

THE SECOND NEIGHBOURHOOD FOR BIPARTITE TOURNAMENTS

RUIJUAN LI

School of Mathematical Sciences
Shanxi University
Taiyuan 030006, PR China

e-mail: ruijuanli@sxu.edu.cn

AND

BIN SHENG

College of Computer Science and Technology
Nanjing University of Aeronautics and Astronautics
Nanjing 211106, PR China

Department of Computer Science, Royal Holloway
University of London, TW20 0EX Egham, UK

e-mail: shengbinhello@nuaa.edu.cn

Abstract

Let $T(X \cup Y, A)$ be a bipartite tournament with partite sets X, Y and arc set A . For any vertex $x \in X \cup Y$, the second out-neighbourhood $N^{++}(x)$ of x is the set of all vertices with distance 2 from x . In this paper, we prove that T contains at least two vertices x such that $|N^{++}(x)| \geq |N^+(x)|$ unless T is in a special class \mathcal{B}_1 of bipartite tournaments; show that T contains at least a vertex x such that $|N^{++}(x)| \geq |N^-(x)|$ and characterize the class \mathcal{B}_2 of bipartite tournaments in which there exists exactly one vertex x with this property; and prove that if $|X| = |Y|$ or $|X| \geq 4|Y|$, then the bipartite tournament T contains a vertex x such that $|N^{++}(x)| + |N^+(x)| \geq 2|N^-(x)|$.

Keywords: second out-neighbourhood, out-neighbourhood, in-neighbourhood, bipartite tournament.

2010 Mathematics Subject Classification: 05C20, 05C12, 05C07.

1. TERMINOLOGY AND INTRODUCTION

We will assume that the reader is familiar with the standard terminology on digraphs and refer to [1] for terminology not discussed here. In this paper, all digraphs have no multiple arcs and no loops.

We denote the vertex set and the arc set of a digraph D by $V(D)$ and $A(D)$, respectively. For a vertex subset X , we denote by $D\langle X \rangle$ the subdigraph of D induced by X , $D\langle V(D) - X \rangle$ by $D - X$. In addition, $D - x = D - \{x\}$ for a vertex x of D .

Let x, y be distinct vertices in D . If there is an arc from x to y then we say that x *dominates* y , write $x \rightarrow y$ and call y (respectively, x) an *out-neighbour* (respectively, an *in-neighbour*) of x (respectively, y). For a subdigraph or simply a vertex subset H of D (possibly, $H = D$), we let $N_H^+(x)$ (respectively, $N_H^-(x)$) denote the set of out-neighbours (respectively, the set of in-neighbours) of x in H and call it *out-neighbourhood* (respectively, *in-neighbourhood*) of x in H . Furthermore, $d_H^+(x) = |N_H^+(x)|$ (respectively, $d_H^-(x) = |N_H^-(x)|$) is called the *out-degree* (respectively, *in-degree*) of x in H . Let

$$N_H^{++}(x) = \bigcup_{u \in N_H^+(x)} N_H^+(u) - N_H^+(x),$$

which is called the *second out-neighbourhood* of x in H . Furthermore, $d_H^{++}(x) = |N_H^{++}(x)|$. We will omit the subscript if $H = D$ is known from the context.

Let X, Y be two disjoint subsets of vertices of D . We let $E(X, Y)$ denote the set of all arcs with head in Y and tail in X . If $E(Y, X) = \emptyset$ and $x \rightarrow y$ for all $x \in X$ and $y \in Y$, then we say that X *completely dominates* Y and denote this by $X \rightarrow Y$.

An *oriented graph* is a digraph with no cycle of length two. One of the most interesting and challenging open questions concerning digraphs is Seymour's Second Neighbourhood Conjecture (SSNC) (see [5] and Problem 325, page 804 in volume 197/198 (1999) of *Discrete Mathematics*), which asserts that one can always find, in an oriented graph D , a vertex x whose second out-neighbourhood is at least as large as its out-neighbourhood.

Conjecture 1 (Seymour's Second Neighbourhood Conjecture). *In every oriented graph D , there exists a vertex x such that $d^{++}(x) \geq d^+(x)$.*

Following [4], we will call such a vertex x a *Seymour vertex*.

