Discussiones Mathematicae
Graph Theory 39 (2019) 489–503
doi:10.7151/dmgt.2088

FAIR DOMINATION NUMBER IN CACTUS GRAPHS

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Abstract

For $k \geq 1$, a k-fair dominating set (or just kFD-set) in a graph G is a dominating set S such that $\big|N(v) \cap S\big| = k$ for every vertex $v \in V \setminus S$. The k-fair domination number of G, denoted by $fd_k(G)$, is the minimum cardinality of a kFD-set. A fair dominating set, abbreviated FD-set, is a kFD-set for some integer $k \geq 1$. The fair domination number, denoted by $fd(G)$, of G that is not the empty graph, is the minimum cardinality of an FD-set in G. In this paper, aiming to provide a particular answer to a problem posed in [Y. Caro, A. Hansberg and M.A. Henning, *Fair domination in graphs*, Discrete Math. 312 (2012) 2905–2914], we present a new upper bound for the fair domination number of a cactus graph, and characterize all cactus graphs G achieving equality in the upper bound of $fd_1(G)$.

Keywords: fair domination, cactus graph, unicyclic graph.

2010 Mathematics Subject Classification: 05C69.

1. Introduction

For notation and graph theory terminology not given here, we follow [10]. Specifically, let G be a graph with vertex set $V(G) = V$ of order $|V| = n$ and let v be a vertex in V. The *open neighborhood* of v is $N_G(v) = \{u \in V \mid uv \in E(G)\}$ and the *closed neighborhood of* v is $N_G[v] = \bigcup_{v \in S} N_G(v)$. If the graph G is clear from the context, we simply write $N(v)$ rather than $N_G(v)$. The *degree* of a vertex v, is $\deg(v) = |N(v)|$. A vertex of degree one is called a *leaf* and its neighbor a *support vertex*. We denote the set of leaves and support vertices of a graph G by $L(G)$ and $S(G)$, respectively. A *strong support vertex* is a support vertex adjacent to at least two leaves, and a *weak support vertex* is a support vertex adjacent to precisely one leaf. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \bigcup_{v \in S} N(v)$, and its *closed neighborhood* is the set $N[S] = N(S) \cup S$. The *corona* graph $cor(G)$ of a graph G is a graph obtained by adding a leaf to every vertex of G. We denote by P_n a path on n vertices. The *distance* $d(u, v)$ between two vertices u and v in a graph G is the minimum number of edges of a path from u to v. The *diameter* $diam(G)$ of G, is $\max_{u,v \in V(G)} d(u, v)$. A path of length $diam(G)$ is called a *diameterical* path. A *cactus graph* is a connected graph in which any two cycles have at most one vertex in common. For a subset S of vertices of G, we denote by $G[S]$ the subgraph of G induced by S.

A subset $S \subseteq V$ is a *dominating set* of G if every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A vertex v is said to be *dominated* by a set S if $N(v) \cap S \neq \emptyset$.

Caro *et al.* [1] studied the concept of fair domination in graphs. For $k \geq 1$, a *k-fair dominating set*, abbreviated kFD-set, in G is a dominating set S such that $\big|N(v) \cap D\big| = k$ for every vertex $v \in V \setminus D$. The k-fair domination number of G, denoted by $fd_k(G)$, is the minimum cardinality of a kFD-set. A kFD-set of G of cardinality $fd_k(G)$ is called a $fd_k(G)$-set. A *fair dominating set*, abbreviated FD-set, in G is a kFD-set for some integer $k \geq 1$. The *fair domination number*, denoted by $fd(G)$, of a graph G that is not the empty graph is the minimum cardinality of an FD-set in G. An FD-set of G of cardinality $fd(G)$ is called a $fd(G)$-set.

A *perfect dominating set* in a graph G is a dominating set S such that every vertex in $V(G) \setminus S$ is adjacent to exactly one vertex in S. Hence a 1FD-set is precisely a perfect dominating set. The concept of perfect domination was introduced by Cockayne *et al.* in [4], and Fellows *et al.* [7] with a different terminology which they called semiperfect domination. This concept was further studied, see for example, [2, 3, 5, 6, 9].

Observation 1 (Caro *et al.* [1])**.** *Every* 1*FD-set in a graph contains all its strong support vertices.*

The following is easily verified.

