ON THE TOTAL ROMAN DOMINATION IN TREES

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Abstract

A total Roman dominating function on a graph \(G\) is a function \(f : V(G) \to \{0, 1, 2\}\) satisfying the following conditions: (i) every vertex \(u\) for which \(f(u) = 0\) is adjacent to at least one vertex \(v\) for which \(f(v) = 2\) and (ii) the subgraph of \(G\) induced by the set of all vertices of positive weight has no isolated vertex. The weight of a total Roman dominating function \(f\) is the value \(f(V(G)) = \sum_{u \in V(G)} f(u)\). The total Roman domination number \(\gamma_{tR}(G)\) is the minimum weight of a total Roman dominating function of \(G\). Ahangar et al. in [H.A. Ahangar, M.A. Henning, V. Samodivkin and I.G. Yero, Total Roman domination in graphs, Appl. Anal. Discrete Math. 10 (2016) 501–517] recently showed that for any graph \(G\) without isolated vertices, \(2\gamma(G) \leq \gamma_{tR}(G) \leq 3\gamma(G)\), where \(\gamma(G)\) is the domination number of \(G\), and they raised the problem of characterizing the graphs \(G\) achieving these upper and lower bounds. In this paper, we provide a constructive characterization of these trees.

Keywords: total Roman dominating function, total Roman domination number, trees.

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1. Introduction

In this paper, \(G\) is a simple graph without isolated vertices, with vertex set \(V = V(G)\) and edge set \(E = E(G)\). The order \(|V|\) of \(G\) is denoted by \(n = n(G)\).

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For every vertex \( v \in V \), the open neighborhood of \( v \) is the set \( N(v) = \{ u \in V(G) \mid uv \in E(G) \} \) and the closed neighborhood of \( v \) is the set \( N[v] = N(v) \cup \{ v \} \). The degree of a vertex \( v \in V \) is \( \deg(v) = \deg_{G}(v) = |N(v)| \). A leaf of \( T \) is a vertex of degree 1, a support vertex of \( T \) is a vertex adjacent to a leaf, a strong support vertex is a support vertex adjacent to at least two leaves and an end support vertex is a support vertex having at most one non-leaf neighbor. A pendant path \( P \) of a graph \( G \) is an induced path such that one of the end points has degree one in \( G \), and its other end point is the only vertex of \( P \) adjacent to some vertex in \( G - P \). The distance \( d_{G}(u, v) \) between two vertices \( u \) and \( v \) in a connected graph \( G \) is the length of a shortest \( uv \)-path in \( G \). The diameter of a graph \( G \), denoted by \( \text{diam}(G) \), is the greatest distance between two vertices of \( G \). For a vertex \( v \) in a (rooted) tree \( T \), let \( C(v) \) and \( D(v) \) denote the set of children and descendants of \( v \), respectively and let \( D[v] = D(v) \cup \{ v \} \). Also, the depth of \( v \), \( \text{depth}(v) \), is the largest distance from \( v \) to a vertex in \( D(v) \). The maximal subtree at \( v \) is the subtree of \( T \) induced by \( D[v] \), and is denoted by \( T_{v} \). We write \( P_{n} \) for the path of order \( n \). A double star is a tree with exactly two vertices that are not leaves. If \( A \subseteq V(G) \) and \( f \) is a mapping from \( V(G) \) into some set of numbers, then \( f(A) = \sum_{x \in A} f(x) \). The sum \( f(V(G)) \) is called the weight \( \omega(f) \) of \( f \).

A vertex set \( S \) of a graph \( G \) is a dominating set if each vertex of \( G \) either belongs to \( S \) or is adjacent to a vertex in \( S \). The domination number \( \gamma(G) \) of \( G \) is the minimum cardinality over all dominating sets of \( G \). A dominating set of \( G \) of cardinality \( \gamma(G) \) is called a \( \gamma(G) \)-set. The domination problem consists of finding the domination number of a graph. The domination problem has many applications and has attracted considerable attention [11, 15]. The literature on the subject of domination parameters in graphs has been surveyed and detailed in the two books [12, 13].

A function \( f : V(G) \to \{0, 1, 2\} \) is a Roman dominating function (RDF) on \( G \) if every vertex \( u \in V \) for which \( f(u) = 0 \) is adjacent to at least one vertex \( v \) for which \( f(v) = 2 \). The weight of an RDF is the value \( f(V(G)) = \sum_{u \in V(G)} f(u) \). The Roman domination number \( \gamma_{R}(G) \) is the minimum weight of an RDF on \( G \). Roman domination was introduced by Cockayne et al. in [10] and was inspired by the work of ReVelle and Rosing [17], Stewart [18]. It is worth mentioning that since 2004, a hundred papers have been published on this topic, where several new variations were introduced: weak Roman domination [14], Roman \( \{2\}\)-domination [9], maximal Roman domination [2], mixed Roman domination [4], double Roman domination [8] and recently total Roman domination introduced by Liu and Chang [16].

A total Roman dominating function of a graph \( G \) with no isolated vertex, abbreviated TRDF, is a Roman dominating function \( f \) on \( G \) with the additional property that the subgraph of \( G \) induced by the set of all vertices of positive weight under \( f \) has no isolated vertex. The total Roman domination number
\( \gamma_{tR}(G) \) is the minimum weight of a TRDF on \( G \). A TRDF of \( G \) with weight \( \gamma_{tR}(G) \) is called a \( \gamma_{tR}(G) \)-function. The concept of the total Roman domination was introduced by Liu and Chang [16] and has been studied in [1, 3, 5–7].

Ahangar et al. [3] showed that for any graph \( G \),
\[
2\gamma(G) \leq \gamma_{tR}(G) \leq 3\gamma(G),
\]
and they posed the following problems.

**Problem 1.** Characterize the graphs \( G \) satisfying \( \gamma_{tR}(G) = 2\gamma(G) \).

**Problem 2.** Characterize the graphs \( G \) satisfying \( \gamma_{tR}(G) = 3\gamma(G) \).

In this paper, we provide a constructive characterization of the trees \( T \) with \( \gamma_{tR}(T) = 2\gamma(T) \) and \( \gamma_{tR}(T) = 3\gamma(T) \) which settles the above problems for trees.

2. Preliminaries

In this section, we provide some results and definitions used throughout the paper. The proof of Observations 1 and 2 can be found in [6].

