*Discussiones Mathematicae*
*Graph Theory 18 (1998) 85–89*

# NEW CLASSES OF CRITICAL KERNEL-IMPERFECT DIGRAPHS

Hortensia Galeana-Sánchez and V. Neumann-Lara

*Instituto de Matemáticas, UNAM*
*Ciudad Universitaria, Circuito Exterior*
*04510 México, D.F., México*

**Abstract**

A kernel of a digraph $D$ is a subset $N \subseteq V(D)$ which is both independent and absorbing. When every induced subdigraph of $D$ has a kernel, the digraph $D$ is said to be kernel-perfect. We say that $D$ is a critical kernel-imperfect digraph if $D$ does not have a kernel but every proper induced subdigraph of $D$ does have at least one. Although many classes of critical kernel-imperfect-digraphs have been constructed, all of them are digraphs such that the block-cutpoint tree of its asymmetrical part is a path. The aim of the paper is to construct critical kernel-imperfect digraphs of a special structure, a general method is developed which permits to build critical kernel-imperfect-digraphs whose asymmetrical part has a prescribed block-cutpoint tree. Specially, any directed cactus (an asymmetrical digraph all of whose blocks are directed cycles) whose blocks are directed cycles of length at least 5 is the asymmetrical part of some critical kernel-imperfect-digraph.

**Keywords:** digraphs, kernel, kernel-perfect, critical kernel-imperfect, block-cutpoint tree.

**1991 Mathematics Subject Classification:** 05C20.

## 1. Introduction

For general concepts we refer the reader to [1]. If $D$ is a digraph, $V(D)$ and $F(D)$ denote the set of vertices and arcs of $D$ respectively. An arc $u_1u_2 \in F(D)$ is *asymmetrical* (resp. symmetrical) if $u_2u_1 \notin F(D)$ (resp. $u_2u_1 \in F(D)$). The asymmetrical part of $D$ (resp. *symmetrical* part of $D$) which is denoted by Asym $D$ (resp. sym $D$) is the spanning subdigraph of $D$ whose arcs are the asymmetrical (resp. symmetrical) arcs of $D$. If $D_1$ and $D_2$ are two digraphs not necessarily disjoint, we will denote by $D_1 \cup D_2$

the digraph $D_1 \cup D_2$ whose arcs are $A(D_1 \cup D_2) = A(D_1) \cup A(D_2)$ and whose vertices are $V(D_1 \cup D_2) = V(D_1) \cup V(D_2)$. A *kernel* $N$ of $D$ is an independent set of vertices such that for every $z \in V(D) - N$ there exists a $zN$-arc in $D$. A digraph $D$ is called kernel-perfect (or $KP$-digraph) if every induced subdigraph of $D$ has a kernel, and critical kernel-imperfect digraph (CKI-digraph) if $D$ has no kernel and every proper induced subdigraph has a kernel. We will say that a digraph $D$ is complete if its underlying graph is a complete graph. Thus every complete subdigraph $\mathbb{C}$ of a kernel-perfect digraph must have an absorbing vertex (i.e., a successor of all other vertices of $\mathbb{C}$). A digraph $D$ is called a normal orientation of its underlying graph $G$ if every complete subdigraph of $D$ has a kernel. A graph $G$ is called solvable if every one of its normal orientations is a kernel-perfect digraph.

Many papers have recently appeared which are devoted to construct CKI-digraphs (see [2, 3, 4, 6, 7]); however, all of those constructions allow only digraphs whose asymmetrical part has a path as its block-cutpoint tree. In this paper, we develop a general method to construct CKI-digraphs whose asymmetrical part has a prescribed block-cutpoint tree. Specially, we prove that any directed cactus all of whose blocks are directed cycles of length at least five is the asymmetrical part of some CKI-digraph.

Define the digraph $C = \overrightarrow{C}_n(j_1, j_2, \ldots, j_k)$ by

$$\begin{array}{rcl} V(C) & = & \{0, 1, \ldots, n-1\} \\ F(C) & = & \{uv \mid v - u \equiv js \ (mod\, n) \text{ for } s = 1, \ldots, k\} \end{array}$$

In particular, we have the digraph

$$D = \overrightarrow{C_n}\left(1, \pm 2, \pm 3, \ldots, \pm \left[\frac{n}{2}\right]\right)$$

defined by

$$V(D) = \{0, 1, \ldots, n-1\}, \quad F(D) = \{uv | v - u \not\equiv -1 \ (\mathrm{mod}\ n)\}.$$

Let $G$ be a connected graph. Denote by $V(G)$ the set of vertices of $G$ and $E(G)$ the set of edges of $G$; $B(G)$ and $c(G)$ the set of blocks and cutpoints of $G$, respectively. The block-cutpoint tree $b_c(G)$ of $G$ is defined by $V(b_c(G)) = B(G) \cup c(G)$ and $E(b_c(G)) = \{[u, x] \mid u \in B(G), x \in c(G)$ and $x \in u\}$. The ordered bipartition $(B(G), c(G))$ is determined since $b_c(G)$ is connected and $B(G)$ contains all endpoints of $b_c(G)$. If $\alpha$ is a block-cutpoint tree, we will denote by $\alpha_b$ and $\alpha_c$ the block-part and the cutpoint-part of

$V(\alpha)$, respectively. We recall that a tree $\alpha$ is a block cutpoint tree if and only if the distance of any two distint endpoints is even (see [8, p. 36]).