Note that if we allow 2-cycles, then SSNC is no longer true as can be seen by taking the complete digraph \overleftrightarrow{K}_n . Note also that SSNC trivially holds for digraphs D which contain a vertex of out-degree zero, e.g. for acyclic digraphs.

A *tournament* is an oriented graph where every pair of distinct vertices are adjacent. SSNC in the case of tournaments was also stated by Dean and Latka [5].

This special case of the conjecture was proved by Fisher [7] using Farkas' Lemma and averaging arguments.

Theorem 2 [7]. *In any tournament, there is a Seymour vertex.*

A more elementary proof of SSNC for tournaments was given by Havet and Thomassé [10] who introduced a median order approach. Their proof also yields the following stronger result.

Theorem 3 [10]. *A tournament with no vertex of out-degree zero has at least two Seymour vertices.*

Kaneko and Locke [11] proved SSNC for oriented graphs with minimum out-degree at most 6. Fidler and Yuster [6] further developed the median order approach and proved that SSNC holds for oriented graphs D with minimum degree $|V(D)| - 2$, tournaments minus a star, and tournaments minus the arc set of a subtournament. The median order approach was also used by Ghazal [8] who proved a weighted version of SSNC for tournaments missing a generalized star. Cohn, Godbole, Wright Harkness, and Zhang [4] proved that the conjecture holds for random oriented graphs. Recently, Gutin and Li [9] proved SSNC for quasi-transitive oriented digraphs which is a superclass of tournaments and transitive acyclic digraphs. Another approach to SSNC is to determine the maximum value γ such that in every oriented graph D , there exists a vertex x such that $d^+(x) \leq \gamma d^{++}(x)$. SSNC asserts that $\gamma = 1$. Chen, Shen, and Yuster [3] proved that $\gamma \geq r$ where $r = 0.657298\dots$ is the unique real root of $2x^3 + x^2 - 1 = 0$. They also claim a slight improvement to $r \geq 0.67815\dots$

Sullivan [13] stated the following "compromise conjectures" on SSNC, where $d^-(v)$ is used instead of or together with $d^+(v)$.

Conjecture 4 [13].

- (1) *Every oriented graph D has a vertex x such that $d^{++}(x) \geq d^-(x)$.*
- (2) *Every oriented graph D has a vertex x such that $d^{++}(x) + d^+(x) \geq 2d^-(x)$.*

For convenience, a vertex x satisfying Conjecture 4(i) is called a *Sullivan- i vertex* for $i = 1, 2$. Recently, we show that these conjectures hold for quasi-transitive oriented graphs. See [14].

A *bipartite tournament* is an oriented graph defined as an orientation of a complete bipartite graph. $T(X \cup Y, A)$ will denote a bipartite tournament with partite sets X, Y and arc set A . When no confusion arises the short form T will be used. In this paper, we consider Conjecture 1 and 4 for bipartite tournaments. It is not difficult to see that each vertex of minimum out-degree is a Seymour vertex in a bipartite tournament. In Section 2, we characterize the class of bipartite tournaments in which there exists exactly one Seymour vertex. In Section 3, we show that any bipartite tournament contains a Sullivan-1 vertex and characterize

the class of bipartite tournaments in which there exists exactly one Sullivan-1 vertex. In Section 4, we prove that if $|X| = |Y|$ or $|X| \geq 4|Y|$, then the bipartite tournament T contains a Sullivan-2 vertex.

2. SSNC FOR BIPARTITE TOURNAMENTS

We consider SSNC for bipartite tournaments. Let $T(X \cup Y, A)$ be a bipartite tournament. For any two vertices x, y of a bipartite tournament T , if $x \rightarrow y$, then $N^+(y) \subseteq N^{++}(x)$. So we can obtain the following observation immediately.

Lemma 5. *Let T be a bipartite tournament and x, y two vertices of T . If $x \rightarrow y$ and $d^+(y) \geq d^+(x)$, then x is a Seymour vertex of T .*

Moreover, SSNC is true for bipartite tournaments. In fact, in a bipartite tournament, each vertex of minimum out-degree is a Seymour vertex due to Lemma 5. Similarly to the Theorem 3 on tournaments, we have the following result on bipartite tournaments.

