Discussiones Mathematicae
Graph Theory 17 (1997) 95–102

PARTITIONS OF SOME PLANAR GRAPHS INTO TWO LINEAR FORESTS

Piotr Borowiecki

and

Mariusz Hałuszczak

Institute of Mathematics
Technical University of Zielona Góra
Podgórna 50, 65–246 Zielona Góra, Poland

e-mail: p.borowiecki@im.pz.zgora.pl
m.haluszczak@im.pz.zgora.pl

Abstract

A *linear forest* is a forest in which every component is a path. It is known that the set of vertices $V(G)$ of any outerplanar graph G can be partitioned into two disjoint subsets V_1, V_2 such that induced subgraphs $\langle V_1\rangle$ and $\langle V_2\rangle$ are linear forests (we say G has an $(\mathcal{LF},\mathcal{LF})$-partition). In this paper, we present an extension of the above result to the class of planar graphs with a given number of *internal vertices* (i.e., vertices that do not belong to the external face at a certain fixed embedding of the graph G in the plane). We prove that there exists an $(\mathcal{LF},\mathcal{LF})$-partition for any plane graph G when certain conditions on the degree of the internal vertices and their neighbourhoods are satisfied.

Keywords: linear forest, bipartition, planar graphs.

1991 Mathematics Subject Classification: 05C15, 05C70.

1. Introduction and Notation

Let $\mathcal{I}$ denote the set of all finite simple graphs. A *graph property* $\mathcal{P}$ is a nonempty isomorphism-closed subclass of $\mathcal{I}$. We also say that a graph G has the property $\mathcal{P}$ if $G \in \mathcal{P}$. A property $\mathcal{P}$ of graphs is said to be (*induced*) *hereditary* if whenever $G \in \mathcal{P}$ and H is a (vertex induced) subgraph of G, then also $H \in \mathcal{P}$. A property $\mathcal{P}$ is called *additive* if for each graph G all of whose components have the property $\mathcal{P}$ it follows that G has the property $\mathcal{P}$, too. A hereditary property $\mathcal{P}$ can be characterized in terms of forbidden subgraphs. The set of *minimal forbidden subgraphs* of $\mathcal{P}$ is defined as follows:

$$\boldsymbol{F}(\mathcal{P}) = \{G \in \mathcal{I} : G \notin \mathcal{P} \text{ but each proper subgraph } H \text{ of } G \text{ belongs to } \mathcal{P}\}.$$

In general, we use the notation and terminology of [1]. Let us mention selected hereditary properties of graphs:

$$\begin{aligned} \mathcal{O} &= \{G \in \mathcal{I} : G \text{ is edgeless, i.e., } E(G) = \emptyset\}, \\ \mathcal{T}_k &= \{G \in \mathcal{I} : G \text{ contains no subgraph homeomorphic to } K_{k+2} \\ &\qquad \text{or } K_{\lfloor \frac{k+3}{2} \rfloor, \lceil \frac{k+3}{2} \rceil}\}, \\ \mathcal{D}_k &= \{G \in \mathcal{I} : G \text{ is } k\text{-degenerate}\}, \\ \mathcal{S}_k &= \{G \in \mathcal{I} : \Delta(G) \leq k\}. \end{aligned}$$

It is easy to see that $\mathcal{D}_1 = \mathcal{T}_1 = \{G : G \text{ is a forest}\}$, $\mathcal{LF} = \mathcal{D}_1 \cap \mathcal{S}_2$ is the linear forest, while $\mathcal{T}_2$ and $\mathcal{T}_3$ are the classes of all outerplanar and all planar graphs, respectively. For $\mathcal{LF}$ the set of minimal forbidden subgraphs is given by

$$\boldsymbol{F}(\mathcal{LF}) = \{K_{1,3}, C_n \text{ with } n \geq 3\}.$$

Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$, $n > 1$ be any properties and let G belong to $\mathcal{I}$. A *vertex* $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$*-partition* of the graph G is a partition $(V_1, V_2, \ldots, V_n)$ of $V(G)$ such that each subgraph $\langle V_i \rangle$ of the graph G induced by V_i has the property $\mathcal{P}_i, i = 1, 2, \ldots, n$. A problem of partitioning planar graphs into linear forests has been extensively studied in many papers. Broere [3], Wang [8] and Mihók [6] proved that any outerplanar graph has an $(\mathcal{LF}, \mathcal{LF})$-partition. Some extensions of the result given above and an algorithm can be found in [2]. The result of Poh [7] and Goddard [4] is that any planar graph has an $(\mathcal{LF}, \mathcal{LF}, \mathcal{LF})$- partition (i.e., into three linear forests).

