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# ORIENTABLE $\mathbb{Z}_N$-DISTANCE MAGIC GRAPHS

Sylwia Cichacz[1]

*AGH University of Science and Technology*
*al. A. Mickiewicza 30, 30-059 Kraków, Poland*

**e-mail:** cichacz@agh.edu.pl

Bryan Freyberg

*Southwest Minnesota State University*
*Marshall, MN, USA*

**e-mail:** bryan.freyberg@smsu.edu

and

Dalibor Froncek

*University of Minnesota Duluth*
*Duluth, MN, USA*

**e-mail:** dalibor@d.umn.edu

**Abstract**

Let $G = (V, E)$ be a graph of order $n$. A distance magic labeling of $G$ is a bijection $\ell\colon V \to \{1, 2, \ldots, n\}$ for which there exists a positive integer $k$ such that $\sum_{x \in N(v)} \ell(x) = k$ for all $v \in V$, where $N(v)$ is the open neighborhood of $v$.

Tuttes flow conjectures are a major source of inspiration in graph theory. In this paper we ask when we can assign $n$ distinct labels from the set $\{1, 2, \ldots, n\}$ to the vertices of a graph $G$ of order $n$ such that the sum of the labels on heads minus the sum of the labels on tails is constant modulo $n$ for each vertex of $G$. Therefore we generalize the notion of distance magic labeling for oriented graphs.

**Keywords:** distance magic graph, digraph, flow graph.

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## 1. Introduction

All graphs considered in this paper are simple finite graphs. Consider a simple graph $G$. We denote by $V(G)$ the vertex set and $E(G)$ the edge set of $G$. We denote the order of $G$ by $|V(G)| = n$. The *open neighborhood* $N(x)$ of a vertex $x$ is the set of vertices adjacent to $x$, and the degree $d(x)$ of $x$ is $|N(x)|$, the size of the neighborhood of $x$. By $C_n$ we denote a cycle on $n$ vertices.

In this paper we investigate distance magic labelings, which belong to a large family of magic-type labelings. Generally speaking, a magic-type labeling of a graph $G = (V, E)$ is a mapping from $V, E$, or $V \cup E$ to a set of labels which most often is a set of integers or group elements. Then the weight of a graph element is typically the sum of labels of the neighboring elements of one or both types. If the weight of each element is required to be equal, then we speak about magic-type labeling; when the weights are all different (or even form an arithmetic progression), then we speak about an antimagic-type labeling. Probably the best known problem in this area is the *antimagic conjecture* by Hartsfield and Ringel [11], which claims that the edges of every graph except $K_2$ can be labeled by integers $1, 2, \ldots, |E|$ so that the weight of each vertex is different. A comprehensive dynamic survey of graph labelings is maintained by Gallian [10]. A more detailed survey related to our topic by Arumugam *et al.* [1] was published recently.

A *distance magic labeling* (also called *sigma labeling*) of a graph $G = (V, E)$ of order $n$ is a bijection $\ell \colon V \to \{1, 2, \ldots, n\}$ with the property that there is a positive integer $k$ (called the *magic constant*) such that

$$w(x) = \sum_{y \in N_G(x)} \ell(y) = k \text{ for every } x \in V(G),$$

where $w(x)$ is the *weight* of vertex $x$. If a graph $G$ admits a distance magic labeling, then we say that $G$ is a *distance magic graph*.

The following observations were proved independently.

**Observation 1** [13, 15–17]**.** *Let $G$ be an $r$-regular distance magic graph on $n$ vertices. Then $k = \frac{r(n+1)}{2}$.*

**Observation 2** [13, 15–17]**.** *There is no distance magic $r$-regular graph with $r$ odd.*

The notion of group distance magic labeling of graphs was introduced in [9]. A $\Gamma$-distance magic labeling of a graph $G = (V, E)$ with $|V| = n$ is an injection from $V$ to an Abelian group $\Gamma$ of order $n$ such that the weight of every vertex evaluated under group operation $x \in V$ is equal to the same element $\mu \in \Gamma$. Some families of graphs that are $\Gamma$-distance magic were studied in [4–6, 9].

An *orientation* of an undirected graph $G = (V, E)$ is an assignment of a direction to each edge, turning the initial graph into a directed graph $\overrightarrow{G} = (V, A)$. An arc $\overrightarrow{xy}$ is considered to be directed from $x$ to $y$, moreover $y$ is called the *head* and $x$ is called the *tail* of the arc. For a vertex $x$, the set of head endpoints adjacent to $x$ is denoted by $N^-(x)$, and the set of tail endpoints adjacent to $x$ denoted by $N^+(x)$. Let $\deg^-(x) = |N^-(x)|$, $\deg^+(x) = |N^+(x)|$ and $\deg(x) = \deg^-(x) + \deg^+(x)$.

Bloom and Hsu defined graceful labelings on directed graphs [2]. Later Bloom *et al.* also defined magic labelings on directed graphs [3]. Probably the biggest challenge (among directed graphs) are Tutte's flow conjectures. An $H$-flow on $D$ is an assignment of values of $H$ to the edges of $D$, such that for each vertex $v$, the sum of the values on the edges going in is the same as the sum of the values on the edges going out of $v$. The 3-flow conjecture says that every 4-edge-connected graph has a nowhere-zero 3-flow (what is equivalent that it has an orientation such that each vertex has the same outdegree and indegree modulo 3). In this paper we ask when we can assign $n$ distinct labels from the set $\{1, 2, \ldots, n\}$ to the vertices of a graph $G$ of order $n$ such that the sum of the labels on heads minus the sum of the labels on tails is constant modulo $n$ for each vertex of $G$. Therefore we introduce a generalization of distance magic labeling on directed graphs.

