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DECOMPOSITION OF MULTIGRAPHS

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Abstract

In this note, we consider the problem of existence of an edge-decomposition of a multigraph into isomorphic copies of 2-edge paths $K_{1,2}$. We find necessary and sufficient conditions for such a decomposition of a multigraph H to exist when

(i) either H does not have incident multiple edges or
(ii) multiplicities of the edges in H are not greater than two.

In particular, we answer a problem stated by Z. Skupień.

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Z. Skupień, at the conference in Zakopane (September 1994), stated a problem of decomposition of the edge set of a multigraph H into stars $K_{1,2}$, if it is assumed that multiplicities of the edges do not exceed 2. This property is denoted by $K_{1,2}|H$. It is known that if H is a simple graph, then $K_{1,2}|H$ if and only if the size of every component of H is even. It is easy to verify that this condition is not sufficient to ensure the decomposition of a multigraph.

Let $\mathcal{M}$ be the class of trees with a perfect matching. Denote by H^* the graph obtained from H by deleting all edges of multiplicity 1 and reducing the multiplicities of all the other edges to 1. Let $\mathcal{M}(H)$ be the set of these components in H^* that belong to $\mathcal{M}$. Clearly, in case (i) members of $\mathcal{M}(H)$

are simple edges and in case (ii) they can be isomorphic to any member of $\mathcal{M}$.

For $\mathcal{A} \subset \mathcal{M}(H)$ denote by $E_{\mathcal{A}}$ the set of edges in H with at least one end in a vertex of a component in $\mathcal{A}$ (multiple edges are counted multiplicity many times). Let $V(\mathcal{A})$ (resp. $E(\mathcal{A})$) stand for the union of the vertex (resp. edge) sets of the components in $\mathcal{A}$. For a component t in $\mathcal{M}(H)$ which corresponds to a multiple edge denote by $m(t)$ the multiplicity of this edge in H. Finally, let $O(H \setminus V(\mathcal{A}))$ be the number of components of $H \setminus V(\mathcal{A})$ with an odd size which are not multiple edges.

Here are our main results.

Theorem 1. *Let H be a multigraph of an even size and with no incident multiple edges. Then $K_{1,2}|H$ if and only if*

(1) *for every set of edges* $\mathcal{A} \subset \mathcal{M}(H),\ |E_{\mathcal{A}}| \geq 2\sum_{t\in\mathcal{A}} m(t) + O(H \setminus V(\mathcal{A}))$.

Clearly, in the above theorem each member of $\mathcal{M}(H)$ is an edge.

A *proper* simple cut-edge in a multigraph is a simple cut-edge whose deletion does not create a component consisting of one vertex.

Corollary 1. *Let H be a multigraph of even size, with no incident multiple edges and with no simple proper cut-edge. If each edge e of multiplicity $m(t) > 1$ is incident to at least $2m(t)$ edges of multiplicity 1 then $K_{1,2}|H$.*

Theorem 2. *Let H be a multigraph of an even size and let the multiplicities of the edges be not greater than 2. Then $K_{1,2}|H$ if and only if*

$$\forall \mathcal{A} \subset \mathcal{M}(H),\ \ |E_{\mathcal{A}}| \geq 2|V(\mathcal{A})| + O(H \setminus V(\mathcal{A})). \tag{2}$$

Before proving these theorems let us make two remarks.

Remark 1. The problem of decomposing a multigraph H into $K_{1,2}$ reduces to the case when H is connected.

Remark 2. Let H be a multigraph. If H contains a pair of incident multiple edges, then we can delete a copy of $K_{1,2}$; we repeat this process for pairs of incident multiple edges until we obtain a multigraph H' (not unique) with no two incident multiple edges. If $K_{1,2}|H'$ for some choice of H', then by adding the deleted edges, we immediately get a decomposition of H.

PROOFS OF RESULTS

For a multigraph H define a graph $G(H)$ whose vertex set is $E(H)$ and a pair of vertices in $G(H)$ is an edge if the corresponding edges in H have exactly one common vertex.

Lemma 1. *$K_{1,2}|H$ if and only if $G(H)$ has a perfect matching.*

Proof. Suppose $K_{1,2}|H$. Since the vertices in $G(H)$ correspond to edges in H, a decomposition of H into stars $K_{1,2}$ defines a perfect matching in $G(H)$.

