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ON HEREDITARY PROPERTIES OF COMPOSITION GRAPHS*

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Abstract

The *composition graph* of a family of $n+1$ disjoint graphs $\{H_i : 0 \leq i \leq n\}$ is the graph H obtained by substituting the n vertices of H_0 respectively by the graphs $H_1, H_2, ..., H_n$. If H has some hereditary property P, then necessarily all its factors enjoy the same property. For some sort of graphs it is sufficient that all factors $\{H_i : 0 \leq i \leq n\}$ have a certain common P to endow H with this P. For instance, it is known that the composition graph of a family of perfect graphs is also a perfect graph (B. Bollobas, 1978), and the composition graph of a family of comparability graphs is a comparability graph as well (M.C. Golumbic, 1980). In this paper we show that the composition graph of a family of co-graphs (i.e., P_4-free graphs), is also a co-graph, whereas for θ_1-perfect graphs (i.e., P_4-free and C_4-free graphs) and for threshold graphs (i.e., P_4-free, C_4-free and $2K_2$-free graphs), the corresponding factors $\{H_i : 0 \leq i \leq n\}$ have to be equipped with some special structure.

Keywords: composition graph, co-graphs, θ_1-perfect graphs, threshold graphs.

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1. Introduction

Let $H = (V, E)$ be a simple graph (i.e., finite, undirected, loopless and without multiple edges). We denote : its vertex set by $V = V(H)$, its edge set by $E = E(H)$. If A, B are two nonempty and disjoint subsets of $V(H)$, by $A \sim B$ we mean that $ab \in E(H)$, for any $a \in A$ and $b \in B$. The *neighborhood* of $v \in V$ is $N(v) = \{w : w \in V \text{ and } vw \in E\}$. By K_n, C_n, P_n, we shall denote the complete graph on $n \geq 1$ vertices, the chordless cycle on $n > 3$ vertices and the chordless path on $n \geq 3$ vertices, respectively.

Let $H_0 = (V_0, E_0)$ be a graph with n vertices $V_0 = \{v_1, v_2, ..., v_n\}$ and let $H_1, H_2, ..., H_n$ be n disjoint graphs. The *composition graph* $H = (V, E)$ will be denoted by $H = H_0[H_1, H_2, ..., H_n]$ and is defined as follows:

$V = \cup\{V_i : 1 \leq i \leq n\}$,

$E = \cup\{E_i : 1 \leq i \leq n\} \cup \{xy : x \in V_i,\, y \in V_j,\, v_i v_j \in E_0, 1 \leq i,\, j \leq n\}$.

This operation was introduced by Sabidussi [17], and is a generalization of both:

(a) *lexicographic product* of two graphs H_0 and H_1, defined as:

$H_0[H_1] = H_0[H_1, H_1, ..., H_1]$, (i.e., $H_1 = H_i$, $2 \leq i \leq n$);

and of

(b) *join* of a family of graphs $H_1, H_2, ..., H_n$, defined as

$*[H_1, H_2, ..., H_n] = K_n[H_1, H_2, ..., H_n]$.

If $H = H_0[H_1, H_2, ..., H_n]$ has some *hereditary property* (e.g., as being a perfect graph, a chordal graph, etc.), then both the outer factor H_0 and the inner factors H_i, $i = 1, ..., n$, enjoy the same property, since they are isomorphic to some subgraphs of H. Usually, the inverse problem is not so easy to solve. Sometimes, if both H_0 and $H_1, H_2, ..., H_n$ have a certain common hereditary property, then their composition graph $H_0[H_1, H_2, ..., H_n]$ will have the same feature. This is true for:

(a) perfectness:
- $*[H_1, H_2, ..., H_n]$ is a perfect graph if and only if all H_i, $1 \leq i \leq n$, are perfect, [10];
- $H_0[H_1]$ is perfect if and only if both H_0 and H_1 are perfect, [16];
- $H_0[H_1, H_2, ..., H_n]$ is perfect if and only if all H_i, $0 \leq i \leq n$, are perfect, [1];

(b) comparability:
- $H_0[H_1, H_2, ..., H_n]$ is a comparability graph if and only if each H_i, $0 \leq i \leq n$, is a comparability graph, [9];

(c) permutation graphs and co-graphs, which will be discussed in Section 2.

