FAIR DOMINATION NUMBER IN CACTUS GRAPHS

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Abstract

For $k \geq 1$, a $k$-fair dominating set (or just $k$FD-set) in a graph $G$ is a dominating set $S$ such that $|N(v) \cap S| = k$ for every vertex $v \in V \setminus S$. The $k$-fair domination number of $G$, denoted by $fd_k(G)$, is the minimum cardinality of a $k$FD-set. A fair dominating set, abbreviated FD-set, is a $k$FD-set for some integer $k \geq 1$. The fair domination number, denoted by $fd(G)$, of $G$ that is not the empty graph, is the minimum cardinality of an FD-set in $G$. In this paper, aiming to provide a particular answer to a problem posed in [Y. Caro, A. Hansberg and M.A. Henning, Fair domination in graphs, Discrete Math. 312 (2012) 2905–2914], we present a new upper bound for the fair domination number of a cactus graph, and characterize all cactus graphs $G$ achieving equality in the upper bound of $fd_1(G)$.

Keywords: fair domination, cactus graph, unicyclic graph.

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1. Introduction

For notation and graph theory terminology not given here, we follow [10]. Specifically, let $G$ be a graph with vertex set $V(G) = V$ of order $|V| = n$ and let $v$ be a vertex in $V$. The open neighborhood of $v$ is $N_G(v) = \{u \in V \mid uv \in E(G)\}$ and
the closed neighborhood of \( v \) is \( N_G(v) = \bigcup_{v \in S} N_G(v) \). If the graph \( G \) is clear from the context, we simply write \( N(v) \) rather than \( N_G(v) \). The degree of a vertex \( v \), is \( \text{deg}(v) = |N(v)| \). A vertex of degree one is called a leaf and its neighbor a support vertex. We denote the set of leaves and support vertices of a graph \( G \) by \( L(G) \) and \( S(G) \), respectively. A strong support vertex is a support vertex adjacent to at least two leaves, and a weak support vertex is a support vertex adjacent to precisely one leaf. For a set \( S \subseteq V \), its open neighborhood is the set \( N(S) = \bigcup_{v \in S} N(v) \), and its closed neighborhood is the set \( N[S] = N(S) \cup S \).

The corona graph \( cor(G) \) of a graph \( G \) is a graph obtained by adding a leaf to every vertex of \( G \). We denote by \( P_n \) a path on \( n \) vertices. The distance \( d(u,v) \) between two vertices \( u \) and \( v \) in a graph \( G \) is the minimum number of edges of a path from \( u \) to \( v \). The diameter \( \text{diam}(G) \) of \( G \) is \( \max_{u,v \in V(G)} d(u,v) \). A path of length \( \text{diam}(G) \) is called a diametrical path. A cactus graph is a connected graph in which any two cycles have at most one vertex in common. For a subset \( S \) of vertices of \( G \), we denote by \( G[S] \) the subgraph of \( G \) induced by \( S \).

A subset \( S \subseteq V \) is a dominating set of \( G \) if every vertex not in \( S \) is adjacent to a vertex in \( S \). The domination number of \( G \), denoted by \( \gamma(G) \), is the minimum cardinality of a dominating set of \( G \). A vertex \( v \) is said to be dominated by a set \( S \) if \( N(v) \cap S \neq \emptyset \).

Caro et al. [1] studied the concept of fair domination in graphs. For \( k \geq 1 \), a \( k \)-fair dominating set, abbreviated \( k \text{FD-set} \), in \( G \) is a dominating set \( S \) such that \( |N(v) \cap D| = k \) for every vertex \( v \in V \setminus D \). The \( k \)-fair domination number of \( G \), denoted by \( fd_k(G) \), is the minimum cardinality of a \( k \text{FD-set} \). A \( k \text{FD-set} \) of \( G \) of cardinality \( fd_k(G) \) is called a \( fd_k(G) \)-set. A fair dominating set, abbreviated \( \text{FD-set} \), in \( G \) is a \( 1 \text{FD-set} \) for some integer \( k \geq 1 \). The fair domination number, denoted by \( fd(G) \), of a graph \( G \) that is not the empty graph is the minimum cardinality of an \( \text{FD-set} \) in \( G \). An \( \text{FD-set} \) of \( G \) of cardinality \( fd(G) \) is called a \( fd(G) \)-set.

A perfect dominating set in a graph \( G \) is a dominating set \( S \) such that every vertex in \( V(G) \setminus S \) is adjacent to exactly one vertex in \( S \). Hence a \( 1 \text{FD-set} \) is precisely a perfect dominating set. The concept of perfect domination was introduced by Cockayne et al. in [4], and Fellows et al. [7] with a different terminology which they called semiperfect domination. This concept was further studied, see for example, [2, 3, 5, 6, 9].

**Observation 1** (Caro et al. [1]). Every \( 1 \text{FD-set} \) in a graph contains all its strong support vertices.

The following is easily verified.

**Observation 2.** Let \( S \) be a \( 1 \text{FD-set} \) in a graph \( G \), \( v \) a support vertex of \( G \) and \( v' \) a leaf adjacent to \( v \). If \( S \) contains a vertex \( u \in N_G(v) \setminus \{v'\} \), then \( v \in S \).
Among other results, Caro et al. [1] proved that $fd(G) \leq n - 2$ for any connected graph $G$ of order $n \geq 3$ with no isolated vertex, and constructed an infinite family of connected graphs achieving equality in this bound. They showed that $fd(G) < 17n/19$ for any maximal outerplanar graph $G$ of order $n$, and $fd(T) \leq n/2$ for any tree $T$ of order $n \geq 2$. They then showed that equality for the bound $fd(T) \leq n/2$ holds if and only if $T$ is the corona of a tree. Among open problems posed by Caro et al. [1], one asks to find $fd(G)$ for other families of graphs.

**Problem 3** (Caro et al. [1]). Find $fd(G)$ for other families of graphs.

In this paper, aiming to study Problem 3, we present a new upper bound for the 1-fair domination number of cactus graphs and characterize all cactus graphs achieving equality for the upper bound. We show that if $G$ is a cactus graph of order $n \geq 5$ with $k \geq 1$ cycles, then $fd_1(G) \leq (n-1)/2 + k$. We also characterize all cactus graphs achieving equality for the upper bound.

### 2. Unicyclic Graphs

Fair domination in unicyclic graphs has been studied in [8]. A vertex $v$ of a cactus graph $G$ is a *special vertex* if $\deg_G(v) = 2$ and $v$ belongs to a cycle of $G$. Let $\mathcal{H}_1$ be the class of all graphs $G$ that can be obtained from the corona $cor(C)$ of a cycle $C$ by removing precisely one leaf of $cor(C)$. Let $\mathcal{G}_1$ be the class of all graphs $G$ that can be obtained from a sequence $G_1, G_2, \ldots, G_s = G$, where $G_1 \in \mathcal{H}_1$, and if $s \geq 2$, then $G_{j+1}$ is obtained from $G_j$ by one of the following Operations $O_1$ or $O_2$, for $j = 1, 2, \ldots, s - 1$.

**Operation $O_1$.** Let $v$ be a vertex of $G_j$ with $\deg(v) \geq 2$ such that $v$ is not a special vertex of $G_j$. Then $G_{j+1}$ is obtained from $G_j$ by adding a path $P_2$ and joining $v$ to a leaf of $P_2$.