Assume $\Gamma$ is an Abelian group of order $n$ with the operation denoted by $+$. For convenience we will write $ka$ to denote $a + a + \cdots + a$ (where the element $a$ appears $k$ times), $-a$ to denote the inverse of $a$ and we will use $a - b$ instead of $a + (-b)$. A *directed* $\Gamma$*-distance magic labeling* of an oriented graph $\overrightarrow{G} = (V, A)$ of order $n$ is a bijection $\overrightarrow{\ell} : V \to \Gamma$ with the property that there is $\mu \in \Gamma$ (called the *magic constant*) such that

$$w(x) = \sum_{y \in N_G^+(x)} \overrightarrow{\ell}(y) - \sum_{y \in N_G^-(x)} \overrightarrow{\ell}(y) = \mu \text{ for every } x \in V(G).$$

If for a graph $G$ there exists an orientation $\overrightarrow{G}$ such that there is a directed $\Gamma$-distance magic labeling $\overrightarrow{\ell}$ for $\overrightarrow{G}$, we say that $G$ is *orientable* $\Gamma$*-distance magic* and the directed $\Gamma$-distance magic labeling $\overrightarrow{\ell}$ we call an *orientable* $\Gamma$*-distance magic labeling.*

The following cycle-related result was proved by Miller, Rodger, and Simanjuntak.

**Theorem 3** [15]**.** *The cycle $C_n$ of length $n$ is distance magic if and only if $n = 4$.*

One can check that $C_n$ is $\Gamma$-distance magic if and only if $n = 4$, however it is not longer true for the case of orientable distance magic labeling (see Figure 1).

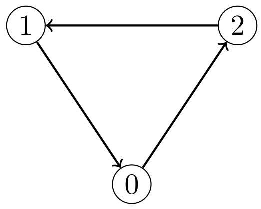


Figure 1. An orientable $\mathbb{Z}_3$-distance magic labeling of $C_3$.

Circulant graphs are an interesting family of vertex-transitive graphs. These graphs arise in various settings; for instance, they are the Cayley graphs over the cyclic group of order $n$. The circulant graph $C_n(s_1, s_2, \ldots, s_k)$ for $0 \leq s_1 < s_2 < \cdots < s_k \leq n/2$ is the graph on the vertex set $V = \{x_0, x_1, \ldots, x_{n-1}\}$ with edges $(x_i, x_{i+s_j})$ for $i = 0, \ldots, n-1$, $j = 1, \ldots, k$ where $i + s_j$ is taken modulo $n$.

We recall three graph products (see [12]). All three, the *Cartesian product* $G \Box H$, *lexicographic product* $G \circ H$, *direct product* $G \times H$ are graphs with the vertex set $V(G) \times V(H)$. Two vertices $(g, h)$ and $(g', h')$ are adjacent in:

- $G \Box H$ if and only if $g = g'$ and $h$ is adjacent to $h'$ in $H$, or $h = h'$ and $g$ is adjacent to $g'$ in $G$;
- $G \times H$ if $g$ is adjacent to $g'$ in $G$ and $h$ is adjacent to $h'$ in $H$;
- $G \circ H$ if and only if either $g$ is adjacent to $g'$ in $G$ or $g = g'$ and $h$ is adjacent to $h'$ in $H$.

For a fixed vertex $g$ of $G$, the subgraph of any of the above products induced by the set $\{(g, h) : h \in V(H)\}$ is called an *H-layer* and is denoted ${}^gH$. Similarly, if $h \in H$ is fixed, then $G^h$, the subgraph induced by $\{(g, h) : g \in V(G)\}$, is a *G-layer*.

In this paper we show some families of orientable $\mathbb{Z}_n$-distance magic graphs.

## 2. Circulant Graphs and Their Products

We start by proving a general theorem for orientable $\Gamma$-distance magic labeling similar to Observation 2.

**Theorem 4.** *Let $G$ have order $n \equiv 2 \pmod 4$ and all vertices of odd degree. There does not exist an orientable $\Gamma$-distance magic labeling of $G$ for any abelian group $\Gamma$ of order $n$.*

***Proof.*** Suppose to the contrary that $G$ is orientable $\Gamma$-distance magic with orientation $\overrightarrow{G}$, orientable $\Gamma$-distance magic labeling $\overrightarrow{\ell}$, and magic constant $\mu$. Since $n \equiv 2 \pmod 4$, say $n = 2n_1n_2 \cdots n_s$ where all $n_i$ are odd, then $\mathbb{Z}_2 \Box \mathbb{Z}_{n_1} \Box \mathbb{Z}_{n_2} \Box$

$\cdots\square\mathbb{Z}_{n_s}$ is isomorphic to any $\mathbb{Z}_{n_1}\square\cdots\square\mathbb{Z}_{2n_i}\square\cdots\square\mathbb{Z}_{n_s}$ as $\gcd(2,n_i)=1$ and it is well known that $\mathbb{Z}_2\square\mathbb{Z}_{n_i}\cong\mathbb{Z}_{2n_i}$. Hence, we may assume that $\Gamma$ is a direct product of cyclic groups containing $\mathbb{Z}_2$. For all $g\in\Gamma$, let $g_0$ denote the $\mathbb{Z}_2$ component of $g$. Similarly, for all $x\in V(G)$, let $w_0(x)$ and $\overrightarrow{\ell_0}(x)$ denote the $\mathbb{Z}_2$ component of $w(x)$ and $\overrightarrow{\ell}(x)$, respectively. Observe that

$$w_0(x)=\sum_{y\in N_G^+(x)}\overrightarrow{\ell_0}(y)-\sum_{y\in N_G^-(x)}\overrightarrow{\ell_0}(y)=\sum_{y\in N_G(x)}\overrightarrow{\ell_0}(y)\text{ for every } x\in V(G).$$

Let $w_0(\overrightarrow{G})=\sum_{x\in V(G)}w_0(x)$. Then clearly $w_0(\overrightarrow{G})=n\mu_0=0$. However, since each vertex has odd degree and $\frac{n}{2}$ is odd, we have $w_0(\overrightarrow{G})=\sum_{x\in V(G)}\sum_{y\in N_G(x)}\overrightarrow{\ell_0}(y)=1$, a contradiction. ■

Notice that the above proof also shows that there exists no Abelian group $\Gamma$ of order $n\equiv 2\pmod 4$ such that $G$ is $\Gamma$-distance magic.