Lemma 6. *A bipartite tournament with no vertex of out-degree zero has at least two Seymour vertices.*

Proof. Let $T = (X \cup Y, A)$ be a bipartite tournament with no vertex of out-degree zero. Without loss of generality, assume that $x \in X$ is a vertex of minimum out-degree in T . Then x is a Seymour vertex of T , so we need to find another vertex with this property. Let $T_r = T - x$ and y a vertex of minimum out-degree in T_r . Then y is a Seymour vertex of the bipartite tournament T_r . We claim that

(1) If $y \in X$ or $y \in Y, x \rightarrow y$, then y is also a Seymour vertex of T .

In fact, in both cases, $d^{++}(y) \geq d_{T_r}^{++}(y) \geq d_{T_r}^+(y) = d^+(y)$. So assume that $y \in Y$ and $y \rightarrow x$.

For the case when $N_{T_r}^+(y) = \emptyset$, we have $d_{T_r}^+(y) = 1$. Recall that the out-degree of x is not zero. Hence $d^{++}(y) \geq d^+(x) = d^+(y)$ and y is another Seymour vertex of T . For the case when $N_{T_r}^+(y) \neq \emptyset$, let $z \in N_{T_r}^+(y)$. Clearly, $z \in X$ and $d_{T_r}^+(z) \geq d_{T_r}^+(y)$. Note that $d_{T_r}^+(z) = d_{T_r}^+(y)$ implies that z is also a vertex of minimum out-degree in T_r . By (1), z is another Seymour vertex of T . So assume that $d_{T_r}^+(z) > d_{T_r}^+(y)$. Since $N_{T_r}^+(z) \subseteq N_{T_r}^{++}(y)$, we have

$$d^{++}(y) = d_{T_r}^{++}(y) \geq d_{T_r}^+(z) \geq d_{T_r}^+(y) + 1 = d^+(y).$$

y is another Seymour vertex. The lemma holds. ■

Let $T = (X \cup Y, A)$ be a bipartite tournament. According to the out-degree of each vertex of T , we give a partition V_1, \dots, V_k of the vertex set $X \cup Y$ of T such that

- (a) $d^+(u) = d^+(v)$ for any $1 \leq i \leq k$ and any $u, v \in V_i$;
- (b) $d^+(u_i) < d^+(u_j)$ for any $1 \leq i < j \leq k$ and any $u_i \in V_i$ and $u_j \in V_j$.

We call the unique sequence V_1, \dots, V_k satisfying the statement (a) and (b) the *out-degree sequence* of T .

Now we consider a special class \mathcal{B}_1 of bipartite tournaments. $T \in \mathcal{B}_1$ if and only if T is a bipartite tournament with the out-degree sequence V_1, \dots, V_k satisfying that

- $|V_1| = 1$ and $|V_1| + |V_3| + \dots + |V_{2i-1}| < |V_2| + |V_4| + \dots + |V_{2i}| < |V_1| + |V_3| + \dots + |V_{2i+1}|$ for any $1 \leq i \leq \lceil \frac{k}{2} \rceil - 1$;
- all V_i 's for i odd are contained in a common partite set and all V_j 's for j even are contained in the other common partite set;
- $V_i \rightarrow V_2, V_4, \dots, V_{i-1}$ for any i odd and $V_j \rightarrow V_1, V_3, \dots, V_{j-1}$ for any j even.

It is not difficult to check that $v \in V_1$ is the only Seymour vertex of T . See two examples of the class \mathcal{B}_1 in Figure 1.

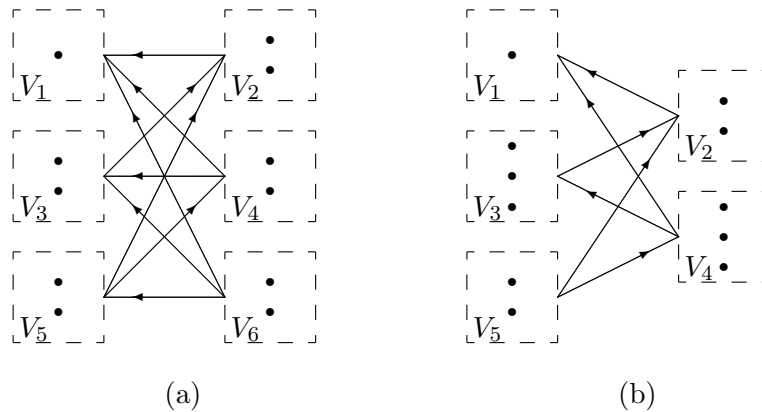


Figure 1. Two bipartite tournaments in \mathcal{B}_1 . The dashed boxes indicate the partition of the vertex set of a bipartite tournament and an arc from a box V_i to a box V_j between two boxes indicates $V_i \rightarrow V_j$.