**Observation 1** [6]. If \( v \) is a strong support vertex in a graph \( G \), then there exists a \( \gamma_{tR}(G) \)-function \( f \) such that \( f(v) = 2 \).

**Observation 2** [6]. If \( u_1, u_2 \) are two adjacent support vertices in a graph \( G \), then there exists a \( \gamma_{tR}(G) \)-function \( f \) such that \( f(u_1) = f(u_2) = 2 \).

**Observation 3.** If \( T \) is a double star, then \( \gamma_{tR}(T) = 2\gamma(T) \).

**Observation 4.** Let \( H \) be a subgraph of a graph \( G \) such that \( G \) and \( H \) have no isolated vertex. If \( \gamma_{tR}(H) = 3\gamma(H) \), \( \gamma(G) \leq \gamma(H) + s \) and \( \gamma_{tR}(G) \geq \gamma_{tR}(H) + 3s \) for some non-negative integer \( s \), then \( \gamma_{tR}(G) = 3\gamma(G) \).

**Proof.** It follows from the assumptions and (1) that
\[
\gamma_{tR}(G) \geq \gamma_{tR}(H) + 3s = 3\gamma(H) + 3s \geq 3\gamma(G) \geq \gamma_{tR}(G),
\]
and this yields \( \gamma_{tR}(G) = 3\gamma(G) \). \( \blacksquare \)

**Observation 5.** Let \( H \) be a subgraph of a graph \( G \) such that \( G \) and \( H \) have no isolated vertex. If \( \gamma_{tR}(G) = 3\gamma(G) \), \( \gamma_{tR}(G) \leq \gamma_{tR}(H) + 3s \) and \( \gamma(G) \geq \gamma(H) + s \) for some non-negative integer \( s \), then \( \gamma_{tR}(H) = 3\gamma(H) \).

**Proof.** By (1) and the assumptions, we have
\[
3\gamma(G) = \gamma_{tR}(G) \leq \gamma_{tR}(H) + 3s \leq 3\gamma(H) + 3s \leq 3\gamma(G),
\]
and this leads to the result. \( \blacksquare \)
Similarly, we have the following results.

**Observation 6.** Let $H$ be a subgraph of a graph $G$ such that $G$ and $H$ have no isolated vertex. If $\gamma_{tR}(H) = 2\gamma(H)$, $\gamma(G) \geq \gamma(H) + s$ and $\gamma_{tR}(G) \leq \gamma_{tR}(H) + 2s$ for some non-negative integer $s$, then $\gamma_{tR}(G) = 2\gamma(G)$.

**Observation 7.** Let $H$ be a subgraph of a graph $G$ such that $G$ and $H$ have no isolated vertex. If $\gamma_{tR}(G) = 2\gamma(G)$, $\gamma_{tR}(G) \geq \gamma_{tR}(H) + 2s$ and $\gamma(G) \leq \gamma(H) + s$ for some non-negative integer $s$, then $\gamma_{tR}(H) = 2\gamma(H)$.

We close this section with some definitions.

**Definition 8.** Let $v$ be a vertex of the graph $G$. A function $f : V(G) \rightarrow \{0, 1, 2\}$ is said to be a nearly total Roman dominating function (nearly TRDF) with respect to $v$, if the following three conditions are fulfilled:

(i) every vertex $x \in V(G) - \{v\}$ for which $f(x) = 0$ is adjacent to at least one vertex $y \in V(G)$ for which $f(y) = 2$,

(ii) every vertex $x \in V(G) - \{v\}$ for which $f(x) \geq 1$ is adjacent to at least one vertex $y \in V(G)$ for which $f(y) \geq 1$ and

(iii) $f(v) \geq 1$ or $f(v) + f(u) \geq 2$ for some $u \in N(v)$. Let

$$\gamma_{tR}(G; v) = \min \{\omega(f) \mid f \text{ is a nearly TRDF with respect to } v\}.$$

Observe that any total Roman dominating function on $G$ is a nearly TRDF with respect to any vertex of $G$. Hence $\gamma_{tR}(G; v)$ is well defined and $\gamma_{tR}(G; v) \leq \gamma_{tR}(G)$ for each $v \in V(G)$. Define $W^1_G = \{v \in V(G) \mid \gamma_{tR}(G; v) = \gamma_{tR}(G)\}$.

**Definition 9.** For a graph $G$ and $v \in V(G)$, we say $v$ has property $P$ in $G$ if there exists a $\gamma_{tR}(G)$-function $f$ such that $f(v) = 2$. Assume that $W^2_G = \{v \mid v \text{ has property } P \text{ in } G\}$, $W^3_G = \{v \mid v \text{ does not have property } P \text{ in } G\}$.

We note that if a vertex $v \in V(G)$ satisfies the condition of Observations 1 or 2, then $v \in W^2_G$.

**Definition 10.** For a graph $G$ and $v \in V(G)$, let

$$\gamma(G, v) = \min \{|S| : S \subseteq V(G) \text{ and each vertex } w \neq v \text{ is dominated by } S\}.$$ Clearly $\gamma(G, v) \leq \gamma(G)$ for each $v \in V(G)$. We define $W^4_G = \{v \mid \gamma(G, v) = \gamma(G)\}$.

For a path $P_4 = v_1v_2v_3v_4$, we have $W^4_{P_4} = W^5_{P_4} = W^6_{P_4} = \{v_2, v_3\}$, $W^7_{P_4} = \{v_1, v_4\}$.

**Definition 11.** For a tree $T$, let $W^8_T = \{v \mid \text{ there exists a function } f : V(T) \rightarrow \{0, 1, 2\} \text{ such that}$$
(i) \( \omega(f) = \gamma_{tR}(T) - 1 \),
(ii) \( f(v) = 1 \),
(iii) every vertex \( x \in V(T) - \{v\} \) for which \( f(x) = 0 \) is adjacent to at least one vertex \( y \in V(T) \) for which \( f(y) = 2 \), and
(iii) every vertex \( x \in V(T) - \{v\} \) for which \( f(x) \geq 1 \) is adjacent to at least one vertex \( y \in V(T) \) for which \( f(y) \geq 1 \).