**Corollary 5.** *Let $G$ be an $r$-regular graph on $n\equiv 2\pmod 4$ vertices, where $r$ is odd. There does not exist an orientable $\mathbb{Z}_n$-distance magic labeling for the graph $G$.*

The following example shows that Theorem 4 is not true when $n\equiv 0\pmod 4$. Consider the graph $G=K_{3,3,3,3}$ with the partite sets $A^1=\{x_0^1,x_1^1,x_2^1\}$, $A^2=\{x_0^2,x_1^2,x_2^2\}$, $A^3=\{x_0^3,x_1^3,x_2^3\}$ and $A^4=\{x_0^4,x_1^4,x_2^4\}$. Let $o(uv)$ be the orientation for the edge $uv\in E(G)$ such that

$$o(x_i^jx_k^p)=\begin{cases}\overrightarrow{x_i^2x_0^1} & \text{for } i=0,1,2,\\ \overrightarrow{x_i^1x_k^2} & \text{for } i=1,2,\ k=0,1,2,\\ \overrightarrow{x_i^1x_k^p} & \text{for } i=0,1,2,\ k=0,1,2,\ p=3,4,\\ \overrightarrow{x_i^jx_k^p} & \text{for } i,k=0,1,2,\ 2\le j<p\le 4.\end{cases}$$

Let now

$$\begin{array}{llll}\overrightarrow{\ell}(x_0^1)=3, & \overrightarrow{\ell}(x_0^2)=6, & \overrightarrow{\ell}(x_0^3)=1, & \overrightarrow{\ell}(x_0^4)=11,\\ \overrightarrow{\ell}(x_1^1)=9, & \overrightarrow{\ell}(x_1^2)=2, & \overrightarrow{\ell}(x_1^3)=4, & \overrightarrow{\ell}(x_1^4)=8,\\ \overrightarrow{\ell}(x_2^1)=0, & \overrightarrow{\ell}(x_2^2)=10, & \overrightarrow{\ell}(x_2^3)=7, & \overrightarrow{\ell}(x_2^4)=5.\end{array}$$

Obviously $w(x)=6$ for any $x\in V(G)$.

**Theorem 6.** *If $G=C_n(s_1,s_2,\dots,s_k)$ is a circulant graph such that $s_k<n/2$, then $pG$ is orientable $\mathbb{Z}_{np}$-distance magic for any $p\ge 1$.*

***Proof.*** Note that $G$ is a $2k$-regular graph, because $s_k < n/2$. Let $V^i = x_0^i, x_1^i, \ldots, x_{n-1}^i$ be the set of vertices of the $i$th copy $G^i$ of the graph $G$, $i = 0, 1, \ldots, p-1$. It is easy to see that we can partition $G$ into disjoint cycles $x_j, x_{j+s_h}, x_{j+2s_h}, \ldots, x_j$ of length of the order of the subgroup $\langle s_h \rangle$ for $h \in \{1, 2, \ldots, k\}$ and $j = 0, 1, \ldots, s_h - 1$. Orient each copy of $G$ such that the orientation is clockwise (in which order the subscripts go) around each cycle $x_j, x_{j+s_h}, x_{j+2s_h}, \ldots, x_j$ for $h \in \{1, 2, \ldots, k\}$ and $j = 0, 1, \ldots, s_h - 1$. Set now $\overrightarrow{\ell}(x_m^i) = mp + i$ for $m = 0, 1, \ldots, n-1$, $i = 0, 1, \ldots, p-1$. Obviously $\overrightarrow{\ell}$ is a bijection. Moreover $w(x) = \sum_{y \in N^+(x)} \overrightarrow{\ell}(y) - \sum_{y \in N^-(x)} \overrightarrow{\ell}(y) = -2p \sum_{j=1}^{k} s_j$ for any $x \in V(pG)$. ■

From the above proof of Theorem 6 it is easy to conclude that in general the magic constant for orientable $\mathbb{Z}_n$-distance magic graphs is not unique (just take counterclockwise orientation in each cycle).

**Theorem 7.** *If $G = C_n(s_1, s_2, \ldots, s_k)$ and $H = C_m(s_1', s_2', \ldots, s_p')$ are circulant graph such that $s_k < n/2$, $s_p' < m/2$ and $\gcd(m, n) = 1$, then the Cartesian product $G \square H$ is orientable $\mathbb{Z}_{nm}$-distance magic.*

***Proof.*** Let $V(G) = \{g_0, g_1, \ldots, g_{n-1}\}$, whereas $V(H) = \{x_0, x_1, \ldots, x_{m-1}\}$. As in the proof of Theorem 6 we orient each copy of $H$ (i.e., ${}^gH$-layer for any $g \in V(G)$) such that the orientation is clockwise around each cycle $(g_i, x_j)$, $(g_i, x_{j+s_a'}), (g_i, x_{j+2s_a'}), \ldots, (g_i, x_j)$ for $a = 1, 2, \ldots, p$, $j = 0, 1, \ldots, s_a' - 1$ and $i = 0, 1, \ldots, n-1$, whereas each copy of $G$ (i.e., $G^h$-layer for any $h \in V(H)$) such that the orientation is clockwise around each cycle $(g_i, x_j)$, $(g_{i+s_b}, x_j)$, $(g_{i+2s_b}, x_j), \ldots, (g_i, x_j)$ for $b = 1, 2, \ldots, k$, $i = 0, 1, \ldots, s_b - 1$ and $j = 0, 1, \ldots, m-1$.