**Operation $O_2$.** Let $v$ be a leaf of $G_j$. Then $G_{j+1}$ is obtained from $G_j$ by adding two leaves to $v$.

**Lemma 4** [8]. If $G \in \mathcal{G}_1$, then every 1FD-set in $G$ contains every vertex of $G$ of degree at least two.

**Theorem 5** [8]. If $G$ is a unicyclic graph of order $n$, then $fd_1(G) \leq (n+1)/2$, with equality if and only if $G = C_5$ or $G \in \mathcal{G}_1$.

### 3. Main Result

Our aim in this paper is to give an upper bound for the fair domination number of a cactus graph $G$ in terms of the number of cycles of $G$, and then characterize
all cactus graphs achieving equality for the proposed bound. For this purpose we first introduce some families of graphs. Let $H_i$ and $G_i$ be the families of unicyclic graphs described in Section 2. For $i = 2, 3, \ldots, k$, we construct a family $H_i$ from $G_{i-1}$, and a family $G_i$ from $H_i$ as follows.

• Family $H_i$. Let $H_i$ be the family of all graphs $H_i$ such that $H_i$ can be obtained from a graph $H_1 \in H_1$ and a graph $G \in G_{i-1}$, by the following Procedure.

Procedure A. Let $w_0 \in V(H_1)$ be a support vertex of $H_1$, and $w \in V(G_{i-1})$ be a support vertex of $G_{i-1}$. We remove precisely one leaf adjacent to $w_0$ and precisely one leaf adjacent to $w$, and then identify the vertices $w_0$ and $w$.

• Family $G_i$. Let $G_i$ be the family of all graphs $G$ that can be obtained from a sequence $G_1, G_2, \ldots, G_s = G$, where $G_1 \in H_i$, and if $s \geq 2$ then $G_{j+1}$ is obtained from $G_j$ by one of the Operations $O_1$ or $O_2$, described in Section 2, for $j = 1, 2, \ldots, s-1$.

Note that $H_i \subseteq G_i$, for $i = 1, 2, \ldots, k$. Figure 1 demonstrates the construction of the family $G_k$.

We will prove the following.

**Theorem 6.** If $G$ is a cactus graph of order $n \geq 5$ with $k \geq 1$ cycles, then $fd_1(G) \leq (n-1)/2 + k$, with equality if and only if $G = C_5$ or $G \in G_k$.

**Corollary 7.** If $G$ is a cactus graph of order $n \geq 5$ with $k \geq 1$ cycles, then $fd(G) \leq (n-1)/2 + k$.

4. Preliminary Results and Observations

4.1. Notation

We call a vertex $w$ in a cycle $C$ of a cactus graph $G$ a *special cut-vertex* if $w$ belongs to a shortest path from $C$ to a cycle $C' \neq C$. We call a cycle $C$ in a cactus graph $G$, a *leaf-cycle* if $C$ contains exactly one special cut-vertex. In the
Figure 2. $C_i$ is a leaf-cycle for $i = 1, 2, 3$ and $v_j$ is a special cut-vertex for $j = 1, 2, \ldots, 8$.

cactus graph presented in Figure 2, $v_i$ is a special cut-vertex, for $i = 1, 2, \ldots, 8$. Moreover, $C_j$ is a leaf-cycle for $j = 1, 2, 3$.

**Observation 8.** Every cactus graph with at least two cycles contains at least two leaf-cycles.

### 4.2. Properties of the family $\mathcal{G}_k$

The following observation can be proved by a simple induction on $k$.

**Observation 9.** If $G \in \mathcal{G}_k$ is a cactus graph of order $n$, then the following conditions are satisfied.

1. No cycle of $G$ contains a strong support vertex. Furthermore, any cycle of $G$ contains precisely one special vertex.
2. $n$ is odd.
3. $|L(G)| = (n + 1)/2 - k$.
4. If a vertex $v$ of $G$ belongs to at least two cycles of $G$, then $v$ is not a support vertex, and $v$ belongs to precisely two cycles of $G$.

**Observation 10.** Let $G \in \mathcal{G}_k$. Let $G$ be obtained from a sequence $G_1, G_2, \ldots, G_s = G$ ($s \geq 2$) such that $G_1 \in \mathcal{H}_1$ and $G_{j+1}$ is obtained from $G_j$ by one of the Operations $O_1$ or $O_2$ or procedure $A$, for $j = 1, 2, \ldots, s - 1$. If $v$ is a vertex of $G$ belonging to two cycles of $G$ then there is an integer $i \in \{2, 3, \ldots, s\}$ such that $G_i$ is obtained from $G_{i-1}$ by applying Procedure $A$ on the vertex $v$ using a graph $H \in \mathcal{H}_1$, such that $v$ belongs to a cycle of $G_{i-1}$.

**Observation 11.** Assume that $G \in \mathcal{G}_k$ and $v \in V(G)$ is a vertex of degree four belonging to two cycles. Let $D_1$ and $D_2$ be the components of $G - v$, $G^*_1$ be the
Let \( G \in \mathcal{G}_k \). Thus \( G \) is obtained from a sequence \( G_1, G_2, \ldots, G_s = G \) (\( s \geq 2 \)) such that \( G_1 \in \mathcal{H}_1 \) and \( G_{j+1} \) is obtained from \( G_j \) by one of the Operations \( O_1 \) or \( O_2 \) or procedure \( A \), for \( j = 1, 2, \ldots, s - 1 \). Note that \( s \geq k \). We define the \( j \)-th Procedure-Operation, or just \( PO_j \) as one of the Operation \( O_1 \), Operation \( O_2 \), or Procedure \( A \) that can be applied to obtain \( G_{j+1} \) from \( G_j \). Thus \( G \) is obtained from \( G_1 \) by Procedure-Operations \( PO_1, PO_2, \ldots, PO_{s-1} \).

Let \( v \) be a vertex of \( G \) of degree four belonging to two cycles of \( G \), and \( D_1 \) and \( D_2 \) be the components of \( G - v \). By Observation 10, there is an integer \( i \in \{2, 3, \ldots, s\} \) such that \( G_i \) is obtained from \( G_{i-1} \) by applying Procedure \( A \) on the vertex \( v \) using a graph \( H \in \mathcal{H}_1 \). Note that \( v \) is a support vertex of \( G_{i-1} \). Let \( v^* \) be the leaf of \( v \) in \( G_{i-1} \) that is removed in Procedure \( A \). Clearly, either \( V(G_{i-1}) \cap D_1 \neq \emptyset \) or \( V(G_{i-1}) \cap D_2 \neq \emptyset \). Without loss of generality, assume that \( V(G_{i-1}) \cap D_1 \neq \emptyset \). Among \( PO_1, PO_{i+1}, \ldots, PO_{s-1} \), let \( PO_{r_1}, PO_{r_2}, \ldots, PO_{r_l} \), be the Procedure-Operations applied on a vertex of \( D_1 \), where \( 1 \leq t \leq s - 1 \). Let \( G_{r_0} = G_{i-1} \) and \( G_{r_{l+1}} \) be obtained from \( G_{r_l} \) by \( PO_{r_{l+1}} \), for \( l = 0, 1, 2, \ldots, t - 1 \). Clearly by an induction on \( t \), we can deduce that there is an integer \( k^* < k \) such that \( G_{r_t} \in \mathcal{G}_{k^*} \). Note that \( G_{r_t} = G_{i-1}^* \).

