SUFFICIENT CONDITIONS FOR MAXIMALLY EDGE-CONNECTED AND SUPER-EDGE-CONNECTED GRAPHS DEPENDING ON THE CLIQUE NUMBER

LUTZ VOLKMANN

Lehrstuhl II für Mathematik
RWTH Aachen University
52056 Aachen, Germany

e-mail: volkm@math2.rwth-aachen.de

Abstract

Let G be a connected graph with minimum degree δ and edge-connectivity λ. A graph is maximally edge-connected if λ = δ, and it is super-edge-connected if every minimum edge-cut is trivial; that is, if every minimum edge-cut consists of edges incident with a vertex of minimum degree. The clique number ω(G) of a graph G is the maximum cardinality of a complete subgraph of G. In this paper, we show that a connected graph G with clique number ω(G) ≤ r is maximally edge-connected or super-edge-connected if the number of edges is large enough. These are generalizations of corresponding results for triangle-free graphs by Volkmann and Hong in 2017.

Keywords: edge-connectivity, clique number, maximally edge-connected graphs, super-edge-connected graphs.

2010 Mathematics Subject Classification: 05C40.

1. Terminology and Introduction

Let G be a finite and simple graph with vertex set V = V(G) and edge set E = E(G). The order and size of G are defined by n = n(G) = |V(G)| and m = m(G) = |E(G)|, respectively. If N(v) = N_G(v) is the neighborhood of the vertex v ∈ V(G), then we denote by d(v) = d_G(v) = |N(v)| the degree of v and by δ = δ(G) the minimum degree of the graph G. For a subset X ⊆ V(G), let G[X] to denote the subgraph of G induced by X. For two subsets X and Y of V(G) let [X,Y] be the set of edges with one endpoint in X and the other one in Y. The clique number ω(G) of a graph G is the maximum cardinality of a
complete subgraph of $G$. An edge-cut of a connected graph $G$ is a set of edges whose removal disconnects $G$. The edge connectivity $\lambda = \lambda(G)$ of a connected graph $G$ is defined as the minimum cardinality of an edge-cut over all edge-cuts of $G$. An edge-cut $S$ is a minimum edge-cut or a $\lambda$-cut if $|S| = \lambda(G)$. The inequality $\lambda(G) \leq \delta(G)$ is immediate. We call a connected graph maximally edge-connected, if $\lambda(G) = \delta(G)$. In 1981, Bauer et al. [1] proposed the concept of super-edge connectedness. A graph is called super-edge-connected or super-$\lambda$ if every minimum edge-cut is trivial; that is, if every minimum edge-cut consists of edges incident with a vertex of minimum degree. Thus every super-edge-connected graph is also maximally edge-connected.

Sufficient conditions for graphs to be maximally edge-connected or super-edge-connected were given by several authors, see for example the survey paper by Hellwig and Volkmann [3]. The starting point was an article by Chartrand [2] in 1966. He observed that if $\delta$ is large enough, then the graph is maximally edge-connected. A similar condition for super-edge-connectivity was given by Kelmans [4] six years later. Over the years, these results have been strengthened many times and in many ways.

Recently, Volkmann and Hong [6] showed that a connected graph or a connected triangle-free graph is maximally edge-connected or super-$\lambda$ if the number of edges is large enough. In particular, they received the following results.

**Theorem 1.** Let $G$ be a connected triangle-free graph of order $n \geq 2$, size $m$, minimum degree $\delta$ and edge-connectivity $\lambda$. If

$$m > \left\lfloor \frac{n^2}{4} \right\rfloor - \delta(n - 1 - 2\delta) - 1,$$

then $\lambda = \delta$.

**Theorem 2.** Let $G$ be a connected triangle-free graph of order $n$, size $m$, minimum degree $\delta \geq 3$ and edge-connectivity $\lambda$. If

$$m > \left\lfloor \frac{(n + 1)^2}{4} \right\rfloor - \delta(n + 1 - 2\delta),$$

then $G$ is super-$\lambda$.

In this paper, we will generalize Theorems 1 and 2 to connected graphs with clique number $\omega(G) \leq r$ for $r \geq 2$. Examples will demonstrate that our results are sharp.
2. Maximally Edge-Connected Graphs

The main tool of our article is the famous theorem of Turán [5].