Recall that $\mathbb{Z}_n \times \mathbb{Z}_m \cong \mathbb{Z}_{nm}$ because $\gcd(n, m) = 1$. Define $\overrightarrow{\ell} : V(G \square H) \to \mathbb{Z}_n \times \mathbb{Z}_m$ as $\overrightarrow{\ell}(g_i, x_j) = (i, j)$ for $i = 0, 1, \ldots, n-1$, $j = 0, 1, \ldots, m-1$. Obviously $\overrightarrow{\ell}$ is a bijection. Notice that $w(g_i, x_j) = \sum_{y \in N^+(g_i, x_j)} \overrightarrow{\ell}(y) - \sum_{y \in N^-(g_i, x_j)} \overrightarrow{\ell}(y) = \left(-2 \sum_{i=1}^{k} s_i, -2 \sum_{j=1}^{p} s_j'\right)$. Hence we obtain that $G \square H$ is orientable $\mathbb{Z}_{nm}$-distance magic. ■

We will show now some sufficient conditions for the lexicographic product to be orientable $\mathbb{Z}_n$-distance magic.

**Theorem 8.** *Let $H = C_{2n}(s_1, s_2, \ldots, s_k)$ be a circulant graph such that $s_k < n$ and $G$ be a graph of order $t$. The lexicographic product $G \circ H$ is orientable $\mathbb{Z}_{2tn}$-distance magic, if one of the following holds:*

- *graph $G$ has all degrees of vertices of the same parity,*
- *$n$ is even.*

***Proof.*** Let $V(G) = \{g_0, g_1, \ldots, g_{t-1}\}$, whereas $V(H) = \{x_0, x_1, \ldots, x_{2n-1}\}$. Let now $(g_i, x_j) = x_j^i$. As in the proof of Theorem 6 we orient each copy of $H$ (i.e., ${}^gH$-layer for any $g \in V(G)$) such that the orientation is clockwise around each cycle $x_j^i, x_{j+s_a}^i, x_{j+2s_a}^i, \ldots, x_j^i$ for $a = 1, 2, \ldots, k$, $j = 0, 1, \ldots, s_a - 1$ and $i = 0, 1, \ldots, t-1$. If $g_i g_p \in E(G)$ $(i < p)$, then the orientation $o\left(x_j^i x_b^p\right)$ for an edge $x_j^i x_b^p \in E(G \circ H)$ is given in the following way

$$o(x_j^i x_b^p) = \begin{cases} \overrightarrow{x_j^i x_b^p}, & \text{for} \quad j, b < n \ \text{ or } \ j, b \geq n, \\ \overrightarrow{x_b^p x_j^i}, & \qquad\qquad \text{otherwise.} \end{cases}$$

Set now $\overrightarrow{\ell}(x_m^i) = mt + i$ for $m = 0, 1, \ldots, 2n-1$, $i = 0, 1, \ldots, t-1$. Obviously $\overrightarrow{\ell}$ is a bijection. Notice that $w(x_j^i) = \sum_{y \in N^+(x_j^i)} \overrightarrow{\ell}(y) - \sum_{y \in N^-(x_j^i)} \overrightarrow{\ell}(y) = -2t \sum_{j=1}^{k} s_j + \deg(g_i) n(tn)$. If now $\deg(g_i) \equiv c \pmod 2$, then we are done. If $n$ is even, then $n(tn) \equiv 0 \pmod{2tn}$. Hence we obtain that $G \circ H$ is orientable $\mathbb{Z}_{2tn}$-distance magic. ∎

Above we have shown that the lexicographic product $G \circ H$ is orientable $\mathbb{Z}_{tm}$-distance magic when $H$ is a circulant of an even order $m$ and $G$ is of order $t$. One can ask if $G \circ H$ is still orientable $\mathbb{Z}_{tm}$-distance magic if the circulant graph $H$ is of an odd order $m$. A partial answer is given in Theorems 10, 11 and 12. Before we proceed, we will need the following theorem.

**Theorem 9** [14]**.** *Let $n = r_1 + r_2 + \cdots + r_q$ be a partition of the positive integer $n$, where $r_i \geq 2$ for $i = 1, 2, \ldots, q$. Let $A = \{1, 2, \ldots, n\}$. Then the set $A$ can be partitioned into pairwise disjoint subsets $A_1, A_2, \ldots, A_q$ such that for every $1 \leq i \leq q$, $|A_i| = r_i$ with $\sum_{a \in A_i} a \equiv 0 \pmod{n+1}$ if $n$ is even and $\sum_{a \in A_i} a \equiv 0 \pmod{n}$ if $n$ is odd.*

**Theorem 10.** *If $G$ is a graph of odd order $t$, then the lexicographic product $G \circ \overline{K}_{2n+1}$ is orientable $\mathbb{Z}_{t(2n+1)}$-distance magic for $n \geq 1$.*

***Proof.*** Let $V(G) = \{g_0, g_1, \ldots, g_{t-1}\}$, whereas $V(\overline{K}_{2n+1}) = \{x_0, x_1, \ldots, x_{2n}\}$. Give first to the graph $G$ any orientation and now orient the graph $G \circ \overline{K}_{2n+1}$ such that each edge $(g_i, x_j)(g_p, x_h) \in E(G \circ \overline{K}_{2n+1})$ has the corresponding orientation of the edge $g_i g_p \in E(G)$.

Since $t, 2n+1$ are odd, there exists a partition $A_1, A_2, \ldots, A_t$ of the set $\{1, 2, \ldots, (2n+1)t\}$ such that for every $1 \leq i \leq t$, $|A_i| = 2n+1$ with $\sum_{a \in A_i} a \equiv 0 \pmod{(2n+1)t}$ by Theorem 9. Label the vertices of the $i$th copy of $\overline{K}_{2n+1}$ using elements from the set $A_i$ for $i = 1, 2, \ldots, t$.