Sometimes, in spite of the fact that all the factors enjoy a common hereditary property, this is not sufficient to endow their composition graph with the same feature. In other words, H_0 and $H_1, H_2, ..., H_n$ have to be equipped with some additional structure. In Section 3 we investigate additional structures of factors induced by θ_1-perfectness and thresholdness.

A number of other hereditary properties of graphs are discussed in [2], [3], [5], [12]. For an excellent survey on this subject we refer a reader to [4].

2. COMPOSITION OF PERMUTATION GRAPHS AND CO-GRAPHS

Let π be a permutation of the numbers $1, 2, ..., n$, and $G[\pi] = (V, E)$ be the graph defined as follows:

$$V = \{1, 2, ..., n\} \text{ and } ij \in E \Leftrightarrow (i - j)(\pi_i^{-1} - \pi_j^{-1}) < 0.$$

A graph G is called a *permutation graph* if there is a permutation π such that G is isomorphic to $G[\pi]$.

Theorem 2.1. (Pnueli, Lempel and Even, [15]) *A graph G is a permutation graph if and only if G and its complement $\overline{G}$ are comparability graphs.*

Remark. If $H = H_0[H_1, H_2, ..., H_n]$, then $\overline{H} = \overline{H}_0[\overline{H}_1, \overline{H}_2, ..., \overline{H}_n]$.

Now, taking into account this simple remark, the Theorem 2.1 and the above mentioned result for comparability graphs, we obtain:

Proposition 2.2. $H_0[H_1, H_2, ..., H_n]$ *is a permutation graph if and only if each H_i, $0 \le i \le n$, is a permutation graph.*

Proof. $H = H_0[H_1, H_2, ..., H_n]$ is a permutation graph $\Leftrightarrow$ H and its complement are comparability graphs $\Leftrightarrow$ each H_i, $0 \le i \le n$, and its complement are comparability graphs $\Leftrightarrow$ each factor H_i, $0 \le i \le n$, is a permutation graph. ■

A graph is called a *co-graph* if it contains no induced subgraph isomorphic to P_4.

Proposition 2.3. *The graph $H_0[H_1, H_2, ..., H_n]$ is a co-graph if and only if each factor H_i, $0 \le i \le n$, is a co-graph.*

Proof. "*if*"-part is clear.

"*only if*". On the contrary, suppose that there exists a P_4 in $H = H_0[H_1, H_2, ..., H_n]$, spanned by the vertices a, b, c, d and having the edges ab, bc, cd. Since none of the factors contains such a subgraph, we may have:

Case I. If there is some i, $1 \leq i \leq n$, such that $a, b \in V(H_i)$, or $b, c \in V(H_i)$, or $c, d \in V(H_i)$, then the subgraph induced by $\{a, b, c, d\}$ in H contains a triangle. (See Figure 1).

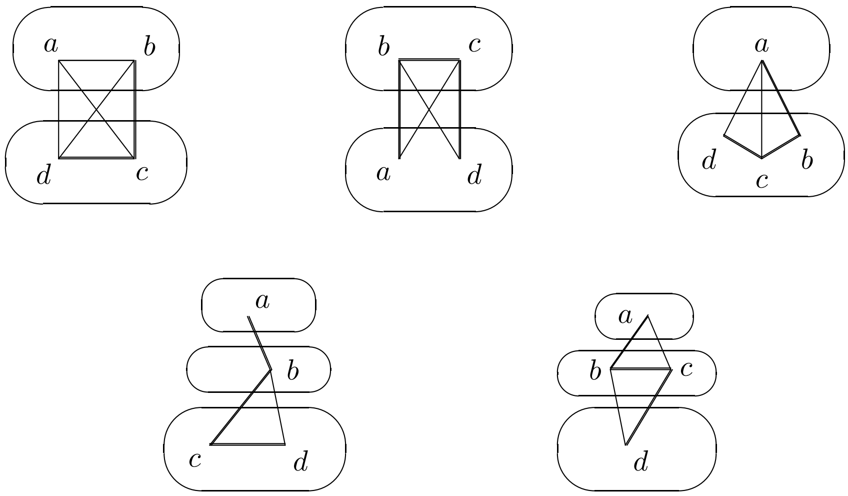

Figure 1

Case II. $a, c \in V(H_i)$ and $b, d \in V(Hj)$; then also $ad \in E(H)$. (See Figure 2.a.)