![Figure 1. The graph \( H \).](image)

Let \( H \) be the graph illustrated in Figure 1. For any \( \gamma_{tR}(H) \)-function \( f \), we have \( f(u) = f(v) = 2 \), \( f(x) = 2 \) or \( f(x) = f(v_1) = 1 \), \( f(y) = 2 \) or \( f(y) = f(v_2) = 1 \), \( f(w) = 2 \) or \( f(w) = f(u_1) = 1 \), and \( f(z) = 0 \) otherwise. It follows that \( W^2_H = \{u, v, x, y, w\} \) and \( W^3_H = \{u_i, v_i \mid i = 1, 2, 3, 4\} \). Now define \( g : V(H) \to \{0, 1, 2\} \) by \( g(u) = g(v) = g(x) = g(y) = 2 \), \( g(w) = 1 \), and \( g(z) = 0 \) otherwise. Clearly, \( g \) is a nearly total Roman dominating function of \( H \) with respect to \( u_1 \) of weight \( \gamma_{tR}(H) - 1 \) yielding \( u_1 \notin W^1_H \). Similarly, \( v_1, v_2 \notin W^1_H \). It is easy to see that \( W^1_H = V(G) - \{u_1, v_1, v_2\} \).

To determine \( W^2_H \), first we note that \( \gamma(H) = 5 \). Obviously, \( \{u, v, x, y\} \) dominates all vertices in \( V(H) - \{u_1\} \) and so \( \gamma(H, u_1) \leq 4 \) yielding \( u_1 \notin W^4_H \). Similarly, \( v_1, v_2 \notin W^4_H \). It is not hard to see that \( W^2_H = V(G) - \{u_1, v_1, v_2\} \).

Now, we determine \( W^3_H \). The function \( h : V(H) \to \{0, 1, 2\} \) defined by \( h(u_1) = 1 \), \( h(u) = h(v) = h(x) = h(y) = 2 \) and \( h(z) = 0 \) otherwise, is a function of weight \( \gamma_{tR}(H) - 1 \) satisfying the conditions of Definition 11 and hence \( u_1 \in W^7_H \). Similarly, we have \( v_1, v_2 \in W^5_H \). It is easy to verify that \( W^5_H = \{u_1, v_1, v_2\} \).

3. A Characterization of Trees \( T \) with \( \gamma_{tR}(T) = 3\gamma(T) \)

In this section we provide a constructive characterization of all trees \( T \) with \( \gamma_{tR}(T) = 3\gamma(T) \). In order to do this, let \( T \) be the family of unlabeled trees \( T \) that can be obtained from a sequence \( T_1, T_2, \ldots, T_m \) \((m \geq 1)\) of trees such that \( T_1 \) is a path \( P_3 \), and, if \( m \geq 2 \), \( T_{i+1} \) can be obtained recursively from \( T_i \) by one of the three operations \( \mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3 \) for \( 1 \leq i \leq m - 1 \).
**Operation O₁.** If \( x \in V(T_i) \) and \( x \) is a strong support vertex, then Operation \( O₁ \) adds a new vertex \( y \) and an edge \( xy \) to obtain \( T_{i+1} \).

**Operation O₂.** If \( x \in W^1_{T_i} \), then Operation \( O₂ \) adds a star \( K_{1,3} \) and joins \( x \) to a leaf of it to obtain \( T_{i+1} \).

**Operation O₃.** If \( x \in W^1_{T_i} \cap W^2_{T_i} \), then Operation \( O₃ \) adds a path \( P_3 \) and joins \( x \) to a leaf of \( P_3 \) to obtain \( T_{i+1} \).

![Figure 2. The operations \( O₁, O₂ \) and \( O₃ \).](image)

**Lemma 12.** If \( T_i \) is a tree with \( \gamma_{tr}(T_i) = 3\gamma(T_i) \) and \( T_{i+1} \) is a tree obtained from \( T_i \) by Operation \( O₁ \), then \( \gamma_{tr}(T_{i+1}) = 3\gamma(T_{i+1}) \).

**Proof.** Clearly \( \gamma(T_{i+1}) = \gamma(T_i) \) and \( \gamma_{tr}(T_{i+1}) = \gamma_{tr}(T_i) \) and so \( \gamma_{tr}(T_{i+1}) = 3\gamma(T_{i+1}) \).

**Lemma 13.** If \( T_i \) is a tree with \( \gamma_{tr}(T_i) = 3\gamma(T_i) \) and \( T_{i+1} \) is a tree obtained from \( T_i \) by Operation \( O₂ \), then \( \gamma_{tr}(T_{i+1}) = 3\gamma(T_{i+1}) \).

**Proof.** Let \( O₂ \) add a star \( K_{1,3} \) with vertex set \( \{ y, y_1, y_2, y_3 \} \) centered in \( y \) and join \( x \) to \( y_1 \). Obviously adding \( y \) to any \( \gamma(T_i) \)-set yields a dominating set of \( T_{i+1} \) and so \( \gamma(T_{i+1}) \leq \gamma(T_i) + 1 \). Let now \( f \) be a \( \gamma_{tr}(T_{i+1}) \)-function such that \( f(y) \) is as large as possible. By Observation 1 we have \( f(y) = 2 \). Since \( f \) is a TRDF of \( G \), we may assume that \( f(y_1) \geq 1 \). If \( f(x) \geq 1 \), then the function \( f \), restricted to \( T_i \) is a nearly TRDF of \( T_i \) of weight at most \( \gamma_{tr}(T_{i+1}) - 3 \) and we deduce from \( x \in W^1_{T_i} \) that \( \gamma_{tr}(T_{i+1}) - 3 \geq \omega(f|_{T_i}) \geq \gamma_{tr}(T_i) \). If \( f(x) = 0 \) and \( f(y_1) = 1 \), then the function \( f \), restricted to \( T_i \) is a TRDF of \( T_i \) of weight \( \gamma_{tr}(T_{i+1}) - 3 \) and so \( \gamma_{tr}(T_{i+1}) - 3 \geq \omega(f|_{T_i}) \geq \gamma_{tr}(T_i) \). If \( f(x) = 0 \) and \( f(y_1) = 2 \), then the function \( g : V(T_i) \to \{0,1,2\} \) defined by \( g(x) = 1 \) and \( g(u) = f(u) \) for each \( u \in V(T_i)-\{x\} \) is a nearly TRDF of \( T_i \) of weight \( \gamma_{tr}(T_{i+1}) - 3 \) and since \( x \in W^1_{T_i} \) we have \( \gamma_{tr}(T_{i+1}) - 3 \geq \omega(f|_{T_i}) \geq \gamma_{tr}(T_i) \). Hence, in all cases \( \gamma_{tr}(T_{i+1}) \geq \gamma_{tr}(T_i) + 3 \) and we conclude from Observation 4 that \( \gamma_{tr}(T_{i+1}) = 3\gamma(T_{i+1}) \).