2. Results

Let W be a subset of the vertex set $V(G)$ such that $\langle W \rangle$ is connected. By the operation of *contraction of the vertex set* W to the vertex u we will understand the removal of all the vertices belonging to W, addition of a new vertex u and all the edges required to satisfy the following condition $\mathrm{N}(u) = \bigcup_{w \in W} \mathrm{N}(w)$, where $\mathrm{N}(v)$ denotes the neighbourhood of the vertex v in G.

Let us define a set $\mathrm{Int}(G)$ of all internal vertices of a planar graph G as a set of vertices not belonging to the external face at a certain fixed embedding of the graph G in the plane. Let $\mathrm{int}(G) = \min |\mathrm{Int}(G)|$ over all embeddings of the graph G in the plane. If $\mathrm{int}(G) = 0$, then the graph G is outerplanar.

Theorem 1. *Let G be a plane graph and $v \in V(G) \backslash \mathrm{Int}(G)$ an arbitrarily chosen vertex. If the following conditions are satisfied:*

(i) *for any $x, y \in \mathrm{Int}(G)$, $(x, y) \notin E(G)$,*
(ii) *for any vertex $x \in \mathrm{Int}(G) \backslash \mathrm{N}(v)$, $d(x) > 4$,*
(iii) *for any vertex $x \in \mathrm{Int}(G) \cap \mathrm{N}(v)$, $d(x) > 3$,*

then there exists a (V_1, V_2)-partition of $V(G)$ such that $\langle V_i \rangle \in \mathcal{LF}$ for $i = 1, 2$ and $v \in V_1$, $\mathrm{N}(v) \subseteq V_2$.

Proof. Without loss of generality, we assume that G is maximal in the sense that graph $G + e$ does not satisfy one of the conditions (i)–(iii). The proof is by induction on the order of G. Let $|V(G)| = 3$. Then the Theorem is true. Assume that the Theorem holds for all graphs of order less than k. Let $|V(G)| = k$. Let the graph G^* be obtained from G by contraction of the set $\mathrm{N}[v] = \mathrm{N}(v) \cup \{v\}$ to the vertex w. We are going to prove that the graph G^* satisfies conditions (i)–(iii).

Claim 1. The graph G^* satisfies conditions (i)–(iii).

Proof. The proof falls into three cases.

Case 1. It is easy to see that for any $x, y \in \mathrm{Int}(G^*)$ if $(x, y) \notin E(G)$, then $(x, y) \notin E(G^*)$, too. Thus the condition (i) is satisfied.

Case 2. From the definition of contraction of the set $\mathrm{N}[v]$ to the vertex w, it immediately follows that a degree of any vertex $x \in \mathrm{Int}(G)$ such that $\mathrm{N}(v) \cap \mathrm{N}(x) = \emptyset$ cannot be affected and $d_G(x) = d_{G^*}(x)$. Thus, for any $x \in \mathrm{Int}(G^*) \backslash \mathrm{N}(w)$, $d(x) > 4$ and the condition (ii) is satisfied.

Case 3. If for the vertex v there exists a vertex $x \in \mathrm{Int}(G)$ such that $d_G(x) > 4$ and $\mathrm{N}(v) \cap \mathrm{N}(x) \neq \emptyset$, then $|\mathrm{N}(v) \cap \mathrm{N}(x)| \leq 2$. If $x \notin \mathrm{N}(v)$, then an operation of contraction of the set $\mathrm{N}(v)$ may decrease the degree of the vertex x by at most 1. If $x \in \mathrm{N}(v)$, then x will be contracted to the vertex $w \notin \mathrm{Int}(G^*)$. Thus, for any $x \in \mathrm{Int}(G^*) \cap \mathrm{N}(w)$, $d(x) > 3$ and the condition (iii) is satisfied.

Hence, we get the graph G^* which satisfies conditions (i)–(iii).

Since $|V(G^*)| < k$ then G^* has a (V_1^*, V_2^*)-partition of $V(G^*)$ such that $\langle V_i^* \rangle \in \mathcal{LF}$ for $i = 1, 2$ and $w \in V_1^*$, $\mathrm{N}(w) \subseteq V_2^*$. Let $V_1 = V_2^* \cup \{v\}$, $V_2 = (V_1^* \backslash \{w\}) \cup \mathrm{N}(v)$. We are going to prove that V_1 and V_2 have the property $\mathcal{LF}$.

Claim 2. $\langle \mathrm{N}(v) \rangle_G$ has the property $\mathcal{LF}$.