Case III. $v_iv_j, v_jv_k \in E(H_0)$, $b \in V(H_i)$, $a, c \in V(H_j)$ and $d \in V(H_k)$; then $ad \in E(H)$. (See Figure 2.b.)

Case IV. $v_iv_j, v_jv_k, v_iv_k \in E(H_0)$, and $a, d \in V(H_i)$, $b \in V(H_j)$, $c \in V(H_k)$; then we get that also $ac, bd \in E(H)$. (See Figure 2.c.)

In fact, as we see in the above Figures 1,2, and according to the definition of graph composition, the subgraph spanned by $\{a, b, c, d\}$ is not a P_4, in contradiction with the assumption on these vertices. Consequently, H is also a co-graph. ∎

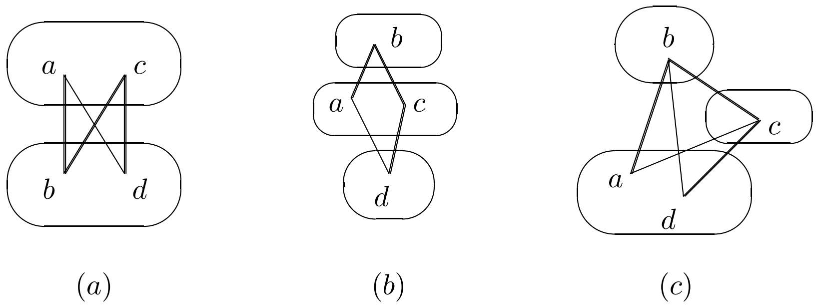

Figure 2

3. Composition of θ_1-Perfect Graphs and of Threshold Graphs

The *stability number* of graph G is the cardinality of a stable set of maximum size of G. If for each $A \subseteq V(G)$, the stability number of the subgraph H induced by A equals:

(a) the minimal number of cliques of H which cover all the edges of H, then G is called θ_1-*perfect* (Parthasarathy, Choudum, Ravindra, [13]);

(b) the number of maximal cliques of H, then G is said to be *trivially perfect* (Golumbic, [8]).

The following theorem emphasizes the connection between these two classes of perfect graphs.

Theorem 3.1. *For a graph G, the following statements are equivalent:*

(i) *G is θ_1-perfect;*

(ii) *G is trivially perfect;*

(iii) *G is both P_4-free and C_4-free.*

Proof. (i) ⇔ (iii) (Parthasarathy, Choudum, Ravindra, [13]);
(ii) ⇔ (iii) (Golumbic, [8]). ■

In order to investigate θ_1-perfectness of composition graphs, we start with the following lemma.

Lemma 3.2. *If H_1 and H_2 are connected graphs, then $H = K_2[H_1, H_2]$ contains C_4 as an induced subgraph if and only if either*

(i) *C_4 is an induced subgraph of H_1 and / or H_2, or*

(ii) *none of H_1, H_2 is a complete graph.*

Proof. "*if*". Suppose H has a C_4 as an induced subgraph, but none of H_1, H_2 contains some C_4. If both H_1, H_2 are complete, then H itself is a complete graph and C_4-free, which is in contradiction with the above assumption. Assume that only H_1 is a complete graph. Let C_4 of H be spanned by $\{a, b, c, d\}$, with the edges ab, bc, cd, da. Then the possible cases are:

Case 1. $a \in V(H_1)$ and $b, c, d \in V(H_2)$, or $a, c \in V(H_1)$ and $b, d \in V(H_2)$; then also $ac \in E(H)$.

Case 2. $a, b \in V(H_1)$ and $c, d \in V(H_2)$, or $a, b, c \in V(H_1)$ and $d \in V(H_2)$; then $ac, bd \in E(H)$.

