PARTITIONS OF SOME PLANAR GRAPHS INTO TWO LINEAR FORESTS

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Abstract

A linear forest is a forest in which every component is a path. It is known that the set of vertices $V(G)$ of any outerplanar graph $G$ can be partitioned into two disjoint subsets $V_1, V_2$ such that induced subgraphs $\langle V_1 \rangle$ and $\langle V_2 \rangle$ are linear forests (we say $G$ has an $(LF, LF)$-partition). In this paper, we present an extension of the above result to the class of planar graphs with a given number of internal vertices (i.e., vertices that do not belong to the external face at a certain fixed embedding of the graph $G$ in the plane). We prove that there exists an $(LF, LF)$-partition for any plane graph $G$ when certain conditions on the degree of the internal vertices and their neighbourhoods are satisfied.

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1. Introduction and Notation

Let $\mathcal{I}$ denote the set of all finite simple graphs. A graph property $\mathcal{P}$ is a nonempty isomorphism-closed subclass of $\mathcal{I}$. We also say that a graph $G$ has the property $\mathcal{P}$ if $G \in \mathcal{P}$. A property $\mathcal{P}$ of graphs is said to be (induced) hereditary if whenever $G \in \mathcal{P}$ and $H$ is a (vertex induced) subgraph of $G$,
then also \( H \in \mathcal{P} \). A property \( \mathcal{P} \) is called additive if for each graph \( G \) all of whose components have the property \( \mathcal{P} \) it follows that \( G \) has the property \( \mathcal{P} \), too. A hereditary property \( \mathcal{P} \) can be characterized in terms of forbidden subgraphs. The set of minimal forbidden subgraphs of \( \mathcal{P} \) is defined as follows:

\[
\mathcal{F}(\mathcal{P}) = \{ G \in \mathcal{I} : G \notin \mathcal{P} \text{ but each proper subgraph } H \text{ of } G \text{ belongs to } \mathcal{P} \}.
\]

In general, we use the notation and terminology of [1]. Let us mention selected hereditary properties of graphs:

- \( \mathcal{O} = \{ G \in \mathcal{I} : G \text{ is edgeless, i.e., } E(G) = \emptyset \} \),
- \( \mathcal{T}_k = \{ G \in \mathcal{I} : G \text{ contains no subgraph homeomorphic to } K_{k+2} \) or \( K_{\left\lfloor \frac{k+3}{2} \right\rfloor, \left\lceil \frac{k+3}{2} \right\rceil} \} \),
- \( \mathcal{D}_k = \{ G \in \mathcal{I} : G \text{ is } k \text{-degenerate} \} \),
- \( \mathcal{S}_k = \{ G \in \mathcal{I} : \Delta(G) \leq k \} \).

It is easy to see that \( \mathcal{D}_1 = \mathcal{T}_1 = \{ G : G \text{ is a forest} \} \), \( \mathcal{LF} = \mathcal{D}_1 \cap \mathcal{S}_2 \) is the linear forest, while \( \mathcal{T}_2 \) and \( \mathcal{T}_3 \) are the classes of all outerplanar and all planar graphs, respectively. For \( \mathcal{LF} \) the set of minimal forbidden subgraphs is given by

\[
\mathcal{F}(\mathcal{LF}) = \{ K_{1,3}, C_n \text{ with } n \geq 3 \}.
\]

Let \( \mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n, n > 1 \) be any properties and let \( G \) belong to \( \mathcal{I} \). A vertex \( (\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n) \)-partition of the graph \( G \) is a partition \( (V_1, V_2, \ldots, V_n) \) of \( V(G) \) such that each subgraph \( \langle V_i \rangle \) of the graph \( G \) induced by \( V_i \) has the property \( \mathcal{P}_i, i = 1, 2, \ldots, n \). A problem of partitioning planar graphs into linear forests has been extensively studied in many papers. Broere [3], Wang [8] and Mihók [6] proved that any outerplanar graph has an \( (\mathcal{LF}, \mathcal{LF}) \)-partition. Some extensions of the result given above and an algorithm can be found in [2]. The result of Poh [7] and Goddard [4] is that any planar graph has an \( (\mathcal{LF}, \mathcal{LF}, \mathcal{LF}) \)-partition (i.e., into three linear forests).

2. Results

Let \( W \) be a subset of the vertex set \( V(G) \) such that \( \langle W \rangle \) is connected. By the operation of contraction of the vertex set \( W \) to the vertex \( u \) we will understand the removal of all the vertices belonging to \( W \), addition of a new vertex \( u \) and all the edges required to satisfy the following condition \( N(u) = \bigcup_{w \in W} N(w) \), where \( N(v) \) denotes the neighbourhood of the vertex \( v \) in \( G \).
Let us define a set $\text{Int}(G)$ of all internal vertices of a planar graph $G$ as a set of vertices not belonging to the external face at a certain fixed embedding of the graph $G$ in the plane. Let $\text{int}(G) = \min |\text{Int}(G)|$ over all embeddings of the graph $G$ in the plane. If $\text{int}(G) = 0$, then the graph $G$ is outerplanar.