Proof. Assuming that $H = \langle \mathrm{N}(v) \rangle_G$ has not the property $\mathcal{LF}$ implies that H contains a cycle or a vertex of degree greater than 2. Thus, we have the following cases:

Case 1. Let us assume that H contains a cycle C_k of length $k \geq 3$. Since, $C_k + \langle \{v\} \rangle$, where $+$ denotes the join, contains a subgraph homeomorphic to K_4, then it is not outerplanar. Thus, there exists a vertex $x \in \mathrm{N}(v)$ such that $x \in \mathrm{Int}(G)$. From (iii) it follows that $d_G(x) > 3$, which implies an existence of the vertex y such that $(x, y) \in E(G)$ and $y \in \mathrm{Int}(G)$, contrary to (i).

Case 2. Let us assume that there exists a vertex $u \in V(H)$ such that $d_H(u) > 2$ (i.e., H contains $K_{1,3}$ as a subgraph). Since $K_{1,3} + \langle \{v\} \rangle$ is not outerplanar, then there exists a vertex $x \in \mathrm{N}(u)$ such that $x \in \mathrm{Int}(G)$. From (iii) it follows that $d_G(x) > 3$, thus there exists a vertex y such that $(x, y) \in E(G)$ and $y \in \mathrm{Int}(G)$, contrary to (i).

Thus, $\langle \mathrm{N}(v) \rangle_G$ has the property $\mathcal{LF}$.

Since $\mathrm{N}(w) \subseteq V_2^*$, then no vertex from $\mathrm{N}(v)$ has the neighbour in the set $V_1^* \backslash \{w\}$. Hence, as V_1^* and $\langle \mathrm{N}(v) \rangle_G$ both have the property $\mathcal{LF}$, it comes out that V_2 has the property $\mathcal{LF}$, too. Obviously, V_1 belongs to $\mathcal{LF}$ and v has no adjacent vertex in V_1. Thus, the partition (V_1, V_2) is the required $(\mathcal{LF}, \mathcal{LF})$-partition of G. ■

Corollary 1. *If a graph G is outerplanar, then for every vertex $v \in V(G)$ there exists an $(\mathcal{LF}, \mathcal{LF})$-partition of G, say (V_1, V_2), such that $v \in V_1$ and* $\mathrm{N}(v) \subseteq V_2$.

Theorem 2. *Let G be a plane graph, $R \subseteq V(G)$ and* $\mathrm{Int}(G)$ *be a proper subset of R. If a subgraph of the graph G induced by R is a path, then the graph G has an $(\mathcal{LF}, \mathcal{LF})$-partition.*

Proof. Contracting the set R to a vertex w, we get an outerplanar graph G^*. Hence, G^* has an $(\mathcal{LF}, \mathcal{LF})$-partition (V_1^*, V_2^*) of $V(G^*)$ such that $w \in V_1^*$ and $\mathrm{N}(w) \subseteq V_2^*$. No vertex from R has a neighbour in the set $V_1^*\backslash\{w\}$. Let $V_1 = V_1^*\backslash\{w\} \cup R$, $V_2 = V_2^*$. Since $\langle V_1^*\rangle_G$ and $\langle R\rangle_G$ both have the $\mathcal{LF}$ property, then V_1 belongs to $\mathcal{LF}$, too. Obviously, (V_1, V_2) is an $(\mathcal{LF}, \mathcal{LF})$-partition of G. ■

Corollary 2. *Let G be a maximal outerplanar graph with an outer-cycle C. Let $P \leq C$ be an induced path of C. Then G has an $(\mathcal{LF}, \mathcal{LF})$-partition (V_1, V_2) of $V(G)$ such that $V(P) \subseteq V_1$.*

Theorem 3. *Every planar graph G with* $\mathrm{int}(G) \leq 2$ *has an $(\mathcal{LF}, \mathcal{LF})$-partition.*

Proof. The proof falls naturally into three cases.

Case 1. $\mathrm{int}(G) = 0$.
If $\mathrm{int}(G) = 0$, then the graph G is outerplanar and it has an $(\mathcal{LF}, \mathcal{LF})$-partition.

Case 2. $\mathrm{int}(G) = 1$.
Let $v \in \mathrm{Int}(G)$ and $u \in \mathrm{N}(v)$. It is easy to notice that $u \notin \mathrm{Int}(G)$. According to Theorem 2, if $R = \{v, u\}$, then the graph G has an $(\mathcal{LF}, \mathcal{LF})$-partition.