So, the completeness of H_1 or the definition of the graph composition implies the existence of at least a chord in the subgraph of H, induced by $\{a, b, c, d\}$, thus contradicting the assumption on these vertices. Therefore, none of H_1, H_2 is a complete graph.

"*only if*". If a, c and b, d are non-adjacent vertices in H_1, H_2 respectively, then it is easy to see that $\{a, b, c, d\}$ spans a C_4 in H, with the edges ab, bc, cd, da. ■

Corollary 3.3. *If H_1, H_2 are connected graphs, then $K_2[H_1, H_2]$ has no C_4 as an induced subgraph if and only if one of H_1, H_2 is complete and the other is C_4-free.*

Lemma 3.4. *If H_1, H_2, H_3 are connected, then $H = P_3[H_1, H_2, H_3]$ is a θ_1-perfect graph if and only if H_2 is a complete graph and H_1, H_3 are θ_1-perfect.*

Proof. "*if*". Clearly, if H is θ_1-perfect, then all H_i, $i = 1, 2, 3$, are θ_1-perfect. Assume that H_2 contains two non-adjacent vertices a_2, b_2; then for $a_1 \in V(H_1)$ and $a_3 \in V(H_3)$, the set $\{a_1, a_2, b_2, a_3\}$ spans a C_4 in H, which is at variance with the θ_1-perfectness of H. Therefore, H_2 must be a complete graph.

"*only if*". By Proposition 2.3, H is a co-graph, since P_3, H_1, H_2, H_3 are, in particular, co-graphs. Suppose now that H contains a C_4 as an induced subgraph. According to Corollary 3.3, it follows $V(C_4) \cap V(H_i) \neq \emptyset$, $i = 1, 2, 3$, and clearly $|V(C_4) \cap V(H_2)| = 2$. Since H_2 is a complete graph, we infer that, actually, the vertices from $V(C_4)$ span a "diamond" in H, (i.e., K_4 without an edge), in contradiction with the assumption on these vertices. So, H is both P_4-free and C_4-free, i.e., H is θ_1-perfect, by Theorem 3.1. ■

For a graph H let us denote:

$EndP_3(H) = \{v : v \in V(H)$ and v is an endpoint of a P_3 in $H\}$,

$MidP_3(H) = \{v : v \in V(H)$ and v is the midpoint of a P_3 in $H\}$.

Proposition 3.5. *Let $\{H_i,\ 0 \le i \le n\}$ be a family of connected and disjoint graphs; then $H = H_0[H_1, H_2, ..., H_n]$ is θ_1-perfect if and only if the following conditions hold:*

(i) *all H_i, $0 \le i \le n$, are θ_1-perfect;*

(ii) *if $v_iv_j \in E(H_0)$, then at least one of H_i, H_j is complete;*

(iii) *if $v_i \in MidP_3(H_0)$, then the corresponding factor H_i is a complete graph.*

Proof. "*if*". If H is θ_1-perfect, then evidently, all H_i, $0 \le i \le n$, are also θ_1-perfect and, by Corollary 3.3 and Lemma 3.4, (ii) and (iii) are clearly fulfilled.

"*only if*". Since all H_i, $0 \le i \le n$, are also co-graphs, by Proposition 2.3 we get that H is a co-graph, too. Suppose that the vertices a, b, c, d span a C_4 in H. Because all the factors C_4-free, using again Corollary 3.3 and Lemma 3.4, we infer that $|V(C_4) \cap V(H_i)| \le 1$, i.e., H_0 contains a C_4, which is contradictory to the fact that H is a co-graph (i.e., P_4-free). Therefore, H is also C_4 -free and, consequently, is θ_1-perfect, by Theorem 3.1. ■

Corollary 3.6. (i) *Let $\{H_i,\ 1 \le i \le n\}$ be a family of connected and disjoint graphs; then $*[H_1, H_2, ..., H_n]$ is θ_1-perfect if and only if all H_i, $1 \le i \le n$, are θ_1-perfect and at least $n-1$ of them are complete graphs.*

(ii) *If H_0, H_1 are connected, then $H_0[H_1]$ is θ_1-perfect if and only if H_0 is θ_1-perfect and H_1 is a complete graph.*

A 4-*graph* is a graph with 4 vertices that can be labeled a, b, c, d such that a is adjacent to b but not to c, and d is adjacent to c but not to b (i.e., either a P_4 or a C_4, or a $2K_2$ graph) (Peled, [14]).