Theorem 7. *A bipartite tournament T has at least two Seymour vertices unless $T \in \mathcal{B}_1$.*

Proof. Let $T(X \cup Y, A)$ be a bipartite tournament. Suppose T has exactly one Seymour vertex. We will show that $T \in \mathcal{B}_1$. Let V_1, \dots, V_k be the out-degree sequence of T . Without loss of generality, assume that k is even since the proof is very similar when k is odd. Recall that a vertex of minimum out-degree is a Seymour vertex and each vertex of V_1 has the minimum out-degree in T . So $|V_1| = 1$. Lemma 6 shows that $V_2, V_4, \dots, V_k \rightarrow V_1$.

We claim that either $V_i \subseteq X$ or $V_i \subseteq Y$ for any $1 \leq i \leq k$. Suppose not. Let $u, v \in V_i$ but $u \in X, v \in Y$. Clearly, $i \geq 2$. By Lemma 5, $u \rightarrow v$ implies that u is a Seymour vertex and $v \rightarrow u$ implies that v is a Seymour vertex. In both cases, T has two Seymour vertices. Hence $V_i \subseteq X$ or $V_i \subseteq Y$ for any $1 \leq i \leq k$.

We also claim that V_i and V_{i+1} are contained in different partite sets. Suppose to the contrary that $V_i, V_{i+1} \subseteq X$. For any $v_i \in V_i$ and $v_{i+1} \in V_{i+1}$, there exists a vertex $y \in Y$ such that $v_{i+1} \rightarrow y \rightarrow v_i$ since $d^+(v_{i+1}) > d^+(v_i)$. Since neither v_{i+1} nor y is a Seymour vertex, we have $d^+(v_{i+1}) > d^+(y) > d^+(v_i)$ by Lemma 5. This contradicts the definition of V_1, V_2, \dots, V_k . Hence V_i and V_{i+1} are contained in different partite sets.

For convenience, assume $V_1 \subseteq X$. The claims above show that $V_i \subseteq X$ for any i odd and $V_j \subseteq Y$ for any j even. Also for any V_i, V_j with $i < j$, either V_i, V_j are nonadjacent or $V_j \rightarrow V_i$ by Lemma 5 and the fact that T has exactly one Seymour vertex. This means that $V_i \rightarrow V_2, V_4, \dots, V_{i-1}$ for any i odd and $V_j \rightarrow V_1, V_3, \dots, V_{j-1}$ for any j even.

Now for any $1 \leq i \leq \lceil \frac{k}{2} \rceil - 1$ and for any $u \in V_{2i+1}$ and $v \in V_{2i+2}$, we see that

$$\begin{aligned} N^+(u) &= V_2 \cup V_4 \cup \dots \cup V_{2i}, & N^{++}(u) &= V_1 \cup V_3 \cup \dots \cup V_{2i-1}, \\ N^+(v) &= V_1 \cup V_3 \cup \dots \cup V_{2i+1}, & N^{++}(v) &= V_2 \cup V_4 \cup \dots \cup V_{2i}. \end{aligned}$$

Since T has exactly one Seymour vertex, we have $d^{++}(u) < d^+(u)$ and $d^{++}(v) < d^+(v)$. This means that

$$|V_1| + |V_3| + \dots + |V_{2i-1}| < |V_2| + |V_4| + \dots + |V_{2i}| < |V_1| + |V_3| + \dots + |V_{2i+1}|.$$

Thus $T \in \mathcal{B}_1$ and the theorem follows. ■

3. SULLIVAN’S CONJECTURE (1) FOR BIPARTITE TOURNAMENTS

We consider Conjecture 4(1) for bipartite tournaments. We begin with two observations.

