Discussiones Mathematicae
Graph Theory 18 (1998) 147–158

ON GENERATING SNARKS

Busiso P. Chisala

Department of Mathematical Sciences, Chancellor College
P.O. Box 280, Zomba, Malawi

Abstract

We discuss the construction of snarks (that is, cyclically 4-edge connected cubic graphs of girth at least five which are not 3-edge colourable) by using what we call colourable snark units and a welding process.

Keywords: snarks, cubic graphs, sirth, edge colouring.

1991 Mathematics Subject Classification: 05C15.

1. Introduction

We define a *pendant* to be a connected graph all of whose vertices have degree either 3 or 1. By a process we call *welding*, two such pendants will often produce a snark. In the reverse direction, *excising* a pendant from a snark produces pendants called *snark units*. We characterise snark units that can be 3-edge coloured and use these to generate other snarks, and close by showing that snarks exist with the property that (1) they have a 1-factor whose corresponding 2-factor has precisely two odd cycles, and (2) for all such 1-factors, the distance between the two odd cycles can be arbitrarily large.

We shall consider the problem of 3-edge colouring of a cubic graph G in terms of a *nowhere zero flow* in the additive 2-group $\Gamma = \mathbf{Z}_2 \times \mathbf{Z}_2$ (see Jaeger's article [3] for further details). We will denote the three nonzero elements of this group by x, y and z, and say that a cubic graph *colourable* when it is 3-edge colourable, and *uncolourable* otherwise. Unless otherwise stated, all graphs will be cubic.

An *edge cut* of G is the set of edges between some proper subset U of $V(G)$ and its complement $\overline{U}$. The edge cut is *cyclic* if both the induced subgraphs $G[U]$ and $G[\overline{U}]$ contain cycles. We shall say that G is *k-edge* *connected* (*cyclically k-edge connected*) if each (cyclic) edge cut of G has order at least k. A *snark* is a cyclically 4-edge connected cubic graph of girth at least five which is not 3-edge colourable. Generally, we will follow the terminology of Bondy and Murty's text [1].

2. Snark Units

Following Isaacs' paper on snarks [4], we shall call a graph with all vertices of degree 1 or 3 a *pendant*, and more precisely, a k-pendant if it has k vertices of degree one. We will also refer to the edges on the vertices of degree one as *free* or *pendant* edges.

To *sever* an edge uv of a graph G is to introduce an new vertex w on this edge and replace it by the two edges uw, wv. Severing all the edges of a k-edge cut produces two k-pendants G_1 and G_2. We shall say that G_2 is the result of *excising* G_1 from G and that G_1 is the *complement* of G_2. To *weld* together two k-pendants with respect to a pairing of their free edges reverses this construction.

A rather special case is the following two 5-pendants from the Petersen graph:

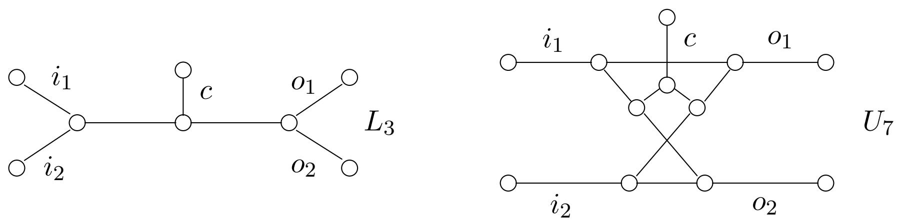


Figure 1

We shall call the first pendant, which is simply a tree, L_3, and the second, is its complement U_7.

The edges of the 5-pendant U_7 naturally fall into three sets: the pair on the left, that on the right and the single free edge in the middle. We label these $I = \{i_1, i_2\}$, $O = \{o_1, o_2\}$ and $\{c\}$. More generally, given any cubic graph G, if $e_1 = uv$, $e_2 = vw \in E(G)$ are incident edges, then e_1, e_2, the other two edges at u and w and the third edge at v determines a subgraph isomorphic to L_3. Excision of this pendant yields a 5-pendant H whose free edges we label i_1, i_2, c, o_1, o_2 as above.

Proposition 1. *Let H be a 5-pendant produced by excising an L_3 from an uncolourable cubic graph G. Suppose that*

(i) *H is connected, and*

(ii) *H admits a colouring $\gamma : E(H) \to \Gamma$,*

then setting $\gamma(I) = \gamma(i_1) + \gamma(i_2)$ and $\gamma(O) = \gamma(o_1) + \gamma(o_2)$, we have

$$\{\, \gamma(I),\, \gamma(O) \,\} = \{\, 0,\, \gamma(c) \,\}.$$

In particular, one of $\gamma(I), \gamma(O)$ is zero, and the other agrees with the colour at c.