A graph $G = (V, E)$ is *threshold* if there is a labeling a of its vertices by non-negative integers and an integer t such that:

$$X \text{ is stable} \Leftrightarrow \sum_{x \in X} a(x) \le t,\ (X \subseteq V).$$

These graphs were defined by Chvátal and Hammer in [6], and extensively studied in the work [11] of Mahadev and Peled. Further we make use of the following characterization of threshold graphs in terms of forbidden induced subgraphs.

Theorem 3.7. (Chvátal and Hammer, [6]) *A graph is threshold if and only if it has no induced subgraph isomorphic to a 4-graph.*

Lemma 3.8. *If H_1, H_2, H_3 are connected graphs, then $P_3[H_1, H_2, H_3]$ is threshold if and only if the next two conditions hold:*
(i) *H_2 is complete,*
(ii) *one of H_1, H_3 is a K_1 graph and the other is a threshold graph.*

Proof. "*if*". If H is a threshold graph, then all H_i, $i = 1, 2, 3$, are also threshold and by Lemma 3.4, we get that H_2 is a complete graph. In addition, only one of the graphs H_1, H_3 may contain K_2 as an induced subgraph (since H is $2K_2$-free), and this ensures that one of H_1, H_3 is a K_1 graph.

"*only if*". According to Lemma 3.4, H must be θ_1-perfect, because P_3, H_1, H_2, H_3 are, in particular, θ_1-perfect and H_2 is complete. In addition, since:
– H_1 is a K_1 graph, H_2 is a complete graph, and H_3 is $2K_2$-free,
– each vertex of H_2 is adjacent to any vertex of both H_1 and H_3,
we infer that H cannot contain a $2K_2$ as an induced subgraph.

Therefore, H is θ_1-perfect and $2K_2$-free. Consequently, by Theorem 3.7, we may conclude that H is a threshold graph. ■

Let us denote by N_3 the 3-*pan* or *paw* graph, i.e., the graph with
$V(N_3) = \{v_1, v_2, v_3, v_4\}$ and $E(N_3) = \{v_1v_2, v_2v_3, v_3v_4, v_2v_4\}$.

Lemma 3.9. *If H_i, $1 \leq i \leq 4$, are connected graphs, then $N_3[H_1, H_2, H_3, H_4]$ is threshold if and only if the following assertions hold:*
(a) *H_1 is a K_1 graph;*
(b) *H_2 is complete;*
(c) *one of H_3, H_4 is a complete graph, and the other is a threshold graph.*

Proof. "*if*". Since v_1, v_2, v_3 and v_1, v_2, v_4 span two P_3's in the outer factor N_3, with the vertices v_1, v_3 and v_1, v_4 as endpoints, respectively, we infer, according to Lemma 3.8, that H_2 must be a complete graph and either:

Case I. H_1 is threshold (with at least an edge, say a_1b_1) and H_3, H_4 are K_1 graphs contrary to thresholdness of H, because if $V(H_i) = \{a_i\}$, $i = 3, 4$, then $\{a_1, b_1, a_3, a_4\}$ spans a $2K_2$ in H; or

Case II. H_1 is a K_1 graph and H_3, H_4 are threshold, but by Corollary 3.3, one of them must be a complete graph.