**Lemma 14.** If \( T_i \) is a tree with \( \gamma_{tr}(T_i) = 3\gamma(T_i) \) and \( T_{i+1} \) is a tree obtained from \( T_i \) by Operation \( O₃ \), then \( \gamma_{tr}(T_{i+1}) = 3\gamma(T_{i+1}) \).
Proof. Let $\mathcal{O}_3$ add a path $yzw$ and the edge $xy$. Obviously any $\gamma(T_i)$-set can be extended to a dominating set of $T_{i+1}$ by adding $z$ and so $\gamma(T_{i+1}) \leq \gamma(T_i) + 1$. Now assume $f$ is a $\gamma_{tR}(T_{i+1})$-function such that $f(y)$ is as large as possible. Clearly $f(z) + f(w) \geq 2$. If $f(y) + f(z) + f(w) \geq 3$, then we may assume that $f(z) = 2$ and $f(y) \geq 1$ and by using an argument similar to that described in the proof of Lemma 13 we obtain $\gamma_{tR}(T_{i+1}) = 3\gamma(T_{i+1})$. Now let $f(y) + f(z) + f(w) = 2$. Then we must have $f(z) = f(w) = 1$ and $f(y) = 0$. Then the function $f$, restricted to $T_i$ is a TRDF of $T_i$ of weight $\gamma_{tR}(T_{i+1}) - 2$ with $f(x) = 2$. Since $x \in W^i_T$, we obtain $\gamma_{tR}(T_{i+1}) - 2 = \omega(f|_{T_i}) \geq \gamma_{tR}(T_i) + 1$ and so $\gamma_{tR}(T_{i+1}) \geq \gamma_{tR}(T_i) + 3$. Now the result follows by Observation 4.

Theorem 15. If $T \in \mathcal{T}$, then $\gamma_{tR}(T) = 3\gamma(T)$.

Proof. Let $T \in \mathcal{T}$. Then there exists a sequence of trees $T_1, T_2, \ldots, T_k$ ($k \geq 1$) such that $T_1$ is $P_3$, and if $k \geq 2$, then $T_{i+1}$ can be obtained recursively from $T_i$ by one of the Operations $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ for $i = 1, 2, \ldots, k - 1$.

We proceed by induction on the number of operations applied to construct $T$. If $k = 1$, then $T = P_3 \in \mathcal{T}$. Suppose that the result is true for each tree $T \in \mathcal{T}$ which can be obtained from a sequence of operations of length $k - 1$ and let $T' = T_{k-1}$.

By the induction hypothesis, we have $\gamma_{tR}(T') = 3\gamma(T')$. Since $T = T_k$ is obtained by one of the Operations $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ from $T'$, we conclude from Lemmas 12, 13 and 14 that $\gamma_{tR}(T) = 3\gamma(T)$.

Now we are ready to prove the main result of this section.

Theorem 16. Let $T$ be a tree of order $n \geq 3$. Then $\gamma_{tR}(T) = 3\gamma(T)$ if and only if $T \in \mathcal{T}$.

Proof. By Theorem 15, we only need to prove the necessity. Let $T$ be a tree with $\gamma_{tR}(T) = 3\gamma(T)$. The proof is by induction on $n$. If $n = 3$, then the only tree $T$ of order 3 with $\gamma_{tR}(T) = 3\gamma(T)$ is $P_3 \in \mathcal{T}$. Let $n \geq 4$ and let the statement hold for all trees $T$ of order less than $n$ and $\gamma_{tR}(T) = 3\gamma(T)$. Assume that $T$ is a tree of order $n$ with $\gamma_{tR}(T) = 3\gamma(T)$ and let $f$ be a $\gamma_{tR}(T)$-function. By Observation 3 we have $\text{diam}(T) \neq 3$. If $\text{diam}(T) = 2$, then $T$ is a star and $T$ can be obtained from $P_3$ iterative application of Operation $\mathcal{O}_1$ and so $T \in \mathcal{T}$. Hence we assume $\text{diam}(T) \geq 4$.

Let $v_1v_2 \cdots v_k$ ($k \geq 5$) be a diametrical path in $T$ and root $T$ at $v_k$. If $\text{deg}(v_2) \geq 4$, then clearly $\gamma_{tR}(T) = \gamma_{tR}(T - v_1)$ and $\gamma(T) = \gamma(T - v_1)$ and hence $\gamma_{tR}(T - v_1) = 3\gamma(T - v_1)$. By the induction hypothesis we have $T - v_1 \in \mathcal{T}$. Now, $T$ can be obtained from $T - v_1$ by Operation $\mathcal{O}_1$ and so $T \in \mathcal{T}$. Suppose that $\text{deg}(v_2) \leq 3$. We consider two cases.

Case 1. $\text{deg}(v_2) = 3$. We claim that $\text{deg}(v_3) = 2$. Suppose, to the contrary, that $\text{deg}(v_3) \geq 3$. Then each child of $v_3$ is a leaf or a support vertex. If $v_3$
has a children other than \( v_2 \) which is a leaf or a strong support vertex, then let \( T' = T - T_{v_2} \). It is not hard to see that \( \gamma(T) = \gamma(T') + 1 \) and \( \gamma_{TR}(T) \leq \gamma_{TR}(T') + 2 \). Then \( \gamma_{TR}(T) \leq \gamma_{TR}(T') + 2 \leq 3\gamma(T') + 2 = 3\gamma(T) - 1 \) which is a contradiction. Assume that each child of \( v_3 \) except \( v_2 \), is a support vertex of degree 2. Let \( v_3, z_2, z_1 \) be a pendant path in \( T \). Suppose \( T' = T - \{ z_1, z_2 \} \). As above we can see that \( \gamma_{TR}(T) \leq 3\gamma(T) - 1 \), a contradiction again. Thus \( \deg(v_3) = 2 \).