Case 3. $\mathrm{int}(G) = 2$.
We can consider a maximal plane graph G with $\mathrm{Int}(G) = \{r_1, r_2\}$.

Subcase 3.1. r_1 is adjacent to r_2.
There exists a vertex u adjacent to r_1 such that u is not adjacent to r_2. According to Theorem 2, if $R = \{u, r_1, r_2\}$, then the graph G has an $(\mathcal{LF}, \mathcal{LF})$-partition.

Subcase 3.2. r_1 is not adjacent to r_2.
Let R contain all the vertices belonging to the shortest path from r_1 to r_2. Since $\langle R\rangle_G$ is a path and R contains at least one vertex not belonging to $\mathrm{Int}(G)$, i.e., $\mathrm{Int}(G)$ is a proper subgraph of R, then according to Theorem 2, the graph G has an $(\mathcal{LF}, \mathcal{LF})$-partition. ■

Theorem 4. *Let G be a planar graph of order $n \leq 9$ with* $\mathrm{int}(G) = 3$. *Then G has an $(\mathcal{LF}, \mathcal{LF})$-partition.*

Proof. If $\mathrm{int}(G) \leq 2$, then by Theorem 3, independently of the order n, the graph G has an $(\mathcal{LF}, \mathcal{LF})$-partition.
Let us consider a planar graph with $\mathrm{int}(G) = 3$, where $\mathrm{Int}(G) = \{r_1, r_2, r_3\}$. Without loss of generality we assume that G is a near-triangulation, i.e., G is a plane graph which consists of an outer-cycle $C_k : v_1v_2\ldots v_kv_1$ in

clockwise order and vertices and edges inside C_k such that each bounded face is bounded by a triangle. Since the graph G is of order $n \leq 9$ and $\mathrm{int}(G) = 3$, then $3 \leq k \leq 6$. Let $S(r_i) = \mathrm{N}(r_i) \cap V(C_k)$ be the set of the vertices of the cycle C_k that are adjacent to the vertex $r_i \in \mathrm{Int}(G)$ and $s_i = |S(r_i)|$. Then we have to consider four cases. Cases 1 and 2 are considered under the assumption that C_k is a chordless cycle. But, if there is a chord, then it divides the graph G into G_1 and G_2, such that $\mathrm{Int}(G_1) = \{r_1, r_2, r_3\}$ and G_2 is outerplanar. Then an $(\mathcal{LF}, \mathcal{LF})$-partition of G_1 can be easily extended to G.

Case 1. $\langle \mathrm{Int}(G) \rangle = K_3$.
Let the vertices r_1, r_2, r_3 be in clockwise order and $s_1 \leq s_2$ and $s_1 \leq s_3$. If $S(r_1) = \{v_1, \ldots, v_{s_1}\}$, $S(r_2) = \{v_{s_1}, \ldots, v_{s_1+s_2-1}\}$, $S(r_3) = \{v_{s_1+s_2-1}, \ldots, v_1\}$, then the partition (V_1, V_2) can be obtained as follows:

$$V_1 = \{r_3, v_1, \ldots, v_{s_1+s_2-2}\}\ ,\ V_2 = \{r_1, r_2, v_{s_1+s_2-1}, \ldots, v_k\}.$$

Case 2. $\langle \mathrm{Int}(G) \rangle = P_3$.
Let $\mathrm{N}(r_2) \supseteq \{r_1, r_3\}$ and $s_1 \leq s_3$. Then we have two subcases.

Case 2.1. $s_3 = 2$.
If $s_3 = 2$, then there exists a vertex $v \in C_k$ such that $v \in \mathrm{N}(r_1) \cap \mathrm{N}(r_2) \backslash \mathrm{N}(r_3)$. Then $V_1 = \{r_2, r_3, v\}$ and $V_2 = \{r_1\} \cup (C_k \backslash \{v\})$.

Case 2.2. $s_3 > 2$.
If $s_3 > 2$, then there exists a vertex $v \in C_k$ such that $v \in \mathrm{N}(r_3) \backslash (\mathrm{N}(r_1) \cup \mathrm{N}(r_2))$. Then $V_1 = \{r_1, r_2, r_3, v\}$ and $V_2 = C_k \backslash \{v\}$.

Case 3. $\langle \mathrm{Int}(G) \rangle = \overline{K_3}$.
It is easy to see that for any graph G considered in this case $|V(G)| \geq 8$ and the cycle C_k has at least two chords. If $|V(G)| = 9$, then we have eight graphs. Their $(\mathcal{LF}, \mathcal{LF})$-partitions are shown in Figure 1.