Proof. Using the same labels i_1, i_2, c, o_1, o_2 on the free edges of L_3, we see that for any colouring σ on L_3, if $\sigma(c) = x$, then $\{\sigma(I),\, \sigma(O)\,\} = \{y,\, z\}$. Since G is not colourable, any colouring on the L_3 is incompatible with the possible colourings on its complement H. So if a colouring γ on H satisfies $\gamma(c) = x$, then the only possible non-zero value for $\gamma(I)$ and $\gamma(O)$ is $x = \gamma(c)$. Since H is connected, $\gamma(I) + \gamma(O) = \gamma(c)$ and the result follows. ■

Of course, $\gamma(I) = 0$ simply means that the colours on the two i-edges are the same. We shall call any connected, colourable 5-pendant together with a labelling of its free edges i_1, i_2, c, o_1, o_2 which has the property that for any colouring, either $\gamma(I) = 0$ or $\gamma(O) = 0$, a *colourable snark unit*, or CSU. It is not apparent (but true), that there are complements of L_3 subgraphs of uncolourable graphs that remain uncolourable, we will give an example of this is Section 3 below.

It is readily checked that U_7 is colourable. From its symmetry, there is a colouring in which $\gamma(I)$ is zero, and one where it is $\gamma(c)$. We will call a CSU *symmetrical* if it shares this property with U_7, and give an example in Section 5 of a non-symmetrical CSU.

Welding an L_3 (or indeed any 5-pendant that admits the same colouring property) onto a CSU by identifying correspondingly labelled edges yields an uncolourable cubic graph. From U_7 this can only be the Petersen graph, so not only is U_7 the unique complement of any L_3 in the Petersen graph, it is also the CSU of least order.

If (G, M) is a cubic graph together with a matching (or 1-factor), the associated 2-factor is a disjoint union of cycles. Since $|V(G)|$ is even, there is an even number of cycles of odd length among these. We will say that G has a *Tait number* n if the smallest number of cycles of odd length in any 2-factor of G is $2n$, and write $\tau(G) = n$. In particular, $\tau(G) = 0$

iff G admits a colouring. A *minimal matching* is one that yields $2\tau(G)$ odd cycles. Referring to the description of L_3 above for notation, we have the following characterisation.

Proposition 2. *The complement H of an L_3 in an uncolourable graph G is a CSU if and only if $\tau(G) = 1$ and G has a matching in which the two odd cycles of the associated 2-factor are bridged by one of the edges e_1, e_2 of the L_3.*

Proof. Let γ be a colouring on H in which $\gamma(c) = x$, and $\gamma(I) = 0$. Up to a renaming of the o-edges, the only possible assignment is:

$$\gamma(c) = x; \qquad (\,\gamma(i_1),\, \gamma(i_2)\,) = (\alpha, \alpha); \qquad (\,\gamma(o_1),\, \gamma(o_2)\,) = (y, z),$$

for some colour α.

We claim that we can further assume that $\alpha = x$: if, say $\alpha = y$, then consider the alternating y-x path beginning at i_1. It terminates at one of c, o_1 or i_2. It cannot end at c, since switching colours along such a path would give colouring $\gamma^*(c) = y$; $(\gamma^*(i_1), \gamma^*(i_2)) = (x, y)$ and $(\gamma^*(o_1), \gamma^*(o_2)) = (y, z)$, which is inadmissible for a CSU. This path also cannot end at o_1 for the same reason, so it terminates at the other input i_2. Switching colours gives the desired result. Clearly, the same argument holds if $\alpha = z$.

We also conclude that the alternating x-y path (resp. x-z) path from c ends at o_1 (resp. o_2). With the assumptions above, we see that in H, there are precisely two x-y paths that are not cycles of even length: that between c and o_1, and the one between the edges i_1 and i_2.