"*only if*". Suppose that $V(H_1) = \{a_1\}$, H_2 and H_3 are complete graphs, while H_4 is a threshold graph. By Proposition 3.5, H is θ_1-perfect. In addition, since:

$$\{a_1\} \sim V(H_2) \sim V(H_3) \sim V(H_4) \sim V(H_2)$$

no $2K_2$ is contained in H, i.e., H is a threshold graph, according to Theorem 3.7. ■

Lemma 3.10. *Let $H_0, H_1, H_2, ..., H_n$ be a family of $n > 1$ disjoint and connected graphs. If $H = H_0[H_1, H_2, ..., H_n]$ is a threshold graph, then all H_i, $0 \leq i \leq n$, are threshold, and at least $n - 1$ of H_i, $1 \leq i \leq n$, are complete graphs.*

Proof. All H_i, $0 \leq i \leq n$, must be threshold, as being isomorphic to some subgraphs of H. If H_0 is complete, then Corollary 3.3 implies that at least $n - 1$ of the inner factors must be also complete. If H_0 is not complete, suppose, on the contrary, that there are two non-complete threshold graphs H_i, H_k as inner factors. Since, by Corollary 3.3, v_i, v_k cannot be adjacent and, on the other hand, H_0 is connected and also P_4-free, there must exist some vertex v_j in H_0 such that $\{v_i, v_j, v_k\}$ spans a P_3 in H_0. By Lemma 3.8, one of H_i, H_k must be K_1, in contradiction with the choice of H_i, H_k. ■

Graph G is called a *split graph* (Foldes and Hammer, [7]) if there exists a partition $V(G) = K \cup S$ of its vertex set into a clique K and a stable set S. From the work of Golumbic [9, Chapter 6, Theorem 6.2] it follows that K may always be chosen maximum. Foldes and Hammer [7] proved that being a split graph is equivalent to containing no induced subgraph isomorphic to $2K_2$, C_4 or C_5. Therefore, according to Theorem 3.7, any threshold graph is a split graph.

For a graph G let us denote:
$EndPan(G) = \{v : v \in V(G),\ v$ is the pendant vertex of an induced N_3 in $G\}$.

Lemma 3.11. *If G is a connected non-complete split graph, then:*

(i) *$V(G) = EndP_3(G) \cup MidP_3(G)$;*

(ii) *$MidP_3(G)$ spans a clique in G and $EndPan(G) \subseteq EndP_3(G) - MidP_3(G)$;*

(iii) *the vertex set of G can be decomposed into pairwise disjoint subsets read as $V(G) = MidP_3(G) \cup EndPan(G) \cup (EndP_3(G) - (MidP_3(G) \cup EndPan(G)))$.*

Proof. G is a split graph. Hence, there exists a partition of $V(G)$ as $V(G) = K \cup S$, where K is a maximum clique and S is a stable set of G. Since G is also a connected non-complete graph, $EndP_3(G)$ and $MidP_3(G)$ are non-empty sets.

(i) If $v \in S$, then there exist $u, w \in K$, such that $uv \in E(G)$ and $vw \notin E(G)$, because G is connected and K is a maximum clique. Hence, we get that $v \in EndP_3(G)$. If $v \in K$ and $N(v) \cap S = \emptyset$, then for $w \in S$ and $u \in N(w)$, we obtain that $u, v \in K$, i.e., $v \in EndP_3(G)$, because $\{v, u, w\}$ spans a P_3. If $v \in K$ and there is some $w \in N(v) \cap S$, then for $u \in K - N(w)$, (such u exists, because K is a maximum clique), we get that v is the midpoint of the P_3 spanned by $\{w, v, u\}$, i.e., $v \in MidP_3(G)$. Hence, $V(G) = EndP_3(G) \cup MidP_3(G)$, but this cover is not necessarily a partition for $V(G)$ (see, for example, graph G in Figure 3).

(ii) If $x \in MidP_3(G)$, then there are $y, z \in V(G)$, such that $\{y, x, z\}$ spans a P_3, with x as its midpoint. Hence, $yz \notin E(G)$ and necessarily $x \in K$. So, we get that $MidP_3(G) \subseteq K$, i.e., $MidP_3(G)$ spans a clique in G.

On the contrary, suppose that there exists some $x \in EndPan(G) \cap MidP_3(G)$. Then also $x \in K$ and there are $a, b, c \in V(G)$, such that $\{x, a, b, c\}$ spans a N_3 in G, with x as its pendant vertex and $xa \in E(G)$. If $a \in K$, then at least one of b, c, say b, is contained in K and hence $xb \in E(G)$, contradicting the fact that $\{x, a, b, c\}$ spans a N_3 in G. If $a \notin K$, then $b, c \in K$, and we get the same contradiction.

