two digraphs not necessarily disjoint, we will denote by $D_1 \cup D_2$...
the digraph $D_1 \cup D_2$ whose arcs are $A(D_1 \cup D_2) = A(D_1) \cup A(D_2)$ and whose vertices are $V(D_1 \cup D_2) = V(D_1) \cup V(D_2)$. A kernel $N$ of $D$ is an independent set of vertices such that for every $z \in V(D) - N$ there exists a $zN$-arc in $D$. A digraph $D$ is called kernel-perfect (or KP-digraph) if every induced subdigraph of $D$ has a kernel, and critical kernel-imperfect digraph (CKI-digraph) if $D$ has no kernel and every proper induced subdigraph has a kernel. We will say that a digraph $D$ is complete if its underlying graph is a complete graph. Thus every complete subdigraph $C$ of a kernel-perfect digraph must have an absorbing vertex (i.e., a successor of all other vertices of $C$). A digraph $D$ is called a normal orientation of its underlying graph $G$ if every complete subdigraph of $D$ has a kernel. A graph $G$ is called solvable if every one of its normal orientations is a kernel-perfect digraph.

Many papers have recently appeared which are devoted to construct CKI-digraphs (see [2, 3, 4, 6, 7]); however, all of those constructions allow only digraphs whose asymmetrical part has a path as its block-cutpoint tree. In this paper, we develop a general method to construct CKI-digraphs whose asymmetrical part has a prescribed block-cutpoint tree. Specially, we prove that any directed cactus all of whose blocks are directed cycles of length at least five is the asymmetrical part of some CKI-digraph.

Define the digraph $C = \overrightarrow{C}_n(j_1, j_2, \ldots, j_k)$ by

$$V(C) = \{0, 1, \ldots, n - 1\}$$
$$F(C) = \{uv \mid v - u \equiv js \pmod{n} \text{ for } s = 1, \ldots, k\}$$

In particular, we have the digraph

$$D = \overrightarrow{C}_n\left(1, \pm 2, \pm 3, \ldots, \pm \left[\frac{n}{2}\right]\right)$$

defined by

$$V(D) = \{0, 1, \ldots, n - 1\}, \quad F(D) = \{uv \mid v - u \not\equiv -1 \pmod{n}\}.$$
V(\alpha), respectively. We recall that a tree \alpha is a block cutpoint tree if and only if the distance of any two distinct endpoints is even (see [8, p. 36]).

The following result was proved in [5, Theorems 3.2, 3.3 and 3.4]

**Theorem 1.1.** Let \(D_1, D_2, H_1, H_2, D\) be digraphs, \(v, u_i \in V(D_i), i = 1, 2\) such that \(u_i v \in F(\text{sym}(D_i)), V(D_1) \cap V(D_2) = \{v\}, H_i = D_i - u_i v - v u_i\) and \(D = H_1 \cup H_2 + u_1 u_2 + u_2 u_1\). Suppose that \(H_1\) and \(H_2\) are KP-digraphs. Then

(a) \(D\) is a CKI-digraph iff \(D_1\) and \(D_2\) are CKI-digraphs. Moreover, if \(D\) is a CKI-digraph, then \(D - u_1 u_2 - u_2 u_1\) is a KP-digraph.

(b) \(D\) is a KP-digraph iff at least one of \(D_1\) and \(D_2\) is a KP-digraph.

Theorem 1.2 was proved in [5, Corollary 2.3]. Its second part is a direct consequence of the fact that for every CKI-digraph \(D\), \(\text{Asym} D\) is strongly connected [5, Theorem 2.2].

**Theorem 1.2.** \(D = \overline{C_n}(1, \pm 2, \pm 3, \ldots, \pm \left\lfloor \frac{n}{2} \right\rfloor)\) is a CKI-digraph for \(n \geq 4\), and if \(\emptyset \neq F_0 \subset F(\text{Asym} D)\), then \(D - F_0\) is a KP-digraph.

**Lemma 1.1.** If \(D = \overline{C_n}(1, \pm 2, \ldots, \pm \left\lfloor \frac{n}{2} \right\rfloor)\) and, \(\alpha\) is a subdigraph of \(\text{sym} D\) and \(\alpha\) is a tree, then \(D - F(\alpha)\) is a KP-digraph.

We omit the proof of Lemma 1.1 since it is a direct consequence of the following result due to M. Blidia, P. Duchet and F. Maffray [2].

**Theorem 1.3.** If the complement of the graph \(G\) is strongly perfect, then \(G\) is solvable.

2. New Classes of Critical Kernel-Imperfect Digraphs

In this section, we develop a method to construct CKI-digraphs whose asymmetrical part has a prescribed block-cutpoint tree. Also we construct CKI-digraphs whose asymmetrical part is a given directed cactus all of whose blocks are directed cycles of length at least five.

