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# A NOTE ON UNIQUELY EMBEDDABLE GRAPHS

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**Abstract**

Let $G$ be a simple graph of order $n$ and size $e(G)$. It is well known that if $e(G) \leq n-2$, then there is an embedding $G$ into its complement $\overline{G}$. In this note, we consider a problem concerning the uniqueness of such an embedding.

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## 1. Result

We shall use the standard graph theory notation. We consider only finite, undirected graphs of order $n = |V(G)|$ and size $e(G) = |E(G)|$. All graphs will be assumed to have neither loops nor multiple edges.

We shall need some additional definitions in order to formulate the results. If a graph $G$ has order $n$ and size $m$, we say that $G$ is an $(n, m)$ graph.

Assume now that $G_1$ and $G_2$ are two graphs with disjoint vertex sets. The *union* $G = G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. If a graph is the union of $n$ $(\geq 2)$ disjoint copies of a graph $H$, then we write $G = nH$.

For our next operation, the conditions are quite different. Let now $G_1$ and $G_2$ be graphs with $V(G_1) = V(G_2)$ and $E(G_1) \cap E(G_2) = \emptyset$. The *edge sum* $G_1 \oplus G_2$ has $V(G) = V(G_1) = V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

An *embedding* of $G$ (in its complement $\overline{G}$) is a permutation $\sigma$ on $V(G)$ such that if an edge $xy$ belongs to $E(G)$, then $\sigma(x)\sigma(y)$ does not belong to $E(G)$. In others words, an embedding is an (edge-disjoint) *placement* (or *packing*) of two copies of $G$ into a complete graph $K_n$.

The following theorem was proved, independently, in [1], [2] and [5].

**Theorem 1.** *Let $G = (V, E)$ be a graph of order $n$. If $|E(G)| \leq n - 2$, then $G$ can be embedded in its complement $\overline{G}$.*

The example of the star $K_{1,n-1}$ shows that Theorem 1 cannot be improved by raising the size of $G$.

This result has been improved in many ways but as far as we know the problem of the uniqueness has not been considered (see e.g., the survey paper [7]).

First, we have to precise what we mean by *distinct* embeddings.

Let $\sigma$ be an embedding of the graph $G = (V, E)$. We denote by $\sigma(G)$ the graph with the vertex set $V$ and the edge set $\sigma^*(E)$ where the map $\sigma^*$ is induced by $\sigma$. Since, by definition of an embedding, the sets $E$ and $\sigma^*(E)$ are disjoint we may form the graph $G \oplus \sigma(G)$.

Two embeddings $\sigma_1$, $\sigma_1$ of a graph $G$ are said to be *distinct* if the graphs $G \oplus \sigma_1(G)$ and $G \oplus \sigma_2(G)$ are not isomorphic. A graph $G$ is called *uniquely embeddable* if for all embeddings $\sigma$ of $G$, all graphs $G \oplus \sigma(G)$ are isomorphic.

Our purpose is to characterize all $(n, n-2)$ graphs that are uniquely embeddable.

**Theorem 2.** *Let $G$ be a graph of order $n$ and size $e(G) = n - 2$. Then either $G$ is not uniquely embeddable or $G$ is isomorphic to one of the seven following graphs* (*see also Figure* 1) : $K_2 \cup K_1$, $2K_2$, $K_3 \cup 2K_1$, $K_3 \cup K_2 \cup K_1$, $C_4 \cup 2K_1$, $K_3 \cup 2K_2$, $2K_3 \cup 2K_1$.

The proof is given in the next section.

In the case where $G$ is a non-star tree, we have the following result. The proof, analogous to the proof of Theorem 2, is omitted.

**Theorem 3.** *Let $T$ be a non-star tree of order $n$. Then either $T$ is not uniquely embeddable or $T$ is isomorphic to the tree $S'_n$ obtained by subdividing one edge of the star $S_{n-1}$ or else $T$ is the balanced double-star on six vertices.* ■

We shall need the following theorem which completely characterizes those graphs with $n$ vertices and $n - 1$ edges which are embeddable ([3], [4])

**Theorem 4.** *Let $G = (V, E)$ be a graph of order $n$. If $|E(G)| \leq n - 1$, then either $G$ is embeddable or $G$ is isomorphic to one of the following graphs:* $K_{1,n-1}$, $K_{1,n-4} \cup K_3$ *for* $n \geq 8$, $K_1 \cup 2K_3$, $K_1 \cup C_4$, $K_1 \cup K_3$, *and* $K_2 \cup K_3$.