There is only one graph H such that $|V(H)| = 8$. It is easy to see that H is a subgraph of two graphs presented at the bottom line of Figure 1.

Case 4. $\langle \mathrm{Int}(G) \rangle = \overline{P_3}$.
In this case C_k has at least one chord. Thus, G is divided by this chord into two graphs G_i with $\mathrm{int}(G_i) = i$, $i = 1, 2$. Each of them has an $(\mathcal{LF}, \mathcal{LF})$-partition which can be extended to the other one. The details are left to the reader. ■

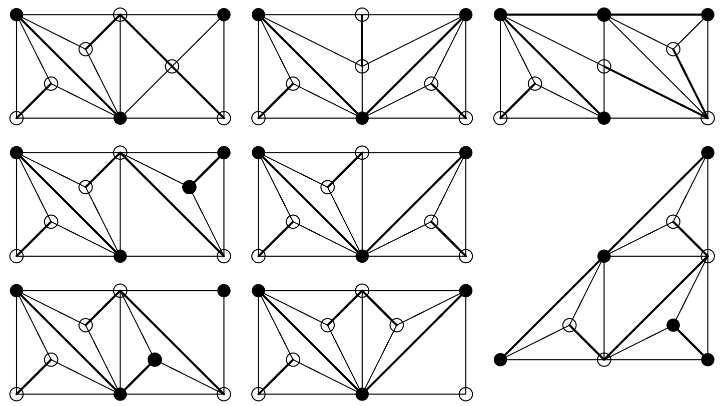

Figure 1

Theorem 5. *For every integer $n \geq 10$ there exists a planar graph G of order n with* $\text{int}(G) = 3$*, which does not have an* $(\mathcal{LF}, \mathcal{LF})$*-partition.*

Proof.

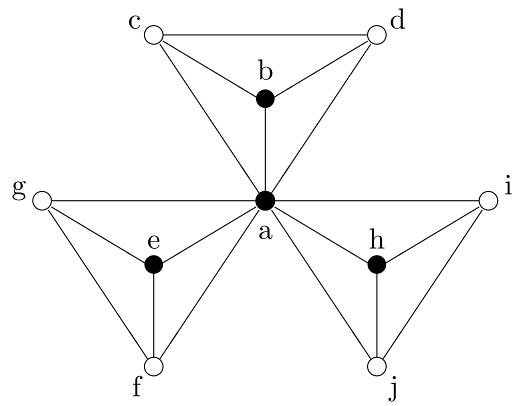

Figure 2

Let us partition the set $V(G)$ of the graph G in Figure 2 into two subsets V_1 and V_2. Assuming that $a \in V_1$, at least one of the vertices from $\{b, c, d\}, \{e, f, g\}$ and $\{h, i, j\}$ must belong to V_1. Otherwise, $\langle V_2 \rangle_G$ would have contained a cycle. But the set V_1 constructed in this way induces a subgraph containing a vertex of a degree greater than 2. Thus, any planar graph containing the graph G as its subgraph does not have an $(\mathcal{LF}, \mathcal{LF})$-partition. ∎

Acknowledgement

The authors of this paper wish to thank the refree for his suggestions and critical comments that were found very helpful.

References

[1] M. Borowiecki, P. Mihók, *Hereditary properties of graphs*, in: Advances in Graph Theory (Vishwa International Publications, 1991) 41–68.

[2] P. Borowiecki, *P-Bipartitions of Graphs*, Vishwa International J. Graph Theory **2** (1993) 109–116.

[3] I. Broere, C.M. Mynhardt, *Generalized colourings of outerplanar and planar graphs*, in: Graph Theory with Applications to Algorithms and Computer Science (Wiley, New York, 1985) 151–161.

[4] W. Goddard, *Acyclic colourings of planar graphs*, Discrete Math. **91** (1991) 91–94.

[5] T.R. Jensen and B. Toft, Graph Colouring Problems (Wiley-Interscience Publications, New York, 1995).

[6] P. Mihók, *On the vertex partition numbers*, in: M. Fiedler, ed., Graphs and Other Combinatorial Topics, Proc. Third Czech. Symp. Graph Theory, Prague, 1982 (Teubner-Verlag, Leipzig, 1983) 183–188.

[7] K.S. Poh, *On the Linear Vertex-Arboricity of a Planar Graph*, J. Graph Theory **14** (1990) 73–75.

[8] J. Wang, *On point-linear arboricity of planar graphs*, Discrete Math. **72** (1988) 381–384.

Received 3 January 1997
Revised 25 February 1997