Lemma 8. *Let T be a bipartite tournament and x, y two vertices of T . If $x \rightarrow y$ and $d^+(y) \geq d^-(x)$, then x is a Sullivan-1 vertex.*

Proof. Note that $N^+(y) \subseteq N^{++}(x)$. Then $d^{++}(x) \geq d^+(y) \geq d^-(x)$. ■

Lemma 9. *Let $T = (X \cup Y, A)$ be a bipartite tournament. If $|E(Y, X)| \geq |E(X, Y)|$, then there exists a vertex $y \in Y$ such that $d^+(y) \geq d^-(y)$.*

Proof. Suppose $d^+(y) < d^-(y)$ for any $y \in Y$. Then

$$E(Y, X) = \sum_{y \in Y} d^+(y) < \sum_{y \in Y} d^-(y) = |E(X, Y)|,$$

a contradiction. Thus there exists a vertex $y \in Y$ such that $d^+(y) \geq d^-(y)$. ■

Now we show that Conjecture 4(1) is true in the case of bipartite tournaments.

Theorem 10. *Any bipartite tournament has a Sullivan-1 vertex.*

Proof. Let $T = (X \cup Y, A)$ be a bipartite tournament. Without loss of generality, assume $|E(Y, X)| \geq |E(X, Y)|$. Then by Lemma 9, there exists a vertex $y \in Y$ such that $d^+(y) \geq d^-(y)$. Let $y_0 \in Y$ such that y_0 has maximum out-degree among the vertices of Y . Clearly, $d^+(y_0) \geq d^-(y_0)$. We give a partition of the vertex set $X \cup Y$ of T . Set

$$V_1 = N^-(y_0), \quad V_2 = N^+(y_0), \quad V_3 = N^{++}(y_0), \quad V_4 = Y - V_3$$

and $t_i = |V_i|$ for $i = 1, 2, 3, 4$. We claim that $V_1 \rightarrow V_4 \rightarrow V_2$. In fact, $V_3 = N^{++}(y_0) = \bigcup_{x \in V_2} N^+(x)$ implies $V_4 \rightarrow V_2$. Moreover, since y_0 has maximum out-degree in Y , we have $d^+(y) \leq d^+(y_0)$ for any $y \in V_4$. Note that $y \rightarrow V_2 = N^+(y_0)$. We have $N^+(y_0) \subseteq N^+(y)$. So $N^+(y) = N^+(y_0)$ and hence $N^-(y) = N^-(y_0)$ for any $y \in V_4$. Thus $V_1 \rightarrow V_4$. See Figure 2(a).

Now we will prove the following claim which directly implies the result.

Claim A. *Either y_0 or $w \in N^-(y_0)$ is a Sullivan-1 vertex. Moreover, if y_0 is not a Sullivan-1 vertex, then $d^{++}(w) > d^-(w)$.*

If $t_3 \geq t_1$, then $d^{++}(y_0) \geq d^-(y_0)$ and y_0 is a Sullivan-1 vertex. We are done. So assume $t_3 < t_1$. Since $d^+(y_0) \geq d^-(y_0)$, we have $t_1 \leq t_2$. For any $w \in V_1$, $N^-(w) \subseteq V_3$ and $V_2 \subseteq N^{++}(w)$. Now

$$d^{++}(w) \geq t_2 \geq t_1 > t_3 \geq d^-(w).$$

w is a Sullivan-1 vertex in T . The theorem follows. ■

We consider a special class \mathcal{B}_2 of bipartite tournaments. $T \in \mathcal{B}_2$ if and only if T is a bipartite tournament with two partite sets X and Y such that $x \rightarrow Y \rightarrow X - x$ (possibly, $X - x = \emptyset$) for some $x \in X$. See Figure 2(b). It is not difficult to check that x is the only Sullivan-1 vertex of T .