(iii) It follows from (i) and (ii). ■

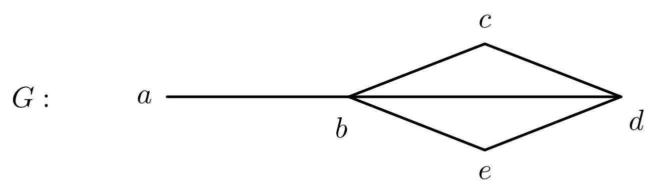

Figure 3. $EndP_3(G) = \{a, c, d, e\}, MidP_3(G) = \{b, d\}, EndPan(G) = \{a\}$

Theorem 3.12. *Let $H_0, H_1, H_2, ..., H_n$ be a family of $n > 1$ disjoint and connected graphs. Then $H = H_0[H_1, H_2, ..., H_n]$ is a threshold graph if and only if one of the two following conditions holds:*

(a) *H_0 is complete, one of H_i, $1 \le i \le n$, may be any threshold graph, while the others must be complete graphs;*

(b) *H_0 is a non-complete threshold graph, and:*

for any $v_j \in EndPan(H_0)$, the corresponding graph H_j is K_1;
for any $v_j \in MidP_3(H_0)$, the corresponding graph H_j is complete;
for any $v_i \in EndP_3(H_0) - (MidP_3(H_0) \cup EndPan(H_0))$, the corresponding graph H_i is K_1, except one, which may be any threshold graph.

Proof. "*if*". If H is a threshold graph, then all its factors, both outer and inner, are also threshold graphs.

Case I. H_0 is a complete graph. Then, according to Lemma 3.10, one of the inner factors may be any threshold graph, but the others must be complete graphs. Thus, the assertion (a) is true.

Case II. H_0 is not a complete graph. By Lemma 3.11 (iii), $V(H_0)$ can be decomposed as follows:

$$\begin{aligned} V(H_0) &= MidP_3(H_0) \cup EndPan(H_0) \cup (EndP_3(H_0) \\ &- (MidP_3(H_0) \cup EndPan(H_0))). \end{aligned}$$

According to Lemma 3.8, for any $v_j \in MidP_3(H_0)$, the corresponding graph H_j must be complete, and by Lemma 3.9, H_j is K_1, for every $v_j \in EndPan(H_0)$. Further, Lemmas 3.8 and 3.10 imply that at most one of the inner factors, corresponding to the vertices in $EndP_3(H_0) - (MidP_3(H_0) \cup EndPan(H_0))$, may be any threshold graph, while the others must be K_1.

"*only if*". Clearly, the conditions (a) imply that H is 4-graph-free, i.e., by Theorem 3.7, H is a threshold graph.

Suppose that the (b)-conditions are fulfilled.

Firstly, H has no $2K_2$ as an induced subgraph. Assuming, on the contrary, that such a subgraph exists in H, we distinguish the three following cases:

Case 1. $2K_2$ is spanned by the edges a_ib_i, a_jb_j from H_i, H_j, respectively. Now, if:

– $v_iv_j \in E(H_0)$, then, $\{a_i, b_i, a_j, b_j\}$ spans a K_4 in H instead of $2K_2$, which brings a contradiction to our assumption;

– v_i, v_j are not adjacent in H_0, then there exists a vertex v_k in H_0, such that the vertices v_i, v_k, v_j span a P_3 in H_0, (since H_0 is a connected and P_4-free graph). Hence, by Lemma 3.11 (ii), $\{v_i, v_k, v_j\} \not\subset MidP_3(H_0)$ and therefore at least one of H_i, H_j must be K_1, contradicting the fact that $E(H_i)$, $E(H_i)$ are non-empty sets.