Welding the L_3 and considering the matching M in H induced by the edges coloured z, we see that $M^* = M \cup \{PQ\}$ is a 1-factor in G, as illustrated below by the bold edges:

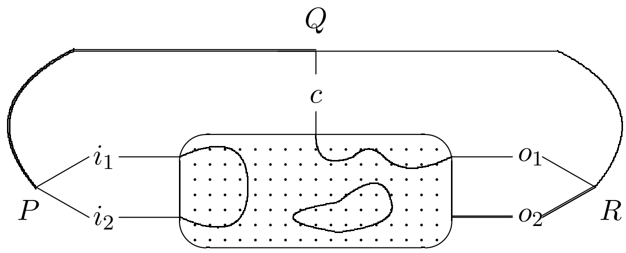


Figure 2

The cycle on i_1, i_2 has odd length, as does the cycle on c and o_1, and these are the **only** odd cycles in the 2-factor given by this matching. Further they are bridged by matching edge PQ.

In the reverse direction, suppose that $\tau(G) = 1$, and that for a minimal matching, the two odd cycles are bridged by an edge. This defines an L_3 by the choice of two edges from one of the cycles, and three from the second, as in the figure above. ∎

3. Linear Constructions

If G is a snark, then the girth condition implies that the distance (in H) between i-edges and that between the c-edge and the o-edges must be at least 5. On the other hand, it is also clear that a minimal cyclic edge cut of the weld of a CSU H and L_3 uses at most one (interior) edge from L_3, so this weld is cyclically 4-edge connected provided that the H is cyclically 3-edge connected.

We shall term a CSU with these two properties *proper*. Evidently, welding the L_3 on a proper CSU produces a snark. By the *order* of a CSU, we shall mean the number of its vertices of degree 3. Thus, U_7 has order 7.

In view of Proposition 1, we obtain CSU's of any order k provided there is a snark of order $k + 3$ with $\tau = 1$ which becomes colourable on removing a single edge. From [5] we have that there are no snarks of orders 12, 14 or 16, so the next order of a proper CSU after U_7 is 15, which can be obtained from either of the two snarks of order 18.

In this Section, we show how to produce (proper) CSU's of any odd order ≥ 15 starting from U_7. The following figure shows a CSU of order 15 created by welding two U_7's, introducing a new vertex on one of the interfacing edges and attaching a free edge c there:

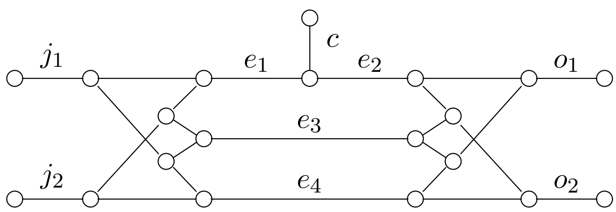


Figure 3

The c-edges of the two U_7's become edge e_3. Suppose that $\gamma(c) = x$. Then edges $\{\gamma(e_1), \gamma(e_2)\} = \{y, z\}$. From the possible colourings for a U_7's o- and i- edges, it follows that $\gamma(e_4)$ is either y or z and that $\gamma(e_3) = x$, so one of $\gamma(I)$, $\gamma(O)$ is zero.

From this CSU, we can produce another of order 17 by subdividing edges e_1, e_4 and introducing a new edge between these new vertices. A further subdivision of e_2 and one of the new edges replacing e_4 then gives a unit of order 19.

An order 21 unit is made from three U_7's in sequence, with the first two welded at the c-edges.

The argument above shows that for any $n \geq 2$, we can produce a CSU of order $7n + (n-1) = 8n - 1$ by using a sequence of n copies of U_7 linked as illustrated below for $n = 4$.

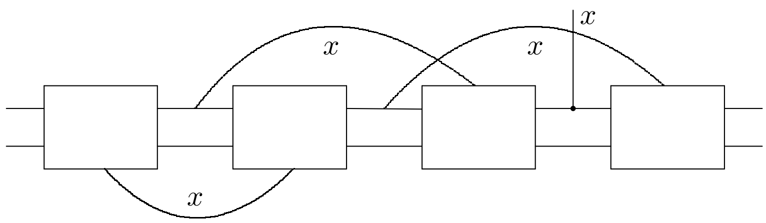


Figure 4

The edges marked x can be successively shown to carry this colour, starting from edge c.

The subdivision process permits a further $4(n-1) = 4n - 4$ vertex additions. For $n \geq 3$ we have $4n - 4 \geq 8$, so that we obtain CSU's of each odd order ≥ 15. It is clear that all these pendants are cyclically 3-edge-connected and by the constructions, satisfy the distance condition, hence they are all proper CSU's. Welding L_3's on them gives an alternative construction to Isaacs' snarks of all even orders ≥ 18 with $\tau = 1$.