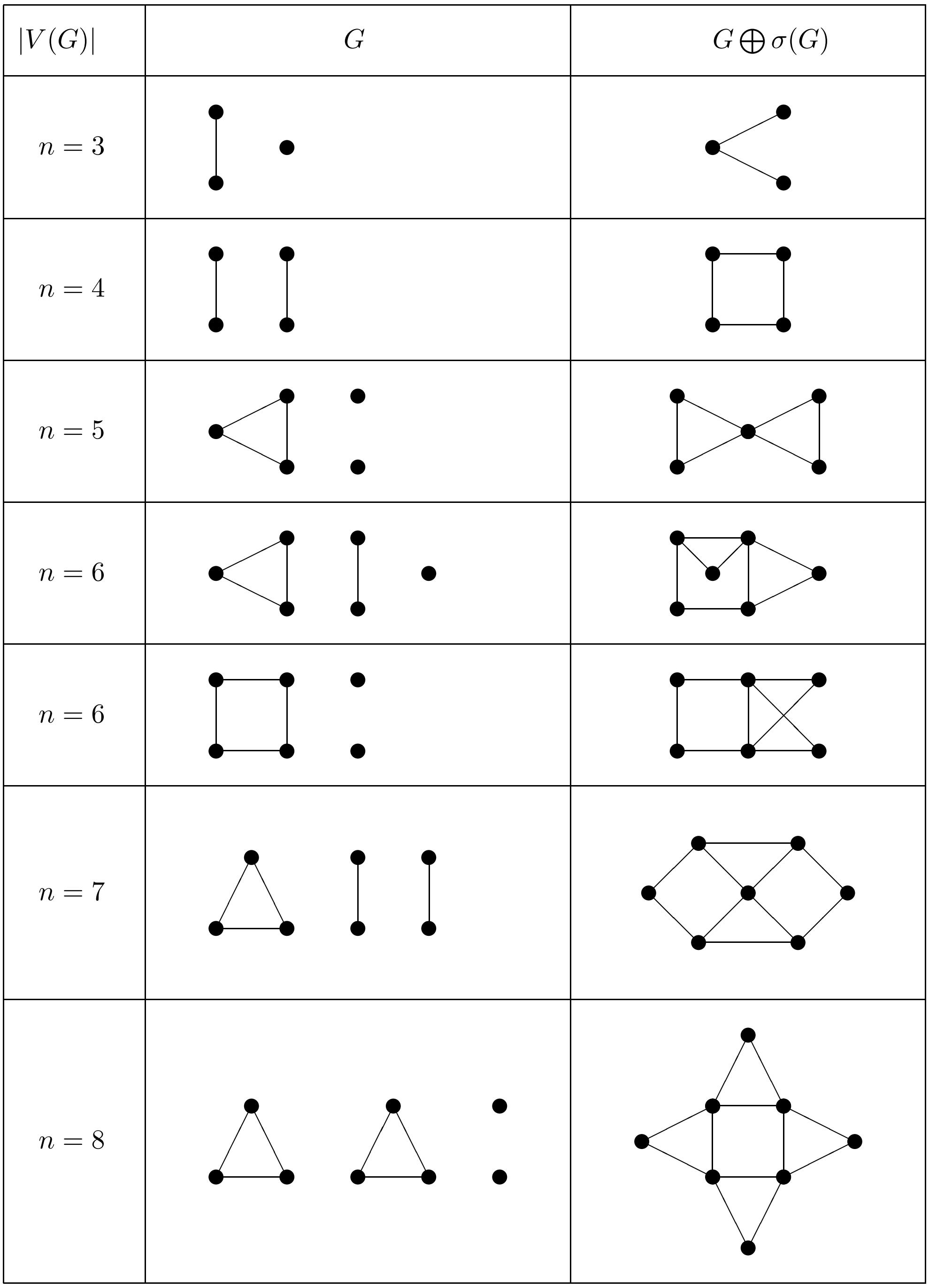


Figure 1

We shall use also the following version of Theorem 1 [6]:

**Theorem 5.** *Let $G = (V, E)$ be a graph of order $n$. If $|E(G)| \leq n-2$, then there exists an embedding $\sigma$ of $G$ such that $\sigma$ is a cyclic permutation.*

## 2. Proof of Theorem 2

Let $G$ be a graph of order $n$ and size $e(G) = n-2$. We shall consider three main cases.