Conversely, suppose $G(H)$ has a perfect matching. Each edge in this matching defines a copy of $K_{1,2}$ in H. Since the matching covers all vertices in $G(H)$, the corresponding copies of $K_{1,2}$ form an edge-decomposition of H. ■

By the result of Tutte [T], $G(H)$ has a perfect matching if and only if

$$\forall S \subset V(G(H)), \quad O_V(G(H) \setminus S) \le |S|, \tag{3}$$

where $O_V(G(H) \setminus S)$ is the number of components of $G(H) \setminus S$ with an odd number of vertices.

When writing this paper, we have been informed that J. Ivančo, M. Meszka and Z. Skupień [IMS] have made the same observation as in our Lemma 1. In particular, they concluded that deciding whether $K_{1,2}|H$ for an instance multigraph H is a polynomial problem.

Assume now that H has no incident multiple edges. Call an edge e in H an *m-bridge* if it is a bridge in the component H_1 of H containing e and if at least one of the components of $H_1 - e$ is a multiple edge.

Lemma 2. *Let H be a multigraph with no incident multiple edges. If e is not an m-bridge in H then*

$$O_V(G(H) - e) \le O_V(G(H)) + 1.$$

Proof. The lemma obviously holds when H is a multiple edge because then $G(H)$ is an edgeless graph. Otherwise it follows from the observation that $G(H_1) - e$ has exactly 2 components (where H_1 is the component of H containing e) and e is not an m-bridge. We leave routine details of this proof to the reader. ■

Let $E(\mathcal{A}, H \setminus V(\mathcal{A}))$ (respectively $E(\mathcal{A}, \mathcal{A})$) be the set of edges with one end-vertex in $\mathcal{A} \subset \mathcal{M}(H)$ and the other one in $V(H) \setminus V(\mathcal{A})$ (respectively with end-vertices in two different members of $\mathcal{A}$).

***Proof of Theorem* 1.** Since H has no incident multiple edges, the set $\mathcal{M}(H)$ represents the set of multiple edges in H. The condition (1) is equivalent to

$$\forall \mathcal{A} \subset \mathcal{M}(H)\,, \quad |E(\mathcal{A}, H \setminus V(\mathcal{A}))| + |E(\mathcal{A}, \mathcal{A})| \geq \textstyle\sum_{t \in \mathcal{A}} m(t) + O(H \setminus V(\mathcal{A})). \tag{4}$$

Suppose there is a decomposition π of H into stars $K_{1,2}$. For every component of $H \setminus V(\mathcal{A})$ of odd size at least one copy of $K_{1,2}$ in π has an edge in $E(\mathcal{A}, H \setminus V(\mathcal{A}))$. Moreover, for every multiple edge $t \in \mathcal{A}$, $m(t)$ copies of $K_{1,2}$ in π have one edge in $E(\mathcal{A}, H \setminus V(\mathcal{A})) \cup E(\mathcal{A}, \mathcal{A})$. Hence

$$\forall \mathcal{A} \subset \mathcal{M}(H), \quad |E(\mathcal{A}, H \setminus V(\mathcal{A}))| + |E(\mathcal{A}, \mathcal{A})| \geq \sum_{t \in \mathcal{A}} m(t) + O(H \setminus V(\mathcal{A})),$$

which completes the proof of necessity.

To show sufficiency suppose that (1) is satisfied and H does not have a decomposition into stars $K_{1,2}$. By Lemma 1 and (3), we get

$$\exists S \subset V(G(H))\,, \quad O_V(G(H) \setminus S) > |S|. \tag{5}$$

Assume that S has the smallest cardinality among the sets satisfying the above inequality.

Supose first that $S = \emptyset$. Then, at least one component of $G(H)$ has an odd number of vertices. By the definition of $G(H)$, either one of the components of H which is not a multiple edge has an odd size or one of the components of H is a multiple edge. In the former case we get a contradiction to (4) because for $\mathcal{A} = \emptyset$ we obtain $O(H) = 0$. To get a contradiction in the latter case, denote by e a multiple edge which is a component in H. The condition (4) yields a contradiction for $\mathcal{A} = \{e\}$. Hence $S \neq \emptyset$.

Suppose now that some $e \in S \subset V(G(H)) = E(H)$ is not an m-bridge in $H \setminus (S \setminus \{e\}) = (H \setminus S) \cup \{e\}$. By minimality of S,

$$O_V((G(H) \setminus S) \cup \{e\}) \leq |S \setminus \{e\}|.$$

The multigraph $(H \setminus S) \cup \{e\}$ satisfies the assumptions of Lemma 2. Consequently, $O_V(G(H) \setminus S) \leq O_V((G(H) \setminus S) \cup \{e\}) + 1$, so

$$O_V(G(H) \setminus S) \leq |S \setminus \{e\}| + 1 = |S|,$$

a contradiction.