Observation 2. *Let S be a* 1*FD-set in a graph G, v a support vertex of G and v' a leaf adjacent to v. If S contains a vertex $u \in N_G(v) \setminus \{v'\}$, then $v \in S$.*

Among other results, Caro *et al.* [1] proved that $fd(G) \leq n-2$ for any connected graph G of order $n \geq 3$ with no isolated vertex, and constructed an infinite family of connected graphs achieving equality in this bound. They showed that $fd(G) < 17n/19$ for any maximal outerplanar graph G of order n, and $fd(T) \leq n/2$ for any tree T of order $n \geq 2$. They then showed that equality for the bound $fd(T) \leq n/2$ holds if and only if T is the corona of a tree. Among open problems posed by Caro *et al.* [1], one asks to find $fd(G)$ for other families of graphs.

Problem 3 (Caro *et al.* [1])**.** *Find $fd(G)$ for other families of graphs.*

In this paper, aiming to study Problem 3, we present a new upper bound for the 1-fair domination number of cactus graphs and characterize all cactus graphs achieving equality for the upper bound. We show that if G is a cactus graph of order $n \geq 5$ with $k \geq 1$ cycles, then $fd_1(G) \leq (n-1)/2+k$. We also characterize all cactus graphs achieving equality for the upper bound.

2. Unicyclic Graphs

Fair domination in unicyclic graphs has been studied in [8]. A vertex v of a cactus graph G is a *special vertex* if $\deg_G(v) = 2$ and v belongs to a cycle of G. Let $\mathcal{H}_1$ be the class of all graphs G that can be obtained from the corona $cor(C)$ of a cycle C by removing precisely one leaf of $cor(C)$. Let $\mathcal{G}_1$ be the class of all graphs G that can be obtained from a sequence $G_1, G_2, \ldots, G_s = G$, where $G_1 \in \mathcal{H}_1$, and if $s \geq 2$, then G_{j+1} is obtained from G_j by one of the following Operations $\mathcal{O}_1$ or $\mathcal{O}_2$, for $j = 1, 2, \ldots, s-1$.

Operation $\mathcal{O}_1$. Let v be a vertex of G_j with $\deg(v) \geq 2$ such that v is not a special vertex of G_j. Then G_{j+1} is obtained from G_j by adding a path P_2 and joining v to a leaf of P_2.

Operation $\mathcal{O}_2$. Let v be a leaf of G_j. Then G_{j+1} is obtained from G_j by adding two leaves to v.

Lemma 4 [8]**.** *If $G \in \mathcal{G}_1$, then every 1FD-set in G contains every vertex of G of degree at least two.*

Theorem 5 [8]**.** *If G is a unicyclic graph of order n, then $fd_1(G) \leq (n+1)/2$, with equality if and only if $G = C_5$ or $G \in \mathcal{G}_1$.*

3. Main Result

Our aim in this paper is to give an upper bound for the fair domination number of a cactus graph G in terms of the number of cycles of G, and then characterize

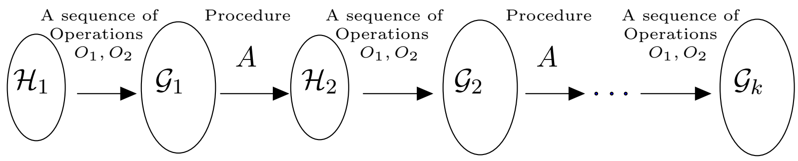

Figure 1. Construction of the family $\mathcal{G}_k$.

all cactus graphs achieving equality for the proposed bound. For this purpose we first introduce some families of graphs. Let $\mathcal{H}_1$ and $\mathcal{G}_1$ be the families of unicyclic graphs described in Section 2. For $i = 2, 3, \ldots, k$, we construct a family $\mathcal{H}_i$ from $\mathcal{G}_{i-1}$, and a family $\mathcal{G}_i$ from $\mathcal{H}_i$ as follows.

• Family $\mathcal{H}_i$. Let $\mathcal{H}_i$ be the family of all graphs H_i such that H_i can be obtained from a graph $H_1 \in \mathcal{H}_1$ and a graph $G \in \mathcal{G}_{i-1}$, by the following Procedure.

Procedure A. Let $w_0 \in V(H_1)$ be a support vertex of H_1, and $w \in V(G_{i-1})$ be a support vertex of G_{i-1}. We remove precisely one leaf adjacent to w_0 and precisely one leaf adjacent to w, and then identify the vertices w_0 and w.

• Family $\mathcal{G}_i$. Let $\mathcal{G}_i$ be the family of all graphs G that can be obtained from a sequence $G_1, G_2, \ldots, G_s = G$, where $G_1 \in \mathcal{H}_i$, and if $s \geq 2$ then G_{j+1} is obtained from G_j by one of the Operations $\mathcal{O}_1$ or $\mathcal{O}_2$, described in Section 2, for $j = 1, 2, \ldots, s-1$.

