SUFFICIENT CONDITIONS FOR MAXIMALLY EDGE-CONNECTED AND SUPER-EDGE-CONNECTED GRAPHS DEPENDING ON THE CLIQUE NUMBER

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Abstract

Let $G$ be a connected graph with minimum degree $\delta$ and edge-connectivity $\lambda$. A graph is maximally edge-connected if $\lambda = \delta$, and it is super-edge-connected if every minimum edge-cut is trivial; that is, if every minimum edge-cut consists of edges incident with a vertex of minimum degree. The clique number $\omega(G)$ of a graph $G$ is the maximum cardinality of a complete subgraph of $G$. In this paper, we show that a connected graph $G$ with clique number $\omega(G) \leq r$ is maximally edge-connected or super-edge-connected if the number of edges is large enough. These are generalizations of corresponding results for triangle-free graphs by Volkmann and Hong in 2017.

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1. Terminology and Introduction

Let $G$ be a finite and simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order and size of $G$ are defined by $n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$, respectively. If $N(v) = N_G(v)$ is the neighborhood of the vertex $v \in V(G)$, then we denote by $d(v) = d_G(v) = |N(v)|$ the degree of $v$ and by $\delta = \delta(G)$ the minimum degree of the graph $G$. For a subset $X \subseteq V(G)$, let $G[X]$ to denote the subgraph of $G$ induced by $X$. For two subsets $X$ and $Y$ of $V(G)$ let $[X,Y]$ be the set of edges with one endpoint in $X$ and the other one in $Y$. The clique number $\omega(G)$ of a graph $G$ is the maximum cardinality of a
complete subgraph of $G$. An edge-cut of a connected graph $G$ is a set of edges whose removal disconnects $G$. The edge connectivity $\lambda = \lambda(G)$ of a connected graph $G$ is defined as the minimum cardinality of an edge-cut over all edge-cuts of $G$. An edge-cut $S$ is a minimum edge-cut or a $\lambda$-cut if $|S| = \lambda(G)$. The inequality $\lambda(G) \leq \delta(G)$ is immediate. We call a connected graph maximally edge-connected, if $\lambda(G) = \delta(G)$. In 1981, Bauer et al. [1] proposed the concept of super-edge connectedness. A graph is called super-edge-connected or super-$\lambda$ if every minimum edge-cut is trivial; that is, if every minimum edge-cut consists of edges incident with a vertex of minimum degree. Thus every super-edge-connected graph is also maximally edge-connected.

Sufficient conditions for graphs to be maximally edge-connected or super-edge-connected were given by several authors, see for example the survey paper by Hellwig and Volkmann [3]. The starting point was an article by Chartrand [2] in 1966. He observed that if $\delta$ is large enough, then the graph is maximally edge-connected. A similar condition for super-edge-connectivity was given by Kelmans [4] six years later. Over the years, these results have been strengthened many times and in many ways.

Recently, Volkmann and Hong [6] showed that a connected graph or a connected triangle-free graph is maximally edge-connected or super-$\lambda$ if the number of edges is large enough. In particular, they received the following results.

**Theorem 1.** Let $G$ be a connected triangle-free graph of order $n \geq 2$, size $m$, minimum degree $\delta$ and edge-connectivity $\lambda$. If

$$m > \left\lfloor \frac{n^2}{4} \right\rfloor - \delta(n - 1 - 2\delta) - 1,$$

then $\lambda = \delta$.

**Theorem 2.** Let $G$ be a connected triangle-free graph of order $n$, size $m$, minimum degree $\delta \geq 3$ and edge-connectivity $\lambda$. If

$$m > \left\lfloor \frac{(n+1)^2}{4} \right\rfloor - \delta(n + 1 - 2\delta),$$

then $G$ is super-$\lambda$.

In this paper, we will generalize Theorems 1 and 2 to connected graphs with clique number $\omega(G) \leq r$ for $r \geq 2$. Examples will demonstrate that our results are sharp.
2. Maximally Edge-Connected Graphs

The main tool of our article is the famous theorem of Turán [5].