Case 2. $2K_2$ is spanned by $a_ib_i \in E(H_i)$ and the edge a_ja_k, where $a_j \in V(H_j)$, $a_k \in V(H_k)$, and $v_jv_k(E(H_0)$.
Now, if:

– $v_iv_j \in E(H_0)$, (or $v_iv_k \in E(H_0)$), then $a_ia_j \in E(H)$, ($a_ia_k \in E(H)$, respectively), in contradiction with the assumption that $\{a_i, b_i, a_j, a_k\}$ spans a $2K_2$;
– v_i is adjacent to none of v_j, v_k; then there exists a vertex v_p in $V(H_0)$, such that v_i, v_p, v_j, v_k span a 3-pan in H_0 with v_i as its pendant vertex. Henceforth, by the (b)-conditions, we infer that H_i must be K_1, in contradiction with $E(H_i) \neq \emptyset$.

Case 3. $2K_2$ is spanned by the edges a_ib_j, a_ka_p, with i, j, k, p distinct. This yields the following contradiction: H_0 is threshold, but contains a $2K_2$, spanned by $\{v_i, v_j, v_k, v_p\}$.

Secondly, by Proposition 3.5, H is also θ_1-perfect. So, according to Theorem 3.7, we may conclude that H is a threshold graph. ■

4. Conclusions

In this paper we present necessary and sufficient conditions for the composition graph $H = H_0[H_1, H_2, ..., H_n]$ of a family of graphs $\{H_i : 0 \leq i \leq n\}$ to have a certain hereditary property P, like being a permutation graph, a co-graph, a θ_1-perfect graph and a threshold graph. It seems to be interesting to answer the inverse question: if a graph H possesses a hereditary property P, how can it be represented as the composition graph of a family of graphs enjoying the same property?

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References

[1] B. Bollobás, Extremal graph theory (Academic Press, London, 1978).

[2] B. Bollobás and A.G. Thomason, *Hereditary and monotone properties of graphs*, in: R.L. Graham and J. Nešetřil, eds., The Mathematics of Paul Erdös, II, Algorithms and Combinatorics 14 (Springer-Verlag, 1997) 70–78.

[3] M. Borowiecki and P. Mihók, *Hereditary properties of graphs*, in: V.R. Kulli ed., Advances in Graph Theory (Vishwa Intern. Publication, Gulbarga,1991) 41–68.

[4] M. Borowiecki, I. Broere, M. Frick, P. Mihók, G. Semanišin, *A Survey of Hereditary Properties of Graphs*, Discussiones Mathematicae Graph Theory **17** (1997) 5–50.

[5] P. Borowiecki and J. Ivančo, *P-bipartitions of minor hereditary properties*, Discussiones Mathematicae Graph Theory **17** (1997) 89–93.

[6] V. Chvátal and P.L. Hammer, Set-packing and threshold graphs, Res. Report CORR 73–21, University Waterloo, 1973.

[7] S. Foldes and P.L. Hammer, *Split graphs*, in: F. Hoffman et al., eds., Proc. 8th Conf. on Combinatorics, Graph Theory and Computing (Louisiana State Univ., Baton Rouge, Louisiana, 1977) 311–315.

[8] M.C. Golumbic, *Trivially perfect graphs*, Discrete Math. **24** (1978) 105–107.

[9] M.C. Golumbic, Algorithmic graph theory and perfect graphs (Academic Press, London, 1980).

[10] J.L. Jolivet, *Sur le joint d' une famille de graphes*, Discrete Math. **5** (1973) 145–158.

[11] N.V.R. Mahadev and U.N. Peled, Threshold graphs and related topics (North-Holland, Amsterdam, 1995).

[12] E. Mandrescu, *Triangulated graph products*, Anal. Univ. Galatzi (1991) 37–44.

[13] K.R. Parthasarathy, S.A. Choudum and G. Ravindra, *Line-clique cover number of a graph*, Proc. Indian Nat. Sci. Acad., Part A **41** (3) (1975) 281–293.

[14] U.N. Peled, *Matroidal graphs*, Discrete Math. **20** (1977) 263–286.

[15] A. Pnueli, A. Lempel and S. Even, *Transitive orientation of graphs and identification of permutation graphs*, Canad. J. Math. **23** (1971) 160–175.

[16] G. Ravindra and K.R. Parthasarathy, *Perfect Product Graphs*, Discrete Math. **20** (1977) 177–186.

[17] G. Sabidussi, *The composition of graphs*, Duke Math. J. **26** (1959) 693–698.

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