Thus all the edges $e \in S \subset E(H)$ are m-bridges in $(H \setminus S) \cup \{e\}$. Let $\mathcal{A}$ be the set of multiple edges which are components in $H \setminus S$. Then, clearly, $S = E(\mathcal{A}, H \setminus V(\mathcal{A})) \cup E(\mathcal{A}, \mathcal{A})$. By the definition of $G(H)$ and (4)

$$O_V(G(H)\setminus S) = \sum_{t\in\mathcal{A}} m(t) + O(H\setminus V(\mathcal{A})) \leq |E(\mathcal{A}, H\setminus V(\mathcal{A}))| + |E(\mathcal{A},\mathcal{A})| = |S|,$$

a contradiction to (5). ■

***Proof of Corollary* 1.** By the assumption of the corollary, for every set of multiple edges $\mathcal{A}$,

$$2|E(\mathcal{A},\mathcal{A})| + |E(\mathcal{A}, H \setminus V(\mathcal{A}))| \geq \sum_{t\in\mathcal{A}} 2m(t). \tag{6}$$

Let $\omega(H \setminus V(\mathcal{A}))$ be the number of components of $H \setminus V(\mathcal{A})$ of order at least 2. Then, since no simple edge in H is a proper cut-edge,

$$|E(\mathcal{A}, H \setminus V(\mathcal{A}))| \geq 2\omega(H \setminus V(\mathcal{A})) \geq 2O(H \setminus V(\mathcal{A})). \tag{7}$$

By adding (6) and (7) we get (4) so by Theorem 1 the proof is complete. ■

***Proof of Theorem* 2.** To show necessity suppose the required decomposition exists. Let M be the perfect matching in the graph formed by the components in $\mathcal{A}$. Denote by B the set of edges obtained from $E_{\mathcal{A}}$ by deletion of the edges of M and their doubles. Clearly, $|B| = |E_{\mathcal{A}}| - 2e(M)$. Note that $O(H \setminus V(\mathcal{A})) = O(H \setminus B)$, where $H \setminus B$ stands for the multigraph obtained from H by removing the edges of B. By the existence of a $K_{1,2}$-decomposition of H at least $O(H \setminus B) + 2e(M)$ different copies of $K_{1,2}$ in the decomposition have one edge in B. Hence

$$|B| \geq O(H \setminus B) + 2e(M)$$

so

$$|E_{\mathcal{A}}| = |B| + 2e(M) \geq 4e(M) + O(H \setminus B) = 2|V(\mathcal{A})| + O(H \setminus V(\mathcal{A}))|.$$

Suppose sufficiency is false. Let H be a multigraph of an even size with the minimum number of doubled edges satisfying (2) and such that $K_{1,2} \not| H$. Assume first that H^* contains a component C of a positive size which is not a member of $\mathcal{M}$.

If the size of C is even, then $K_{1,2}|C$. Therefore, if we delete copies of every edge in C from H, then the resulting multigraph H' still has an even size, satisfies (2) and $K_{1,2} \not| H'$ contradicting to the minimality of H.

Let the size of C be odd. Suppose first C contains a cycle and let e be one of its edges. It is routine to show that the multigraph C' obtained from C by doubling the edge e has a decomposition into stars $K_{1,2}$. Moreover, the multigraph H' obtained from H by deleting the edges of C' has an even size, satisfies (2) and $K_{1,2} \not| H'$ contradicting to the minimality of H again.

Let now C be a tree of an odd size. One can easily show that since $C \notin \mathcal{M}$, C can be decomposed into graphs A and B such that $K_{1,2}|A$ and B is isomorphic to $K_{1,3}$. If the size of A is positive then as before we can delete from H the edges of A and obtain a multigraph H' contradicting to the minimality of H.

Thus, we can assume that C is isomorphic to $K_{1,3}$. Let e and f be two of the edges of C and let e_1 and e_2 (respectively f_1 and f_2) be the parallel edges corresponding to e (resp. f) in H. Subdivide the edges e_1 and e_2 by inserting two new vertices v_1 and v_2 into e_1 and two new vertices u_1 and u_2 into e_2. Let e'_1 (resp. e'_2) denote the edge v_1v_2 (resp. u_1u_2). The resulting multigraph H' has an even size, satisfies (2) and, by the minimality of H, H' admits a decomposition π' into stars $K_{1,2}$. Contract the copies of $K_{1,2}$ in π' containing e'_1 and e'_2. We get the multigraph H again. The decomposition π' of H' defines in H a decomposition π which (by $K_{1,2} \not| H$) is a decomposition into copies of $K_{1,2}$ and the multigraph induced by the parallel edges e_1 and e_2. In the latter case, consider the multigraph F induced by e_1 and e_2 and the edges of copies of $K_{1,2}$ in π containing f_1 and f_2. It is routine to show that $K_{1,2}|F$, so consequently $K_{1,2}|H$, a contradiction.