**Theorem 3.** Let $r \geq 1$ be an integer, and let $G$ be a graph of order $n$. If the clique number $\omega(G) \leq r$, then

$$|E(G)| \leq \left\lfloor \frac{r - 1}{2r} \cdot n^2 \right\rfloor.$$

**Theorem 4.** Let $r \geq 2$ be an integer, and let $G$ be a connected graph of order $n$, size $m$, minimum degree $\delta \geq 1$, edge-connectivity $\lambda$ and clique number $\omega(G) \leq r$. If

$$m > \left( \frac{r - 1}{2r} \left( n^2 + 2 \left( \frac{r\delta}{r - 1} \right)^2 - 2n \left( \frac{r\delta}{r - 1} \right) \right) \right) + \delta - 1,$$

then $\lambda = \delta$.

**Proof.** If $\delta = 1$, then $\lambda = \delta$ in every case. Thus assume in the following that $\delta \geq 2$. Suppose to the contrary that $\lambda \leq \delta - 1$. Then there exist two disjoint sets $X, Y \subset V(G)$ with $X \cup Y = V(G)$ and $|[X,Y]| = \lambda$. Assume, without loss of generality, that $|X| \leq |Y|$.

We first show that $X$ contains at least $\delta + 1$ vertices. Otherwise, suppose that $X$ contains at most $\delta$ vertices. Then we obtain

$$\delta |X| \leq \sum_{x \in X} d_G(x) \leq |X|(|X| - 1) + \lambda \leq \delta(|X| - 1) + \delta - 1.$$

Obviously, this is a contradiction and thus $|X| \geq \delta + 1$. Using Theorem 3, we conclude that

(1) $$|E(G[X])| \leq \left\lfloor \frac{(r - 1)|X|^2}{2r} \right\rfloor$$

and (2) $$|E(G[Y])| \leq \left\lfloor \frac{(r - 1)|Y|^2}{2r} \right\rfloor.$$

Next we show that $|X| \geq \left\lfloor (r\delta)/(r - 1) \right\rfloor$. Suppose to the contrary that $|X| \leq \left\lfloor (r\delta)/(r - 1) \right\rfloor - 1$. Since $2|E(G[X])| = \sum_{x \in X} d_G(x) - \lambda$, (1) implies that

$$|X|\delta \leq \sum_{x \in X} d_G(x) \leq 2 \left( \frac{(r - 1)|X|^2}{2r} \right) + \lambda \leq \frac{(r - 1)|X|^2}{r} + \delta - 1$$

$$\leq |X| \frac{r - 1}{r} \left( \left\lfloor \frac{r\delta}{r - 1} \right\rfloor - 1 \right) + \delta - 1 \leq |X| \frac{r - 1}{r} \left( \frac{r\delta}{r - 1} - 1 \right) + \delta - 1$$

$$= |X|\delta - \frac{r - 1}{r} |X| + \delta - 1$$
and thus \(|X| \leq \frac{r(\delta - 1)}{r} - 1\). Using this argument once more, we arrive at

\[ |X| \delta \leq \frac{(r - 1)|X|^2}{r} + \delta - 1 \leq |X| \frac{r - 1}{r} \cdot \frac{r(\delta - 1)}{r} + \delta - 1 \]

\[ = |X|(\delta - 1) + \delta - 1 \]

and thus \(|X| \leq \delta - 1\), which contradicts the fact that \(|X| \geq \delta + 1\). Hence \(|X| \geq \frac{(r\delta)}{(r - 1)}\). Since \(|X| + |Y| = n\) and \(|X| \leq n/2\), the inequalities (1) and (2) lead to

\[
m = |E(G[X])| + |E(G[Y])| + \lambda \\
\leq \left( \frac{r - 1}{2r} |X|^2 \right) + \left( \frac{r - 1}{2r} |Y|^2 \right) + \delta - 1 \\
= \left( \frac{r - 1}{2r} |X|^2 \right) + \left( \frac{r - 1}{2r} (n - |X|)^2 \right) + \delta - 1 \\
\leq \left( \frac{r - 1}{2r} \left( |X|^2 + (n - |X|)^2 \right) \right) + \delta - 1 \\
= \left( \frac{r - 1}{2r} \left( n^2 + 2(|X|^2 - n|X|) \right) \right) + \delta - 1 \\
\leq \left( \frac{r - 1}{2r} \left( n^2 + 2 \left( \frac{r\delta}{r - 1} \right)^2 - 2n \left( \frac{r\delta}{r - 1} \right) \right) \right) + \delta - 1,
\]

a contradiction to the hypothesis. Thus \(\lambda = \delta\).