**Theorem 3.** Let $r \geq 1$ be an integer, and let $G$ be a graph of order $n$. If the clique number $\omega(G) \leq r$, then

$$|E(G)| \leq \left\lfloor \frac{r-1}{2r} \cdot n^2 \right\rfloor.$$ 

**Theorem 4.** Let $r \geq 2$ be an integer, and let $G$ be a connected graph of order $n$, size $m$, minimum degree $\delta \geq 1$, edge-connectivity $\lambda$ and clique number $\omega(G) \leq r$. If

$$m > \left\lfloor \frac{r-1}{2r} \left( n^2 + 2 \left\lfloor \frac{r\delta}{r-1} \right\rfloor - 2n \left\lfloor \frac{r\delta}{r-1} \right\rfloor \right) \right\rfloor + \delta - 1,$$

then $\lambda = \delta$.

**Proof.** If $\delta = 1$, then $\lambda = \delta$ in every case. Thus assume in the following that $\delta \geq 2$. Suppose to the contrary that $\lambda \leq \delta - 1$. Then there exist two disjoint sets $X, Y \subset V(G)$ with $X \cup Y = V(G)$ and $|X,Y| = \lambda$. Assume, without loss of generality, that $|X| \leq |Y|$.

We first show that $X$ contains at least $\delta + 1$ vertices. Otherwise, suppose that $X$ contains at most $\delta$ vertices. Then we obtain

$$\delta |X| \leq \sum_{x \in X} d_G(x) \leq |X|(|X| - 1) + \lambda \leq \delta(|X| - 1) + \delta - 1.$$

Obviously, this is a contradiction and thus $|X| \geq \delta + 1$. Using Theorem 3, we conclude that

(1) $$|E(G[X])| \leq \left\lfloor \frac{(r-1)|X|^2}{2r} \right\rfloor$$

and

(2) $$|E(G[Y])| \leq \left\lfloor \frac{(r-1)|Y|^2}{2r} \right\rfloor.$$

Next we show that $|X| \geq \left\lfloor (r\delta)/(r-1) \right\rfloor$. Suppose to the contrary that $|X| \leq \left\lfloor (r\delta)/(r-1) \right\rfloor - 1$. Since $2|E(G[X])| = \sum_{x \in X} d_G(x) - \lambda$, (1) implies that

$$|X|\delta \leq \sum_{x \in X} d_G(x) \leq 2 \left\lfloor \frac{(r-1)|X|^2}{2r} \right\rfloor + \lambda \leq \frac{(r-1)|X|^2}{r} + \delta - 1$$

$$\leq |X| \frac{r-1}{r} \left( \left\lfloor \frac{r\delta}{r-1} \right\rfloor - 1 \right) + \delta - 1 \leq |X| \frac{r-1}{r} \left( \frac{r\delta}{r-1} - 1 \right) + \delta - 1$$

$$= |X| \delta - \frac{r-1}{r} |X| + \delta - 1.$$
and thus $|X| \leq \frac{r(\delta-1)}{r}$.

Using this argument once more, we arrive at

$$|X| \leq \frac{(r-1)|X|^2}{r} + \delta - 1 \leq |X| \frac{r-1}{r} \cdot \frac{r(\delta-1)}{r-1} + \delta - 1$$

$$= |X|((\delta-1) + \delta - 1)$$

and thus $|X| \leq \delta - 1$, which contradicts the fact that $|X| \geq \delta + 1$. Hence $|X| \geq \lfloor (r\delta)/(r-1) \rfloor$.

Since $|X| + |Y| = n$ and $|X| \leq n/2$, the inequalities (1) and (2) lead to

$$m = |E(G[X])| + |E(G[Y])| + \lambda$$

$$\leq \left[ \frac{(r-1)|X|^2}{2r} \right] + \left[ \frac{(r-1)|Y|^2}{2r} \right] + \delta - 1$$

$$= \left[ \frac{(r-1)|X|^2}{2r} \right] + \left[ \frac{(r-1)(n-|X|)^2}{2r} \right] + \delta - 1$$

$$\leq \left[ \frac{(r-1)}{2r} (|X|^2 + (n-|X|)^2) \right] + \delta - 1$$

$$= \left[ \frac{(r-1)}{2r} (n^2 + 2(|X|^2 - n|X|)) \right] + \delta - 1$$

$$\leq \left[ \frac{(r-1)}{2r} \left( n^2 + 2 \left( \frac{r\delta}{r-1} \right)^2 - 2n \left( \frac{r\delta}{r-1} \right) \right) \right] + \delta - 1,$$

a contradiction to the hypothesis. Thus $\lambda = \delta$.