**Theorem 1.** Let $G$ be a plane graph and $v \in V(G) \setminus \text{Int}(G)$ an arbitrarily chosen vertex. If the following conditions are satisfied:

(i) for any $x, y \in \text{Int}(G)$, $(x, y) \notin E(G)$,
(ii) for any vertex $x \in \text{Int}(G) \setminus N(v)$, $d(x) > 4$,
(iii) for any vertex $x \in \text{Int}(G) \cap N(v)$, $d(x) > 3$,

then there exists a $(V_1, V_2)$-partition of $V(G)$ such that $\langle V_i \rangle \in \mathcal{LF}$ for $i = 1, 2$ and $v \in V_1$, $N(v) \subseteq V_2$.

**Proof.** Without loss of generality, we assume that $G$ is maximal in the sense that graph $G + e$ does not satisfy one of the conditions (i)–(iii). The proof is by induction on the order of $G$. Let $|V(G)| = 3$. Then the Theorem is true. Assume that the Theorem holds for all graphs of order less than $k$. Let $|V(G)| = k$. Let the graph $G^*$ be obtained from $G$ by contraction of the set $N[v] = N(v) \cup \{v\}$ to the vertex $w$. We are going to prove that the graph $G^*$ satisfies conditions (i)–(iii).

**Claim 1.** The graph $G^*$ satisfies conditions (i)–(iii).

**Proof.** The proof falls into three cases.

**Case 1.** It is easy to see that for any $x, y \in \text{Int}(G^*)$ if $(x, y) \notin E(G)$, then $(x, y) \notin E(G^*)$, too. Thus the condition (i) is satisfied.

**Case 2.** From the definition of contraction of the set $N[v]$ to the vertex $w$, it immediately follows that a degree of any vertex $x \in \text{Int}(G)$ such that $N(v) \cap N(x) = \emptyset$ cannot be affected and $d_G(x) = d_{G^*}(x)$. Thus, for any $x \in \text{Int}(G^*) \setminus N(w)$, $d(x) > 4$ and the condition (ii) is satisfied.

**Case 3.** If for the vertex $v$ there exists a vertex $x \in \text{Int}(G)$ such that $d_G(x) > 4$ and $N(v) \cap N(x) \neq \emptyset$, then $|N(v) \cap N(x)| \leq 2$. If $x \notin N(v)$, then an operation of contraction of the set $N(v)$ may decrease the degree of the vertex $x$ by at most 1. If $x \in N(v)$, then $x$ will be contracted to the vertex $w \notin \text{Int}(G^*)$. Thus, for any $x \in \text{Int}(G^*) \cap N(w)$, $d(x) > 3$ and the condition (iii) is satisfied.

Hence, we get the graph $G^*$ which satisfies conditions (i)–(iii).
Since $|V(G^*)| < k$ then $G^*$ has a $(V'_1, V'_2)$-partition of $V(G^*)$ such that $\langle V'_i \rangle \in \mathcal{LF}$ for $i = 1, 2$ and $w \in V'_1$, $N(w) \subseteq V'_2$. Let $V_1 = V'_2 \cup \{v\}$, $V_2 = (V'_1 \setminus \{w\}) \cup N(v)$. We are going to prove that $V_1$ and $V_2$ have the property $\mathcal{LF}$.

**Claim 2.** $\langle N(v) \rangle_G$ has the property $\mathcal{LF}$.

**Proof.** Assuming that $H = \langle N(v) \rangle_G$ has not the property $\mathcal{LF}$ implies that $H$ contains a cycle or a vertex of degree greater than 2. Thus, we have the following cases:

**Case 1.** Let us assume that $H$ contains a cycle $C_k$ of length $k \geq 3$. Since, $C_k + \langle \{v\} \rangle$, where $+$ denotes the join, contains a subgraph homeomorphic to $K_4$, then it is not outerplanar. Thus, there exists a vertex $x \in N(v)$ such that $x \in \text{Int}(G)$. From (iii) it follows that $d_G(x) > 3$, which implies an existence of the vertex $y$ such that $(x, y) \in E(G)$ and $y \in \text{Int}(G)$, contrary to (i).