Theorem 11. *Any bipartite tournament has at least two Sullivan-1 vertices unless $T \in \mathcal{B}_2$.*

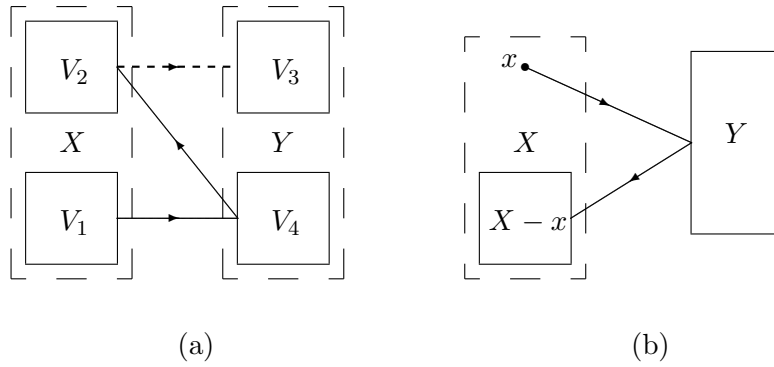


Figure 2. (a) A partition of the vertex set of a bipartite tournament $T = (X \cup Y, A)$. For any vertex $y \in V_1$, $d^+(y) \geq d^-(y)$ and y has the maximum out-degree among all vertices in Y . $V_1 = N^-(y), V_2 = N^+(y), V_3 = N^{++}(y), V_4 = Y - V_3$. An dotted arc from a box V_2 to a box V_3 indicates $N^+(V_2) = V_3$. $V_1 \rightarrow V_4 \rightarrow V_2$. (b) A bipartite tournament in \mathcal{B}_2 . $x \rightarrow Y \rightarrow X - x$.

Proof. Let $T = (X \cup Y, A)$ be a bipartite tournament. Suppose T has exactly one Sullivan-1 vertex. It is sufficient to show that $T \in \mathcal{B}_2$. Without loss of generality, assume $|E(Y, X)| \geq |E(X, Y)|$. Let y_0, V_i and t_i be defined as in the proof of Theorem 10. Then $d^+(y_0) \geq d^-(y_0)$ and $V_1 \rightarrow V_4 \rightarrow V_2$. We consider the following two cases.

Case 1. $t_3 \geq t_1$. Clearly, each vertex of V_4 is a Sullivan-1 vertex. So $|V_4| = 1$ and $V_4 = \{y_0\}$. Let $T_r = T - y_0$.

Subcase 1.1. There is a vertex $y \in Y - y_0$ such that $d^+(y) \geq d^-(y)$. Let y_1 be the vertex of maximum out-degree in $Y - y_0$. Then $d^+(y_1) \geq d^-(y_1)$. Clearly, $y_1 \in V_3$ and $d_{T_r}^+(y_1) = d^+(y_1) \geq d^-(y_1) = d_{T_r}^-(y_1)$. Applying Claim A of the proof of Theorem 10 to the bipartite tournament T_r , either y_1 is a Sullivan-1 vertex or $w \in N_{T_r}^-(y_1)$ is a Sullivan-1 vertex of T_r . And if y_1 is not a Sullivan-1 vertex of T_r , then $d_{T_r}^{++}(w) > d_{T_r}^-(w)$.

For the case when y_1 is a Sullivan-1 vertex of T_r , we have $d^{++}(y_1) \geq d_{T_r}^{++}(y_1) \geq d_{T_r}^-(y_1) = d^-(y_1)$. So y_1 is also Sullivan-1 vertex of T . For the case when y_1 is not a Sullivan-1 vertex of T_r , we have $w \in N_{T_r}^-(y_1)$ is a Sullivan-1 vertex and $d_{T_r}^{++}(w) > d_{T_r}^-(w)$. Now $d^{++}(w) \geq d_{T_r}^{++}(w) \geq d_{T_r}^-(w) + 1 \geq d^-(w)$. So w is also Sullivan-1 vertex of T .

Subcase 1.2. For any vertex $y \in Y - y_0$, $d^+(y) < d^-(y)$. In the bipartite tournament T_r , we see that

$$|E(X, Y - y_0)| = \sum_{y \in Y - y_0} d^-(y) > \sum_{y \in Y - y_0} d^+(y) = |E(Y - y_0, X)|.$$

By Lemma 9, there exists a vertex $x \in X$ such that $d_{T_r}^+(x) \geq d_{T_r}^-(x)$. Let $x_0 \in X$ be the vertex of maximum out-degree among the vertices of X in T_r . Clearly, $d_{T_r}^+(x_0) \geq d_{T_r}^-(x_0)$. Similarly to the proof of Theorem 10, set

$$V'_1 = N_{T_r}^-(x_0), \quad V'_2 = N_{T_r}^+(x_0), \quad V'_3 = N_{T_r}^{++}(x_0), \quad V'_4 = Y - y_0 - V'_3.$$