4. CYCLIC CONSTRUCTIONS

Given two CSU's H_1, H_2, let the only admissible weldings be

(1) the o-edges of H_1 to the i-edges of H_2 and

(2) the c-edges of the two.

Starting with n (possibly different) CSU's $H_1, \ldots, H_n$, let the graph G be the result of effecting a set of admissible welds between the H_i's.

Let G^* be the graph whose vertices consist of

(1) a set $\{g_1, \ldots, g_n\}$ in one-to-one correspondence with the units G_i;

(2) one vertex f_j for each end vertex of a free edge c_j of a unit G_j that is not welded to any other edge;

(3) one vertex d_j for each *pair* of *o*- or *i*- edges of a unit G_j that is not welded to any other pair from another unit.

The edge set of G^* is given by: $g_i g_j \in E(G^*)$ iff the pendant G_i is welded to the pendant G_j in G. For all vertices g_j, f_j and d_j, we also demand that $f_j g_j,\ d_j g_j \in E(G^*)$. Evidently, G^* is a k-pendant for some $k \geq 0$.

We also equip G^* with a *weight function* $w : E(G^*) \to \{1, 2\}$ defined by

$$\begin{aligned} w(g_i g_j) &= \text{the number of edges welded between } G_i, G_j; \\ w(f_j g_j) &= 1; \\ w(d_j g_j) &= 2 \quad \text{for each relevant index } j. \end{aligned}$$

In contrast to the weight functions in [2], this weight function has an odd sum at each vertex of degree 3.

If G has a colouring γ, there is an induced flow γ^* on G^* given by taking the sum of the colours on the edges in G that the edge in G^* arises from: this flow is zero on edges arising from pairs with the same colour.

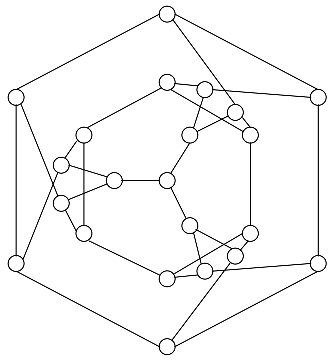
If we take n copies of U_7 and weld then together in a cycle, so that the edge o_1, o_2 of the k-th CSU are welded to the edges i_1, i_2 of the $(k+1)$-st unit, and the *o*-edges of the n-th unit are welded to the *i*-edges of the first, we obtain an n-pendant U_7^n with free edges $c_1, c_2, \ldots, c_n$. These are all admissible welds.

Proposition 3. *If $n \geq 3$ is odd, then U_7^n is uncolourable.*

Proof. Taking $G = U_7^n$, we see that G^* is an n-pendant consisting of a free edge at each vertex of an n-cycle. Its weight function takes the value 2 around the cycle, and 1 on the free edges. If G was coloured, then for each U_7, exactly one of its *o*- / *i*- pair of edges receives a colour sum 0. Thus $\gamma^*(e) = 0$ for a set of edges in G^* that forms a matching of the vertices in the cycle. Since this is only possible if the cycle has even length, G is uncolourable. ■

For example, welding an L_3 on U_7^5 gives an example of a snark where the excision of this L_3 does **not** produce a CSU.

In the construction above, if we identify the three end vertices of U_7^3 together, we obtain the following uncolourable graph G_3 on 22 vertices, for which $G_3^* = K_4$ (see Figure 5).



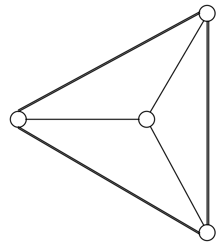


Figure 5

G_3 has girth 5, is cyclically 5-edge connected and hence is a snark.

We can turn this construction around. Starting with any cubic graph G and a (1,2)-weight function w such that the sum over the edges incident at any vertex is odd, the set of edges where $w(e) = 2$ forms a disjoint union of cycles Σ. Replacing each such cycle with a cycle of CSU's as above produces a graph H such that $H^* = G$. By Proposition 3, if at least one of the cycles in Σ has odd length, H is uncolourable.

As with Isaacs' J class, G_3 is the smallest in an infinite family $\{ G_{2k+1} \}$, with members produced from the prisms on $2k + 1$-cycles together with the weight function that is 2 only on one of the copies of this cycle. All are cyclically 5-edge connected. A more general method is to take an existing snark together with any 1-factor. This matching determines the edges assigned weight 2 from which a snark may be built.