**Case 2.** Let us assume that there exists a vertex $u \in V(H)$ such that $d_H(u) > 2$ (i.e., $H$ contains $K_{1,3}$ as a subgraph). Since $K_{1,3} + \langle \{v\} \rangle$ is not outerplanar, then there exists a vertex $x \in N(u)$ such that $x \in \text{Int}(G)$. From (iii) it follows that $d_G(x) > 3$, thus there exists a vertex $y$ such that $(x, y) \in E(G)$ and $y \in \text{Int}(G)$, contrary to (i).

Thus, $\langle N(v) \rangle_G$ has the property $\mathcal{LF}$.

Since $N(w) \subseteq V'_2$, then no vertex from $N(v)$ has the neighbour in the set $V'_1 \setminus \{w\}$. Hence, as $V'_1$ and $\langle N(v) \rangle_G$ both have the property $\mathcal{LF}$, it comes out that $V_2$ has the property $\mathcal{LF}$, too. Obviously, $V_1$ belongs to $\mathcal{LF}$ and $v$ has no adjacent vertex in $V_1$. Thus, the partition $(V_1, V_2)$ is the required $(\mathcal{LF}, \mathcal{LF})$-partition of $G$. ■

**Corollary 1.** If a graph $G$ is outerplanar, then for every vertex $v \in V(G)$ there exists an $(\mathcal{LF}, \mathcal{LF})$-partition of $G$, say $(V_1, V_2)$, such that $v \in V_1$ and $N(v) \subseteq V_2$.

**Theorem 2.** Let $G$ be a plane graph, $R \subseteq V(G)$ and $\text{Int}(G)$ be a proper subset of $R$. If a subgraph of the graph $G$ induced by $R$ is a path, then the graph $G$ has an $(\mathcal{LF}, \mathcal{LF})$-partition.

**Proof.** Contracting the set $R$ to a vertex $w$, we get an outerplanar graph $G^*$. Hence, $G^*$ has an $(\mathcal{LF}, \mathcal{LF})$-partition $(V'_1, V'_2)$ of $V(G^*)$ such that $w \in V'_1$ and $N(w) \subseteq V'_2$. No vertex from $R$ has a neighbour in the
set $V_1^* \setminus \{w\}$. Let $V_1 = V_1^* \setminus \{w\} \cup R$, $V_2 = V_2^*$. Since $\langle V_1^* \rangle_G$ and $\langle R \rangle_G$ both have the $\mathcal{LF}$ property, then $V_1$ belongs to $\mathcal{LF}$, too. Obviously, $(V_1, V_2)$ is an $(\mathcal{LF}, \mathcal{LF})$-partition of $G$.

**Corollary 2.** Let $G$ be a maximal outerplanar graph with an outer-cycle $C$. Let $P \leq C$ be an induced path of $C$. Then $G$ has an $(\mathcal{LF}, \mathcal{LF})$-partition $(V_1, V_2)$ of $V(G)$ such that $V(P) \subseteq V_1$.

**Theorem 3.** Every planar graph $G$ with $\text{int}(G) \leq 2$ has an $(\mathcal{LF}, \mathcal{LF})$-partition.

**Proof.** The proof falls naturally into three cases.

*Case 1.* $\text{int}(G) = 0$.

If $\text{int}(G) = 0$, then the graph $G$ is outerplanar and it has an $(\mathcal{LF}, \mathcal{LF})$-partition.

*Case 2.* $\text{int}(G) = 1$.

Let $v \in \text{Int}(G)$ and $u \in N(v)$. It is easy to notice that $u \notin \text{Int}(G)$. According to Theorem 2, if $R = \{v, u\}$, then the graph $G$ has an $(\mathcal{LF}, \mathcal{LF})$-partition.

*Case 3.* $\text{int}(G) = 2$.

We can consider a maximal plane graph $G$ with $\text{Int}(G) = \{r_1, r_2\}$.

*Subcase 3.1.* $r_1$ is adjacent to $r_2$.

There exists a vertex $u$ adjacent to $r_1$ such that $u$ is not adjacent to $r_2$. According to Theorem 2, if $R = \{u, r_1, r_2\}$, then the graph $G$ has an $(\mathcal{LF}, \mathcal{LF})$-partition.

*Subcase 3.2.* $r_1$ is not adjacent to $r_2$.

Let $R$ contain all the vertices belonging to the shortest path from $r_1$ to $r_2$. Since $\langle R \rangle_G$ is a path and $R$ contains at least one vertex not belonging to $\text{Int}(G)$, i.e., $\text{Int}(G)$ is a proper subgraph of $R$, then according to Theorem 2, the graph $G$ has an $(\mathcal{LF}, \mathcal{LF})$-partition.

**Theorem 4.** Let $G$ be a planar graph of order $n \leq 9$ with $\text{int}(G) = 3$. Then $G$ has an $(\mathcal{LF}, \mathcal{LF})$-partition.