*Case* 1. $G$ has an isolated vertex $v$ and the $(n-1, n-2)$ graph $G' = G-v$ is embeddable in its complement.
Let $\sigma$ be an embedding of $G'$. We define an embedding of $G$, $\sigma_1 : V(G) \mapsto V(G)$ by putting $\sigma_1(x) = \sigma(x)$ for $x \in V(G')$ and $\sigma_1(v) = v$. Note that the graph $G \oplus \sigma_1(G)$ has at least one isolated vertex, namely $v$. Now, let $a$ and $b$ be two vertices of $G'$ such that $d(a) \neq 0$, $d(b) \neq 0$ and $\sigma(a) = b$. If we change slightly the permutation $\sigma_1$ by putting the vertex $a$ on $v$ and $v$ on $b$, we obtain a permutation $\sigma_2$ with one isolated vertex in $G \oplus \sigma_2(G)$ less than in $G \oplus \sigma_1(G)$. So, the graphs $G \oplus \sigma_1(G)$ and $G \oplus \sigma_2(G)$ cannot be isomorphic. Suppose now that $\sigma$ is such that for all vertices $a$ with $d(a) \neq 0$, $\sigma(a)$ is isolated in $G$. Denote by $B$ the set of isolated vertices of $G$ and by $A$ the set of nonisolated vertices. Let $A = \{a_1, a_2, \ldots, a_p\}$ and $B = \{b_1, b_2, \ldots, b_r\}$.

Since $v$ is an isolated vertex of $G$ we have, of course, $r > p$. It is easy to see that the permutation $\sigma_1$ consisting of $p$ transpositions $(a_i, b_i)$ and $r-p$ fixed points $b_{p+1}, \ldots, b_r$ defines an embedding of $G$.

Observe that the graph $G \oplus \sigma_1(G)$ has exactly $r-p$ isolated vertices. The second embedding is defined as follows: We put $\sigma_2(a_1) = a_1$, $\sigma_2(b_1) = b_1$, and $\sigma_2(x) = \sigma_1(x)$ for other vertices. Observe that the graph $G \oplus \sigma_2(G)$ has one isolated vertex more than the graph $G \oplus \sigma_1(G)$.

*Case* 2. $G$ has an isolated vertex $v$ but $G - v$ is not embeddable.
By Theorem 4, $G$ is one of the following six graphs: $K_{1,n-2} \cup K_1$, $K_3 \cup K_{1,n-5} \cup K_1$ with $n \geq 9$, $K_3 \cup K_2 \cup K_1$, $C_4 \cup 2K_1$, $K_3 \cup 2K_1$, and $2K_3 \cup 2K_1$. If $G = K_{1,n-2} \cup K_1$, then for $n = 3$ we have an exceptional graph. In order to obtain two different packings for $n \geq 4$ we proceed as follows: First we map the center of the star on the isolated vertex and the leaves on the leaves and we get a graph with two vertices of degree $n-2$ and $n-2$ vertices of degree two.

In the second packing we map the center of the star on a leaf. We get a graph with one vertex of degree one.

Also in the case where $G = K_3 \cup K_{1,n-5} \cup K_1$, $n \geq 9$, two distinct embeddings are easy to find. The details are left to the reader. Note that the remaining cases lead to exceptional graphs.

*Case* 3. $G$ has no isolated vertex.
We shall consider three subcases.

*Subcase* 3a. $G$ has no isolated edge.
Denote by $X$ the set of end-vertices of $G$ and by $Y$ the set $V(G) \setminus X$. Let $G' = G \setminus X$. The graph $G'$ is a $(k, k-2)$ graph, so, by Theorem 5 there exists a cyclic permutation $\sigma$ which embeds $G'$ into $\overline{G'}$. This permutation can be easily extended to an embedding $\sigma_1$ of $G$ by adding $|X|$ fixed points corresponding to $|X|$ end-vertices of $G$. Observe that the graph $G \oplus \sigma_1(G)$ has exactly $|X|$ vertices of degree two because all vertices of $Y$ are of degree at least four (in $G \oplus \sigma_1(G)$). The permutation $\sigma_2$ is defined in an analogous way but this time we do not remove all end-vertices of $G$ but all but one, say $x$. Now, the permutation $\sigma_2$ has $|X| - 1$ fixed points which generate the vertices of degree two in the graph $G \oplus \sigma_2(G)$. Since in the set $Y$ we have only one end-vertex of $G$, all vertices of $Y$ are of degree at least three in $G \oplus \sigma_2(G)$. Hence, the packing $\sigma_2$ is different from $\sigma_1$.