The following result was proved in [5, Theorems 3.2, 3.3 and 3.4]

**Theorem 1.1.** *Let $D_1, D_2, H_1, H_2, D$ be digraphs, $v, u_i \in V(D_i)$, $i = 1, 2$ such that $u_i v \in F(sym\ (D_i))$, $V(D_1) \cap V(D_2) = \{v\}$, $H_i = D_i - u_i v - v u_i$ and $D = H_1 \cup H_2 + u_1 u_2 + u_2 u_1$. Suppose that $H_1$ and $H_2$ are $KP$-digraphs. Then*

(a) *$D$ is a CKI-digraph iff $D_1$ and $D_2$ are CKI-digraphs. Moreover, if $D$ is a CKI-digraph, then $D - u_1 u_2 - u_2 u_1$ is a $KP$-digraph.*

(b) *$D$ is a $KP$-digraph iff at least one of $D_1$ and $D_2$ is a $KP$-digraph.*

Theorem 1.2 was proved in [5, Corollary 2.3]. Its second part is a direct consequence of the fact that for every CKI-digraph $D$, Asym $D$ is strongly connected [5, Theorem 2.2].

**Theorem 1.2.** *$D = \overrightarrow{C_n}(1, \pm 2, \pm 3, \ldots, \pm[\frac{n}{2}])$ is a CKI-digraph for $n \geq 4$, and if $\emptyset \neq F_0 \subset F(Asym\ D)$, then $D - F_0$ is a $KP$-digraph.*

**Lemma 1.1.** *If $D = \overrightarrow{C_n}(1, \pm 2, \ldots, \pm[\frac{n}{2}])$ and, $\alpha$ is a subdigraph of sym $D$ and $\alpha$ is a tree, then $D - F(\alpha)$ is a $KP$-digraph.*

We omit the proof of Lemma 1.1 since it is a direct consequence of the following result due to M. Blidia, P. Duchet and F. Maffray [2].

**Theorem 1.3.** *If the complement of the graph $G$ is strongly perfect, then $G$ is solvable.*

## 2. New Classes of Critical Kernel-Imperfect Digraphs

In this section, we develop a method to construct CKI-digraphs whose asymmetrical part has a prescribed block-cutpoint tree. Also we construct CKI-digraphs whose asymmetrical part is a given directed cactus all of whose blocks are directed cycles of length at least five.

**Theorem 2.1** *Let $\alpha$ be any block-cutpoint tree and $\mathcal{N} : \alpha_b \to \mathbb{N}$ a function satisfying $\mathcal{N}(u) \geq \max\{\delta_\alpha(u), 5\}$ ($\delta_\alpha(u)$ is the degree of $u$ in $\alpha$). Then there exists a CKI-digraph $D_{(\alpha,\mathcal{N})}$ and an isomorphism $h : V(\alpha) \to V(b_c(Asym\, D_{(\alpha,\mathcal{N})}))$ satisfying the following conditions:*

(i) *For each $u \in \alpha_b$, $h(u)$ is a directed cycle in $Asym\, D$ of length $\mathcal{N}(u)$.*

(ii) *$D - F_0$ is a $KP$-digraph for every non empty set $F_0 \subset F$ $(Asym\, D)$.*

(iii) *For every $z \in V(D)$ there exists $w \in V(D)$ such that $wz \in F$ (sym $D$) and $D - \{wz, zw\}$ is a $KP$-digraph.*

***Proof.*** We proceed by induction on $n = |V(\alpha)|$. First let $n = 1$, $\mathcal{N}(u) = k \geq 5$. Take $D = D_{(\alpha,\mathcal{N})} = \overrightarrow{C}_k(1, \pm 2, \ldots, \pm[\frac{k}{2}])$. By Theorem 1.2 and Lemma 1.1 $D$ satisfies conditions (ii) and (iii), condition (i) is trivial. Suppose the assertion of Theorem 2.1 holds for $n < s$ and consider any block-cutpoint tree $\alpha$ with $n = s$. Let $u$ be any endpoint of $\alpha$ (therefore $u \in \alpha_b$) and call $c$ the neighbour of $u$ ($c \in \alpha_c$). Two cases are possible:

*Case* 1. If $\delta_\alpha(c) = 2$, call $u' \in \alpha_b$ the neighbour of $c$ different from $u$. Notice that $\alpha - c$ decomposes into two connected components one (say $\alpha'$) containing $u'$ and the other consisting of the single point $u$.