Assume \( T' = T - T_{v_3} \). Let \( S \) be a \( \gamma(T) \)-set containing support vertices, and define \( S' = S - \{ v_2 \} \) if \( v_3 \not\in S \) and \( S' = (S - \{ v_2, v_3 \}) \cup \{ v_4 \} \) when \( v_3 \in S \). Clearly, \( S' \) is a dominating set of \( T' \) and so \( \gamma(T') \leq |S'| = \gamma(T) - 1 \). On the other hand, any \( \gamma_{TR}(T') \)-function can be extended to a TRDF of \( T \) by assigning 1 to \( v_3 \), 2 to \( v_2 \) and 0 to the leaves adjacent to \( v_2 \). This yields \( \gamma_{TR}(T) \leq \gamma_{TR}(T') + 3 \). It follows from Observation 5 that \( \gamma_{TR}(T') = 3\gamma(T') \) and by the induction hypothesis we have \( T' \in T \). If \( v_4 \not\in W_{T'}^3 \), then let \( g \) be a nearly TRDF of \( T' \) with respect to \( v_4 \) of weight at most \( \gamma_{TR}(T') - 1 \) and define \( h : V(T) \to \{ 0, 1, 2 \} \) by \( h(u) = g(u) \) for \( u \in V(T') \), \( h(v_3) = 1, h(v_2) = 2 \) and \( h(u) = 0 \) otherwise. Clearly \( h \) is a TRDF of \( T \) of weight \( \gamma_{TR}(T') + 2 \) which leads to a contradiction. Hence \( v_4 \in W_{T'}^3 \) and \( T \) can be obtained from \( T' \) by Operation \( O_2 \). Thus \( T \in T \) in this case.

Case 2. \( \deg(v_2) = 2 \). Considering Case 1, we may assume that each child of \( v_3 \) is a support vertex of degree 2. If \( \deg(v_3) \geq 3 \), then let \( T' = T - T_{v_3} \). Any \( \gamma(T') \)-set can be extended to a dominating set of \( T \) by adding \( C(v_3) \) and so \( \gamma(T) \leq \gamma(T') + |C(v_3)| \). On the other hand, let \( S \) be a \( \gamma(T) \)-set containing no leaves. To dominate the leaves of \( T_{v_3} \), we must have \( C(v_3) \subseteq S \). Then the set \( S' = S \setminus C(v_3) \) if \( v_3 \not\in S \) and \( S' = (S - (C(v_3) \cup \{ v_3 \})) \cup \{ v_4 \} \) if \( v_3 \in S \), is a dominating set set of \( T' \) and this implies that \( \gamma(T') \leq \gamma(T) - |C(v_3)| \). Hence \( \gamma(T) = \gamma(T') + |C(v_3)| \).

Also, any \( \gamma_{TR}(T') \)-function can be extended to a TRDF of \( T \) by assigning 1 to \( v_3 \), 2 to the children of \( v_3 \) and 0 to all leaves of \( T_{v_3} \), and so

\[
\gamma_{TR}(T) \leq \gamma_{TR}(T') + 2|C(v_3)| + 1 \\
\leq 3\gamma(T') + 2|C(v_3)| + 1 \\
= 3(\gamma(T') + |C(v_3)|) - |C(v_3)| + 1 \\
= 3\gamma(T) - |C(v_3)| + 1 \\
< 3\gamma(T) \quad \text{(since } |C(v_3)| \geq 2),
\]

a contradiction. Henceforth, we assume \( \deg(v_3) = 2 \). Suppose \( T' = T - T_{v_3} \). Clearly, \( \gamma(T) = \gamma(T') + 1 \). Analogously as in Case 1, we can see that \( \gamma_{TR}(T') = 3\gamma(T') \) and \( v_4 \in W_{T'}^3 \). Thus \( T' \in T \) by the induction hypothesis. If \( v_4 \not\in W_{T'}^3 \), then let \( g \) be a \( \gamma_{TR}(T') \)-function with \( g(v_4) = 2 \) and define \( h : V(T) \to \{ 0, 1, 2 \} \) by \( h(u) = g(u) \) for \( u \in V(T') \), \( h(v_3) = 0, h(v_2) = 1 \). Clearly \( h \) is an TRDF of \( T \) of weight \( \gamma_{TR}(T') + 2 \) which leads to a contradiction. Hence \( v_4 \in W_{T'}^3 \).
and $T$ can be obtained from $T'$ by Operation $O_3$. It follows that $T \in \mathcal{T}$ and the proof is complete.

\[\text{Operation } T_1. \text{ If } x \in W^2_{T_i} \text{ is a support vertex, then the Operation } T_1 \text{ adds a new vertex } y \text{ and an edge } xy \text{ to obtain } T_{i+1}.\]

\[\text{Operation } T_2. \text{ If } x \in V(T_i) \text{ is at distance 2 from a leaf } w, \text{ then the Operation } T_2 \text{ adds a path } yz \text{ and joins } x \text{ to } y \text{ to obtain } T_{i+1}.\]

\[\text{Operation } T_3. \text{ If } x \in W^4_{T_i}, \text{ then the Operation } T_3 \text{ adds a path } z_4z_3z_2z_1 \text{ and joins } x \text{ to } z_3 \text{ to obtain } T_{i+1}.\]

\[\text{Operation } T_4. \text{ If } x \in W^2_{T_i} \cup W^3_{T_i}, \text{ then the Operation } T_4 \text{ adds a path } P_3 = zyw \text{ and joins } x \text{ to } z \text{ to obtain } T_{i+1}.\]

\[\text{Figure 3. The operations } T_1, T_2, T_3 \text{ and } T_4.\]

**Lemma 17.** If $T_i$ is a tree with $\gamma_{tR}(T_i) = 2\gamma(T_i)$ and $T_{i+1}$ is a tree obtained from $T_i$ by Operation $T_1$, then $\gamma_{tR}(T_{i+1}) = 2\gamma(T_{i+1})$. 

In this section we present a constructive characterization of all trees $T$ with $\gamma_{tR}(T) = 2\gamma(T)$.

Let $\mathcal{F}$ be the family of unlabeled trees $T$ that can be obtained from a sequence $T_1, T_2, \ldots, T_m$ ($m \geq 1$) of trees such that $T_1$ is a path $P_2$ or $P_4$, and, if $m \geq 2$, $T_{i+1}$ can be obtained recursively from $T_i$ by one of the following four operations for $1 \leq i \leq m - 1$.

**Operation $T_1$.** If $x \in W^2_{T_i}$ is a support vertex, then the Operation $T_1$ adds a new vertex $y$ and an edge $xy$ to obtain $T_{i+1}$.

**Operation $T_2$.** If $x \in V(T_i)$ is at distance 2 from a leaf $w$, then the Operation $T_2$ adds a path $yz$ and joins $x$ to $y$ to obtain $T_{i+1}$.

**Operation $T_3$.** If $x \in W^4_{T_i}$, then the Operation $T_3$ adds a path $z_4z_3z_2z_1$ and joins $x$ to $z_3$ to obtain $T_{i+1}$.

**Operation $T_4$.** If $x \in W^2_{T_i} \cup W^3_{T_i}$, then the Operation $T_4$ adds a path $P_3 = zyw$ and joins $x$ to $z$ to obtain $T_{i+1}$.
Proof. It is easy to see that $\gamma(T_{i+1}) = \gamma(T_i)$ and $\gamma_{tR}(T_{i+1}) = \gamma_{tR}(T_i)$ and so $\gamma_{tR}(T_{i+1}) = 2\gamma(T_{i+1})$. \hfill \blacksquare

Lemma 18. If $T_i$ is a tree with $\gamma_{tR}(T_i) = 2\gamma(T_i)$ and $T_{i+1}$ is a tree obtained from $T_i$ by Operation $T_2$, then $\gamma_{tR}(T_{i+1}) = 2\gamma(T_{i+1})$.

Proof. Let $w'$ be the support vertex of $w$. If $S$ is a $\gamma(T_{i+1})$-set, then clearly $y, w' \in S$ and $S - \{y\}$ is a dominating set of $T_i$ yielding $\gamma(T_{i+1}) \geq \gamma(T_i) + 1$. Also, if $f$ is a $\gamma_{tR}(T_i)$-function such that $f(x) \geq 1$, then $f$ can be extended to a TRDF of $T_{i+1}$ by assigning the weight 1 to $y, z$. Hence $\gamma_{tR}(T_{i+1}) \leq \gamma_{tR}(T_i) + 2$. Now the result follows by Observation 6. \hfill \blacksquare

Lemma 19. If $T_i$ is a tree with $\gamma_{tR}(T_i) = 2\gamma(T_i)$ and $T_{i+1}$ is a tree obtained from $T_i$ by Operation $T_3$, then $\gamma_{tR}(T_{i+1}) = 2\gamma(T_{i+1})$.

Proof. If $S$ is a $\gamma(T_{i+1})$-set containing no leaves, then $z_3, z_2 \in S$ and we deduce from $x \in W_{T_i}^2$ that $|S - \{z_3, z_2\}| \geq \gamma(T_i)$ yielding $\gamma(T_{i+1}) \geq \gamma(T_i) + 2$. On the other hand, any $\gamma_{tR}(T_i)$-function can be extended to a TRDF of $T$ by assigning the weight 2 to $z_3, z_2$ and the weight 0 to $z_1, z_4$ and so $\gamma_{tR}(T_{i+1}) \leq \gamma_{tR}(T_i) + 4$. It follows from Observation 6 that $\gamma_{tR}(T_{i+1}) = 2\gamma(T_{i+1})$. \hfill \blacksquare

Lemma 20. If $T_i$ is a tree with $\gamma_{tR}(T_i) = 2\gamma(T_i)$ and $T_{i+1}$ is a tree obtained from $T_i$ by Operation $T_4$, then $\gamma_{tR}(T_{i+1}) = 2\gamma(T_{i+1})$.

Proof. Let $T_i$ add a path $zyw$ and joins $x$ to $z$. If $S$ is a $\gamma(T_{i+1})$-set, then $y \in S$ and the set $S' = S - \{y\}$ if $z \not\in S$ and $S' = (S - \{y, z\}) \cup \{x\}$ if $z \in S$, is a dominating set of $T_i$ yielding $\gamma(T_{i+1}) \geq \gamma(T_i) + 1$. Now we show that $\gamma_{tR}(T_{i+1}) \leq \gamma_{tR}(T_i) + 2$. If $x \in W_{T_i}^2$, then let $f$ be a $\gamma_{tR}(T_i)$-function with $f(x) = 2$. Clearly $f$ can be extended to an TRDF of $T_{i+1}$ by assigning the weight 1 to $w, y$ and the weight 0 to $z$ and so $\gamma_{tR}(T_{i+1}) \leq \gamma_{tR}(T_i) + 2$. If $x \in W_{T_i}^2$, then let $f$ be a function satisfying the conditions of Definition 11. Clearly $f$ can be extended to a TRDF of $T_{i+1}$ by assigning the weight 1 to $z, y, w$ and so $\gamma_{tR}(T_{i+1}) \leq \gamma_{tR}(T_i) + 2$. Now the result follows by Observation 6. \hfill \blacksquare

Theorem 21. If $T \in F$, then $\gamma_{tR}(T) = 2\gamma(T)$.

Proof. Let $T \in F$. Then there exists a sequence of trees $T_1, T_2, \ldots, T_k$ ($k \geq 1$) such that $T_1 = P_2$ or $P_4$, and if $k \geq 2$, then $T_{i+1}$ can be obtained recursively from $T_i$ by one of the Operations $T_1, T_2, T_3, T_4$ for $i = 1, 2, \ldots, k - 1$.

We proceed by induction on the number of operations used to construct $T$. If $k = 1$, then $T = P_2$ or $P_4$ and the result is trivial. Suppose the statement holds for each tree $T \in F$ which can be obtained from a sequence of operations of length $k - 1$ and let $T' = T_{k-1}$. By the induction hypothesis, we have $\gamma_{tR}(T') = 2\gamma(T')$. Since $T = T_k$ is obtained by one of the Operations $T_1, T_2, T_3, T_4$ we conclude from previous lemmas that $\gamma_{tR}(T) = 2\gamma(T)$. \hfill \blacksquare
Now we prove the main result of this section.

**Theorem 22.** Let \( T \) be a tree of order \( n \geq 2 \). Then \( \gamma_{tR}(T) = 2\gamma(T) \) if and only if \( T \in \mathcal{F} \).

**Proof.** According to Theorem 21, we only need to prove the necessity. Let \( T \) be a tree with \( \gamma_{tR}(T) = 2\gamma(T) \). Since \( \gamma_{tR}(K_{1,s}) = 3 = 3\gamma(K_{1,s}) \) for \( s \geq 2 \), \( T \) is not a star of order \( n(T) \geq 3 \). We proceed by induction on \( n \). If \( n \in \{2, 4\} \), then the only trees \( T \) of order 2 or 4 with \( \gamma_{tR}(T) = 2\gamma(T) \) are \( P_2, P_4 \in \mathcal{F} \). Assume \( n \geq 5 \) and let the statement hold for all trees \( T \) of order less than \( n \) and \( \gamma_{tR}(T) = 2\gamma(T) \).

Assume that \( T \) is a tree of order \( n \) with \( \gamma_{tR}(T) = 2\gamma(T) \) and let \( f \) be a \( \gamma_{tR}(T) \)-function. Since \( T \) is not a star, we have \( \text{diam}(T) \geq 3 \). If \( \text{diam}(T) = 3 \), then \( T \) is a double star and \( T \) can be obtained from \( P_4 \) by iterative application of Operation \( T_1 \) because the support vertices of \( P_4 \) belong to \( W^2_{P_4} \) and so \( T \in \mathcal{F} \). Hence we assume \( \text{diam}(T) \geq 4 \).

Let \( v_1v_2 \cdots v_k \) \( (k \geq 5) \) be a diametrical path in \( T \) such that \( \text{deg}(v_2) \) is as large as possible and root \( T \) at \( v_k \). First let \( \text{deg}(v_2) \geq 3 \). Clearly \( \gamma_{tR}(T) \geq \gamma_{tR}(T-v_1) \) and \( \gamma(T) = \gamma(T-v_1) \). If \( \gamma_{tR}(T) \geq \gamma_{tR}(T-v_1)+1 \), then we have

\[
2\gamma(T) = \gamma_{tR}(T) \geq \gamma_{tR}(T-v_1)+1 \geq 2\gamma(T-v_1)+1 = 2\gamma(T)+1
\]

which is a contradiction. Thus \( \gamma_{tR}(T) = \gamma_{tR}(T-v_1) \). By Observation 1, there exists a \( \gamma_{tR}(T) \)-function \( f \) such that \( f(v_2) = 2 \). Then clearly \( f \) is a \( \gamma_{tR}(T-v_1) \)-function yielding \( v_2 \in W^2_{T-v_1} \). Now, \( T \) can be obtained from \( T-v_1 \) by Operation \( T_1 \) and so \( T \in \mathcal{F} \). Suppose that \( \text{deg}(v_2) = 2 \).

Consider the following cases.

**Case 1.** \( \text{deg}(v_3) = 2 \). Let \( T' = T-T_{v_3} \). Clearly

\[
(2) \quad \gamma(T') = \gamma(T) - 1.
\]

Now let \( f \) be a \( \gamma_{tR}(T) \)-function. Clearly \( f(v_1)+f(v_2) \geq 2 \) if \( f(v_1)+f(v_2) \geq 3 \), then clearly \( f(v_3) = 0 \) and the function \( f \), restricted to \( T' \) is a TRDF of \( T' \) yielding \( \gamma_{tR}(T') \geq \gamma_{tR}(T') + 3 \). But then

\[
2\gamma(T) = \gamma_{tR}(T) \geq \gamma_{tR}(T') + 3 \geq 2\gamma(T') + 3 = 2(\gamma(T) - 1) + 3 = 2\gamma(T) + 1,
\]

a contradiction. Thus \( f(v_1)+f(v_2) = 2 \). If \( f(v_3) = 1 \) and \( f(v_4) = 0 \), then we get a contradiction as above. If \( f(v_3) = 1 \) and \( f(v_4) \geq 1 \), then the function \( g : V(T') \to \{0, 1, 2\} \) defined by \( g(v_3) = \min\{2, f(v_3) + 1\} \) and \( g(u) = f(u) \) otherwise, is a TRDF of \( T' \) of weight \( \gamma_{tR}(T) - 2 \). Assume that \( f(v_3) \neq 1 \). If \( f(v_3) = 2 \), then \( f(v_4) = 0 \) and the function \( g : V(T') \to \{0, 1, 2\} \) defined by \( g(v_4) = 1, g(v_3) = \min\{2, f(v_3) + 1\} \) and \( g(u) = f(u) \) otherwise, is a TRDF of \( T' \) of weight \( \gamma_{tR}(T) - 2 \). We conclude from

\[
2\gamma(T) = \gamma_{tR}(T) \geq \gamma_{tR}(T') + 2 \geq 2\gamma(T') + 2 \geq 2(\gamma(T) - 1) + 2 = 2\gamma(T)
\]
that
\[ \gamma_{tR}(T) = \gamma_{tR}(T') + 2. \]
By (2) and (3), we obtain \( \gamma_{tR}(T') = 2\gamma(T') \) and by the induction hypothesis we have \( T' \in \mathcal{F} \). Now we show that \( v_4 \in W^3_T \cup W^3_{T'} \). Let \( f \) be a \( \gamma_{tR}(T) \)-function. As above we can see that \( f(v_1) + f(v_2) = 2 \). If \( f(v_3) = 0 \), then the function \( f \) restricted to \( T' \) is a \( \gamma_{tR}(T') \)-function with \( f(v_4) = 2 \) implying that \( v_4 \in W^3_{T'} \). If \( f(v_3) = 2 \) and \( v_4 \) has a neighbor with positive weight under \( f \), then the function \( g : V(T') \rightarrow \{0, 1, 2\} \) defined by \( g(v_4) = 1 \) and \( g(x) = f(x) \) otherwise, is a TRDF of \( T' \) of weight \( \gamma_{tR}(T) - 3 \) contradicting (3). If \( f(v_3) = 2 \) and \( v_4 \) has no neighbor other than \( v_3 \) with positive weight under \( f \), then the function \( g : V(T') \rightarrow \{0, 1, 2\} \) defined by \( g(v_4) = 1 \) and \( g(x) = f(x) \) otherwise, is a function of weight \( \gamma_{tR}(T) - 3 = \gamma_{tR}(T') - 1 \) satisfying the conditions of Definition 11 and so \( v_4 \in W^3_{T'} \). Suppose that \( f(v_3) = 1 \). We can see as above that \( f(v_4) \geq 1 \). If \( f(v_4) = 2 \), then the function \( g : V(T') \rightarrow \{0, 1, 2\} \) defined by \( g(v_4) = \min\{2, f(v_5) + 1\} \) and \( g(x) = f(x) \) otherwise, is a \( \gamma_{tR}(T') \)-function with \( g(v_4) = 2 \) implying that \( v_4 \in W^3_{T'} \). If \( f(v_4) = 1 \) and \( v_4 \) has a neighbor different from \( v_3 \) with positive weight under \( f \), then the function \( f \) restricted to \( T' \) is a TRDF of \( T' \) of weight \( \gamma_{tR}(T) - 3 \) which contradicts (3). Finally if \( f(v_3) = 1 \) and \( v_4 \) has no neighbor other than \( v_3 \) with positive weight, then the function \( f \) restricted to \( T' \) fulfilled the conditions of Definition 11 and so \( v_4 \in W^3_{T'} \). Thus \( v_4 \in W^3_{T'} \cup W^3_{T} \) and \( T \) can be obtained from \( T' \) by Operation \( T_2 \) and so \( T \in \mathcal{F} \).

**Case 2.** \( \text{deg}(v_3) \geq 3 \). By the choice of diametrical path, we may assume that all the children of \( v_3 \) with depth one have degree 2. We consider three subcases.

**Subcase 2.1.** \( v_3 \) is a support vertex and is at distance 2 from some leaves different from \( v_1 \). Let \( T' = T - \{v_1, v_2\} \). Then clearly \( \gamma(T) = \gamma(T') + 1 \) and \( \gamma_{tR}(T) \geq \gamma_{tR}(T') + 2 \). Hence \( \gamma_{tR}(T') = 2\gamma(T') \) by Observation 7. By the induction hypothesis we have \( T' \in \mathcal{F} \) and hence \( T \) can be obtained from \( T' \) by Operation \( T_2 \) and so \( T \in \mathcal{F} \).

**Subcase 2.2.** All children of \( v_3 \) have degree 2. Let \( v_3 z_2 z_1 \) be a pendant path and let \( T' = T - \{v_1, v_2\} \). Clearly \( \gamma(T) = \gamma(T') + 1 \). Now let \( f \) be a \( \gamma_{tR}(T) \)-function. Then \( f(v_2) \geq 1 \), \( f(v_1) + f(v_2) \geq 2 \) and \( f(z_1) + f(z_2) \geq 2 \). If \( f(v_3) \geq 1 \) or \( f(v_3) = 0 \) and \( f(v_2) = 1 \), then the function \( f \) restricted to \( T' \) is a TRDF of \( T' \) of weight \( \omega(f) - 2 \) and so \( \gamma_{tR}(T) \geq \gamma_{tR}(T') + 2 \). Assume that \( f(v_3) = 0 \) and \( f(v_2) = 2 \). Since \( f \) is a TRDF of \( T \), we have \( f(v_1) = 1 \). Then the function \( g : V(T) \rightarrow \{0, 1, 2\} \) defined by \( g(z_1) = g(v_2) = g(v_1) = 1 \) and \( g(x) = f(x) \) otherwise, is a \( \gamma_{tR}(T) \)-function and as above we obtain \( \gamma_{tR}(T) \geq \gamma_{tR}(T') + 2 \). Hence \( \gamma_{tR}(T') = 2\gamma(T') \) by Observation 7. By the induction hypothesis we have \( T' \in \mathcal{F} \) and so \( T \) can be obtained from \( T' \) by Operation \( T_2 \). Thus \( T \in \mathcal{F} \).

**Subcase 2.3.** All children of \( v_3 \) except \( v_2 \) are leaves. Let \( w \) be a leaf adjacent to \( v_3 \). First let \( v_3 \) be a strong support vertex. It is easy to see that \( \gamma(T) = \gamma(T - w) \)
and $\gamma_{tR}(T) = \gamma_{tR}(T - w)$ yielding $\gamma_{tR}(T - w) = 2\gamma(T - w)$. By the induction hypothesis we have $T - w \in \mathcal{F}$ and by Observation 2 we obtain $v_3 \in W^*_T$. Thus $T$ can be obtained from $T - w$ by Operation $T_1$ and so $T \in \mathcal{F}$. Suppose next that $v_3$ is not a strong support vertex. Then by the assumption we have $\deg(v_3) = 3$. Consider the following.

(a) $v_4$ is a support vertex. Let $T' = T - T_{v_3}$. It is easy to see that $\gamma_{tR}(T) = \gamma_{tR}(T') + 2$ and $\gamma(T) = \gamma(T') + 1$. It follows that $\gamma_{tR}(T') = 2\gamma(T')$ and by the induction hypothesis we have $T' \in \mathcal{F}$. Then $T$ can be obtained from $T'$ by Operation $T_2$ and so $T \in \mathcal{F}$.

(b) $v_4$ has a child $z_2$ with depth 1. As above we may assume that $\deg(z_2) = 2$. Let $z_1$ be the leaf adjacent to $z_2$ and let $T' = T - \{z_1, z_2\}$. Clearly $\gamma(T) = \gamma(T') + 1$. By Observation 2, there exists a $\gamma_{tR}(T)$-function $f$ such that $f(v_3) = 2$. Also we have $f(z_1) + f(z_2) \geq 2$. Obviously the function $f$ restricted to $T'$ is a TRDF of $T'$ and so $\gamma_{tR}(T) \geq \gamma_{tR}(T') + 2$. We conclude from $2\gamma(T) = \gamma_{tR}(T) \geq \gamma_{tR}(T') + 2 \geq 2\gamma(T') + 2 = 2\gamma(T)$ that $\gamma_{tR}(T') = 2\gamma(T')$ and by the induction hypothesis we have $T' \in \mathcal{F}$. Now $T$ can be obtained from $T'$ by Operation $T_2$ and so $T \in \mathcal{F}$.

(c) $v_4$ has a child $z_3$ with depth 2. Let $v_4z_3z_2z_1$ be a path in $T$. Using the above argument we may assume that $\deg(z_2) = 2$ and either $\deg(z_3) = 2$ or $\deg(z_3) = 3$ and $z_3$ is a support vertex. If $\deg(z_3) = 2$, then as in Case 1 we can see that $T \in \mathcal{F}$.

Let $\deg(z_3) = 3$ and $z_3$ is a support vertex. Let $T' = T - T_{z_3}$. It is not hard to see that $\gamma(T) = \gamma(T') + 2$ and $\gamma_{tR}(T) = \gamma_{tR}(T') + 4$. This implies that $\gamma_{tR}(T') = 2\gamma(T')$ and by the induction hypothesis we have $T' \in \mathcal{F}$. Since $v_4$ is adjacent to a support vertex, we deduce that $v_4 \in W^*_T$. Now $T$ can be obtained from $T'$ by Operation $T_3$ and so $T \in \mathcal{F}$.

This completes the proof.

\section*{References}


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