Notice that $\sum_{j=1}^{2n+1} \overrightarrow{\ell}(g_i, x_j) = 0$ for $i = 1, 2, \ldots, t$. Therefore $w(g_i, x_j) = \sum_{y \in N^+(g_i, x_j)} \overrightarrow{\ell}(y) - \sum_{y \in N^-(g_i, x_j)} \overrightarrow{\ell}(y) = 0$. ∎

**Theorem 11.** *If $G = C_n(s_1, s_2, \ldots, s_k)$ and $H = C_m\big(s'_1, s'_2, \ldots, s'_p\big)$ are circulant graph such that $s_k < n/2$, $s'_p < m/2$ and $\gcd(m, n) = 1$, then lexicographic product $G \circ H$ is orientable $\mathbb{Z}_{nm}$-distance magic.*

***Proof.*** Let $V(G) = \{g_0, g_1, \ldots, g_{n-1}\}$, whereas $V(H) = \{x_0, x_1, \ldots, x_{m-1}\}$. Give first to the graph $G$ the orientation as in the proof of Theorem 6, i.e., $g_i, g_{i+s_b}, g_{i+2s_b}, \ldots, g_i$ for $b = 1, 2, \ldots, k$, $i = 0, 1, \ldots, s_b - 1$. For $i \neq p$ orient now each edge $(g_i, x_j)(g_p, x_h) \in E(G \circ H)$ such that it has the corresponding orientation of the edge $g_i g_p \in E(G)$. Recall that for each vertex $g \in V(G)$ we have $\deg^+(g) = \deg^-(g)$. Each copy of $H$ (i.e., ${}^gH$-layer for any $g \in V(G)$) we orient such that the orientation is clockwise around each cycle $(g_i, x_j), (g_i, x_{j+s'_a}), (g_i, x_{j+2s'_a}), \ldots, (g_i, x_j)$ for $a = 1, 2, \ldots, p$, $j = 0, 1, \ldots, s'_a - 1$ and $i = 0, 1, \ldots, n-1$. Recall that $\mathbb{Z}_n \times \mathbb{Z}_m \cong \mathbb{Z}_{nm}$ because $\gcd(n, m) = 1$. Then define $\overrightarrow{\ell} : V(G \circ H) \to \mathbb{Z}_n \times \mathbb{Z}_m$ as $\overrightarrow{\ell}(g_i, x_j) = (i, j)$ for $i = 0, 1, \ldots, n-1$, $j = 0, 1, \ldots, m-1$. Obviously $\overrightarrow{\ell}$ is a bijection. Notice that $w(g_i, x_j) = \sum_{y \in N^+(g_i, x_j)} \overrightarrow{\ell}(y) - \sum_{y \in N^-(g_i, x_j)} \overrightarrow{\ell}(y) = \Big(-2m \sum_{i=1}^{k} s_i, -2 \sum_{j=1}^{p} s'_j\Big)$. Hence we obtain that $G \circ H$ is orientable $\mathbb{Z}_{nm}$-distance magic. ∎

**Theorem 12.** *The lexicographic product $C_n \circ C_m$ is orientable $\mathbb{Z}_{nm}$-distance magic for all $n$, $m \geq 3$.*

***Proof.*** Let $G = C_n = (g_0, g_1, \ldots, g_{n-1})$ and $H = C_m = (x_0, x_1, \ldots, x_{m-1})$. Give first to the graph $G$ the orientation counter-clockwise around the cycle $g_0, g_1, g_2, \ldots, g_0$. For each $i$ orient now each edge $(g_i, x_j)(g_{i+1}, x_h) \in E(G \circ H)$ such that it has the corresponding orientation to the edge $g_i g_{i+1} \in E(G)$. Each copy of $H$ (i.e., ${}^gH$-layer for any $g \in V(G)$) we orient such that the orientation is counter-clockwise around each cycle $(g_i, x_0), (g_i, x_1), (g_i, x_2), \ldots, (g_i, x_0)$ for $i = 0, 1, \ldots, n-1$. Define $\overrightarrow{\ell} : V(G \circ H) \to \mathbb{Z}_{mn}$ as $\overrightarrow{\ell}(g_i, x_j) = jn + i$ for $i = 0, 1, \ldots, n-1$, $j = 0, 1, \ldots, m-1$.

$$\begin{aligned} w(g_i, x_j) = & \sum_{h=0}^{m-1} \Big(\overrightarrow{\ell}(g_{i+1}, x_h) - \overrightarrow{\ell}(g_{i-1}, x_h)\Big) \\ & + \overrightarrow{\ell}\big(g_i, x_{j+1}\big) - \overrightarrow{\ell}\big(g_i, x_{j-1}\big) = 2n + 2m. \end{aligned}$$

Hence $G \circ H$ is orientable $\mathbb{Z}_{nm}$-distance magic. ∎

An analogous theorem is also true for a direct product of cycles as shown in the following theorem.

**Theorem 13.** *The direct product $C_n \times C_m$ is orientable $\mathbb{Z}_{nm}$-distance magic for all $n$, $m \geq 3$.*

***Proof.*** Let $G \cong C_n \cong g_0, g_1, \ldots, g_{n-1}$ and $H \cong C_m \cong x_0, x_1, \ldots, x_{m-1}$. For all $i$ and $j$, orient counter-clockwise with respect to $j$ each cycle of the form $(g_i, x_j), (g_{i-1}, x_{j+1}), (g_{i-2}, x_{j+2}), \ldots, (g_i, x_j)$ and each cycle of the form $(g_i, x_j), (g_{i+1}, x_{j+1}), (g_{i+2}, x_{j+2}), \ldots, (g_i, x_j)$, where the arithmetic in the indices is performed modulo $n$ and $m$, respectively. Then define $\overrightarrow{\ell} : V(G \times H) \to \mathbb{Z}_{nm}$ as $\overrightarrow{\ell}(g_i, x_j) = jn + i$ for $i = 0, 1, \ldots, n-1$, $j = 0, 1, \ldots, m-1$. Therefore for all $i$ and $j$ we have,

$$w(g_i, x_j) = \overrightarrow{\ell}(g_{i-1}, x_{j+1}) + \overrightarrow{\ell}(g_{i+1}, x_{j+1}) - \overrightarrow{\ell}(g_{i-1}, x_{j-1}) - \overrightarrow{\ell}(g_{i+1}, x_{j-1}) = 4n.$$