5. Asymmetric CSU's

The constructions in the last section carry over if we replace the unit U_7 by any symmetric one. It is not hard to characterise symmetric units in terms of the possible alternating paths between the *i*- / *o*- edges. The following is an example of a unit that is not symmetric: the pair of edges that accepts a sum flow of zero is distinguished.

The first diagram in the following figure shows G_5, the snark on 40 vertices of the family mentioned in Section 3.

The bold subgraph in that diagram shows an L_3 whose complement is a CSU we will call U_{37}: the second diagram is of the associated colouring γ^* on the prism G_5^*, with the corresponding L_3 excised. The bold edge indicates the two input edges i_1, i_2. The edges labelled 0 are those where the sum flow on the corresponding U_7's i-/ o- pair of edges is zero. In essence, this is a zero flow at the 'double edge' given by the two inputs. It is easily checked that this colouring extends to U_{37}.

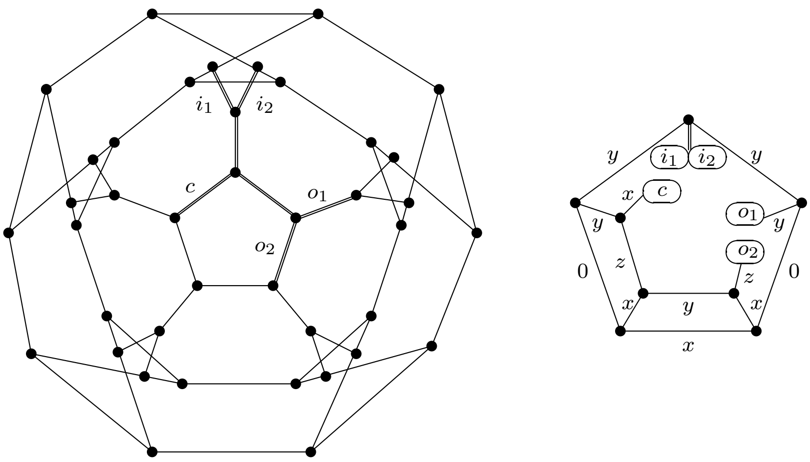


Figure 6

To show that U_{37} is not symmetrical, it suffices to note that setting $\gamma(i_1) = y$, $\gamma(i_2) = z$ (while fixing $\gamma(c) = x$) would permit an extension of γ to a colouring of the outer cycle U_7^5, which is not possible. So only $\gamma(I) = 0$ is possible.

6. Tait Number One

In this Section, we give constructions of snarks with $\tau = 1$ that have the property that in a minimal matching, the two cycles of odd length are not bridged by an edge. For $n \geq 2$, we denote by L_n the tree with the $2n + 2$ vertices in the set $\{v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n, z_1, z_n\}$. The edge set consists of $v_i w_i$, $i = 1, \ldots, n$ together with $v_1 z_1$, $v_n z_n$ and $v_i v_{i+1}$ for $1 \leq i \leq n - 1$. So L_n is an $(n + 2)$-pendant: the 5-pendant L_3 is then an example.

Let G be the cubic graph that is the weld of the 6-pendant H on the left and the L_4 beside it in the following figure. The boxes represent two copies of the 'directed' CSU in Section 5. The arc within each box indicates the distinguished pair of edges that always carry the same colour in any colouring of the unit. In essence, there is no longer a distinction between the unit's o-edges and the c-edge.

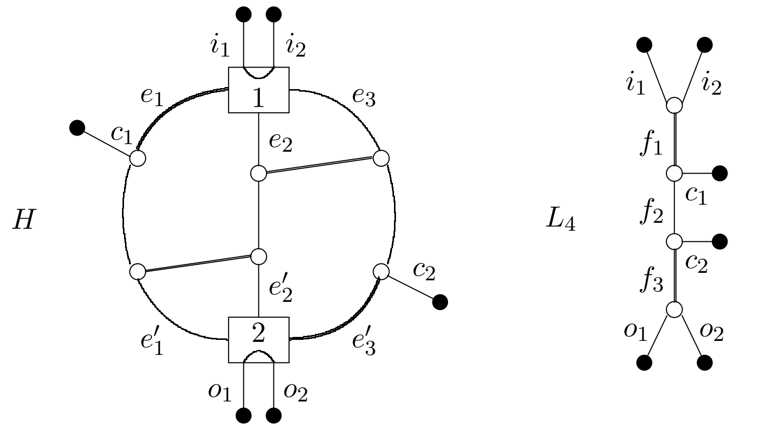


Figure 7

The welding is along equivalently labelled edges. In view of the property of the directed unit, the following is readily checked by considering alternating paths.