Note that $\mathcal{H}_i \subseteq \mathcal{G}_i$, for $i = 1, 2, \ldots, k$. Figure 1 demonstrates the construction of the family $\mathcal{G}_k$.

We will prove the following.

Theorem 6. *If G is a cactus graph of order $n \geq 5$ with $k \geq 1$ cycles, then $fd_1(G) \leq (n-1)/2 + k$, with equality if and only if $G = C_5$ or $G \in \mathcal{G}_k$.*

Corollary 7. *If G is a cactus graph of order $n \geq 5$ with $k \geq 1$ cycles, then $fd(G) \leq (n-1)/2 + k$.*

4. Preliminary Results and Observations

4.1. Notation

We call a vertex w in a cycle C of a cactus graph G a *special cut-vertex* if w belongs to a shortest path from C to a cycle $C' \neq C$. We call a cycle C in a cactus graph G, a *leaf-cycle* if C contains exactly one special cut-vertex. In the

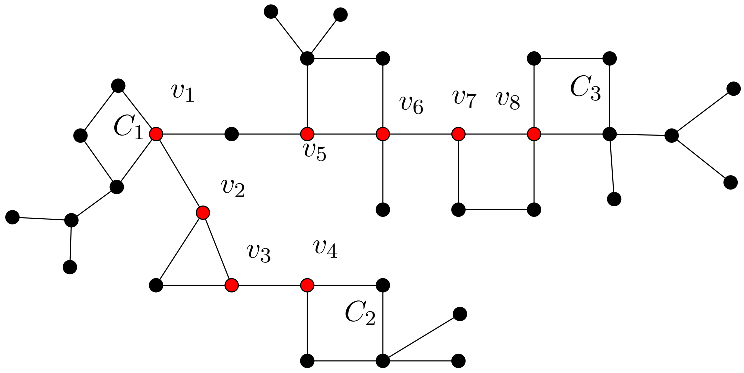

Figure 2. C_i is a leaf-cycle for $i = 1, 2, 3$ and v_j is a special cut-vertex for $j = 1, 2, \ldots, 8$.

cactus graph presented in Figure 2, v_i is a special cut-vertex, for $i = 1, 2, \ldots, 8$. Moreover, C_j is a leaf-cycle for $j = 1, 2, 3$.

Observation 8. *Every cactus graph with at least two cycles contains at least two leaf-cycles.*

4.2. Properties of the family $\mathcal{G}_k$

The following observation can be proved by a simple induction on k.

Observation 9. *If $G \in \mathcal{G}_k$ is a cactus graph of order n, then the following conditions are satisfied.*

(1) *No cycle of G contains a strong support vertex. Furthermore, any cycle of G contains precisely one special vertex.*
(2) *n is odd.*
(3) $|L(G)| = (n + 1)/2 - k$.
(4) *If a vertex v of G belongs to at least two cycles of G, then v is not a support vertex, and v belongs to precisely two cycles of G.*

Observation 10. *Let $G \in \mathcal{G}_k$. Let G be obtained from a sequence $G_1, G_2, \ldots, G_s = G$ $(s \geq 2)$ such that $G_1 \in \mathcal{H}_1$ and G_{j+1} is obtained from G_j by one of the Operations O_1 or O_2 or procedure A, for $j = 1, 2, \ldots, s - 1$. If v is a vertex of G belonging to two cycles of G then there is an integer $i \in \{2, 3, \ldots, s\}$ such that G_i is obtained from G_{i-1} by applying Procedure A on the vertex v using a graph $H \in \mathcal{H}_1$, such that v belongs to a cycle of G_{i-1}.*

Observation 11. *Assume that $G \in \mathcal{G}_k$ and $v \in V(G)$ is a vertex of degree four belonging to two cycles. Let D_1 and D_2 be the components of $G - v$, G_1^* be the* *graph obtained from* $G[D_1 \cup \{v\}]$ *by adding a leaf* v_1^* *to* v*, and* G_2^* *be the graph obtained from* $G[D_2 \cup \{v\}]$ *by adding a leaf* v_2^* *to* v*. Then there exists an integer* $k' < k$ *such that* $G_1^* \in \mathcal{G}_{k'}$ *or* $G_2^* \in \mathcal{G}_{k'}$*.*

Proof. Let $G \in \mathcal{G}_k$. Thus G is obtained from a sequence $G_1, G_2, \ldots, G_s = G$ ($s \geq 2$) such that $G_1 \in \mathcal{H}_1$ and G_{j+1} is obtained from G_j by one of the Operations O_1 or O_2 or procedure A, for $j = 1, 2, \ldots, s-1$. Note that $s \geq k$. We define the *j-th Procedure-Operation* or just PO_j as one of the Operation O_1, Operation O_2, or Procedure A that can be applied to obtain G_{j+1} from G_j. Thus G is obtained from G_1 by Procedure-Operations PO_1, PO_2,...,PO_{s-1}.