Theorem 1 is the special case $r = 2$ of Theorem 4. The next family of graphs shows that Theorem 4 is best possible in the sense that

$$m = \left[ \frac{r-1}{2r} \left( n^2 + 2 \left( \frac{r\delta}{r-1} \right)^2 - 2n \left( \frac{r\delta}{r-1} \right) \right) \right] + \delta - 1$$

does not guarantee $\lambda = \delta$.

**Example 5.** Let $r \geq 2$ and $q \geq 1$ be integers. Let $H_1$ and $H_2$ be two disjoint copies of the complete $r$-partite graph with $q$ vertices in each partite set. Define $H$ as the union of $H_1$ and $H_2$ by adding $\delta - 1 = q(r-1) - 1$ edges between $H_1$ and $H_2$ such that $\omega(H) \leq r$. Then $H$ has order $n = 2qr$, minimum degree $\delta = q(r-1)$ such that

$$m(H) = q^2r(r-1) + q(r-1) - 1$$

$$= \left[ \frac{r-1}{2r} \left( n^2 + 2 \left( \frac{r\delta}{r-1} \right)^2 - 2n \left( \frac{r\delta}{r-1} \right) \right) \right] + \delta - 1,$$

but obviously, $\lambda(H) = \delta(H) - 1$. 

3. Super Edge-Connected Graphs

**Theorem 6.** Let $r \geq 2$ be an integer, and let $G$ be a connected graph of order $n$, size $m$, minimum degree $\delta \geq 2$, edge-connectivity $\lambda$ and $\omega(G) \leq r$. If $\delta \geq 3$ or $r \geq 3$ and

$$m > \left\lceil \frac{r - 1}{2r} \left(n^2 + 2 \left(\left\lceil \frac{r\delta}{r - 1} \right\rceil - 1\right)^2 - 2n \left(\left\lceil \frac{r\delta}{r - 1} \right\rceil - 1\right)\right) \right\rceil + \delta,$$

then $G$ is super-$\lambda$.

**Proof.** Suppose to the contrary that $G$ is not super-$\lambda$. Then there exist two disjoint sets $X, Y \subset V(G)$ such that $X \cup Y = V(G)$, $|X|, |Y| \geq 2$ and $|[X, Y]| = \lambda$. Assume, without loss of generality, that $2 \leq |X| \leq |Y|$.

We first show that $X$ contains at most $\delta - 1$ vertices. Otherwise, suppose that $X$ contains at most $\delta - 1$ vertices. Then we obtain

$$\delta |X| \leq \sum_{x \in X} d_G(x) \leq |X|(|X| - 1) + \lambda \leq (\delta - 1)(|X| - 1) + \delta,$$

which implies that $|X| \leq 1$, contradicting that $|X| \geq 2$. Thus $|X| \geq \delta$.

Next we show that $|X| \geq \lfloor (r\delta)/(r - 1) \rfloor - 1$. If $\delta = 2$ and $r \geq 3$, then

$$|X| \geq \delta = 2 \geq \lfloor (2r)/(r - 1) \rfloor - 1 = \lfloor (r\delta)/(r - 1) \rfloor - 1.$$

Let now $\delta \geq 3$. Suppose to the contrary that $X$ contains at most $\lfloor (r\delta)/(r - 1) \rfloor - 2$ vertices. Since $2|E(G[X])| = \sum_{x \in X} d_G(x) - \lambda$, we conclude from (1) that

$$|X|\delta \leq \sum_{x \in X} d_G(x) \leq 2 \left\lceil \frac{(r - 1)|X|^2}{2r} \right\rceil + \lambda \leq \frac{(r - 1)|X|^2}{r} + \delta$$

$$\leq |X| \frac{r - 1}{r} \left(\left\lceil \frac{r\delta}{r - 1} \right\rceil - 1\right) + \delta \leq |X| \frac{r - 1}{r} \left(\frac{r\delta}{r - 1} - 2\right) + \delta$$

$$= |X|\delta - \frac{2(r - 1)}{r}|X| + \delta$$

and thus $|X| \leq \frac{r\delta}{2(r - 1)}$. Using this argument once more, we arrive at