Claim 1. *The* 6*-pendant* H *is colourable, and in every such colouring* γ*, we have* $\gamma(i_1) = \gamma(i_2)$*,* $\gamma(o_1) = \gamma(o_2)$ *and* $\gamma(c_1) = \gamma(c_2)$*. Consequently,* G *is uncolourable.* ■

In fact, the colours at the o-, i- and c-edges of H can be arbitrarily assigned, so may be taken to be all equal to x.

Claim 2. $\tau(G) = 1$*. In any minimal matching, the two odd cycles are linked by a path of length three.* ■

Colouring all the pendants on H by x, let M be the matching of the z-coloured edges (shown above as the bold edges). This can be completed to a matching $M^* = M \cup \{f_1, f_3\}$ of G in which the only odd cycles are the two x-y paths between i_1 and i_2 and that between o_1 and o_2. The path between c_1 and c_2 together with edge f_2 becomes a cycle of even length. This is clearly a minimal matching with the requisite property.

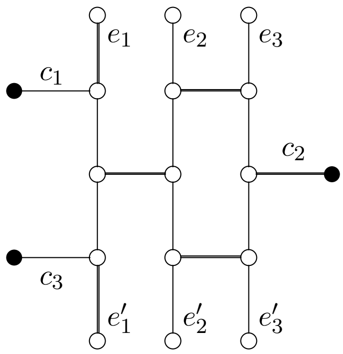
By Proposition 2, we need only show that there is no edge whose removal leaves a colourable graph. We concentrate on the two units numbered 1 and 2. Removal of any one of f_1, f_2, f_3 produces a subgraph with either or both of these two units intact, hence uncolourable. The same holds if we remove an edge **within** either of the units: the other remains unaffected.

The remaining case is of the other edges in H. With respect to unit 1, it is readily checked that removal of any of e_1, e_2 or e_3 does not affect unit 2, unless it is together with one of e'_1, e'_2, e'_3. In any event, the shortest path between these edges is evidently 3. No other single edge will do. Notice that the resulting graphs are cyclically 4-edge-connected snarks.

We close this Section by showing that, perhaps not surprisingly, 'anything goes'. If a cubic graph is uncolourable with Tait number one, then either it becomes colourable on removal of exactly one edge, as covered in Proposition 2, or else removal of two edges suffices. In this latter case, among all matchings, there is one where the distance between the two odd cycles, measured by the length of a path from a vertex on one to a vertex on the other, is minimised.

Proposition 4. *Snarks with Tait number* 1 *exist for which the distance between the two odd cycles is any integer* ≥ 3*.*

Proof. Since the first and last edge of such a minimal length path must belong to the matching, the distance cannot be 2. The example in figure 7 has distance 3. An example with distance 4 (resp. 5) is produced by excising the two numbered CSU's and replacing the resulting subgraph by the pendants in the figure below, and the L_4 by an L_5 (resp. L_6)



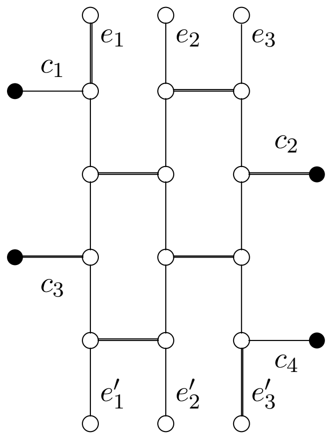


Figure 8

These pendants illustrate the difference between even and odd distance cases. If the bold edges are coloured z, there is a unique way of ensuring that edges e_1, e_2, e_3 (and edges e'_1, e'_2, e'_3) have a sum flow of zero. In the even cases, e_1 and e'_1 are both coloured z, and c_n is the edge adjacent to e'_1; in the odd case, it is the edge adjacent to e'_3 that is labelled c_n. The edges $c_2, c_3, \ldots, c_{n-1}$ are all coloured z, and assigned arbitrarily.

The labelling of the c-edges is the same as on the corresponding L_5 and L_6 pendants, on which edges f_1, f_4 (resp. f_5) are given colour z. The rest of the edges lie on a single alternating x-y path between c_1 and c_3 (resp. c_4). The assertion about distance and the fact that these are all snarks should be clear. ■

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Received 22 August 1997
Revised 2 May 1998