Theorem 1 is the special case \(r = 2\) of Theorem 4. The next family of graphs shows that Theorem 4 is best possible in the sense that

\[
m = \left[ \frac{r - 1}{2r} \left( n^2 + 2 \left( \frac{r\delta}{r - 1} \right)^2 - 2n \left( \frac{r\delta}{r - 1} \right) \right) \right] + \delta - 1
\]

does not guarantee \(\lambda = \delta\).

**Example 5.** Let \(r \geq 2\) and \(q \geq 1\) be integers. Let \(H_1\) and \(H_2\) be two disjoint copies of the complete \(r\)-partite graph with \(q\) vertices in each partite set. Define \(H\) as the union of \(H_1\) and \(H_2\) by adding \(\delta - 1 = q(r - 1) - 1\) edges between \(H_1\) and \(H_2\) such that \(\omega(H) \leq r\). Then \(H\) has order \(n = 2qr\), minimum degree \(\delta = q(r - 1)\) such that

\[
m(H) = q^2r(r - 1) + q(r - 1) - 1 \\
= \left[ \frac{r - 1}{2r} \left( n^2 + 2 \left( \frac{r\delta}{r - 1} \right)^2 - 2n \left( \frac{r\delta}{r - 1} \right) \right) \right] + \delta - 1,
\]

but obviously, \(\lambda(H) = \delta(H) - 1\).
3. Super Edge-Connected Graphs

**Theorem 6.** Let \( r \geq 2 \) be an integer, and let \( G \) be a connected graph of order \( n \), size \( m \), minimum degree \( \delta \geq 2 \), edge-connectivity \( \lambda \) and \( \omega(G) \leq r \). If \( \delta \geq 3 \) or \( r \geq 3 \) and

\[
m > \left\lfloor \frac{r-1}{2r} \left( n^2 + 2 \left( \left\lfloor \frac{r\delta}{r-1} \right\rfloor - 1 \right)^2 - 2n \left( \frac{r\delta}{r-1} - 1 \right) \right) \right\rfloor + \delta,
\]
then \( G \) is super-\( \lambda \).

**Proof.** Suppose to the contrary that \( G \) is not super-\( \lambda \). Then there exist two disjoint sets \( X, Y \subset V(G) \) such that \( X \cup Y = V(G) \), \( |X| \geq 2 \) and \( |[X,Y]| = \lambda \). Assume, without loss of generality, that \( 2 \leq |X| \leq |Y| \).

We first show that \( X \) contains at most \( \delta - 1 \) vertices. Otherwise, suppose that \( X \) contains at most \( \delta - 1 \) vertices. Then we obtain

\[
\delta |X| \leq \sum_{x \in X} d_G(x) \leq |X|(|X| - 1) + \lambda \leq (\delta - 1)(|X| - 1) + \delta,
\]
which implies that \( |X| \leq 1 \), contradicting that \( |X| \geq 2 \). Thus \( |X| \geq \delta \).

Next we show that \( |X| \geq \left\lfloor \frac{(r\delta)}{(r-1)} \right\rfloor - 1 \). If \( \delta = 2 \) and \( r \geq 3 \), then

\[
|X| \geq \delta = 2 \geq \left\lfloor \frac{(2r)}{(r-1)} \right\rfloor - 1 = \left\lfloor \frac{(r\delta)}{(r-1)} \right\rfloor - 1.
\]

Let now \( \delta \geq 3 \). Suppose to the contrary that \( X \) contains at most \( \left\lfloor \frac{(r\delta)}{(r-1)} \right\rfloor - 2 \) vertices. Since \( 2|E(G[X])| = \sum_{x \in X} d_G(x) - \lambda \), we conclude from (1) that

\[
|X| \delta \leq \sum_{x \in X} d_G(x) \leq 2 \left\lfloor \frac{(r-1)|X|^2}{2r} \right\rfloor + \lambda \leq \left( \frac{r-1}{r} \right)|X|^2 + \delta
\]

\[
\leq |X| \left( \frac{r-1}{r} \left( \frac{r\delta}{r-1} \right) - 2 \right) + \delta \leq |X| \left( \frac{r-1}{r} \frac{r\delta}{r-1} - 2 \right) + \delta
\]

\[
= |X| \delta - \frac{2(r-1)}{r} |X| + \delta
\]

and thus \( |X| \leq \frac{r\delta}{2(r-1)} \). Using this argument once more, we arrive at