**Lemma 12.** If \( G \in \mathcal{G}_k \), then every 1FD-set in \( G \) contains every vertex of \( G \) of degree at least two.

**Proof.** Let \( G \in \mathcal{G}_k \), and \( S \) be a 1FD-set in \( G \). We prove by an induction on \( k \), namely first-induction, to show that \( S \) contains every vertex of \( G \) of degree at least two. For the base step, if \( k = 1 \) then \( G \in \mathcal{G}_1 \), and the result follows by Lemma 4. Assume the result holds for all graphs \( G' \in \mathcal{G}_{k'} \) with \( k' < k \). Now consider the graph \( G \in \mathcal{G}_k \), where \( k > 1 \). Clearly, \( G \) is obtained from a sequence \( G_1, G_2, \ldots, G_l = G \), of cactus graphs such that \( G_1 \in \mathcal{H}_k \), and if \( l \geq 2 \), then \( G_{i-1} \) is obtained from \( G_i \) by one of the operations \( O_1 \) or \( O_2 \) for \( i = 1, 2, \ldots, l - 1 \).

We employ an induction on \( l \), namely second-induction, to show that \( S \) contains every vertex of \( G \) of degree at least two.

For the base step of the second-induction, let \( l = 1 \). Thus \( G \in \mathcal{H}_k \). By the construction of graphs in the family \( \mathcal{H}_k \), there are graphs \( H \in \mathcal{H}_1 \) and \( G' \in \mathcal{G}_{k-1} \) such that \( G \) is obtained from \( H \) and \( G' \) by Procedure \( A \). Clearly, \( H \) is obtained from the corona \( \text{cor}(C) \) of a cycle \( C \), by removing precisely one leaf of \( \text{cor}(C) \). Let \( C = c_0c_1 \cdots c_rc_0 \), where \( c_0 \) is the support vertex of \( H \) that its leaf is removed according to Procedure \( A \). Since \( H \) has precisely one special vertex, let \( c_t \) be the special vertex of \( H \). Let \( w \in V(G') \) be a support vertex of \( G' \) that its leaf, say \( w' \), is removed to obtain \( G \) according to Procedure \( A \). First we show that \( \{c_1, c_r\} \cap S \neq \emptyset \). Clearly \( S \cap \{c_{t-1}, c_t, c_{t+1}\} \neq \emptyset \), since \( \text{deg}_G(c_t) = 2 \). Assume that
c_t \in S$. Since at least one of $c_{t-1}$ or $c_{t+1}$ is a support vertex, by Observation 2, 
\{c_{t-1}, c_{t+1}\} \cap S \neq \emptyset$. By applying Observation 2, we obtain that 
\{c_1, c_r\} \cap S \neq \emptyset, since any vertex of \{c_1, \ldots, c_r\} \setminus \{c_t\} is a support vertex of $G$. Thus assume that 
c_t \notin S$. Then \{c_{t-1}, c_{t+1}\} \cap S \neq \emptyset, and so \{c_1, c_r\} \cap S \neq \emptyset, since any vertex of 
\{c_1, \ldots, c_r\} \setminus \{c_t\} is a support vertex of $G$. Hence, \{c_1, c_r\} \cap S \neq \emptyset. If $c_0 \notin S$, then 
\(S \cap V(G')) \cup \{w'\}\) is a 1FD-set for $G'$, and thus by the first-inductive hypothesis, 
$S$ contains $w = c_0$, a contradiction. Thus $c_0 \in S$. By Observation 2, $V(C) \subseteq S$, 
since any vertex of \{c_1, \ldots, c_r\} \setminus \{c_t\} is a support vertex of $G$. Thus $S \cap V(G')$ is 
a 1FD-set for $G'$. By the first-inductive hypothesis, \(S \cap V(G')\) contains 
every vertex of $G'$ of degree at least two. Consequently, $S$ contains every vertex of $G$ 
of degree at least two. We conclude that the base step of the second-induction 
holds.

Assume that the result (for the second-induction) holds for $2 \leq l' < l$. Now let 
$G = G_l$. Clearly $G$ is obtained from $G_{l-1}$ by applying one of the Operations 
$O_1$ or $O_2$.

Assume that $G$ is obtained from $G_{l-1}$ by applying Operation $O_2$. Let $x$ be a 
leaf of $G_{l-1}$ and $G$ be obtained from $G_{l-1}$ by adding two leaves $x_1$ and $x_2$ to $x$. 
By Observation 1, $x \in S$. Thus $S$ is a 1FD-set for $G_{l-1}$. By the second-inductive hypothesis 
$S$ contains all vertices of $G_{l-1}$ of degree at least two. Consequently, $S$ contains every vertex of $G_k$ of degree at least two.

Next assume that $G$ is obtained from $G_{l-1}$ by applying Operation $O_1$. Let 
x_1x_2 be a path and $x_1$ is joined to $y \in V(G_{l-1})$, where $\deg_{G_{l-1}}(y) \geq 2$ and $y$ 
is not a special vertex of $G_{l-1}$. Observe that \{x_1, x_2\} \cap S \neq \emptyset. If $x_1 \notin S$, then 
x_2 \in S and $y \notin S$. Then $S \setminus \{x_2\}$ is a 1FD-set for $G_{l-1}$ that does not contain $y$, 
a contradiction by the second-inductive hypothesis. Thus assume that $x_1 \in S$. 
Suppose that $y \notin S$. Clearly $N_{G_{l-1}}(y) \cap S = \emptyset$.

Assume that there exists a component $G'_l$ of $G_{l-1} - y$ such that $|V(G'_l) \cap 
N_{G_{l-1}}(y)| = 1$. Then clearly $S' = (S \cap V(G_{l-1})) \cup V(G'_l)$ is a 1FD-set for 
$G_{l-1}$, and by the second-inductive hypothesis $S'$ contains every vertex of $G_{l-1}$ of degree at least two. Thus $y \in S'$, and so $y \in S$, a contradiction. Next assume that 
every component of $G_{l-1} - y$ has at least two vertices in $N_{G_{l-1}}(y)$. Since $y$ 
is a non-special vertex of $G_{l-1}$, $y$ belongs to at least two cycles of $G_{l-1}$. By 
Observation 9(4), $y$ belongs to exactly two cycles of $G_{l-1}$. Thus $\deg_{G_{l-1}}(y) = 4$. 
By Observation 11, $G_{l-1} - y$ has exactly two components $D_1$ and $D_2$. Let $G^*$ 
be a graph obtained from $D_1 \cup \{v\}$ or $D_2 \cup \{v\}$, by adding a leaf $v^*$ to $y$. Then 
there exists $k' \leq k$ such that $G^* \in G_{k'}$. Evidently, $S^* = (S \cap V(G^*)) \cup \{v^*\}$ 
is a 1FD-set for $G^*$, and so by the first-inductive hypothesis, $S^*$ contains every vertex of $G^*$ of degree at least two (since $G^* \in G_{k'}$). Thus $y \in S^*$, and so $y \in S$, a contradiction. We conclude that $y \in S$. Observe that $S \cap V(G_{l-1})$ is a 1FD-set 
for $G_{l-1}$, and so by the second-inductive hypothesis, $S \cap V(G_{l-1})$ contains every 
vertex of $G_{l-1}$ of degree at least two. Consequently $S$ contains every vertex of $G$.
of degree at least two.

As a consequence of Observation 9(3) and Lemma 12, we obtain the following.

**Corollary 13.** If $G \in \mathcal{G}_k$ is a cactus graph of order $n$, then $V(G) \setminus L(G)$ is the unique $fd_1(G)$-set.

## 5. Proof of Theorem 6

We first establish the upper bound by proving the following.

**Theorem 14.** If $G$ is a cactus graph of order $n$ with $k \geq 1$ cycles, then $fd_1(G) \leq (n(G) - 1)/2 + k$.

**Proof.** The result follows by Theorem 5 if $k = 1$. Thus assume that $k \geq 2$. Suppose to the contrary that $fd_1(G) > (n(G) - 1)/2 + k$. Assume that $G$ has the minimum order, and among all such graphs, we may assume that the size of $G$ is minimum. Let $C_1, C_2, \ldots, C_k$ be the $k$ cycles of $G$. Let $C_i$ be a leaf-cycle of $G$, where $i \in \{1, 2, \ldots, k\}$. Let $C_i = u_0u_1 \cdots u_ku_0$, where $u_0$ is a special cut-vertex of $G$. Assume that $deg_{G}(u_j) = 2$ for each $j = 1, 2, \ldots, l$. Let $G' = G - u_1u_2$. Then by the choice of $G$, $fd_1(G') \leq (n(G') - 1)/2 + k - 1 = (n(G) - 1)/2 + k - 1$. Let $S'$ be a $fd_1(G')$-set. Now if $|S' \cap \{u_1, u_2\}| \in \{0, 2\}$, then $S'$ is a 1FD-set for $G$, a contradiction. Thus $|S' \cap \{u_1, u_2\}| = 1$. Assume that $u_1 \in S'$. Then $u_3 \in S'$, and so $\{u_2\} \cup S'$ is a 1FD-set in $G$ of cardinality at most $(n(G) - 1)/2 + k$, a contradiction. If $u_2 \in S'$, then $u_0 \in S'$, and $\{u_1\} \cup S'$ is a 1FD-set in $G$ of cardinality at most $(n(G) - 1)/2 + k$, a contradiction. We deduce that $deg_{G}(u_i) \geq 3$ for some $i \in \{1, 2, \ldots, l\}$. Let $v_d$ be a leaf of $G$ such that $d(v_d, C_i - u_0)$ is as maximum as possible, and the shortest path from $v_d$ to $C_i$ does not contain $u_0$. Let $v_0v_1 \cdots v_{d-1}$ be the shortest path from $v_d$ to $C_i$ with $v_0 \in C_i$. Assume that $d \geq 2$. Assume that $deg_{G}(v_{d-1}) = 2$. Let $G' = G - \{v_d, v_{d-1}\}$. By the choice of $G$, $fd_1(G') \leq (n(G') - 1)/2 + k$. Let $S'$ be a $fd_1(G')$-set. If $v_{d-2} \in S'$, then $S' \cup \{v_{d-1}\}$ is a 1FD-set in $G$, and if $v_{d-2} \notin S'$, then $S' \cup \{v_d\}$ is a 1FD-set in $G$. Thus $fd_1(G) \leq (n - 1)/2 + k$, a contradiction. Thus assume that $deg_{G}(v_{d-1}) \geq 3$. Clearly any vertex of $N_G(v_{d-1}) \setminus \{v_{d-2}\}$ is a leaf. Let $G'$ be obtained from $G$ by removing all leaves adjacent to $v_{d-1}$. By the choice of $G$, $fd_1(G') \leq (n(G') - 1)/2 + k$, since $G$ has the minimum order among all graphs $H$ with 1-fair domination number more than $(n(H) - 1)/2 + k$. Let $S'$ be a $fd_1(G')$-set. If $v_{d-1} \in S'$, then $S'$ is a 1FD-set in $G$, a contradiction. Thus assume that $v_{d-1} \notin S'$. Then $S' \cup \{v_{d-1}\}$ is a 1FD-set in $G$ of cardinality at most $(n(G') - 1)/2 + k + 1 \leq (n(G) - 1)/2 + k$, a contradiction.

We thus assume that $d = 1$. Assume that $u_i$ is a vertex of $C_i$ such that $deg_{G}(u_i) = 2$. Assume that $deg_{G}(u_{i+1}) = 2$. Let $G' = G - u_iu_{i+1}$. By the
If $n \geq f_d(G)$, we prove by induction on $n$. Let $S'$ be a $f_d(G)$-set. If $|S' \cap \{u_1, u_{i+1}\}| \in \{0, 2\}$, then $S'$ is a 1FD-set for $G$, a contradiction. Then $|S' \cap \{u_i, u_{i+1}\}| = 1$. Assume that $u_i \in S'$. Then $u_{i+1} \in S'$ and so $\{u_{i+1}\} \cup S'$ is a 1FD-set in $G$ of cardinality at most $\frac{n(G) - 1}{2} + k$, a contradiction. Thus $\deg_G(u_{i+1}) \geq 3$, and similarly $\deg_G(u_{i-1}) \geq 3$. Since $C_i$ is a leaf-cycle, it has precisely one special cut-vertex. Thus we may assume, without loss of generality, that $u_{i+1}$ is a support vertex of $G$. Let $G' = G - u_{i-1}$. By the choice of $G$, $f_d(G') \leq (n(G') - 1)2 + k - 1$. Let $S'$ be a $f_d(G')$-set. By Observation 1, $u_{i+1} \in S'$. If $u_{i-1} \notin S'$, then $S'$ is a 1FD-set in $G$ of cardinality at most $\frac{n(G) - 1}{2} + k - 1$, a contradiction. Thus $u_{i-1} \in S'$. Then $S' \cup \{u_i\}$ is a 1FD-set in $G$ of cardinality at most $\frac{n(G) - 1}{2} + k$, a contradiction.

We conclude that $\deg_G(u_i) \geq 3$ for $i = 0, 1, \ldots, l$. Furthermore, $u_i$ is a support vertex for $i = 1, 2, \ldots, l$. Assume that $u_i$ is a strong support vertex for some $i \in \{1, 2, \ldots, l\}$. Let $G'$ be obtained from $G$ by removal of all vertices in $\bigcup_{i=1}^l (N[u_i]) \setminus \{u_0, u_1, u_l\}$. Clearly $u_0$ is a strong support vertex of $G'$. By the choice of $G$, $f_d(G') \leq (n(G') - 1)2 + k - 1 \leq (n(G) - 2l + 1 + 2 - 1)/2 + k - 1$, since $u_i$ is a strong support vertex of $G$. By Observation 1, $u_0 \in S'$, and so $S' \cup \{u_1, \ldots, u_l\}$ is a 1FD-set in $G'$ of cardinality at most $\frac{n(G) - (2l + 1)2 - 1}{2} + k - 1 + l = n(G)/2 - k - 1$, a contradiction. Thus $u_i$ is a weak support vertex, for each $i = 1, 2, \ldots, l$. Let $G'$ be obtained from $G$ by removal of any vertex in $\bigcup_{i=1}^l (N[u_i]) \setminus \{u_0\}$. By the choice of $G$, $f_d(G') \leq (n(G') - 1)2 + k - 1$. Let $S'$ be a $f_d(G')$-set. If $u_0 \notin S'$, then $S' \cup \{u_1, \ldots, u_l\}$ is a 1FD-set in $G$ of cardinality at most $\frac{n(G) - 1}{2} + k$, where $u_i$ is the leaf adjacent to $u_i$, for $i = 1, 2, \ldots, l$. This is a contradiction. Thus $u_0 \in S'$. Then $S' \cup \{u_1, \ldots, u_l\}$ is a 1FD-set in $G$ of cardinality at most $\frac{n(G) - 1}{2} + k - 1$, a contradiction.

If $G$ is a cactus graph of order $n$ with $k \geq 1$ cycles and $f_d(G) = (n - 1)/2 + k$, then clearly $n \geq 3$ is odd, and since $f_d(C_3) \neq 2$, we have $n \geq 5$. It is obvious that $f_d(C_3) = 3 = (5 - 1)/2 + 1$.

Theorem 15. If $G \neq C_5$ is a cactus graph of order $n \geq 5$ with $k \geq 1$ cycles, then $f_d(G) = (n - 1)/2 + k$ if and only if $G \in \mathcal{G}_k$.

Proof. We prove by an induction on $k$ to show that any cactus graph $G$ of order $n \geq 5$ with $k \geq 1$ cycles and $f_d(G) = (n - 1)/2 + k$ belongs to $\mathcal{G}_k$. The base step of the induction follows by Theorem 5. Assume the result holds for all cactus graphs $G'$ with $k' < k$ cycles. Now let $G$ be a cactus graph of order $n$ with $k \geq 2$ cycles and $f_d(G) = (n - 1)/2 + k$. Clearly $n$ is odd. Suppose to the contrary that $G \notin \mathcal{G}_k$. Assume that $G$ has the minimum order, and among all such graphs, assume that the size of $G$ is minimum. By Observation 8, $G$ has at least two leaf-cycles. Let $C_1 = c_0c_1 \cdots c_rc_0$ and $C_2 = c'_0c'_1 \cdots c'_rc'_0$, be two leaf-cycles of
$G$, where $c_0$ and $c'_0$ are two special cut-vertices of $G$. Let $G'_1$ be the component of $G - c_0 c_1 - c_0 c_r$ containing $c_1$, and $G''_r$ be the component of $G - c'_0 c'_1 - c'_0 c'_r$, containing $c'_1$.

**Claim 1.** $V(G'_1) \neq \{c_1, \ldots, c_r\}$, and $V(G''_r) \neq \{c'_1, \ldots, c'_r\}$.

**Proof.** Suppose that $V(G'_1) = \{c_1, \ldots, c_r\}$. Then $\deg_G(c_i) = 2$ for $i = 1, 2, \ldots, r$. Let $G^* = G-c_1 c_2$, and $S^*$ be a $d_1(G^*)$-set. By Theorem 14, $d_1(G^*) \leq (n(G^*)-1)/2+k-1 = (n(G)-1)/2+k-1$. Assume that $r = 2$. Then $c_0$ is a strong support vertex of $G^*$, and by Observation 1, $c_0 \in S^*$. Thus $|S^* \cap \{c_1, c_2\}| = 0$, and so $S^*$ is a 1FD-set in $G$ of cardinality at most $(n(G)-1)/2+k-1 < (n(G)-1)/2+k$, a contradiction. Assume that $r = 3$. If $|S^* \cap \{c_1, c_2\}| \in \{0, 2\}$, then $S^*$ is a 1FD-set in $G$ of cardinality at most $(n(G)-1)/2+k-1 < (n(G)-1)/2+k$, a contradiction. Thus $|S^* \cap \{c_1, c_2\}| = 1$. If $c_1 \in S^*$, then $c_3 \in S^*$, and so $c_0 \in S^*$. Then $S^* \setminus \{c_1\}$ is a 1FD-set in $G^*$, a contradiction. Thus $c_1 \notin S^*$, and so $c_2 \in S^*$. Since $c_1$ is dominated by $S^*$, we obtain that $c_0 \in S^*$, and so $c_3 \in S^*$. Then $S^* \setminus \{c_2\}$ is a 1FD-set in $G^*$, a contradiction. Assume that $r = 4$. Suppose that $d_1(G^*) = (n(G^*)-1)/2+k-1$. Let $G''_r = G^* - \{c_2, c_3, c_4\}$. By Theorem 14, $d_1(G''_r) \leq (n(G'')-1)/2+k-3$, and thus $d_1(G''_r) \leq (n-1)/2+k-3$, since $n$ is odd. Let $S''_r$ be a $d_1(G''_r)$-set. If $c_0 \notin S''_r$, then $S''_r \cup \{c_3\}$ is a 1FD-set for $G^*$ and if $c_0 \notin S''_r$, then $S''_r \cup \{c_3\}$ is a 1FD-set for $G^*$, thus $d_1(G''_r) \leq |S''_r| + 1 \leq (n-1)/2+k-2$, a contradiction. Thus $d_1(G''_r) < (n(G''_r)-1)/2+k-1 = (n(G)-1)/2+k-1$. If $|S''_r \cap \{c_1, c_2\}| \in \{0, 2\}$, then $S''_r$ is a 1FD-set in $G$ of cardinality at most $(n(G)-1)/2+k-1 < (n(G)-1)/2+k$, a contradiction. Thus $|S''_r \cap \{c_1, c_2\}| = 1$. Without loss of generality, assume that $c_1 \in S''_r$. Then $S''_r \cup \{c_3\}$ is a 1FD-set in $G$, and so $d_1(G) \leq |S''_r| + 1 < (n(G)-1)/2+k$, a contradiction. It remains to assume that $r \geq 5$. Suppose that $d_1(G^*) = (n(G^*)-1)/2+k-1$. Let $G''_2 = G^* - \{c_2, c_3, c_4\}$. By Theorem 14, $d_1(G''_2) \leq (n(G''_2)-1)/2+k-3$, and thus $d_1(G''_2) \leq (n-1)/2+k-3$, since $n$ is odd. Let $S''_2$ be a $d_1(G''_2)$-set. If $c_5 \notin S''_2$, then $S''_2 \cup \{c_3\}$ is a 1FD-set for $G''_2$ and if $c_5 \notin S''_2$, then $S''_2 \cup \{c_3\}$ is a 1FD-set for $G''_2$. Thus $d_1(G''_2) \leq |S''_2| + 1 \leq (n-1)/2+k-2$, a contradiction. Thus $d_1(G''_2) < (n(G''_2)-1)/2+k-1 = (n(G)-1)/2+k-1$. If $|S''_2 \cap \{c_1, c_2\}| \in \{0, 2\}$, then $S''_2$ is a 1FD-set in $G$ of cardinality at most $(n(G)-1)/2+k-1 < (n(G)-1)/2+k$, a contradiction. Thus $|S''_2 \cap \{c_1, c_2\}| = 1$. Without loss of generality, assume that $c_1 \in S''_2$. Then $S''_2 \cup \{c_2\}$ is a 1FD-set in $G$, and so $d_1(G) \leq |S''_2| + 1 < (n(G)-1)/2+k$, a contradiction. We conclude that $V(G'_1) \neq \{c_1, \ldots, c_r\}$. Similarly $V(G''_r) \neq \{c'_1, \ldots, c'_r\}$. □

Let $v_d \in V(G'_1) \setminus \{c_1, \ldots, c_r\}$ be a leaf of $G'_1$ at maximum distance from $\{c_1, \ldots, c_r\}$, and assume that $v_0 v_1 \cdots v_d$ is the shortest path from $v_d$ to $\{c_1, \ldots, c_r\}$, where $v_0 \in \{c_1, \ldots, c_r\}$. Likewise, let $v'_d \in V(G''_r) \setminus \{c'_1, \ldots, c'_r\}$ be a leaf of $G''_r$ at maximum distance from $\{c'_1, \ldots, c'_r\}$, and assume that $v'_0 v'_1 \cdots v'_d$ is the shortest
path from $v'_d$ to $\{c'_1, \ldots, c'_r\}$, where $v'_0 \in \{c'_1, \ldots, c'_r\}$. Without loss of generality, assume that $d' \leq d$.

**Claim 2.** Every support vertex of $G$ is adjacent to at most two leaves.

**Proof.** Suppose that there is a support vertex $v \in S(G)$ such that $v$ is adjacent to at least three leaves $v_1, v_2$ and $v_3$. Let $G' = G - \{v_1\}$, and let $S'$ be a $fd_1(G')$-set. By Observation 1, $v \in S'$, and thus we may assume that $S' \cap \{v_2, v_3\} = \emptyset$. By Theorem 14, $|S'| \leq (n(G') - 1)/2 + k = (n-2)/2 + k$. Clearly $S'$ is a 1FD-set for $G$, a contradiction. 

**Claim 3.** If $d \geq 2$, then $G \in \mathcal{G}_k$.

**Proof.** Let $d \geq 2$. By Claim 2, $2 \leq deg_G(v_{d-1}) \leq 3$. Assume first that $deg_G(v_{d-1}) = 3$. Let $x \neq v_d$ be a leaf adjacent to $v_{d-1}$. Let $G' = G - \{x, v_d\}$. By Theorem 14, $fd_1(G') \leq (n(G') - 1)/2 + k$. Suppose that $fd_1(G') < (n(G') - 1)/2 + k$. Let $S'$ be a $fd_1(G')$-set. If $v_{d-1} \in S'$, then $S'$ is a 1FD-set for $G$ and if $v_{d-1} \notin S'$, then $S' \cup \{v_{d-1}\}$ is a 1FD-set for $G$. Thus $fd_1(G') \leq fd_1(G') + 1 < (n-1)/2 + k$, a contradiction. Hence, $fd_1(G') = (n(G') - 1)/2 + k$. By the choice of $G$, $G' \in \mathcal{G}_k$. Therefore $G$ is obtained from $G'$ by Operation $O_2$. Consequently, $G \in \mathcal{G}_k$. Next assume that $deg_G(v_{d-1}) = 2$. We consider the following cases.

**Case 1.** $d \geq 3$. Suppose that $deg_G(v_{d-2}) = 2$. Let $G' = G - \{v_{d-2}, v_{d-1}, v_d\}$. By Theorem 14, $fd_1(G') \leq (n(G') - 1)/2 + k = n/2 + k - 2$, and thus $fd_1(G') \leq (n-1)/2 + k - 2$, since $n$ is odd. Let $S'$ be a $fd_1(G')$-set. If $v_{d-3} \notin S'$, then $S' \cup \{v_d\}$ is a 1FD-set for $G$ and if $v_{d-3} \notin S'$, then $S' \cup \{v_{d-1}\}$ is a 1FD-set for $G$. Thus $fd_1(G) \leq |S'| + 1 \leq (n-1)/2 + k - 1$, a contradiction. Thus $deg_G(v_{d-2}) \geq 3$. Let $G' = G - \{v_{d-1}, v_d\}$. By Theorem 14, $fd_1(G') \leq (n(G') - 1)/2 + k$. Suppose that $fd_1(G') < (n(G') - 1)/2 + k$. Let $S'$ be a $fd_1(G')$-set. If $v_{d-2} \notin S'$, then $S' \cup \{v_{d-1}\}$ is a 1FD-set for $G$ and if $v_{d-2} \notin S'$, then $S' \cup \{v_d\}$ is a 1FD-set for $G$. Thus $fd_1(G) \leq |S'| + 1 \leq fd_1(G') + 1 < (n-1)/2 + k$, a contradiction. We deduce that $fd_1(G') = (n(G') - 1)/2 + k$. By the choice of $G$, $G' \in \mathcal{G}_k$. Since $d \geq 3$, $v_{d-2}$ is not a special vertex of $G'$. Thus $G$ is obtained from $G'$ by Operation $O_1$, and so $G \in \mathcal{G}_k$.

**Case 2.** $d = 2$. As noted, $deg(v_1) = 2$. Clearly $deg(v_0) \geq 3$. Assume first that $deg(v_0) \geq 4$. Let $G' = G - \{v_2, v_1\}$. By Theorem 14, $fd_1(G') \leq (n(G') - 1)/2 + k$. Suppose that $fd_1(G') < (n(G') - 1)/2 + k$. Let $S'$ be a $fd_1(G')$-set. If $v_0 \in S'$, then $S' \cup \{v_1\}$ is a 1FD-set for $G$, and if $v_0 \notin S'$, then $S' \cup \{v_2\}$ is a 1FD-set for $G$. Thus $fd_1(G) \leq |S'| + 1 < (n-1)/2 + k$, a contradiction. Thus, $fd_1(G') = (n(G') - 1)/2 + k$. By the choice of $G$, $G' \in \mathcal{G}_k$. Since $deg_G(v_0) \geq 3$, $v_0$ is not a special vertex of $G'$. Hence $G$ is obtained from $G'$ by Operation $O_1$. Consequently, $G \in \mathcal{G}_k$. Thus we assume that $deg(v_0) = 3$. Let $G' = G - \{v_2, v_1\}$. By Theorem 14, $fd_1(G') \leq (n(G') - 1)/2 + k$. Suppose
that \( fd_1(G') < (n(G') - 1)/2 + k \). Let \( S' \) be a \( fd_1(G') \)-set. If \( v_0 \in S' \), then 
\( S' \cup \{v_1\} \) is a 1FD-set for \( G \), and if \( v_0 \notin S' \), then 
\( S' \cup \{v_2\} \) is a 1FD-set for \( G \). Thus 
\( fd_1(G) \leq |S'| + 1 \leq fd_1(G') + 1 < (n-1)/2 + k \), a contradiction. Thus we 
obtain that 
\( fd_1(G') = (n(G') - 1)/2 + k \). By the choice of \( G, G' \in \mathcal{G}_k \). Then 
\( v_0 \) is a special vertex of \( G' \). From Observation 9(1), we obtain that 
\( \deg_{G'}(c_i) \geq 3 \) for each \( i \in \{1, \ldots, r\} \).

Suppose that \( N_G(c_j) \setminus V(C_1) \) contains no strong support vertex for each 
\( j \in \{1, \ldots, r\} \). Observation 9(1) implies that \( c_j \) is not a strong support vertex of 
\( G, \) since \( G' \in \mathcal{G}_k \). Assume that there is a vertex \( c_j \in \{ c_1, \ldots, c_r \} \) such that \( c_j \) has a 
neighbor \( a \) which is a support vertex. By assumption, \( a \) is a weak support vertex. 
If \( a' \) is the leaf adjacent to \( a \), then \( a' \) plays the role of \( v_d \). Since \( \deg(v_0) = 3 \), we 
may assume that \( \deg(c_j) = 3 \). Thus by Observation 9(1), we may assume that 
\( \deg_G(c_i) = 3 \) for each \( c_i \in \{ c_1, \ldots, c_r \} \). Let 
\( F = \bigcup_{i=1}^{r}(N[c_i]) \setminus \{ c_0, \ldots, c_r \} \). Clearly 
\( |F| = r \), since \( \deg_G(c_i) = 3 \) for each \( c_i \in \{ c_1, \ldots, c_r \} \). Let 
\( F = \{ u_1, u_2, \ldots, u_r \}, \) 
\( F' = \{ u_i \in F \mid \deg_G(u_i) = 1 \} \), and 
\( F'' = F \setminus F' \). Then every vertex of \( F'' \) is a 
weak support vertex. Since \( v_i \in F'' \), \( |F''| \geq 1 \). Now let 
\( G' = G - v_0c_1 - c_0c_r, \) and \( G_1^* \) and \( G_2^* \) be the components of \( G^* \), where 
\( c_1 \in V(G_2^*) \). By Theorem 14, 
\( fd_1(G_2^*) \leq (n(G_2^*) - 1)/2 + k - 1 \). Clearly 
\( n(G_2^*) = n - 2r - |F''| \). Let 
\( S_2^* \) be a \( fd_1(G_2^*) \)-set. If \( c_0 \notin S_2^* \), then 
\( S_2^* \cup F \) is a 1FD-set for \( G \), and so 
\( fd_1(G) \leq (n(G) - 2r - |F''| - 1)/2 + k - 1 < (n-1)/2 + k \), a contradiction. Thus 
\( c_0 \in S_2^* \). If \( |F''| = 1 \), then 
\( S_2^* \cup C_1 \cup \{v_1\} \) is a 1FD-set for \( G \) and thus 
\( fd_1(G) \leq fd_1(G_2^*) + r + 1 \leq (n-2)/2 + k \), a contradiction. Thus assume that 
\( |F''| \geq 2 \). Let \( \{ u_t, u_{t'} \} \subseteq F'' \) (assume without loss of generality that 
\( t < t' \)) such that \( \deg_G(u_t) = 1 \) for \( 1 \leq i < t \) and 
\( t' < i \leq r \). Let \( u_t' \) and \( u_{t'}' \) be the leaves of \( u_t \) and \( u_{t'} \), respectively. Clearly 
\( S_2^* \cup \{ c_1, \ldots, c_{t-1} \} \cup \{ c_{t+1}, \ldots, c_r \} \cup \{ u_{t+1}, \ldots, u_{t-1} \} \cup \{ u_t', u_{t'}' \} \) is a 1FD-set for 
\( G \) and thus 
\( fd_1(G) \leq fd_1(G_2^*) + r < (n-1)/2 + k - 1 \), a contradiction.

Thus we may assume that \( N(c_j) \setminus C_1 \) contains at least one strong support 
vertex for some \( c_j \in \{ c_1, \ldots, c_r \} \). Let \( u_j \) be a strong support vertex in 
\( N(c_j) \setminus C_1 \). By Claim 2, there are precisely two leaves adjacent to \( u_j \). Let 
\( u' \) and \( u'' \) be the leaves adjacent to \( u_j \), and \( G^* = G - \{ u', u'' \} \). By Theorem 14, 
\( fd_1(G^*) \leq (n(G^*) - 1)/2 + k \). Assume that 
\( fd_1(G^*) < (n(G^*) - 1)/2 + k \). Let \( S' \) be a 
\( fd_1(G^*) \)-set. If \( u_j \in S' \), then \( S' \) is a 1FD-set for \( G \), and if \( u_j \notin S' \), then 
\( S' \cup \{ u_j \} \) is a 1FD-set for \( G \). Thus 
\( fd_1(G) \leq fd_1(G^*) + 1 < (n-1)/2 + k \), a contradiction. 
We deduce that 
\( fd_1(G^*) = (n(G^*) - 1)/2 + k \). By the choice of \( G, G^* \in \mathcal{G}_k \). Thus 
\( G \) is obtained from \( G^* \) by Operation \( O_2 \). Consequently, \( G \in \mathcal{G}_k \). \( \square \)

By Claim 3, we assume that \( d = d' = 1 \).

**Claim 4.** \( C_i \) has precisely one special vertex, for \( i = 1, 2 \).

**Proof.** We first show that \( C_i \) has at least one special vertex, for \( i = 1, 2 \). Suppose 
that \( C_1 \) has no special vertex. Thus \( \deg_{G}(c_i) \geq 3 \) for \( i = 1, \ldots, r \). Clearly, \( c_i \) is a
support vertex for \( i = 1, 2, \ldots, r \). Suppose that \( c_j \) is a strong support vertex for some \( j \in \{1, 2, \ldots, r\} \). Let \( G' \) be obtained from \( G \) by removal of all vertices in \( \bigcup_{i=1}^{r} (N[c_i]) \setminus \{c_0, c_1, c_r\} \). Clearly, \( c_0 \) is a strong support vertex of \( G' \). By Theorem 14, \( fd_1(G') \leq (n(G') - 1)/2 + k - 1 \). Since \( c_j \) is a strong support vertex of \( G \), we have \( n(G') \leq n(G) - (2r+1) + 2. \) Thus, \( fd_1(G') \leq (n(G) - (2r+1) + 2 - 1)/2 + k - 1 \).

By Observation 1, \( c_0 \in S' \), and so \( S' \cup \{c_1, \ldots, c_r\} \) is a 1FD-set in \( G \) of cardinality at most \( (n(G) - (2r+1) + 2 - 1)/2 + k - 1 + r = n(G)/2 + k - 1 < (n(G) - 1)/2 + k \), a contradiction. Thus \( c_0 \) is a weak support vertex for each \( i = 1, 2, \ldots, r \). Let \( G' \) be obtained from \( G \) by removal of any vertex in \( \bigcup_{i=1}^{r} (N[c_i]) \setminus \{c_0\} \). By Theorem 14, \( fd_1(G') \leq (n(G') - 1)/2 + k - 1 \). Let \( S' \) be a \( fd_1(G') \)-set. If \( c_0 \notin S' \), then \( S' \cup \{u_1, \ldots, u_r\} \) is a 1FD-set in \( G \) of cardinality at most \( (n(G) - 1)/2 + k - 1 < (n(G) - 1)/2 + k \), a contradiction. Thus \( c_0 \in S' \). Then \( S' \cup \{c_1, \ldots, c_r\} \) is a 1FD-set in \( G \) of cardinality at most \( (n(G) - 1)/2 + k - 1 < (n(G) - 1)/2 + k \), a contradiction.

Thus \( C_1 \) has at least one special vertex. Similarly, \( C_2 \) has at least one special vertex. Let \( c_t \) be a special vertex of \( C_1 \) and \( c'_t \) be a special vertex of \( C_2 \).

We show that \( c_t \) is the unique special vertex of \( C_1 \). Suppose to the contrary that \( C_1 \) has at least two special vertices. Assume that \( \deg_G(c_{h+1}') \geq 2 \). Let \( G' = G - c_t c_{h+1}' \), and \( S' \) be a \( fd_1(G') \)-set. By Theorem 14, \( fd_1(G') \leq (n(G') - 1)/2 + k - 1 \). If \( fd_1(G') = (n(G') - 1)/2 + k - 1 \), then by the inductive hypothesis, \( G' \in G_{k-1} \). This is a contradiction by Observation 9(1), since \( C_1 \) has at least two special vertices. Thus \( fd_1(G') < (n(G') - 1)/2 + k - 1 \). If \( |S' \cap \{c_h, c_{h+1}''\}| \notin \{0, 2\} \), then \( S' \) is a 1FD-set in \( G \) of cardinality at most \( (n(G) - 1)/2 + k - 1 \), a contradiction. Thus \( |S' \cap \{c_h, c_{h+1}''\}| = 1 \). Without loss of generality, assume that \( c_{h+1}' \in S' \). Then \( \{c_{h+1}', c_h\} \cup S' \) is a 1FD-set in \( G \), and so \( fd_1(G') < (n(G) - 1)/2 + k - 1 \), a contradiction. We thus assume that \( \deg_G(c_{h+1}') \geq 3 \). Likewise, we may assume that \( \deg_G(c_{h-1}') \geq 3 \). Since \( C_2 \) is a leaf-cycle, \( c_0' \) is its unique special cut-vertex. Thus we may assume, without loss of generality, that \( c_{h+1}' \neq c_{h-1}' \). Clearly, \( c_{h+1}' \) is a support vertex of \( G \). Let \( G' = G - c_t c_{h+1}' \), and \( S' \) be a \( fd_1(G') \)-set. Clearly \( c_{h+1}' \) is a strong support vertex of \( G' \). By Theorem 14, \( fd_1(G') \leq (n(G') - 1)/2 + k - 1 \). If \( fd_1(G') = (n(G') - 1)/2 + k - 1 \), then by the inductive hypothesis \( G' \in G_{k-1} \). This is a contradiction by Observation 9(1), since \( C_1 \) has at least two special vertices. Thus \( fd_1(G') < (n(G') - 1)/2 + k - 1 \). By Observation 1, \( c_{h+1}' \in S' \). If \( c_{h-1}' \notin S' \), then \( S' \) is a 1FD-set in \( G \) of cardinality at most \( (n(G) - 1)/2 + k - 1 \), a contradiction. Thus \( c_{h-1}' \in S' \). Now, \( S' \cup \{c_h\} \) is a 1FD-set in \( G \), and thus \( fd_1(G) \leq |S'| + 1 < (n(G) - 1)/2 + k \), a contradiction. Thus \( c_t \) is the unique special vertex of \( C_1 \). Similarly, \( c_h' \) is the unique special vertex of \( C_2 \).

□

Let \( c_t \) be the unique special vertex of \( C_1 \), and \( c_h' \) be the unique special vertex of \( C_2 \), and note that Claim 4 guarantees the existence of \( c_t \) and \( c_h' \).

**Claim 5.** No vertex of \( C_i \) is a strong support vertex, for \( i = 1, 2 \).
Proof. Suppose that $c_j \in C_1$ is a strong support vertex. Since $C_2$ is a leaf-cycle, $c_0'$ is its unique special cut-vertex. Thus, we may assume, without loss of generality, that $c_{h+1}'$ is a support vertex of $G$. Let $G' = G - c_{h+1}'$, and $S'$ be a $fd_1(G')$-set. Clearly $c_{h+1}'$ is a strong support vertex of $G'$. By Theorem 14, $fd_1(G') \leq (n(G') - 1)/2 + k - 1$. If $fd_1(G') = (n(G') - 1)/2 + k - 1$, then by the inductive hypothesis $G' \in \mathcal{G}_{k-1}$. This is a contradiction by Observation 9(1), since $C_1$ has a strong support vertex. Thus $fd_1(G') < (n(G') - 1)/2 + k - 1$. By Observation 1, $c_{h+1}' \in S'$. If $c_{h-1}' \notin S'$, then $S'$ is a 1FD-set in $G$ of cardinality at most $(n(G) - 1)/2 + k - 1$, a contradiction. Thus $c_{h-1}' \in S'$. Then $S' \cup \{c_{h}'\}$ is a 1FD-set in $G$, and so $fd_1(G) \leq |S'| + 1 < (n(G) - 1)/2 + k$, a contradiction. We deduce that $C_1$ has no strong support vertex. Similarly, $C_2$ has no strong support vertex.

We deduce that $c_i$ is a weak support vertex for each $i \in \{1, 2, \ldots, r\} \setminus \{t\}$, and similarly $c_i'$ is a weak support vertex for each $i \in \{1, 2, \ldots, r'\} \setminus \{h\}$. For each $i \in \{1, 2, \ldots, r\} \setminus \{t\}$, let $u_i$ be the leaf adjacent to $c_i$.

Let $G_2'$ be the component of $G - c_0c_1 - c_0c_{r'}$ that contains $c_0$, and $G^*$ be a graph obtained from $G_2'$ by adding a leaf $v^*$ to $c_0$. Clearly $n(G^*) = n(G) - 2r + 2$. By Theorem 14, $fd_1(G^*) < (n(G^*) - 1)/2 + k - 1$. Suppose that $fd_1(G^*) < (n(G^*) - 1)/2 + k - 1$. Let $S^*$ be a $fd_1(G^*)$-set. If $c_0 \in S^*$, then $S^* \cup \{c_1, c_2, \ldots, c_r\}$ is a 1FD-set in $G$, so we obtain that $fd_1(G) < (n - 1)/2 + k$, a contradiction. Thus $c_0 \notin S^*$. Then $v^* \in S^*$. If $t > 1$, then $S^* \cup \{c_1, \ldots, c_{t-1}\} \cup \{u_{t+1}, \ldots, u_r\} \setminus \{v^*\}$ is a 1FD-set in $G$ of cardinality at most $(n(G^*) - 1)/2 + k - 1 + 1 + k - 1 = (n(G) - 1)/2 + k - 2$, a contradiction. Thus assume that $t = 1$. Then $S^* \cup \{c_2, \ldots, c_r\} \setminus \{v^*\}$ is a 1FD-set in $G$ of cardinality at most $(n(G^*) - 1)/2 + k - 2$, a contradiction. Thus $fd_1(G^*) = (n(G^*) - 1)/2 + k - 1$. By the inductive hypothesis, $G^* \in \mathcal{G}_{k-1}$. Let $H^*$ be the graph obtained from $G'[\{c_0, c_1, \ldots, c_r, u_1, \ldots, u_{t-1}, u_{t+1}, \ldots, u_r\}]$ by adding a leaf to $c_0$. Clearly $H^* \in \mathcal{H}_1$. Thus $G$ is obtained from $G^* \in \mathcal{G}_{k-1}$ and $H^* \in \mathcal{H}_1$ by Procedure A. Consequently, $G \in \mathcal{H}_k \subseteq \mathcal{G}_k$.

For the converse, by Corollary 13, $V(G) \setminus L(G)$ is the unique $fd_1(G)$-set. Now Observation 9 implies that $fd_1(G) = (n - 1)/2 + k$.

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References


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