Let $t'_i = |V'_i|$ for $i = 1, 2, 3, 4$. By Claim A of the proof of Theorem 10, either x_0 is a Sullivan-1 vertex in T_r or $z \in N_{T_r}^-(x_0)$ is a Sullivan-1 vertex of T_r . For the case when $z \in N_{T_r}^-(x_0)$ is a Sullivan-1 vertex of T_r , we have $d^{++}(z) \geq d_{T_r}^{++}(z) \geq d_{T_r}^-(z) = d^-(z)$. Then z is also a Sullivan-1 vertex of T . For the case when x_0 is a Sullivan-1 vertex in T_r , we have $t'_3 \geq t'_1$. Note that $t'_3 > t'_1$ implies that $d^{++}(x_0) \geq d_{T_r}^{++}(x_0) \geq d_{T_r}^-(x_0) + 1 \geq d^-(x_0)$. Then x_0 is also a Sullivan-2 vertex of T_r . So assume $t'_3 = t'_1$. Recall that $t'_1 \leq t'_2$. So $t'_3 \leq t'_2$. On the other hand, z is not a Sullivan-1 vertex of T_r implies that $t'_2 \leq d_{T_r}^{++}(z) < d_{T_r}^-(z) \leq t'_3$, a contradiction.

In any case, we get a contradiction. Thus Case 1 is impossible.

Case 2. $t_3 < t_1$. Clearly, any vertex $y \in V_4$ is not a Sullivan-1 vertex. So any vertex $w \in V_1$ is a Sullivan-1 vertex and $d^{++}(w) > d^-(w)$ by Claim A of the proof of Theorem 10. Since T has exactly one Sullivan-1 vertex, we have $t_1 = 1$. So $t_3 = 0$ and V_3 is an empty set. Thus $w \rightarrow Y \rightarrow X - w$ (possibly, $X - w = \emptyset$) and $T \in \mathcal{B}_2$. The theorem follows. ■

4. SUPPORT FOR SULLIVAN'S CONJECTURE (2) ON BIPARTITE TOURNAMENTS

The results in Section 4 provide support for Conjecture 4(2) on bipartite tournaments.

Lemma 12. *Let $T = (X \cup Y, A)$ be a bipartite tournament with $|X| \leq |Y|$. If there exists a vertex $y \in Y$ such that $d^+(y) \geq d^-(y)$, then T has a Sullivan-2 vertex.*

Proof. Choose $y_0 \in Y$ such that y_0 has maximum out-degree among the vertices of Y . By the assumption, $d^+(y_0) \geq d^-(y_0)$. Let V_i and t_i be defined as in the proof of Theorem 10. Then $|X| \leq |Y|$ implies that $t_1 + t_2 \leq t_3 + t_4$. Recall that $t_2 \geq t_1$. If y_0 is a Sullivan-2 vertex of T , we are done. So assume that $d^{++}(y_0) + d^+(y_0) < 2d^-(y_0)$, i.e., $t_2 + t_3 < 2t_1$. So $t_3 < t_1 \leq t_2$. For any $w \in N^-(y_0)$, suppose that w is also not a Sullivan-2 vertex of T . We have $d^{++}(w) + d^+(w) < 2d^-(w)$, which means $t_2 + t_4 < 2t_3$. So $t_4 < t_3$. Now $t_3 + t_4 < 2t_3 < t_1 + t_2$, a contradiction. Thus either y_0 or $w \in N^-(y_0)$ is a Sullivan-2 vertex. The lemma follows. ■

Corollary 13. *Any balance bipartite tournament has a Sullivan-2 vertex.*

Proof. Let $T = (X \cup Y, A)$ be a balance bipartite tournament. Then $|X| = |Y|$. By Lemma 9, there exists a vertex $u \in X \cup Y$ such that $d^+(u) \geq d^-(u)$. Now Lemma 12 yields the result. ■

Lemma 14. *Let $T = (X \cup Y, A)$ be a bipartite tournament. If there exists a vertex $x \in X$ such that $d^+(x) \geq 2|X| - 3$, then any $y \in N^-(x)$ is a Sullivan-2 vertex.*