\[
|X| \delta \leq \frac{(r-1)|X|^2}{r} + \delta \leq |X| \left( \frac{r-1}{r} \right) \frac{r\delta}{2(r-1)} + \delta = \frac{|X|\delta}{2} + \delta
\]

and thus \( |X| \leq 2 \), which contradicts the fact that \( 3 \leq \delta \leq |X| \). Hence we have shown that \( |X| \geq \left\lfloor \frac{(r\delta)}{(r-1)} \right\rfloor - 1 \) when \( \delta \geq 3 \) or \( r \geq 3 \). Since \( |X| + |Y| = n \)
and $|X| \leq n/2$, the inequalities (1) and (2) lead to

$$m = |E(G[X])| + |E(G[Y])| + \lambda \leq \left\lfloor \frac{(r-1)|X|^2}{2r} \right\rfloor + \left\lfloor \frac{(r-1)|Y|^2}{2r} \right\rfloor + \delta$$

$$= \left\lfloor \frac{(r-1)|X|^2}{2r} \right\rfloor + \left\lfloor \frac{(r-1)(n-|X|)^2}{2r} \right\rfloor + \delta$$

$$\leq \left\lfloor \frac{(r-1)}{2r} \left(|X|^2 + (n-|X|)^2\right) \right\rfloor + \delta$$

$$= \left\lfloor \frac{(r-1)}{2r} \left(n^2 + 2(|X|^2 - n|X|)\right) \right\rfloor + \delta$$

$$\leq \left\lfloor \frac{(r-1)}{2r} \left(n^2 + 2\left(\left\lfloor \frac{r\delta}{r-1} \right\rfloor - 1\right)^2 - 2n \left(\left\lfloor \frac{r\delta}{r-1} \right\rfloor - 1\right)\right) \right\rfloor + \delta,$$

a contradiction to the hypothesis. Thus $G$ is super-$\lambda$.

Theorem 2 is the special case $r = 2$ of Theorem 6. The next family of graphs shows that Theorem 6 is best possible in the sense that $m$ does not guarantee that the graph is super-$\lambda$.

**Example 7.** Let $r \geq 2$ and $q \geq 3$ be integers. Let $H_1$ be the complete $r$-partite graph with $q-1$ vertices in one partite set and $q$ vertices in $r-1$ partite sets, and let $H_2$ be the complete $r$-partite graph with $q$ vertices in each partite set. Define $H$ as the union of $H_1$ and $H_2$ by adding $\delta = q(r-1)$ edges between $H_1$ and $H_2$ such that $\omega(H) \leq r$ and $\delta(H) = \delta = q(r-1)$. Then $H$ has order $n = 2qr - 1$, minimum degree $\delta = q(r-1)$ such that

$$m(H) = q^2 r^2 - q^2 r$$

$$= \left\lfloor \frac{r-1}{2r} \left(n^2 + 2\left(\left\lfloor \frac{r\delta}{r-1} \right\rfloor - 1\right)^2 - 2n \left(\left\lfloor \frac{r\delta}{r-1} \right\rfloor - 1\right)\right) \right\rfloor + \delta$$

but obviously, $H$ is not super-$\lambda$.

Our last example demonstrates that Theorem 6 is not valid for $\delta = 2$ and $r = 2$ in general.

**Example 8.** Let $q \geq 2$ be an integer, and let $K_{q,q}$ be the complete bipartite graph with the partite sets $X = \{x_1, x_2, \ldots, x_q\}$ and $Y = \{y_1, y_2, \ldots, y_q\}$, and let $u$ and $v$ be two further vertices. Define the graph $H$ as the union of $K_{q,q}$, $u$ and
v together with the edges $uv$, $ux_1$ and $vx_2$. Then $H$ has order $n(H) = 2q + 2$, minimum degree $\delta(H) = 2$ and $\omega(H) \leq 2$. Furthermore,

$$m(H) = q^2 + 3 > q^2 - q + 4$$

$$= \left\lfloor \frac{r - 1}{2r} \left( n^2 + 2 \left( \left\lfloor \frac{r\delta}{r - 1} \right\rfloor - 1 \right) \right)^2 - 2n \left( \left\lfloor \frac{r\delta}{r - 1} \right\rfloor - 1 \right) \right\rfloor + \delta$$

but $H$ is not super-$\lambda$.

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