*Subcase* 3b. $G$ has only one isolated edge $x_1y_1$.
The argument is similar as in Case 3a. We assign to the set $X$ all end-vertices except for $y_1$. We put $G' = G \setminus X$ and $Y = V(G) \setminus X$. Let $\sigma' = (y_1, y_2, \ldots, y_r)$ be a cyclic permutation which embeds $G'$ into $\overline{G'}$. As previously, this permutation can be easily extended to an embedding $\sigma_1$ of $G$, by adding $|X|$ fixed points corresponding to end-vertices belonging to $X$. Since $Y$ contains only one vertex of degree one (in $G$), all vertices of $Y$ are of degree at least three (in $G \oplus \sigma_1(G)$). Hence, the graph $G \oplus \sigma_1(G)$ has exactly $|X|$ vertices of degree two. In order to define the permutation $\sigma_2$ we proceed as follows. Let $x_2$ be an end-vertex of $G$, $x_2 \neq x_1$, $x_2 \neq y_1$. Consider the graph $G'' = G \setminus (X \setminus \{x_1\})$. Suppose that $x_2$ is adjacent to $y_i$. Observe that if $y_iy_{i-1} \notin E$, then the cyclic permutation $(y_1, y_2, \ldots, y_i, x_2, y_{i+1}, \ldots, y_r)$ is an embedding of $G''$, and if $y_iy_{i+1} \notin E$, then the cyclic permutation $(y_1, y_2, \ldots, y_{i-1}, x_2, y_i, \ldots, y_r)$ is an embedding of $G''$.

As previously, we extend this permutation by considering other vertices as fixed points. We obtain an embedding of $G$, say $\sigma_2$. Moreover, since $y_iy_1$ is not in $E$, this embedding can be chosen in such a way that neither the vertex $x_2$ is mapped on $y_1$ nor $y_1$ is mapped on $x_2$. So, the graph $G \oplus \sigma_2(G)$ has exactly $|X| - 1$ vertices of degree two.

*Subcase* 3c. $G$ has two isolated edges $x_1y_1$ and $x_2y_2$.
Recall that $G$ has exactly $n-2$ edges. The reader may check easily that for $n \leq 7$ we have necessarily $n=4$ or $n=7$ and $G$ is an exceptional graph. So, let $n \geq 8$. Denote by $z_1, z_2$ two nonadjacent vertices of $G$ of degree at least two, if they exist. Consider the graph $G' = G \setminus \{x_1, x_2, y_1, y_2, z_1, z_2\}$. By Theorem 1, there exists an embedding $\sigma'$ of $G'$. We shall extend $\sigma'$ by two ways. We put:

$$\sigma_1 = \sigma'(x_1y_1z_1x_2y_2z_2), \qquad \sigma_2 = \sigma'(x_1z_1y_1x_2y_2z_2).$$

It is easy to see that $\sigma_1$ and $\sigma_2$ are embeddings of $G$. Observe that in both cases the graphs $G \oplus \sigma_i(G)$, $i = 1, 2$, have the same number of vertices of degree two. However, in the first case the maximum number of independent vertices of degree two is greater than in the second case. Thus $G \oplus \sigma_2(G)$ is not isomorphic to $G \oplus \sigma_2(G)$.

There remains the case where every two vertices of degree greater than one are adjacent. Then $G$ is of the form $G = pK_2 \cup H$ where $H$ is either a complete graph, say $K_k$ or a graph obtained from $K_k$ by adding some vertices of degree one connected by an edge to $V(K_k)$. Observe that $p = \frac{1}{2}k(k-3)+2$.

Consider first the other possibility and let $a'$ be one of such vertices of degree one adjacent to the vertex $a \in V(K_k)$. We define $G' = G \setminus \{x_1, y_1, a, a'\}$. As above, by Theorem 1, there exists an embedding $\sigma'$ of $G'$. We extend it by two ways by putting:

$$\sigma_1 = \sigma'(aa'x_1y_1), \qquad \sigma_2 = \sigma'(x_1a).$$

By a similar argument as above, it is easy to see that $\sigma_1$ and $\sigma_2$ are embeddings of $G$ but $G \oplus \sigma_2(G)$ is not isomorphic to $G \oplus \sigma_2(G)$.

The case where $H = K_k$, $k \geq 4$, is straightforward and is left to the reader. ■

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