We have shown that all components of H^* are isomorphic to members of $\mathcal{M}$.

If all the components in $\mathcal{M}(H)$ are single edges then by Theorem 1 the proof is complete. Suppose now that at least one of the components, say C, in $\mathcal{M}(H)$ is a tree with a perfect matching different from a single edge. It is easy to notice that then there are edges e and f in C such that e is a pendant edge in C and f is the only edge in C incident to e. Denote by e_1, e_2 (respectively f_1, f_2) the parallel edges in H corresponding to e (respectively f) in C. Subdivide f_1 and f_2 by inserting 2 new vertices x_1, x_2 into f_1 and y_1, y_2 into f_2. Let f'_1 (respectively f'_2) denote the edge x_1x_2 (respectively y_1y_2).

Let us check the inequality (2) for H'. Note that $\mathcal{M}(H') = (\mathcal{M}(H) \setminus \{C\}) \cup \{C_1, C_2\}$, where C_1 is the edge e and $C_2 = C \setminus \{e, f\}$.

Let $\mathcal{A} \in \mathcal{M}(H')$. The condition (2) is easy to verify when $C_1, C_2 \in \mathcal{A}$ and when $C_1, C_2 \notin \mathcal{A}$. Thus suppose that $C_2 \in \mathcal{A}$ and $C_1 \notin \mathcal{A}$ (the case $C_1 \in \mathcal{A}$ and $C_2 \notin \mathcal{A}$ is analogous and we leave it to the reader).

Let $\mathcal{A}' = \mathcal{A} \setminus \{C_2\}$. Then by our assumption for H

$$|E_{\mathcal{A}'}| \geq 2|V(\mathcal{A}')| + O(H \setminus V(\mathcal{A}')) = 2|V(\mathcal{A}')| + O(H' \setminus V(\mathcal{A}')).$$

Let k be the number of odd-sized components in $H' \setminus V(\mathcal{A})$ which are not odd-sized components in $H' \setminus V(\mathcal{A}')$. Clearly each of them is joined to a vertex in C_2 by at least one edge. Moreover, the component of $H' \setminus V(\mathcal{A})$ containing C_1 is joined to a vertex [4] of C_2 by at least 2 edges. Hence

$$|E_{\mathcal{A}}| \geq |E_{\mathcal{A}'}| + 2|E(C_2)| + k + 1 = |E_{\mathcal{A}'}| + 2|V(C_2)| - 2 + k + 1$$

$$\geq 2|V(\mathcal{A}')| + O(H' \setminus V(\mathcal{A}')) + 2|V(C_2)| + k - 1 \geq 2|V(\mathcal{A})| + O(H' \setminus V(\mathcal{A})) - 1.$$

Note that $|E_{\mathcal{A}}|$ and $O(H' \setminus V(\mathcal{A}))$ have the same parity. Indeed,

$$0 \equiv e(H') = |E_{\mathcal{A}}| + \sum_{C \in EV} e(C) + \sum_{C \in OD} e(C) \equiv |E_{\mathcal{A}}| + |OD|$$

$$= |E_{\mathcal{A}}| + O(H' \setminus V(\mathcal{A})) \pmod 2,$$

where EV (resp. OD) stands for the set of even-sized (resp. odd-sized) components in $H' \setminus V(\mathcal{A})$. Consequently $|E_{\mathcal{A}}| \geq 2|V(\mathcal{A})| + O(H' \setminus V(\mathcal{A}))$. By the minimality of H, H' admits a $K_{1,2}$-decomposition π'.

Contract the copies of $K_{1,2}$ in π' containing f_1' and f_2'. We get again the multigraph H. The decomposition π' of H' defines in H a decomposition π which is either a $K_{1,2}$-decomposition (in this case the proof is complete) or a decomposition into copies of $K_{1,2}$ and the multigraph induced by the parallel edges f_1, f_2. In the latter case consider the multigraph induced by f_1, f_2 and the copies of $K_{1,2}$ in π containing e_1 and e_2. It is routine to show that this multigraph admits a $K_{1,2}$-decomposition. This contradiction completes our proof. ■

Remark 3. One can easily deduce from Theorem 2 that a multigraph H with multiplicities of all edges equal to 2 is $K_{1,2}$-decomposable if and only if H^* is not a tree with a perfect matching. This result was earlier proved by Bondy [B].

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