**Theorem 2.1** Let \(\alpha\) be any block-cutpoint tree and \(\mathcal{N} : \alpha_b \rightarrow \mathbb{N}\) a function satisfying \(\mathcal{N}(u) \geq \max\{\delta_\alpha(u), 5\}\) (\(\delta_\alpha(u)\) is the degree of \(u\) in \(\alpha\)). Then there exists a CKI-digraph \(D_{(\alpha, \mathcal{N})}\) and an isomorphism \(h : V(\alpha) \rightarrow V(b_c(\text{Asym} D_{(\alpha, \mathcal{N})}))\) satisfying the following conditions:

(i) For each \(u \in \alpha_b\), \(h(u)\) is a directed cycle in \(\text{Asym} D\) of length \(\mathcal{N}(u)\).
(ii) $D - F_0$ is a KP-digraph for every non empty set $F_0 \subset F$ (Asym $D$).
(iii) For every $z \in V(D)$ there exists $w \in V(D)$ such that $wz \in F$ (sym $D$) and $D - \{wz, zw\}$ is a KP-digraph.

**Proof.** We proceed by induction on $n = |V(\alpha)|$. First let $n = 1$, $\mathcal{N}(u) = k \geq 5$. Take $D = D_{(\alpha, \mathcal{N})} = \overline{C}_k(1, \pm 2, \ldots, \pm k)$. By Theorem 1.2 and Lemma 1.1 $D$ satisfies conditions (ii) and (iii), condition (i) is trivial. Suppose the assertion of Theorem 2.1 holds for $n < s$ and consider any block-cutpoint tree $\alpha$ with $n = s$. Let $u$ be any endpoint of $\alpha$ (therefore $u \in \alpha_0$) and call $c$ the neighbour of $u$ ($c \in \alpha_c$). Two cases are possible:

**Case 1.** If $\delta_\alpha(c) = 2$, call $u' \in \alpha_0$ the neighbour of $c$ different from $u$. Notice that $\alpha - c$ decomposes into two connected components one (say $\alpha'$) containing $u'$ and the other consisting of the single point $u$.

**Case 2.** If $\delta_\alpha(c) > 2$, take $\alpha' = \alpha - u$. In both cases $\alpha'$ is a block-cutpoint tree. Take the restriction $\mathcal{N}' = \mathcal{N}/\alpha'_0$. Then we obtain by induction a CKI-digraph $D' = D_{(\alpha', \mathcal{N}')}\mathcal{N}'$ and an isomorphism $h' : V(\alpha') \rightarrow V(b_c$ Asym $D'$) satisfying (i), (ii) and (iii). Choose $t \in V(D')$ as follows: In Case 1, $h'(u')$ induces a directed cycle $\overline{\gamma}'_{u'}$ in Asym $D'$ of length $N'(u') = N(u') \geq \delta_\alpha(u') = \delta_{\alpha'}(u') + 1$. Therefore $\overline{\gamma}'_{u'}$ contains some vertex $t$ which is not a cutpoint of $D'$.

In Case 2, let $t = h'(c)$ be any cutpoint of $D'$ which corresponds to $c$. Take any isomorphic copy $D''$ of $\overline{C}_{\mathcal{N}(u)}(1, \pm 2, \ldots, \pm k(\mathcal{N}(u)))$ such that $V(D'') \cap V(D') = \{t\}$ and choose $t_1 \in V(D')$ such that $tt_1 \in F$(sym $D'$) and $D' - \{tt_1, t_1t\}$ is a KP-digraph (condition (iii)). Choose also $t_2 \in V(D'')$ such that $tt_2 \in F$(sym $D''$) and $D'' - \{tt_2, t_2t\}$ is a KP-digraph (Theorem 1.2).

By Theorem 1.1 (a) $D = (D' - \{tt_1, t_1t\}) \cup (D'' - \{tt_2, t_2t\}) + t_1t + t_2t$ is a CKI-digraph. Extend $h'$ to $h : V(\alpha) \rightarrow V(b_c$ (Asym $D$)) in an obvious way. Condition (i) is obvious, condition (ii) follows from Theorem 1.1 (b), and by the fact that for every CKI-digraph $D$, Asym $D$ is strongly connected [5, Theorem 7.2]. Finally, condition (iii) follows from the induction hypothesis and Theorem 1.1 (b) in case $z \neq t_1, t_2, t$; in case $z = t_1, t_2$, from the fact that $D - \{t_1t_2, t_2t_1\}$ is a KP-digraph (Theorem 1.1 (a)) and, in case $z = t$, by taking $w \in V(D'') w \neq t_2$, such that $tw \in F$(sym $D''$). By Lemma 1.1, $D'' - \{tt_2, t_2t\}$, $D'' - \{tw, wt\}$ and $D'' - \{tt_2, t_2t, tw, wt\}$ are KP-digraphs. Applying Theorem 1.1 (b), $D - \{tw, wt\}$ is a KP-digraph and the proof is complete. 

\[\blacksquare\]
Theorem 2.2. Let $H$ be an asymmetrical digraph each one of whose blocks is a directed cycle of length at least five. Then there exists a critical kernel-imperfect digraph $D$ satisfying the following properties:

(i) $\text{Asym } D$ is isomorphic to $H$.
(ii) $D - F_0$ is a $\text{KP}$-digraph for every non empty set $F_0 \subset F (\text{Asym } D)$.
(iii) For every $z \in V(D)$ there exists $w \in V(D)$ such that $wz \in F (\text{sym } D)$ and $D - \{wz, zw\}$ is a $\text{KP}$-digraph.

The proof of Theorem 2.2 is similar to that of Theorem 2.1.

References


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