*Case* 2. If $\delta_\alpha(c) > 2$, take $\alpha' = \alpha - u$. In both cases $\alpha'$ is a block-cutpoint tree. Take the restriction $\mathcal{N}' = \mathcal{N}/\alpha'_b$. Then we obtain by induction a $CKI$-digraph $D' = D_{(\alpha',\mathcal{N}')}$ and an isomorphism $h' : V(\alpha') \to V(b_c \text{ Asym}\, D')$ satisfying (i), (ii) and (iii). Choose $t \in V(D')$ as follows: In Case 1, $h'(u')$ induces a directed cycle $\overrightarrow{\gamma}'_u$ in Asym $D'$ of length $\mathcal{N}'(u') = \mathcal{N}(u') \geq \delta_\alpha(u') = \delta_{\alpha'}(u') + 1$.

Therefore $\overrightarrow{\gamma}_{u'}$ contains some vertex $t$ which is not a cutpoint of $D'$.

In Case 2, let $t = h'(c)$ be any cutpoint of $D'$ which corresponds to $c$. Take any isomorphic copy $D''$ of $\overrightarrow{C}_{\mathcal{N}(u)}$ $(1, \pm 2, \ldots, \pm[\frac{1}{2}\mathcal{N}(u)])$ such that $V(D'') \cap V(D') = \{t\}$ and choose $t_1 \in V(D')$ such that $tt_1 \in F(\text{sym } D')$ and $D' - \{tt_1, t_1 t\}$ is a $KP$-digraph (condition (iii)). Choose also $t_2 \in V(D'')$ such that $tt_2 \in F(\text{sym } D'')$ and $D'' - \{tt_2, t_2 t\}$ is a $KP$-digraph (Theorem 1.2). By Theorem 1.1 (a) $D = (D' - \{tt_1, t_1 t\}) \cup (D'' - \{tt_2, t_2 t\}) + t_1 t_2 + t_2 t_1$ is a CKI-digraph. Extend $h'$ to $h : V(\alpha) \to V(b_c \text{ (Asym}\, D))$ in an obvious way. Condition (i) is obvious, condition (ii) follows from Theorem 1.1 (b), and by the fact that for every $CKI$-digraph $D$, Asym $D$ is strongly connected [5, Theorem 7.2]. Finally, condition (iii) follows from the induction hypothesis and Theorem 1.1 (b) in case $z \neq t_1, t_2, t$; in case $z = t_1, t_2$, from the fact that $D - \{t_1 t_2, t_2 t_1\}$ is a $KP$-digraph (Theorem 1.1 (a)) and, in case $z = t$, by taking $w \in V(D'')$ $w \neq t_2$, such that $tw \in F(\text{sym } D'')$. By Lemma 1.1, $D'' - \{tt_2, t_2 t\}$, $D'' - \{tw, wt\}$ and $D'' - \{tt_2, t_2 t, tw, wt\}$ are $KP$-digraphs. Applying Theorem 1.1 (b), $D - \{tw, wt\}$ is a $KP$-digraph and the proof is complete. ■

**Theorem 2.2.** *Let $H$ be an asymmetrical digraph each one of whose blocks is a directed cycle of length at least five. Then there exists a critical kernel-imperfect-digraph $D$ satisfying the following properties:*

(i) *$Asym D$ is isomorphic to $H$.*

(ii) *$D - F_0$ is a $KP$-digraph for every non empty set $F_0 \subset F$ $(Asym\ D)$.*

(iii) *For every $z \in V(D)$ there exists $w \in V(D)$ such that $wz \in F$ $(sym\ D)$ and $D - \{wz, zw\}$ is a $KP$-digraph.*

The proof of Theorem 2.2 is similar to that of Theorem 2.1.

## References


[1] C. Berge, Graphs (North-Holland, Amsterdam, 1985).

[2] M. Blidia, P. Duchet, F. Maffray, *On orientations of perfect graphs*, in preparation.

[3] P. Duchet, *Graphes Noyau-Parfaits*, Annals Discrete Math. **9** (1980) 93–101.

[4] H. Galeana-Sánchez, *A new method to extend kernel-perfect graphs to kernel-perfect critical graphs*, Discrete Math. **69** (1988) 207–209.

[5] H. Galeana-Sánchez and V. Neumann-Lara, *On kernel-perfect critical digraphs*, Dicrete Math. **59** (1986) 257–265.

[6] H. Galeana-Sánchez and V. Neumann-Lara, *Extending kernel-perfect digraphs to kernel-perfect critical digraphs*, Discrete Math. **94** (1991) 181–187.

[7] H. Galeana-Sánchez and V. Neumann-Lara, *New extensions of kernel-perfect digraphs to critical kernel-imperfect digraphs*, Graphs & Combinatorics **10** (1994) 329–336.

[8] F. Harary, Graph Theory (Addison-Wesley Publishing Company, New York, 1969).


Received 22 April 1997
Revised 3 November 1997