**Proof.** If $\text{int}(G) \leq 2$, then by Theorem 3, independently of the order $n$, the graph $G$ has an $(\mathcal{LF}, \mathcal{LF})$-partition.

Let us consider a planar graph with $\text{int}(G) = 3$, where $\text{Int}(G) = \{r_1, r_2, r_3\}$. Without loss of generality we assume that $G$ is a near-triangulation, i.e., $G$ is a plane graph which consists of an outer-cycle $C_k : v_1v_2 \ldots v_kv_1$ in
clockwise order and vertices and edges inside $C_k$ such that each bounded face is bounded by a triangle. Since the graph $G$ is of order $n \leq 9$ and $\text{int}(G) = 3$, then $3 \leq k \leq 6$. Let $S(r_i) = N(r_i) \cap V(C_k)$ be the set of the vertices of the cycle $C_k$ that are adjacent to the vertex $r_i \in \text{Int}(G)$ and $s_i = |S(r_i)|$. Then we have to consider four cases. Cases 1 and 2 are considered under the assumption that $C_k$ is a chordless cycle. But, if there is a chord, then it divides the graph $G$ into $G_1$ and $G_2$, such that $\text{Int}(G_1) = \{r_1, r_2, r_3\}$ and $G_2$ is outerplanar. Then an $(\mathcal{LF}, \mathcal{LF})$-partition of $G_1$ can be easily extended to $G$.

**Case 1.** $\langle \text{Int}(G) \rangle = K_3$.

Let the vertices $r_1, r_2, r_3$ be in clockwise order and $s_1 \leq s_2$ and $s_1 \leq s_3$. If $S(r_1) = \{v_1, \ldots, v_{s_1}\}$, $S(r_2) = \{v_{s_1}, \ldots, v_{s_1+s_2-1}\}$, $S(r_3) = \{v_{s_1+s_2-1}, \ldots, v_1\}$, then the partition $(V_1, V_2)$ can be obtained as follows:

$$V_1 = \{r_3, v_1, \ldots, v_{s_1+s_2-2}\}, \quad V_2 = \{r_1, r_2, v_{s_1+s_2-1}, \ldots, v_k\}.$$

**Case 2.** $\langle \text{Int}(G) \rangle = P_3$.

Let $N(r_2) \supseteq \{r_1, r_3\}$ and $s_1 \leq s_3$. Then we have two subcases.

**Case 2.1.** $s_3 = 2$.

If $s_3 = 2$, then there exists a vertex $v \in C_k$ such that $v \in N(r_1) \cap N(r_2) \setminus N(r_3)$. Then $V_1 = \{r_2, r_3, v\}$ and $V_2 = \{r_1\} \cup (C_k \setminus \{v\})$.

**Case 2.2.** $s_3 > 2$.

If $s_3 > 2$, then there exists a vertex $v \in C_k$ such that $v \in N(r_3) \setminus (N(r_1) \cup N(r_2))$. Then $V_1 = \{r_1, r_2, r_3, v\}$ and $V_2 = C_k \setminus \{v\}$.

**Case 3.** $\langle \text{Int}(G) \rangle = K_3$.

It is easy to see that for any graph $G$ considered in this case $|V(G)| \geq 8$ and the cycle $C_k$ has at least two chords. If $|V(G)| = 9$, then we have eight graphs. Their $(\mathcal{LF}, \mathcal{LF})$-partitions are shown in Figure 1.

There is only one graph $H$ such that $|V(H)| = 8$. It is easy to see that $H$ is a subgraph of two graphs presented at the bottom line of Figure 1.

**Case 4.** $\langle \text{Int}(G) \rangle = P_3$.

In this case $C_k$ has at least one chord. Thus, $G$ is divided by this chord into two graphs $G_i$ with $\text{int}(G_i) = i$, $i = 1, 2$. Each of them has an $(\mathcal{LF}, \mathcal{LF})$-partition which can be extended to the other one. The details are left to the reader. \[\blacksquare\]
**Theorem 5.** For every integer $n \geq 10$ there exists a planar graph $G$ of order $n$ with $\text{int}(G) = 3$, which does not have an $(\mathcal{L}F, \mathcal{L}F)$-partition.

**Proof.**

Let us partition the set $V(G)$ of the graph $G$ in Figure 2 into two subsets $V_1$ and $V_2$. Assuming that $a \in V_1$, at least one of the vertices from $\{b, c, d\}, \{e, f, g\}$ and $\{h, i, j\}$ must belong to $V_1$. Otherwise, $(V_2)_G$ would have contained a cycle. But the set $V_1$ constructed in this way induces a subgraph containing a vertex of a degree greater than 2. Thus, any planar graph containing the graph $G$ as its subgraph does not have an $(\mathcal{L}F, \mathcal{L}F)$-partition.
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References


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