$$|X|\delta \leq \frac{(r - 1)|X|^2}{r} + \delta \leq |X| \frac{r - 1}{r} \cdot \frac{r\delta}{2(r - 1)} + \delta = \frac{|X|\delta}{2} + \delta$$

and thus $|X| \leq 2$, which contradicts the fact that $3 \leq \delta \leq |X|$. Hence we have shown that $|X| \geq \lfloor (r\delta)/(r - 1) \rfloor - 1$ when $\delta \geq 3$ or $r \geq 3$. Since $|X| + |Y| = n$
and $|X| \leq n/2$, the inequalities (1) and (2) lead to

$$m = |E(G[X])| + |E(G[Y])| + \lambda \leq \left\lfloor \frac{(r-1)|X|^2}{2r} \right\rfloor + \left\lfloor \frac{(r-1)|Y|^2}{2r} \right\rfloor + \delta$$

$$\leq \left\lfloor \frac{(r-1)}{2r} \left( |X|^2 + (n-|X|)^2 \right) \right\rfloor + \delta$$

$$= \left\lfloor \frac{(r-1)}{2r} \left( n^2 + 2(|X|^2 - n|X|) \right) \right\rfloor + \delta$$

$$\leq \left\lfloor \frac{(r-1)}{2r} \left( n^2 + 2 \left( \left\lfloor \frac{r\delta}{r-1} \right\rfloor - 1 \right)^2 - 2n \left( \left\lfloor \frac{r\delta}{r-1} \right\rfloor - 1 \right) \right) \right\rfloor + \delta,$$

a contradiction to the hypothesis. Thus $G$ is super-$\lambda$.

Theorem 2 is the special case $r = 2$ of Theorem 6. The next family of graphs shows that Theorem 6 is best possible in the sense that $m = \left\lfloor \frac{r-1}{2r} \left( n^2 + 2 \left( \left\lfloor \frac{r\delta}{r-1} \right\rfloor - 1 \right)^2 - 2n \left( \left\lfloor \frac{r\delta}{r-1} \right\rfloor - 1 \right) \right) \right\rfloor + \delta$
does not guarantee that the graph is super-$\lambda$.

**Example 7.** Let $r \geq 2$ and $q \geq 3$ be integers. Let $H_1$ be the complete $r$-partite graph with $q-1$ vertices in one partite set and $q$ vertices in each partite set, and let $H_2$ be the complete $r$-partite graph with $q$ vertices in each partite set. Define $H$ as the union of $H_1$ and $H_2$ by adding $\delta = q(r-1)$ edges between $H_1$ and $H_2$ such that $\omega(H) \leq r$ and $\delta(H) = \delta = q(r-1)$. Then $H$ has order $n = 2qr - 1$, minimum degree $\delta = q(r-1)$ such that

$$m(H) = \left\lfloor \frac{r-1}{2r} \left( n^2 + 2 \left( \left\lfloor \frac{r\delta}{r-1} \right\rfloor - 1 \right)^2 - 2n \left( \left\lfloor \frac{r\delta}{r-1} \right\rfloor - 1 \right) \right) \right\rfloor + \delta$$

but obviously, $H$ is not super-$\lambda$.

Our last example demonstrates that Theorem 6 is not valid for $\delta = 2$ and $r = 2$ in general.

**Example 8.** Let $q \geq 2$ be an integer, and let $K_{q,q}$ be the complete bipartite graph with the partite sets $X = \{x_1, x_2, \ldots, x_q\}$ and $Y = \{y_1, y_2, \ldots, y_q\}$, and let $u$ and $v$ be two further vertices. Define the graph $H$ as the union of $K_{q,q}$, $u$ and
v together with the edges uv, ux_1 and vx_2. Then H has order n(H) = 2q + 2, minimum degree δ(H) = 2 and ω(H) ≤ 2. Furthermore,

\[ m(H) = q^2 + 3 > q^2 - q + 4 \]

\[ = \left\lfloor \frac{r - 1}{2r} \left( n^2 + 2 \left( \left\lfloor \frac{r\delta}{r - 1} \right\rfloor - 1 \right)^2 - 2n \left( \left\lfloor \frac{r\delta}{r - 1} \right\rfloor - 1 \right) \right) \right\rfloor + \delta \]

but H is not super-λ.

REFERENCES


doi:10.1137/0114065

doi:10.1016/j.disc.2007.06.035

doi:10.1137/1117029


Received 8 June 2017
Revised 19 October 2017
Accepted 23 October 2017