Let v be a vertex of G of degree four belonging to two cycles of G, and D_1 and D_2 be the components of $G - v$. By Observation 10, there is an integer $i \in \{2, 3, \ldots, s\}$ such that G_i is obtained from G_{i-1} by applying Procedure A on the vertex v using a graph $H \in \mathcal{H}_1$. Note that v is a support vertex of G_{i-1}. Let v^* be the leaf of v in G_{i-1} that is removed in Procedure A. Clearly, either $V(G_{i-1}) \cap D_1 \neq \emptyset$ or $V(G_{i-1}) \cap D_2 \neq \emptyset$. Without loss of generality, assume that $V(G_{i-1}) \cap D_1 \neq \emptyset$. Among PO_i, $PO_{i+1}, \ldots, PO_{s-1}$, let PO_{r_1}, $PO_{r_2}, \ldots, PO_{r_t}$, be the Procedure-Operations applied on a vertex of D_1, where $i \leq t \leq s-1$. Let $G_{r_0} = G_{i-1}$ and $G_{r_{l+1}}$ be obtained from G_{r_l} by $PO_{r_{l+1}}$, for $l = 0, 1, 2, \ldots, t-1$. Clearly by an induction on t, we can deduce that there is an integer $k^* < k$ such that $G_{r_t} \in \mathcal{G}_{k^*}$. Note that $G_{r_t} = G_1^*$. ∎

Lemma 12. *If* $G \in \mathcal{G}_k$*, then every* 1*FD-set in* G *contains every vertex of* G *of degree at least two.*

Proof. Let $G \in \mathcal{G}_k$, and S be a 1FD-set in G. We prove by an induction on k, namely ***first-induction***, to show that S contains every vertex of G of degree at least two. For the base step, if $k = 1$ then $G \in \mathcal{G}_1$, and the result follows by Lemma 4. Assume the result holds for all graphs $G' \in \mathcal{G}_{k'}$ with $k' < k$. Now consider the graph $G \in \mathcal{G}_k$, where $k > 1$. Clearly, G is obtained from a sequence $G_1, G_2, \ldots, G_l = G$, of cactus graphs such that $G_1 \in \mathcal{H}_k$, and if $l \geq 2$, then G_{i+1} is obtained from G_i by one of the operations $\mathcal{O}_1$ or $\mathcal{O}_2$ for $i = 1, 2, \ldots, l-1$.

We employ an induction on l, namely ***second-induction***, to show that S contains every vertex of G of degree at least two.

For the base step of the second-induction, let $l = 1$. Thus $G \in \mathcal{H}_k$. By the construction of graphs in the family $\mathcal{H}_k$, there are graphs $H \in \mathcal{H}_1$ and $G' \in \mathcal{G}_{k-1}$ such that G is obtained from H and G' by Procedure A. Clearly, H is obtained from the corona $cor(C)$ of a cycle C, by removing precisely one leaf of $cor(C)$. Let $C = c_0c_1 \cdots c_rc_0$, where c_0 is the support vertex of H that its leaf is removed according to Procedure A. Since H has precisely one special vertex, let c_t be the special vertex of H. Let $w \in V(G')$ be a support vertex of G' that its leaf, say w', is removed to obtain G according to Procedure A. First we show that $\{c_1, c_r\} \cap S \neq \emptyset$. Clearly $S \cap \{c_{t-1}, c_t, c_{t+1}\} \neq \emptyset$, since $\deg_G(c_t) = 2$. Assume that

$c_t \in S$. Since at least one of c_{t-1} or c_{t+1} is a support vertex, by Observation 2, $\{c_{t-1}, c_{t+1}\} \cap S \neq \emptyset$. By applying Observation 2, we obtain that $\{c_1, c_r\} \cap S \neq \emptyset$, since any vertex of $\{c_1, \ldots, c_r\} \setminus \{c_t\}$ is a support vertex of G. Thus assume that $c_t \notin S$. Then $\{c_{t-1}, c_{t+1}\} \cap S \neq \emptyset$, and so $\{c_1, c_r\} \cap S \neq \emptyset$, since any vertex of $\{c_1, \ldots, c_r\} \setminus \{c_t\}$ is a support vertex of G. Hence, $\{c_1, c_r\} \cap S \neq \emptyset$. If $c_0 \notin S$, then $\big(S \cap V(G')\big) \cup \{w'\}$ is a 1FD-set for G', and thus by the first-inductive hypothesis, S contains $w = c_0$, a contradiction. Thus $c_0 \in S$. By Observation 2, $V(C) \subseteq S$, since any vertex of $\{c_1, \ldots, c_r\} \setminus \{c_t\}$ is a support vertex of G. Thus $S \cap V(G')$ is a 1FD-set for G'. By the first-inductive hypothesis, $\big(S \cap V(G')\big) \cup \{w\}$ contains every vertex of G' of degree at least two. Consequently, S contains every vertex of G of degree at least two. We conclude that the base step of the second-induction holds.