Proof. Note that $N^+(x) \subseteq N^{++}(y)$. So $d^{++}(y) \geq 2|X| - 3$. Thus $d^{++}(y) + d^+(y) \geq 2|X| - 3 + 1 \geq 2d^-(y)$ and y is a Sullivan-2 vertex of T . ■

Corollary 15. *Let $T = (X \cup Y, A)$ be a bipartite tournament. If $|E(X, Y)| \geq 2|X|^2$, then there is a vertex $x \in X$ such that any $y \in N^-(x)$ is a Sullivan-2 vertex.*

Proof. Since $|E(X, Y)| = \sum_{x \in X} d^+(x) \geq 2|X|^2$, there is a vertex $x \in X$ such that $d^+(x) \geq 2|X|$. By Lemma 14, any $y \in N^-(x)$ is a Sullivan-2 vertex. ■

Corollary 16. *A bipartite tournament $T = (X \cup Y, A)$ with $|Y| \geq 4|X|$ has a Sullivan-2 vertex.*

Proof. By Lemma 9, there exists a vertex $u \in X \cup Y$ such that $d^+(u) \geq d^-(u)$. If $u \in Y$, by Lemma 12, T has a Sullivan-2 vertex and we are done. So assume $u \in X$. Now $d^+(u) \geq \frac{|Y|}{2} \geq 2|X|$. By Lemma 14, any $y \in N^-(u)$ is a Sullivan-2 vertex of T . ■

REFERENCES

- [1] J. Bang-Jensen and G. Gutin, *Digraphs: Theory, Algorithms and Applications*, 2nd Ed. (Springer-Verlag, London, 2009).
- [2] J. Bang-Jensen and J. Huang, *Quasi-transitive digraphs*, *J. Graph Theory* **20** (1995) 141–161.
doi:10.1002/jgt.3190200205
- [3] G. Chen, J. Shen and R. Yuster, *Second neighborhood via first neighborhood in digraphs*, *Ann. Combin.* **7** (2003) 15–20.
doi:10.1007/s000260300001
- [4] Z. Cohn, A. Godbole, E. Wright Harkness and Y. Zhang, *The number of Seymour vertices in random tournaments and digraphs*, *Graphs Combin.* **32** (2016) 1805–1816.
doi:10.1007/s00373-015-1672-9
- [5] N. Dean and B.J. Latka, *Squaring the tournament—an open problem*, *Congr. Numer.* **109** (1995) 73–80.

- [6] D. Fidler and R. Yuster, *Remarks on the second neighborhood problem*, J. Graph Theory **55** (2007) 208–220.
doi:10.1002/jgt.v55:3
- [7] D.C. Fisher, *Squaring a tournament: a proof of Dean’s conjecture*, J. Graph Theory **23** (1996) 43–48.
doi:10.1002/(SICI)1097-0118(199609)23:1(43::AID-JGT4)3.0.CO;2-K
- [8] S. Ghazal, *Seymour’s second neighbourhood conjecture for tournaments missing a generalized star*, J. Graph Theory **71** (2012) 89–94.
doi:10.1002/jgt.20634
- [9] G. Gutin and R. Li, *Seymour’s second neighbourhood conjecture for quasi-transitive oriented graphs*.
arxiv.org/abs/1704.01389.
- [10] F. Havet and S. Thomassé, *Median orders of tournaments: a tool for the second neighborhood problem and Sumner’s conjecture*, J. Graph Theory **35** (2000) 244–256.
doi:10.1002/1097-0118(200012)35:4(244::AID-JGT2)3.0.CO;2-H
- [11] Y. Kaneko and S.C. Locke, *The minimum degree approach for Paul Seymour’s distance 2 conjecture*, Congr. Numer. **148** (2001) 201–206.
- [12] J.W. Moon, *Solution to problem 463*, Math. Mag. **35** (1962) 189.
doi:10.2307/2688555
- [13] B. Sullivan, *A summary of results and problems related to the Caccetta-Häggkvist conjecture*.
arxiv.org/abs/math/0605646.
- [14] R. Li and B. Sheng, *The second neighbourhood for quasi-transitive oriented graphs*, submitted.

Received 12 June 2017

Revised 13 October 2017

Accepted 13 October 2017