Since $\overrightarrow{\ell}$ is obviously a bijection, it follows that $G \times H$ is orientable $\mathbb{Z}_{nm}$-distance magic. ■

**Theorem 14.** *Let $H$ be the circulant graph $C_{2n}\big(1, 3, 5, \ldots, 2\left\lceil \frac{n}{2} \right\rceil - 1\big)$. If $G$ is an Eulerian graph of order $t$, then the direct product $G \times H$ is orientable $\mathbb{Z}_{2nt}$-distance magic.*

***Proof.*** Let $V(G) = \{g_0, g_1, \ldots, g_{t-1}\}$, whereas $V(H) = \{x_0, x_1, \ldots, x_{2n-1}\}$. Give first to the graph $G$ the orientation according to Fleury's Algorithm for finding Eulerian trail in $G$ and now orient the graph $G \times H$ such that each edge $(g_i, x_j)(g_p, x_h) \in E(G \times H)$ has the corresponding orientation to the edge $g_i g_p \in E(G)$. Recall that for each vertex $g \in V(G)$ we have $\deg^+(g) = \deg^-(g)$. Observe that $H \cong K_{n,n}$ with the partite sets $A = \{x_0, x_2, \ldots, x_{2n-2}\}$ and $B = \{x_1, x_3, \ldots, x_{2n-1}\}$. Define

$$\overrightarrow{\ell}(g_i, x_j) = \begin{cases} ti + j & \text{for } \quad j = 0, 2, \ldots, 2n-2, \\ 2tn - 1 - \overrightarrow{\ell}(g_i, x_{j-1}) & \text{for } \quad j = 1, 3, \ldots, 2n-1, \end{cases}$$

for $i = 0, 1, \ldots, t-1$.

Notice that $\overrightarrow{\ell}(g_i, x_j) + \overrightarrow{\ell}(g_i, x_{j-1}) = 2tn - 1$ for $i = 0, 1, \ldots, t-1$, $j = 1, 3, \ldots, 2n-1$. Therefore $w(g_i, x_j) = \sum_{y \in N^+(g_i, x_j)} \overrightarrow{\ell}(y) - \sum_{y \in N^-(g_i, x_j)} \overrightarrow{\ell}(y) = \frac{\deg^+(g_i)}{2} 2n(2nt-1) - \frac{\deg^-(g_i)}{2} 2n(2nt-1) = 0$. ■

## 3. COMPLETE $t$-PARTITE GRAPHS

**Theorem 15.** *The complete graph $K_n$ is orientable $\mathbb{Z}_n$-distance magic if and only if $n$ is odd.*

***Proof.*** Suppose first that $n$ is odd. Then $K_n \cong C_n(1, 2, \ldots, (n-1)/2)$ and thus it is orientable $\mathbb{Z}_n$-distance magic by Theorem 6. By Theorem 4 we can consider now only the case when $n \equiv 0 \pmod 4$. Suppose that $K_n$ is orientable $\mathbb{Z}_n$-distance magic. Let $\overrightarrow{\ell}(x) = 1$, $\overrightarrow{\ell}(u) = 0$. Then it is easy to see that $w(x) = \sum_{y\in N^+(x)} \overrightarrow{\ell}(y) - \sum_{y\in N^-(x)} \overrightarrow{\ell}(y) \equiv 1 \pmod 2$, whereas $w(u) = \sum_{y\in N^+(u)} \overrightarrow{\ell}(y) - \sum_{y\in N^-(u)} \overrightarrow{\ell}(y) \equiv 0 \pmod 2$, a contradiction. ∎

**Proposition 16.** *Let $G = K_{n_1,n_2,n_3,\ldots,n_k}$ be a complete $k$-partite graph such that $1 \le n_1 \le n_2 \le \cdots \le n_k$ and $n = n_1 + n_2 + \cdots + n_k$ is odd. The graph $G$ is orientable $\mathbb{Z}_n$-distance magic graph if $n_2 \ge 2$.*

***Proof.*** Give first to the graph $G$ an orientation such that all arcs from the set of lower index go to the set of higher index. Since $n$ is odd, there exists a partition $A_0, A_1, \ldots, A_{k-1}$ of $\{1, 2, \ldots, n\}$ such that for every $0 \le i \le k-1$, $|A_i| = n_i$ with $\sum_{a\in A_i} a \equiv 0 \pmod n$ by Theorem 9. Label the vertices from $i$th partition set of $G$ using elements from the set $A_i$ for $i = 0, 1, \ldots, k-1$.

Notice that $w(x) = 0$ for any $x \in V(G)$. ∎

**Proposition 17.** *$K_{n,n}$ is orientable $\mathbb{Z}_{2n}$-distance magic if and only if $n$ is even.*

***Proof.*** Suppose first that $n$ is even. Then $K_{n,n} \cong C_{2n}(1, 3, 5 \ldots, n-1)$ and is orientable $\mathbb{Z}_{2n}$-distance magic by Theorem 6. If $n$ is odd, then because $2n \equiv 2 \pmod 4$, $K_{n,n}$ is not orientable $\mathbb{Z}_{2n}$-distance magic by Theorem 4. ∎

Recall that if $n = n_1 + n_2 \equiv 2 \pmod 4$ and $n_1, n_2$ are both odd, then $K_{n_1,n_2}$ is not orientable $\mathbb{Z}_n$-distance magic by Theorem 4. It was proved in [7] that if $K_{n_1,n_2}$ is orientable $\mathbb{Z}_n$-distance magic, then $n \not\equiv 2 \pmod 4$. The next theorem shows that the converse is also true.

**Theorem 18.** *Let $G = K_{n_1,n_2}$ and $n = n_1 + n_2$. If $n \not\equiv 2 \pmod 4$, then $G$ is orientable $\mathbb{Z}_n$-distance magic.*

***Proof.*** Let $G = K_{n_1,n_2}$ with the partite sets $A^i = \{x_0^i, x_1^i, \ldots, x_{n_i-1}^i\}$ for $i = 1, 2$. Without loss of generality we can assume that $n_1 \ge n_2$.

Let $\mathbb{Z}_n = \{a_0, a_1, a_2, \ldots, a_{n-1}\}$ such that $a_0 = 0$, $a_1 = n/4$, $a_2 = n/2$, $a_3 = 3n/4$ and $a_{i+1} = -a_i$ for $i = 4, 6, 8, \ldots, n-2$. Let $o(uv)$ be the orientation for the edge $uv \in E(G)$ such that

$$o(x_i^j x_k^p) = \begin{cases} \overrightarrow{x_i^2 x_0^1} & \text{for} \quad i = 0, 1, \ldots, n_2 - 1, \\ \overrightarrow{x_i^1 x_k^2} & \text{for} \quad i = 1, 2, \ldots, n_1 - 1,\ k = 0, 1, \ldots, n_2 - 1. \end{cases}$$

*Case* 1. $n_1, n_2$ are both odd.
$\overrightarrow{\ell}(x_0^1) = a_1$, $\overrightarrow{\ell}(x_1^1) = a_3$, $\overrightarrow{\ell}(x_2^1) = a_0$ and $\overrightarrow{\ell}(x_i^1) = a_{1+i}$ for $i = 3, 4, \ldots, n_1 - 1$.
$\overrightarrow{\ell}(x_0^2) = a_2$ and $\overrightarrow{\ell}(x_i^2) = a_{n_1+i}$ for $i = 1, 2, \ldots, n_2 - 1$.

*Case* 2. $n_1, n_2$ are both even.
$\overrightarrow{\ell}(x_0^1) = a_1$, $\overrightarrow{\ell}(x_1^1) = a_3$ and $\overrightarrow{\ell}(x_i^1) = a_{2+i}$ for $i = 2, 3, \ldots, n_1 - 1$.
$\overrightarrow{\ell}(x_0^2) = a_2$, $\overrightarrow{\ell}(x_1^2) = a_0$ and $\overrightarrow{\ell}(x_i^2) = a_{n_1+i}$ for $i = 2, 3, \ldots, n_2 - 1$.

Note that in both cases $w(x) = n/2$ for any $x \in V(G)$. ■

**Theorem 19.** *Let $G = K_{n_1,n_2,n_3}$ and $n = n_1 + n_2 + n_3$. Then $G$ is orientable $\mathbb{Z}_n$-distance magic for all $n_1, n_2, n_3$.*

***Proof.*** Let $G = K_{n_1,n_2,n_3}$ with the partite sets $A^i = \left\{x_0^i, x_1^i, \ldots, x_{n_i-1}^i\right\}$ for $i = 1, 2, 3$.

Assume first that $n$ is odd. We have to consider only the case $n_1 = n_2 = 1$ by Proposition 16. If $n_3 = 1$, then $G \cong C_3$ is orientable $\mathbb{Z}_n$-distance magic, so assume $n_3 \geq 3$ is odd. Set the orientation $o(uv)$ for the edge $uv \in E(G)$ such that

$$o\left(x_i^j x_k^p\right) = \begin{cases} \overrightarrow{x_0^1 x_0^2}, & \\ \overrightarrow{x_i^3 x_0^2} & i = 0, 1, \ldots, n_3 - 1. \end{cases}$$

We will orient the remaining edges of the form $x_0^1 x_i^3$ for $i = 0, 1, \ldots, n_3 - 1$ later. Now let $\overrightarrow{\ell}(x_0^1) = 0$, $\overrightarrow{\ell}(x_0^2) = n - 1$, and $\overrightarrow{\ell}(x_i^3) = i + 1$ for $i = 0, 1, \ldots, n_3 - 1$. Notice that $\sum_{i=0}^{n_3-1} \overrightarrow{\ell}(x_i^3) = 1$. Observe now that $w(x_0^2)$ and $w(x_i^3)$ for $i = 0, 1, \ldots, n_3 - 1$ are independent of the yet-to-be oriented edges and hence $w(x_0^2) = w(x_i^3) = 1$. So all that remains is to orient the edges of the form $x_0^1 x_i^3$ for $i = 0, 1, \ldots, n_3 - 1$ so that $w(x_0^1) = 1$. It is easy to see that this is equivalent to finding $a, b \in \{1, 2, \ldots, n-2\} \subseteq \mathbb{Z}_n$ such that $a + b = \frac{n+1}{2}$, $a \neq b$. Clearly such $a$ and $b$ exist for all odd $n \geq 5$ since the group table for $\mathbb{Z}_n$ is a latin square. Therefore, set the orientation

$$o\left(x_i^j x_k^p\right) = \begin{cases} \overrightarrow{x_0^1 x_i^3}, & i = a - 1, b - 1, \\ \overrightarrow{x_i^3 x_0^1}, & \text{otherwise,} \end{cases}$$

which implies that $w(v) = 1$ for any $v \in V(G)$.

From now $n$ is even. Without loss of generality we assume that $n_1$ is even. Let $\mathbb{Z}_n = \{a_0, a_1, a_2, \ldots, a_{n-1}\}$. We will consider now two cases.

*Case* 1. $n \equiv 0 \pmod 4$. Let $a_0 = 0$, $a_1 = n/4$, $a_2 = n/2$, $a_3 = 3n/4$ and $a_{i+1} = -a_i$ for $i = 4, 6, 8, \ldots, n - 2$. Set the orientation $o(uv)$ for the edge $uv \in E(G)$ such that

$$o(x_i^j x_k^p) = \begin{cases} \overrightarrow{x_i^2 x_0^1} & \text{for} \quad i = 0, 1, \ldots, n_2 - 1, \\ \overrightarrow{x_i^1 x_k^2} & \text{for} \quad i = 1, 2, \ldots, n_1 - 1, \ k = 0, 1, \ldots, n_2 - 1, \\ \overrightarrow{x_i^1 x_k^3} & \text{for} \quad i = 0, 1, \ldots, n_1 - 1, \ k = 0, 1, \ldots, n_3 - 1, \\ \overrightarrow{x_i^2 x_k^3} & \text{for} \quad i = 0, 1, \ldots, n_2 - 1, \ k = 0, 1, \ldots, n_3 - 1. \end{cases}$$

Let now $\overrightarrow{\ell}(x_0^1) = a_1$, $\overrightarrow{\ell}(x_1^1) = a_3$ and $\overrightarrow{\ell}(x_i^1) = a_{i+2}$ for $i = 2, 3, \ldots, n_1 - 1$.

*Case* 1.1. $n_2, n_3$ are both odd.
$\overrightarrow{\ell}(x_0^2) = a_2$ and $\overrightarrow{\ell}(x_i^2) = a_{n_1+1+i}$ for $i = 1, 2, \ldots, n_2 - 1$.
$\overrightarrow{\ell}(x_0^3) = a_0$ and $\overrightarrow{\ell}(x_i^3) = a_{n_1+n_2+i}$ for $i = 1, 2, \ldots, n_3 - 1$.

*Case* 1.2. $n_2, n_3$ are both even.
$\overrightarrow{\ell}(x_0^2) = a_0$, $\overrightarrow{\ell}(x_1^2) = a_2$ and $\overrightarrow{\ell}(x_i^2) = a_{n_1+i}$ for $i = 2, 3 \ldots, n_2 - 1$.
$\overrightarrow{\ell}(x_i^3) = a_{n_1+n_2+i}$ for $i = 0, 1, \ldots, n_3 - 1$.

Note that in both subcases $w(v) = n/2$ for any $v \in V(G)$.

*Case* 2. $n \equiv 2 \pmod 4$. Without loss of generality we can assume that $n_2 \geq n_3$. Let $a_0 = 0$, $a_1 = n/2$, $a_2 = 1$, $a_3 = n/2 - 1$, $a_4 = n - 1$, $a_5 = n/2 + 1$ and $a_{i+1} = -a_i$ for $i = 6, 8, 10, \ldots, n - 2$. Set the orientation $o(uv)$ for the edge $uv \in E(G)$ such that

$$o\left(x_i^j x_k^p\right) = \left\{ \overrightarrow{x_i^j x_k^p} \quad \text{for} \quad j < p. \right.$$

Let now $\overrightarrow{\ell}(x_0^1) = a_2$, $\overrightarrow{\ell}(x_1^1) = a_3$ and $\overrightarrow{\ell}(x_i^1) = a_{i+4}$ for $i = 2, 3, \ldots, n_1 - 1$.

*Case* 2.1. $n_2, n_3$ are both even.
$\overrightarrow{\ell}(x_0^2) = a_4$, $\overrightarrow{\ell}(x_1^2) = a_5$ and $\overrightarrow{\ell}(x_i^2) = a_{n_1+2+i}$ for $i = 2, 3, \ldots, n_2 - 1$.
$\overrightarrow{\ell}(x_0^3) = a_0$, $\overrightarrow{\ell}(x_1^3) = a_1$ and $\overrightarrow{\ell}(x_i^3) = a_{n_1+n_2+i}$ for $i = 2, 3, \ldots, n_3 - 1$.

Note that $\sum_{x \in A^i} \overrightarrow{\ell}(x) = n/2$ for $i = 1, 2, 3$, thus $w(v) = 0$ for any $v \in V(G)$.

*Case* 2.2. $n_2, n_3$ are both odd. Assume first that $n_2 \geq 3$. Set $\overrightarrow{\ell}(x_0^2) = a_0$, $\overrightarrow{\ell}(x_1^2) = a_4$, $\overrightarrow{\ell}(x_2^2) = a_5$ and $\overrightarrow{\ell}(x_i^1) = a_{n_1+1+i}$ for $i = 3, 4, \ldots, n_2 - 1$. $\overrightarrow{\ell}(x_0^3) = a_1$ and $\overrightarrow{\ell}(x_i^3) = a_{n_1+n_2+i}$ for $i = 1, 2, \ldots, n_3 - 1$. As in Case 2.1 $\sum_{x \in A^i} \overrightarrow{\ell}(x) = n/2$ for $i = 1, 2, 3$, thus $w(v) = 0$ for any $v \in V(G)$.

Let now $n_2 = n_3 = 1$, then $n_1 \equiv 0 \pmod 4$. Set the orientation $o(uv)$ for the edge $uv \in E(G)$ such that

$$o\left(x_i^j x_k^p\right) = \begin{cases} \overrightarrow{x_0^2 x_i^1}, & i \text{ even}, \\ \overrightarrow{x_i^1 x_0^2}, & i \text{ odd}, \\ \overrightarrow{x_0^3 x_i^1}, & i = 0, 1, \ldots, n_1 - 1, \\ \overrightarrow{x_0^3 x_0^2}. \end{cases}$$

Then let $\overrightarrow{\ell}(x_0^2) = \frac{n}{2}$, $\overrightarrow{\ell}(x_0^3) = \frac{n}{2} + 2$, $\overrightarrow{\ell}(x_{n/2}^1) = \frac{n}{2} + 1$, and

$$\overrightarrow{\ell}(x_i^1) = \begin{cases} i, & i = 0, 1, \dots, \frac{n}{2} - 1, \\ i + 2, & i = \frac{n}{2} + 1, \frac{n}{2} + 2, \dots, n_1 - 1. \end{cases}$$

Observe that $\sum_{g \in \mathbb{Z}_n} g = \frac{n}{2}$ since $n \equiv 2 \pmod 4$, and also $\sum_{i \text{ odd}} \overrightarrow{\ell}(x_i^1) - \sum_{i \text{ even}} \overrightarrow{\ell}(x_i^1) = \frac{n}{2}$, so $w(v) = 2$ for any $v \in V(G)$. ■

We finish this section with the following conjecture.

**Conjecture 20.** *If $G$ is a $2r$-regular graph of order $n$, then $G$ is orientable $\mathbb{Z}_n$-distance magic.*


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