Assume that the result (for the second-induction) holds for $2 \leq l' < l$. Now let $G = G_l$. Clearly G is obtained from G_{l-1} by applying one of the Operations $\mathcal{O}_1$ or $\mathcal{O}_2$.

Assume that G is obtained from G_{l-1} by applying Operation $\mathcal{O}_2$. Let x be a leaf of G_{l-1} and G be obtained from G_{l-1} by adding two leaves x_1 and x_2 to x. By Observation 1, $x \in S$. Thus S is a 1FD-set for G_{l-1}. By the second-inductive hypothesis S contains all vertices of G_{l-1} of degree at least two. Consequently, S contains every vertex of G_k of degree at least two.

Next assume that G is obtained from G_{l-1} by applying Operation $\mathcal{O}_1$. Let x_1x_2 be a path and x_1 is joined to $y \in V(G_{l-1})$, where $\deg_{G_{l-1}}(y) \geq 2$ and y is not a special vertex of G_{l-1}. Observe that $\{x_1, x_2\} \cap S \neq \emptyset$. If $x_1 \notin S$, then $x_2 \in S$ and $y \notin S$. Then $S \setminus \{x_2\}$ is a 1FD-set for G_{l-1} that does not contain y, a contradiction by the second-inductive hypothesis. Thus assume that $x_1 \in S$. Suppose that $y \notin S$. Clearly $N_{G_{l-1}}(y) \cap S = \emptyset$.

Assume that there exists a component G'_1 of $G_{l-1} - y$ such that $\big|V(G'_1) \cap N_{G_{l-1}}(y)\big| = 1$. Then clearly $S' = \big(S \cap V(G_{l-1})\big) \cup V(G'_1)$ is a 1FD-set for G_{l-1}, and by the second-inductive hypothesis S' contains every vertex of G_{l-1} of degree at least two. Thus $y \in S'$, and so $y \in S$, a contradiction. Next assume that every component of $G_{l-1} - y$ has at least two vertices in $N_{G_{l-1}}(y)$. Since y is a non-special vertex of G_{l-1}, y belongs to at least two cycles of G_{l-1}. By Observation 9(4), y belongs to exactly two cycles of G_{l-1}. Thus $\deg_{G_{l-1}}(y) = 4$. By Observation 11, $G_{l-1} - y$ has exactly two components D_1 and D_2. Let G^* be a graph obtained from $D_1 \cup \{v\}$ or $D_2 \cup \{v\}$, by adding a leaf v^* to y. Then there exists $k' \leq k$ such that $G^* \in \mathcal{G}_{k'}$. Evidently, $S^* = \big(S \cap V(G^*)\big) \cup \{v^*\}$ is a 1FD-set for G^*, and so by the first-inductive hypothesis, S^* contains every vertex of G^* of degree at least two (since $G^* \in \mathcal{G}_{k'}$). Thus $y \in S^*$, and so $y \in S$, a contradiction. We conclude that $y \in S$. Observe that $S \cap V(G_{l-1})$ is a 1FD-set for G_{l-1}, and so by the second-inductive hypothesis, $S \cap V(G_{l-1})$ contains every vertex of G_{l-1} of degree at least two. Consequently S contains every vertex of G

of degree at least two. ■

As a consequence of Observation 9(3) and Lemma 12, we obtain the following.

Corollary 13. *If $G \in \mathcal{G}_k$ is a cactus graph of order n, then $V(G) \setminus L(G)$ is the unique $fd_1(G)$-set.*

5. Proof of Theorem 6

We first establish the upper bound by proving the following.

Theorem 14. *If G is a cactus graph of order n with $k \geq 1$ cycles, then $fd_1(G) \leq \big(n(G) - 1\big)/2 + k$.*

Proof. The result follows by Theorem 5 if $k = 1$. Thus assume that $k \geq 2$. Suppose to the contrary that $fd_1(G) > \big(n(G)-1\big)/2+k$. