ON THE TOTAL ROMAN DOMINATION IN TREES

JAFAR AMJADI¹, SEYED MAHMoud SHEIKHOESLAMI

AND

MARZIEH SORoudI

Department of Mathematics
Azarbaijan Shahid Madani University
Tabriz, I.R. Iran

e-mail: \{j-amjadi;s.m.sheikholeslami;m.soroudi\}@azaruniv.ac.ir

Abstract

A total Roman dominating function on a graph $G$ is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the following conditions: (i) every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$ and (ii) the subgraph of $G$ induced by the set of all vertices of positive weight has no isolated vertex. The weight of a total Roman dominating function $f$ is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The total Roman domination number $\gamma_{tR}(G)$ is the minimum weight of a total Roman dominating function of $G$. Ahangar et al. in [H.A. Ahangar, M.A. Henning, V. Samodivkin and I.G. Yero, Total Roman domination in graphs, Appl. Anal. Discrete Math. 10 (2016) 501–517] recently showed that for any graph $G$ without isolated vertices, $2\gamma(G) \leq \gamma_{tR}(G) \leq 3\gamma(G)$, where $\gamma(G)$ is the domination number of $G$, and they raised the problem of characterizing the graphs $G$ achieving these upper and lower bounds. In this paper, we provide a constructive characterization of these trees.

Keywords: total Roman dominating function, total Roman domination number, trees.

2010 Mathematics Subject Classification: 05C69.

1. Introduction

In this paper, $G$ is a simple graph without isolated vertices, with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of $G$ is denoted by $n = n(G)$.

¹Corresponding author.
For every vertex \( v \in V \), the \textit{open neighborhood} of \( v \) is the set \( N(v) = \{ u \in V(G) \mid uv \in E(G) \} \) and the \textit{closed neighborhood} of \( v \) is the set \( N[v] = N(v) \cup \{ v \} \). The \textit{degree} of a vertex \( v \in V \) is \( \text{deg}(v) = \text{deg}_G(v) = |N(v)| \). A \textit{leaf} of \( T \) is a vertex of degree 1, a \textit{support vertex} of \( T \) is a vertex adjacent to a leaf, a \textit{strong support vertex} is a support vertex adjacent to at least two leaves and an \textit{end support vertex} is a support vertex having at most one non-leaf neighbor. A \textit{pendant path} \( P \) of a graph \( G \) is an induced path such that one of the end points has degree one in \( G \), and its other end point is the only vertex of \( P \) adjacent to some vertex in \( G - P \). The \textit{distance} \( d_G(u,v) \) between two vertices \( u \) and \( v \) in a connected graph \( G \) is the length of a shortest \( uv \)-path in \( G \). The \textit{diameter} of a graph \( G \), denoted by \( \text{diam}(G) \), is the greatest distance between two vertices of \( G \). For a vertex \( v \) in a (rooted) tree \( T \), let \( C(v) \) and \( D(v) \) denote the set of children and descendants of \( v \), respectively and let \( D[v] = D(v) \cup \{ v \} \). Also, the \textit{depth} of \( v \), \( \text{depth}(v) \), is the largest distance from \( v \) to a vertex in \( D(v) \). The \textit{maximal subtree} at \( v \) is the subtree of \( T \) induced by \( D[v] \), and is denoted by \( T_v \). We write \( P_n \) for the \textit{path} of order \( n \). A \textit{double star} is a tree with exactly two vertices that are not leaves. If \( A \subseteq V(G) \) and \( f \) is a mapping from \( V(G) \) into some set of numbers, then \( f(A) = \sum_{x \in A} f(x) \). The sum \( f(V(G)) \) is called the \textit{weight} \( \omega(f) \) of \( f \).

A vertex set \( S \) of a graph \( G \) is a dominating set if each vertex of \( G \) either belongs to \( S \) or is adjacent to a vertex in \( S \). The domination number \( \gamma(G) \) of \( G \) is the minimum cardinality over all dominating sets of \( G \). A dominating set of \( G \) of cardinality \( \gamma(G) \) is called a \( \gamma(G) \)-set. The domination problem consists of finding the domination number of a graph. The domination problem has many applications and has attracted considerable attention [11, 15]. The literature on the subject of domination parameters in graphs has been surveyed and detailed in the two books [12, 13].

A function \( f : V(G) \rightarrow \{0, 1, 2\} \) is a \textit{Roman dominating function} (RDF) on \( G \) if every vertex \( u \in V \) for which \( f(u) = 0 \) is adjacent to at least one vertex \( v \) for which \( f(v) = 2 \). The weight of an RDF is the value \( f(V(G)) = \sum_{u \in V(G)} f(u) \). The \textit{Roman domination number} \( \gamma_R(G) \) is the minimum weight of an RDF on \( G \). Roman domination was introduced by Cockayne \textit{et al.} in [10] and was inspired by the work of ReVelle and Rosing [17], Stewart [18]. It is worth mentioning that since 2004, a hundred papers have been published on this topic, where several new variations were introduced: weak Roman domination [14], Roman \{2\}-domination [9], maximal Roman domination [2], mixed Roman domination [4], double Roman domination [8] and recently total Roman domination introduced by Liu and Chang [16].

A \textit{total Roman dominating function} of a graph \( G \) with no isolated vertex, abbreviated \( \text{TRDF} \), is a Roman dominating function \( f \) on \( G \) with the additional property that the subgraph of \( G \) induced by the set of all vertices of positive weight under \( f \) has no isolated vertex. The \textit{total Roman domination number}
\( \gamma_{tR}(G) \) is the minimum weight of a TRDF on \( G \). A TRDF of \( G \) with weight \( \gamma_{tR}(G) \) is called a \( \gamma_{tR}(G) \)-function. The concept of the total Roman domination was introduced by Liu and Chang [16] and has been studied in [1,3,5–7].

Ahangar et al. [3] showed that for any graph \( G \),

\[
2\gamma(G) \leq \gamma_{tR}(G) \leq 3\gamma(G),
\]

(1)

and they posed the following problems.

**Problem 1.** Characterize the graphs \( G \) satisfying \( \gamma_{tR}(G) = 2\gamma(G) \).

**Problem 2.** Characterize the graphs \( G \) satisfying \( \gamma_{tR}(G) = 3\gamma(G) \).

In this paper, we provide a constructive characterization of the trees \( T \) with \( \gamma_{tR}(T) = 2\gamma(T) \) and \( \gamma_{tR}(T) = 3\gamma(T) \) which settles the above problems for trees.

2. Preliminaries

In this section, we provide some results and definitions used throughout the paper. The proof of Observations 1 and 2 can be found in [6].

**Observation 1** [6]. If \( v \) is a strong support vertex in a graph \( G \), then there exists a \( \gamma_{tR}(G) \)-function \( f \) such that \( f(v) = 2 \).

**Observation 2** [6]. If \( u_1, u_2 \) are two adjacent support vertices in a graph \( G \), then there exists a \( \gamma_{tR}(G) \)-function \( f \) such that \( f(u_1) = f(u_2) = 2 \).

**Observation 3.** If \( T \) is a double star, then \( \gamma_{tR}(T) = 2\gamma(T) \).

**Observation 4.** Let \( H \) be a subgraph of a graph \( G \) such that \( G \) and \( H \) have no isolated vertex. If \( \gamma_{tR}(H) = 3\gamma(H) \), \( \gamma(G) \leq \gamma(H) + s \) and \( \gamma_{tR}(G) \geq \gamma_{tR}(H) + 3s \) for some non-negative integer \( s \), then \( \gamma_{tR}(G) = 3\gamma(G) \).

**Proof.** It follows from the assumptions and (1) that

\[
\gamma_{tR}(G) \geq \gamma_{tR}(H) + 3s = 3\gamma(H) + 3s \geq 3\gamma(G) \geq \gamma_{tR}(G),
\]

and this yields \( \gamma_{tR}(G) = 3\gamma(G) \). \( \square \)

**Observation 5.** Let \( H \) be a subgraph of a graph \( G \) such that \( G \) and \( H \) have no isolated vertex. If \( \gamma_{tR}(G) = 3\gamma(G) \), \( \gamma_{tR}(G) \leq \gamma_{tR}(H) + 3s \) and \( \gamma(G) \geq \gamma(H) + s \) for some non-negative integer \( s \), then \( \gamma_{tR}(H) = 3\gamma(H) \).

**Proof.** By (1) and the assumptions, we have

\[
3\gamma(G) = \gamma_{tR}(G) \leq \gamma_{tR}(H) + 3s \leq 3\gamma(H) + 3s \leq 3\gamma(G),
\]

and this leads to the result. \( \square \)
Similarly, we have the following results.

**Observation 6.** Let $H$ be a subgraph of a graph $G$ such that $G$ and $H$ have no isolated vertex. If $\gamma_{IR}(H) = 2\gamma(H)$, $\gamma(G) \geq \gamma(H) + s$ and $\gamma_{IR}(G) \leq \gamma_{IR}(H) + 2s$ for some non-negative integer $s$, then $\gamma_{IR}(G) = 2\gamma(G)$.

**Observation 7.** Let $H$ be a subgraph of a graph $G$ such that $G$ and $H$ have no isolated vertex. If $\gamma_{IR}(G) = 2\gamma(G)$, $\gamma_{IR}(G) \geq \gamma_{IR}(H) + 2s$ and $\gamma(G) \leq \gamma(H) + s$ for some non-negative integer $s$, then $\gamma_{IR}(H) = 2\gamma(H)$.

We close this section with some definitions.

**Definition 8.** Let $v$ be a vertex of the graph $G$. A function $f : V(G) \rightarrow \{0, 1, 2\}$ is said to be a nearly total Roman dominating function (nearly TRDF) with respect to $v$, if the following three conditions are fulfilled:

(i) every vertex $x \in V(G) - \{v\}$ for which $f(x) = 0$ is adjacent to at least one vertex $y \in V(G)$ for which $f(y) = 2$,

(ii) every vertex $x \in V(G) - \{v\}$ for which $f(x) \geq 1$ is adjacent to at least one vertex $y \in V(G)$ for which $f(y) \geq 1$ and

(iii) $f(v) \geq 1$ or $f(v) + f(u) \geq 2$ for some $u \in N(v)$. Let

$$\gamma_{IR}(G; v) = \min\{\omega(f) \mid f \text{ is a nearly TRDF with respect to } v\}.$$ 

Observe that any total Roman dominating function on $G$ is a nearly TRDF with respect to any vertex of $G$. Hence $\gamma_{IR}(G; v)$ is well defined and $\gamma_{IR}(G; v) \leq \gamma_{IR}(G)$ for each $v \in V(G)$. Define $W^1_G = \{v \in V(G) \mid \gamma_{IR}(G; v) = \gamma_{IR}(G)\}$.

**Definition 9.** For a graph $G$ and $v \in V(G)$, we say $v$ has property $P$ in $G$ if there exists a $\gamma_{IR}(G)$-function $f$ such that $f(v) = 2$. Assume that $W^2_G = \{v \mid v \text{ has property } P \text{ in } G\}$, $W^3_G = \{v \mid v \text{ does not have property } P \text{ in } G\}$.

We note that if a vertex $v \in V(G)$ satisfies the condition of Observations 1 or 2, then $v \in W^2_G$.

**Definition 10.** For a graph $G$ and $v \in V(G)$, let

$$\gamma(G, v) = \min\{|S| : S \subseteq V(G) \text{ and each vertex } w \neq v \text{ is dominated by } S\}.$$ 

Clearly $\gamma(G, v) \leq \gamma(G)$ for each $v \in V(G)$. We define $W^4_G = \{v \mid \gamma(G, v) = \gamma(G)\}$.

For a path $P_4 = v_1v_2v_3v_4$, we have $W^4_{P_4} = W^3_{P_4} = W^3_{v_4} = \{v_2, v_3\}$, $W^3_{P_4} = \{v_1, v_4\}$.

**Definition 11.** For a tree $T$, let $W^5_T = \{v \mid \text{ there exists a function } f : V(T) \rightarrow \{0, 1, 2\} \text{ such that}$$
(i) \( \omega(f) = \gamma_{tR}(T) - 1 \),
(ii) \( f(v) = 1 \),
(iii) every vertex \( x \in V(T) - \{ v \} \) for which \( f(x) = 0 \) is adjacent to at least
one vertex \( y \in V(T) \) for which \( f(y) = 2 \), and
(iii) every vertex \( x \in V(T) - \{ v \} \) for which \( f(x) \geq 1 \) is adjacent to at least
one vertex \( y \in V(T) \) for which \( f(y) \geq 1 \).

Let \( H \) be the graph illustrated in Figure 1. For any \( \gamma_{tR}(H) \)-function \( f \), we
have \( f(u) = f(v) = 2 \), \( f(x) = 2 \) or \( f(x) = f(v_1) = 1 \), \( f(y) = 2 \) or \( f(y) = f(v_2) =
1 \), \( f(w) = 2 \) or \( f(w) = f(u_1) = 1 \), and \( f(z) = 0 \) otherwise. It follows that \( W^2_H =
\{ u, v, x, y, w \} \) and \( W^3_H = \{ u_i, v_1 | i = 1, 2, 3, 4 \} \). Now define \( g : V(H) \to \{ 0, 1, 2 \} \)
by \( g(u) = g(v) = g(x) = g(y) = 2 \), \( g(w) = 1 \), and \( g(z) = 0 \) otherwise. Clearly, \( g \)
is a nearly total Roman dominating function of \( H \) with respect to \( u_1 \) of weight \( \gamma_{tR}(H) - 1 \)
yielding \( u_1 \notin W^1_H \). Similarly, \( v_1, v_2 \notin W^1_H \). It is easy to see that
\( W^1_H = V(G) - \{ u_1, v_1, v_2 \} \).

To determine \( W^2_H \), first we note that \( \gamma(H) = 5 \). Obviously, \( \{ u, v, x, y \} \) domi-
nates all vertices in \( V(H) - \{ u_1 \} \) and so \( \gamma(H, u_1) \leq 4 \) yielding \( u_1 \notin W^4_H \). Similarly,
\( v_1, v_2 \notin W^4_H \). It is not hard to see that \( W^2_H = V(G) - \{ u_1, v_1, v_2 \} \).

Now, we determine \( W^3_H \). The function \( h : V(H) \to \{ 0, 1, 2 \} \) defined by
\( h(u_1) = 1 \), \( h(u) = h(v) = h(x) = h(y) = 2 \) and \( h(z) = 0 \) otherwise, is a function
of weight \( \gamma_{tR}(H) - 1 \) satisfying the conditions of Definition 11 and hence \( u_1 \in W^5_H \).
Similarly, we have \( v_1, v_2 \in W^5_H \). It is easy to verify that \( W^3_H = \{ u_1, v_1, v_2 \} \).

3. A Characterization of Trees \( T \) with \( \gamma_{tR}(T) = 3\gamma(T) \)

In this section we provide a constructive characterization of all trees \( T \) with
\( \gamma_{tR}(T) = 3\gamma(T) \). In order to do this, let \( T \) be the family of unlabeled trees \( T \)
that can be obtained from a sequence \( T_1, T_2, \ldots, T_m \) (\( m \geq 1 \)) of trees such that
\( T_1 \) is a path \( P_3 \), and, if \( m \geq 2 \), \( T_{i+1} \) can be obtained recursively from \( T_i \) by one
of the three operations \( O_1, O_2, O_3 \) for \( 1 \leq i \leq m - 1 \).
**Operation \( \mathcal{O}_1 \).** If \( x \in V(T_i) \) and \( x \) is a strong support vertex, then Operation \( \mathcal{O}_1 \) adds a new vertex \( y \) and an edge \( xy \) to obtain \( T_{i+1} \).

**Operation \( \mathcal{O}_2 \).** If \( x \in W^1_{T_i} \), then Operation \( \mathcal{O}_2 \) adds a star \( K_{1,3} \) and joins \( x \) to a leaf of it to obtain \( T_{i+1} \).

**Operation \( \mathcal{O}_3 \).** If \( x \in W^1_{T_i} \cap W^2_{T_i} \), then Operation \( \mathcal{O}_3 \) adds a path \( P_3 \) and joins \( x \) to a leaf of \( P_3 \) to obtain \( T_{i+1} \).

\[
\begin{array}{ccc}
\text{Operation } \mathcal{O}_1 & \text{Operation } \mathcal{O}_2 & \text{Operation } \mathcal{O}_3 \\
\includegraphics[width=1.5in]{operation1} & \includegraphics[width=1.5in]{operation2} & \includegraphics[width=1.5in]{operation3}
\end{array}
\]

Figure 2. The operations \( \mathcal{O}_1, \mathcal{O}_2 \) and \( \mathcal{O}_3 \).

**Lemma 12.** If \( T_i \) is a tree with \( \gamma_{tR}(T_i) = 3\gamma(T_i) \) and \( T_{i+1} \) is a tree obtained from \( T_i \) by Operation \( \mathcal{O}_1 \), then \( \gamma_{tR}(T_{i+1}) = 3\gamma(T_{i+1}) \).

**Proof.** Clearly \( \gamma(T_{i+1}) = \gamma(T_i) \) and \( \gamma_{tR}(T_{i+1}) = \gamma_{tR}(T_i) \) and so \( \gamma_{tR}(T_{i+1}) = 3\gamma(T_{i+1}) \).

**Lemma 13.** If \( T_i \) is a tree with \( \gamma_{tR}(T_i) = 3\gamma(T_i) \) and \( T_{i+1} \) is a tree obtained from \( T_i \) by Operation \( \mathcal{O}_2 \), then \( \gamma_{tR}(T_{i+1}) = 3\gamma(T_{i+1}) \).

**Proof.** Let \( \mathcal{O}_2 \) add a star \( K_{1,3} \) with vertex set \( \{y, y_1, y_2, y_3\} \) centered in \( y \) and join \( x \) to \( y_1 \). Obviously adding \( y \) to any \( \gamma(T_i) \)-set yields a dominating set of \( T_{i+1} \) and so \( \gamma(T_{i+1}) \leq \gamma(T_i) + 1 \). Let now \( f \) be a \( \gamma_{tR}(T_{i+1}) \)-function such that \( f(y) \) is as large as possible. By Observation 1 we have \( f(y_1) = 2 \). Since \( f \) is a TRDF of \( G \), we may assume that \( f(y_1) \geq 1 \). If \( f(x) \geq 1 \), then the function \( f \), restricted to \( T_i \) is a nearly TRDF of \( T_i \) of weight at most \( \gamma_{tR}(T_{i+1}) - 3 \) and we deduce from \( x \in W^1_{T_i} \) that \( \gamma_{tR}(T_{i+1}) - 3 \geq \omega(f|T_i) \geq \gamma_{tR}(T_i) \). If \( f(x) = 0 \) and \( f(y_1) = 1 \), then the function \( f \), restricted to \( T_i \) is a TRDF of \( T_i \) of weight \( \gamma_{tR}(T_{i+1}) - 3 \) and so \( \gamma_{tR}(T_{i+1}) - 3 \geq \omega(f|T_i) \geq \gamma_{tR}(T_i) \). If \( f(x) = 0 \) and \( f(y_1) = 2 \), then the function \( g : V(T_i) \to \{0, 1, 2\} \) defined by \( g(x) = 1 \) and \( g(u) = f(u) \) for each \( u \in V(T_i) - \{x\} \) is a nearly TRDF of \( T_i \) of weight \( \gamma_{tR}(T_{i+1}) - 3 \) and since \( x \in W^1_{T_i} \) we have \( \gamma_{tR}(T_{i+1}) - 3 \geq \omega(f|T_i) \geq \gamma_{tR}(T_i) \). Hence, in all cases \( \gamma_{tR}(T_{i+1}) \geq \gamma_{tR}(T_i) + 3 \) and we conclude from Observation 4 that \( \gamma_{tR}(T_{i+1}) = 3\gamma(T_{i+1}) \).

**Lemma 14.** If \( T_i \) is a tree with \( \gamma_{tR}(T_i) = 3\gamma(T_i) \) and \( T_{i+1} \) is a tree obtained from \( T_i \) by Operation \( \mathcal{O}_3 \), then \( \gamma_{tR}(T_{i+1}) = 3\gamma(T_{i+1}) \).
**Proof.** Let $O_3$ add a path $yzw$ and the edge $xy$. Obviously any $\gamma(T_i)$-set can be extended to a dominating set of $T_{i+1}$ by adding $z$ and so $\gamma(T_{i+1}) \leq \gamma(T_i)+1$. Now assume $f$ is a $\gamma_{tR}(T_{i+1})$-function such that $f(y)$ is as large as possible. Clearly $f(z)+f(w) \geq 2$. If $f(y)+f(z)+f(w) \geq 3$, then we may assume that $f(z) = 2$ and $f(y) \geq 1$ and by using an argument similar to that described in the proof of Lemma 13 we obtain $\gamma_{tR}(T_{i+1}) = 3\gamma(T_{i+1})$. Now let $f(y)+f(z)+f(w) = 2$. Then we must have $f(z) = f(w) = 1$ and $f(y) = 0$. Then the function $f$, restricted to $T_i$ is a TRDF of $T_i$ of weight $\gamma_{tR}(T_{i+1}) - 2$ with $f(x) = 2$. Since $x \in W_{T_i}^3$, we obtain $\gamma_{tR}(T_{i+1}) - 2 = \omega(f|_{T_i}) \geq \gamma_{tR}(T_i) + 1$ and so $\gamma_{tR}(T_{i+1}) \geq \gamma_{tR}(T_i) + 3$. Now the result follows by Observation 4.

**Theorem 15.** If $T \in \mathcal{T}$, then $\gamma_{tR}(T) = 3\gamma(T)$.

**Proof.** Let $T \in \mathcal{T}$. Then there exists a sequence of trees $T_1, T_2, \ldots, T_k$ ($k \geq 1$) such that $T_1$ is $P_3$, and if $k \geq 2$, then $T_{i+1}$ can be obtained recursively from $T_i$ by one of the Operations $O_1, O_2, O_3$ for $i = 1, 2, \ldots, k-1$.

We proceed by induction on the number of operations applied to construct $T$. If $k = 1$, then $T = P_3 \in \mathcal{T}$. Suppose that the result is true for each tree $T \in \mathcal{T}$ which can be obtained from a sequence of operations of length $k-1$ and let $T' = T_{k-1}$. By the induction hypothesis, we have $\gamma_{tR}(T') = 3\gamma(T')$. Since $T = T_k$ is obtained by one of the Operations $O_1, O_2, O_3$ from $T'$, we conclude from Lemmas 12, 13 and 14 that $\gamma_{tR}(T) = 3\gamma(T)$.

Now we are ready to prove the main result of this section.

**Theorem 16.** Let $T$ be a tree of order $n \geq 3$. Then $\gamma_{tR}(T) = 3\gamma(T)$ if and only if $T \in \mathcal{T}$.

**Proof.** By Theorem 15, we only need to prove the necessity. Let $T$ be a tree with $\gamma_{tR}(T) = 3\gamma(T)$. The proof is by induction on $n$. If $n = 3$, then the only tree $T$ of order 3 with $\gamma_{tR}(T) = 3\gamma(T)$ is $P_3 \in \mathcal{T}$. Let $n \geq 4$ and let the statement hold for all trees $T$ of order less than $n$ and $\gamma_{tR}(T) = 3\gamma(T)$. Assume that $T$ is a tree of order $n$ with $\gamma_{tR}(T) = 3\gamma(T)$ and let $f$ be a $\gamma_{tR}(T)$-function. By Observation 3 we have diam$(T) \neq 3$. If diam$(T) = 2$, then $T$ is a star and $T$ can be obtained from $P_3$ iterative application of Operation $O_1$ and so $T \in \mathcal{T}$. Hence we assume diam$(T) \geq 4$.

Let $v_1 v_2 \cdots v_k$ ($k \geq 5$) be a diametrical path in $T$ and root $T$ at $v_k$. If $\deg(v_2) \geq 4$, then clearly $\gamma_{tR}(T) = \gamma_{tR}(T - v_1)$ and $\gamma(T) = \gamma(T - v_k)$ and hence $\gamma_{tR}(T - v_1) = 3\gamma(T - v_1)$. By the induction hypothesis we have $T - v_1 \in \mathcal{T}$. Now, $T$ can be obtained from $T - v_1$ by Operation $O_1$ and so $T \in \mathcal{T}$. Suppose that $\deg(v_2) \leq 3$. We consider two cases.

**Case 1.** $\deg(v_2) = 3$. We claim that $\deg(v_3) = 2$. Suppose, to the contrary, that $\deg(v_3) \geq 3$. Then each child of $v_3$ is a leaf or a support vertex. If $v_3$
has a children other than $v_2$ which is a leaf or a strong support vertex, then let $T' = T - T_{v_2}$. It is not hard to see that $\gamma(T) = \gamma(T') + 1$ and $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 2$. Then $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 2 \leq 3\gamma(T') + 2 = 3\gamma(T) - 1$ which is a contradiction. Assume that each child of $v_3$ except $v_2$, is a support vertex of degree 2. Let $v_3, z_2, z_1$ be a pendant path in $T$. Suppose $T' = T - \{z_1, z_2\}$. As above we can see that $\gamma_{tR}(T) \leq 3\gamma(T) - 1$, a contradiction again. Thus $\deg(v_3) = 2$.

Assume $T' = T - T_{v_3}$. Let $S$ be a $\gamma(T)$-set containing support vertices, and define $S' = S - \{v_2\}$ if $v_3 \notin S$ and $S' = (S - \{v_3, v_3\}) \cup \{v_4\}$ when $v_3 \in S$. Clearly, $S'$ is a dominating set of $T'$ and so $\gamma(T') \leq |S'| = \gamma(T) - 1$. On the other hand, any $\gamma_{tR}(T')$-function can be extended to a TRDF of $T$ by assigning 1 to $v_3$, 2 to $v_2$ and 0 to the leaves adjacent to $v_2$. This yields $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 3$. It follows from Observation 5 that $\gamma_{tR}(T') = 3\gamma(T')$ and by the induction hypothesis we have $T' \in T$. If $v_4 \notin W_3^{T'}$, then let $g$ be a nearly TRDF of $T'$ with respect to $v_4$ of weight at most $\gamma_{tR}(T') - 1$ and define $h : V(T) \to \{0, 1, 2\}$ by $h(u) = g(u)$ for $u \in V(T')$, $h(v_3) = 1$, $h(v_2) = 2$ and $h(u) = 0$ otherwise. Clearly $h$ is a TRDF of $T$ of weight $\gamma_{tR}(T') + 2$ which leads to a contradiction. Hence $v_4 \in W_3^{T'}$, and $T$ can be obtained from $T'$ by Operation $O_2$. Thus $T \in T$ in this case.

Case 2. $\deg(v_2) = 2$. Considering Case 1, we may assume that each child of $v_3$ is a support vertex of degree 2. If $\deg(v_3) \geq 3$, then let $T' = T - T_{v_3}$. Any $\gamma(T)$-set can be extended to a dominating set of $T$ by adding $C(v_3)$ and so $\gamma(T) \leq \gamma(T') + |C(v_3)|$. On the other hand, let $S$ be a $\gamma(T)$-set containing no leaves. To dominate the leaves of $T_{v_3}$, we must have $C(v_3) \subseteq S$. Then the set $S' = S \setminus C(v_3)$ if $v_3 \notin S$ and $S' = (S - (C(v_3) \cup \{v_3\})) \cup \{v_4\}$ if $v_3 \in S$, is a dominating set of $T'$ and this implies that $\gamma(T') \leq \gamma(T) - |C(v_3)|$. Hence $\gamma(T) = \gamma(T') + |C(v_3)|$.

Also, any $\gamma_{tR}(T')$-function can be extended to a TRDF of $T$ by assigning 1 to $v_3$, 2 to the children of $v_3$ and 0 to all leaves of $T_{v_3}$, and so

$$
\gamma_{tR}(T) \leq \gamma_{tR}(T') + 2|C(v_3)| + 1
\leq 3\gamma(T') + 2|C(v_3)| + 1
= 3\gamma(T') + |C(v_3)| - |C(v_3)| + 1
= 3\gamma(T) - |C(v_3)| + 1
< 3\gamma(T) \quad \text{(since } |C(v_3)| \geq 2),
$$
a contradiction. Henceforth, we assume $\deg(v_3) = 2$. Suppose $T' = T - T_{v_3}$. Clearly, $\gamma(T) = \gamma(T') + 1$. Analogously as in Case 1, we can see that $\gamma_{tR}(T') = 3\gamma(T')$ and $v_4 \in W_3^{T'}$. Thus $T' \in T$ by the induction hypothesis. If $v_4 \notin W_3^{T'}$, then let $g$ be a $\gamma_{tR}(T')$-function with $g(v_4) = 2$ and define $h : V(T) \to \{0, 1, 2\}$ by $h(u) = g(u)$ for $u \in V(T')$, and $h(v_3) = 0$, $h(v_2) = h(v_1) = 1$. Clearly $h$ is a TRDF of $T$ of weight $\gamma_{tR}(T') + 2$ which leads to a contradiction. Hence $v_4 \in W_3^{T'}$. 

and $T$ can be obtained from $T'$ by Operation $O_3$. It follows that $T \in \mathcal{T}$ and the proof is complete.

4. A Characterization of Trees $T$ with $\gamma_{tR}(T) = 2\gamma(T)$

In this section we present a constructive characterization of all trees $T$ with $\gamma_{tR}(T) = 2\gamma(T)$.

Let $\mathcal{F}$ be the family of unlabeled trees $T$ that can be obtained from a sequence $T_1, T_2, \ldots, T_m$ ($m \geq 1$) of trees such that $T_1$ is a path $P_2$ or $P_4$, and, if $m \geq 2$, $T_{i+1}$ can be obtained recursively from $T_i$ by one of the following four operations for $1 \leq i \leq m - 1$.

**Operation $T_1$.** If $x \in W_2^T$ is a support vertex, then the Operation $T_1$ adds a new vertex $y$ and an edge $xy$ to obtain $T_{i+1}$.

**Operation $T_2$.** If $x \in V(T_i)$ is at distance 2 from a leaf $w$, then the Operation $T_2$ adds a path $yz$ and joins $x$ to $y$ to obtain $T_{i+1}$.

**Operation $T_3$.** If $x \in W_4^T$, then the Operation $T_3$ adds a path $z_4z_3z_2z_1$ and joins $x$ to $z_3$ to obtain $T_{i+1}$.

**Operation $T_4$.** If $x \in W_2^T \cup W_4^T$, then the Operation $T_4$ adds a path $P_3 = zw$ and joins $x$ to $z$ to obtain $T_{i+1}$.

![Figure 3. The operations $T_1, T_2, T_3$ and $T_4$.](image)

**Lemma 17.** If $T_i$ is a tree with $\gamma_{tR}(T_i) = 2\gamma(T_i)$ and $T_{i+1}$ is a tree obtained from $T_i$ by Operation $T_1$, then $\gamma_{tR}(T_{i+1}) = 2\gamma(T_{i+1})$. 
Proof. It is easy to see that $\gamma(T_{i+1}) = \gamma(T_i)$ and $\gamma_{tR}(T_{i+1}) = \gamma_{tR}(T_i)$ and so $\gamma_{tR}(T_{i+1}) = 2\gamma(T_{i+1})$.

Lemma 18. If $T_i$ is a tree with $\gamma_{tR}(T_i) = 2\gamma(T_i)$ and $T_{i+1}$ is a tree obtained from $T_i$ by Operation $T_2$, then $\gamma_{tR}(T_{i+1}) = 2\gamma(T_{i+1})$.

Proof. Let $w'$ be the support vertex of $w$. If $S$ is a $\gamma(T_{i+1})$-set, then clearly $y, w' \in S$ and $S - \{y\}$ is a dominating set of $T_i$ yielding $\gamma(T_{i+1}) \geq \gamma(T_i) + 1$. Also, if $f$ is a $\gamma_{tR}(T_i)$-function such that $f(x) \geq 1$, then $f$ can be extended to a TRDF of $T_{i+1}$ by assigning the weight 1 to $y, z$. Hence $\gamma_{tR}(T_{i+1}) \leq \gamma_{tR}(T_i) + 2$. Now the result follows by Observation 6.

Lemma 19. If $T_i$ is a tree with $\gamma_{tR}(T_i) = 2\gamma(T_i)$ and $T_{i+1}$ is a tree obtained from $T_i$ by Operation $T_3$, then $\gamma_{tR}(T_{i+1}) = 2\gamma(T_{i+1})$.

Proof. If $S$ is a $\gamma(T_{i+1})$-set containing no leaves, then $z_3, z_2 \in S$ and we deduce from $x \in W^2_{T_1}$ that $|S - \{z_3, z_2\}| \geq \gamma(T_i)$ yielding $\gamma(T_{i+1}) \geq \gamma(T_i) + 2$. On the other hand, any $\gamma_{tR}(T_i)$-function can be extended to a TRDF of $T$ by assigning the weight 2 to $z_3, z_2$ and the weight 0 to $z_1, z_4$ and so $\gamma_{tR}(T_{i+1}) \leq \gamma_{tR}(T_i) + 4$. It follows from Observation 6 that $\gamma_{tR}(T_{i+1}) = 2\gamma(T_{i+1})$.

Lemma 20. If $T_i$ is a tree with $\gamma_{tR}(T_i) = 2\gamma(T_i)$ and $T_{i+1}$ is a tree obtained from $T_i$ by Operation $T_4$, then $\gamma_{tR}(T_{i+1}) = 2\gamma(T_{i+1})$.

Proof. Let $T_4$ add a path $zyw$ and joins $x$ to $z$. If $S$ is a $\gamma(T_{i+1})$-set, then $y \in S$ and the set $S' = S - \{y\}$ if $z \notin S$ and $S' = (S - \{y, z\}) \cup \{x\}$ if $z \in S$, is a dominating set of $T_i$ yielding $\gamma(T_{i+1}) \geq \gamma(T_i) + 1$. Now we show that $\gamma_{tR}(T_{i+1}) \leq \gamma_{tR}(T_i) + 2$. If $x \in W^2_{T_1}$, then let $f$ be a $\gamma_{tR}(T_i)$-function with $f(x) = 2$. Clearly $f$ can be extended to an TRDF of $T_{i+1}$ by assigning the weight 1 to $w, y$ and the weight 0 to $z$ and so $\gamma_{tR}(T_{i+1}) \leq \gamma_{tR}(T_i) + 2$. If $x \in W^2_{T_1}$, then let $f$ be a function satisfying the conditions of Definition 11. Clearly $f$ can be extended to a TRDF of $T_{i+1}$ by assigning the weight 1 to $z, y, w$ and so $\gamma_{tR}(T_{i+1}) \leq \gamma_{tR}(T_i) + 2$. Now the result follows by Observation 6.

Theorem 21. If $T \in F$, then $\gamma_{tR}(T) = 2\gamma(T)$.

Proof. Let $T \in F$. Then there exists a sequence of trees $T_1, T_2, \ldots, T_k$ ($k \geq 1$) such that $T_1$ is $T_2$ or $P_3$, and if $k \geq 2$, then $T_{i+1}$ can be obtained recursively from $T_i$ by one of the Operations $T_1, T_2, T_3, T_4$ for $i = 1, 2, \ldots, k - 1$.

We proceed by induction on the number of operations used to construct $T$. If $k = 1$, then $T = T_2$ or $P_3$ and the result is trivial. Suppose the statement holds for each tree $T \in F$ which can be obtained from a sequence of operations of length $k - 1$ and let $T' = T_{k-1}$. By the induction hypothesis, we have $\gamma_{tR}(T') = 2\gamma(T')$. Since $T = T_k$ is obtained by one of the Operations $T_1, T_2, T_3, T_4$ we conclude from previous lemmas that $\gamma_{tR}(T) = 2\gamma(T)$.
Now we prove the main result of this section.

**Theorem 22.** Let $T$ be a tree of order $n \geq 2$. Then $\gamma_{tR}(T) = 2\gamma(T)$ if and only if $T \in \mathcal{F}$.

**Proof.** According to Theorem 21, we only need to prove the necessity. Let $T$ be a tree with $\gamma_{tR}(T) = 2\gamma(T)$. Since $\gamma_{tR}(K_{1,s}) = 3 = 3\gamma(K_{1,s})$ for $s \geq 2$, $T$ is not a star of order $n(T) \geq 3$. We proceed by induction on $n$. If $n \in \{2, 4\}$, then the only trees $T$ of order 2 or 4 with $\gamma_{tR}(T) = 2\gamma(T)$ are $P_2, P_4 \in \mathcal{F}$. Assume $n \geq 5$ and let the statement hold for all trees $T$ of order less than $n$ and $\gamma_{tR}(T) = 2\gamma(T)$.

Assume that $T$ is a tree of order $n$ with $\gamma_{tR}(T) = 2\gamma(T)$ and let $f$ be a $\gamma_{tR}(T)$-function. Since $T$ is not a star, we have $\text{diam}(T) \geq 3$. If $\text{diam}(T) = 3$, then $T$ is a double star and $T$ can be obtained from $P_4$ by iterative application of Operation $T_1$ because the support vertices of $P_4$ belong to $W_2^{P_4}$ and so $T \in \mathcal{F}$. Hence we assume $\text{diam}(T) \geq 4$.

Let $v_1v_2 \cdots v_k$ ($k \geq 5$) be a diametrical path in $T$ such that $\deg(v_2)$ is as large as possible and root $T$ at $v_k$. First let $\deg(v_2) \geq 3$. Clearly $\gamma_{tR}(T) \geq \gamma_{tR}(T-v_1)$ and $\gamma(T) = \gamma(T-v_1)$. If $\gamma_{tR}(T) \geq \gamma_{tR}(T-v_1) + 1$, then we have

$$2\gamma(T) = \gamma_{tR}(T) \geq \gamma_{tR}(T-v_1) + 1 \geq 2\gamma(T-v_1) + 1 = 2\gamma(T) + 1$$

which is a contradiction. Thus $\gamma_{tR}(T) = \gamma_{tR}(T-v_1)$. By Observation 1, there exists a $\gamma_{tR}(T)$-function $f$ such that $f(v_2) = 2$. Then clearly $f$ is a $\gamma_{tR}(T-v_1)$-function yielding $v_2 \in W_{T-v_1}^2$. Now, $T$ can be obtained from $T-v_1$ by Operation $T_1$ and so $T \in \mathcal{F}$. Suppose that $\deg(v_2) = 2$.

Consider the following cases.

**Case 1.** $\deg(v_3) = 2$. Let $T' = T - T_{v_3}$. Clearly

$$\gamma(T') = \gamma(T) - 1.$$  \hspace{1cm} (2)

Now let $f$ be a $\gamma_{tR}(T)$-function. Clearly $f(v_1) + f(v_2) \geq 2$. If $f(v_1) + f(v_2) \geq 3$, then clearly $f(v_3) = 0$ and the function $f$, restricted to $T'$, is a TRDF of $T'$ yielding $\gamma_{tR}(T') \geq \gamma_{tR}(T') + 3$. But then

$$2\gamma(T) = \gamma_{tR}(T) \geq \gamma_{tR}(T') + 3 \geq 2\gamma(T') + 3 = 2(\gamma(T) - 1) + 3 = 2\gamma(T) + 1,$$

a contradiction. Thus $f(v_1) + f(v_2) = 2$. If $f(v_3) = 1$ and $f(v_4) = 0$, then we get a contradiction as above. If $f(v_3) = 1$ and $f(v_4) \geq 1$, then the function $g : V(T') \to \{0, 1, 2\}$ defined by $g(v_3) = \min\{2, f(v_3) + 1\}$ and $g(u) = f(u)$ otherwise, is a TRDF of $T'$ of weight $\gamma_{tR}(T) - 2$. Assume that $f(v_3) \neq 1$. If $f(v_3) = 2$, then $f(v_4) = 0$ and the function $g : V(T') \to \{0, 1, 2\}$ defined by

$$g(v_4) = 1, g(v_5) = \min\{2, f(v_5) + 1\} \text{ and } g(u) = f(u) \text{ otherwise, is a TRDF of } T'$$

of weight $\gamma_{tR}(T) - 2$. We conclude from

$$2\gamma(T) = \gamma_{tR}(T) \geq \gamma_{tR}(T') + 2 \geq 2\gamma(T') + 2 \geq 2(\gamma(T) - 1) + 2 = 2\gamma(T).$$
that
\[ \gamma_{tR}(T) = \gamma_{tR}(T') + 2. \]

By (2) and (3), we obtain \( \gamma_{tR}(T') = 2\gamma(T') \) and by the induction hypothesis we have \( T' \in \mathcal{F} \). Now we show that \( v_4 \in W^2_{T'} \cup W^5_{T'} \). Let \( f \) be a \( \gamma_{tR}(T) \)-function. As above we can see that \( f(v_1) + f(v_2) = 2 \). If \( f(v_3) = 0 \), then the function \( f \) restricted to \( T' \) is a \( \gamma_{tR}(T') \)-function with \( f(v_4) = 2 \) implying that \( v_4 \in W^5_{T'} \). If \( f(v_3) = 2 \) and \( v_4 \) has a neighbor with positive weight under \( f \), then the function \( g : V(T') \to \{0, 1, 2\} \) defined by \( g(v_4) = 1 \) and \( g(x) = f(x) \) otherwise, is a TRDF of \( T' \) of weight \( \gamma_{tR}(T) - 3 \) contradicting (3). If \( f(v_3) = 2 \) and \( v_4 \) has no neighbor other than \( v_3 \) with positive weight under \( f \), then the function \( g : V(T') \to \{0, 1, 2\} \) defined by \( g(v_4) = 1 \) and \( g(x) = f(x) \) otherwise, is a function of weight \( \gamma_{tR}(T) - 3 = \gamma_{tR}(T') - 1 \) satisfying the conditions of Definition 11 and so \( v_4 \in W^5_{T'} \). Suppose that \( f(v_3) = 1 \). We can see as above that \( f(v_4) \geq 1 \). If \( f(v_4) = 2 \), then the function \( g : V(T') \to \{0, 1, 2\} \) defined by \( g(v_5) = \min\{2, f(v_5) + 1\} \) and \( g(x) = f(x) \) otherwise, is a \( \gamma_{tR}(T') \)-function with \( g(v_4) = 2 \) implying that \( v_4 \in W^5_{T'} \). If \( f(v_4) = 1 \) and \( v_4 \) has a neighbor different from \( v_3 \) with positive weight under \( f \), then the function \( f \) restricted to \( T' \) is a TRDF of \( T' \) of weight \( \gamma_{tR}(T) - 3 \) which contradicts (3). Finally if \( f(v_3) = 1 \) and \( v_4 \) has no neighbor other than \( v_3 \) with positive weight, then the function \( f \) restricted to \( T' \) fulfilled the conditions of Definition 11 and so \( v_4 \in W^5_{T'} \). Thus \( v_4 \in W^2_{T'} \cup W^5_{T'} \) and \( T \) can be obtained from \( T' \) by Operation \( T_4 \) and so \( T \in \mathcal{F} \).

**Case 2.** \( \deg(v_3) \geq 3 \). By the choice of diametrical path, we may assume that all the children of \( v_3 \) with depth one have degree 2. We consider three subcases.

**Subcase 2.1.** \( v_3 \) is a support vertex and is at distance 2 from some leaves different from \( v_1 \). Let \( T' = T - \{v_1, v_2\} \). Then clearly \( \gamma(T) = \gamma(T') + 1 \) and \( \gamma_{tR}(T) \geq \gamma_{tR}(T') + 2 \). Hence \( \gamma_{tR}(T') = 2\gamma(T') \) by Observation 7. By the induction hypothesis we have \( T' \in \mathcal{F} \) and hence \( T \) can be obtained from \( T' \) by Operation \( T_2 \) and so \( T \in \mathcal{F} \).

**Subcase 2.2.** All children of \( v_3 \) have degree 2. Let \( v_3' \) be a pendant path and let \( T' = T - \{v_1, v_2\} \). Clearly \( \gamma(T) = \gamma(T') + 1 \). Now let \( f \) be a \( \gamma_{tR}(T) \)-function. Then \( f(v_2) \geq 1 \), \( f(v_1) + f(v_2) \geq 2 \) and \( f(z_1) + f(z_2) \geq 2 \). If \( f(v_3) \geq 1 \) or \( f(v_3) = 0 \) and \( f(v_2) = 1 \), then the function \( f \) restricted to \( T' \) is a TRDF of \( T' \) of weight \( \omega(f) - 2 \) and so \( \gamma_{tR}(T) \geq \gamma_{tR}(T') + 2 \). Assume that \( f(v_3) = 0 \) and \( f(v_2) = 2 \). Since \( f \) is a TRDF of \( T \), we have \( f(v_1) = 1 \). Then the function \( g : V(T) \to \{0, 1, 2\} \) defined by \( g(v_3) = g(v_2) = g(v_1) = 1 \) and \( g(x) = f(x) \) otherwise, is a \( \gamma_{tR}(T) \)-function and as above we obtain \( \gamma_{tR}(T) \geq \gamma_{tR}(T') + 2 \). Hence \( \gamma_{tR}(T') = 2\gamma(T') \) by Observation 7. By the induction hypothesis we have \( T' \in \mathcal{F} \) and so \( T \) can be obtained from \( T' \) by Operation \( T_2 \). Thus \( T \in \mathcal{F} \).

**Subcase 2.3.** All children of \( v_3 \) except \( v_2 \) are leaves. Let \( w \) be a leaf adjacent to \( v_3 \). First let \( v_3 \) be a strong support vertex. It is easy to see that \( \gamma(T) = \gamma(T - w) \)
and \( \gamma_{tR}(T) = \gamma_{tR}(T - w) \) yielding \( \gamma_{tR}(T - w) = 2\gamma(T - w) \). By the induction hypothesis we have \( T - w \in \mathcal{F} \) and by Observation 2 we obtain \( v_3 \in W^2_{T - w} \). Thus \( T \) can be obtained from \( T - w \) by Operation \( T_1 \) and so \( T \in \mathcal{F} \). Suppose next that \( v_3 \) is not a strong support vertex. Then by the assumption we have \( \deg(v_3) = 3 \).

Consider the following.

(a) \( v_4 \) is a support vertex. Let \( T' = T - T_{v_3} \). It is easy to see that \( \gamma_{tR}(T) = \gamma_{tR}(T') + 2 \) and \( \gamma(T) = \gamma(T') + 1 \). It follows that \( \gamma_{tR}(T') = 2\gamma(T') \) and by the induction hypothesis we have \( T' \in \mathcal{F} \). Then \( T \) can be obtained from \( T' \) by Operation \( T_2 \) and so \( T \in \mathcal{F} \).

(b) \( v_4 \) has a child \( z_2 \) with depth 1. As above we may assume that \( \deg(z_2) = 2 \). Let \( z_1 \) be the leaf adjacent to \( z_2 \) and let \( T' = T - \{z_1, z_2\} \). Clearly \( \gamma(T) = \gamma(T') + 1 \). By Observation 2, there exists a \( \gamma_{tR}(T) \)-function \( f \) such that \( f(v_3) = 2 \). Also we have \( f(z_1) + f(z_2) \geq 2 \). Obiously the function \( f \) restricted to \( T' \) is a TRDF of \( T' \) and so \( \gamma_{tR}(T) \geq \gamma_{tR}(T') + 2 \). We conclude from \( 2\gamma(T) = \gamma_{tR}(T) \geq \gamma_{tR}(T') + 2 \geq 2\gamma(T') + 2 = 2\gamma(T) \) that \( \gamma_{tR}(T') = 2\gamma(T') \) and by the induction hypothesis we have \( T' \in \mathcal{F} \). Now \( T \) can be obtained from \( T' \) by Operation \( T_2 \) and so \( T \in \mathcal{F} \).

(c) \( v_4 \) has a child \( z_3 \) with depth 2. Let \( v_4z_3z_2z_1 \) be a path in \( T \). Using the above argument we may assume that \( \deg(z_2) = 2 \) and either \( \deg(z_3) = 2 \) or \( \deg(z_3) = 3 \) and \( z_3 \) is a support vertex. If \( \deg(z_3) = 2 \), then as in Case 1 we can see that \( T \in \mathcal{F} \).

Let \( \deg(z_3) = 3 \) and \( z_3 \) is a support vertex. Let \( T' = T - T_{z_3} \). It is not hard to see that \( \gamma(T) = \gamma(T') + 2 \) and \( \gamma_{tR}(T) = \gamma_{tR}(T') + 4 \). This implies that \( \gamma_{tR}(T') = 2\gamma(T') \) and by the induction hypothesis we have \( T' \in \mathcal{F} \). Since \( v_4 \) is adjacent to a support vertex, we deduce that \( v_4 \in W^4_{T'} \). Now \( T \) can be obtained from \( T' \) by Operation \( T_3 \) and so \( T \in \mathcal{F} \).

This completes the proof.

\[ \blacksquare \]

References


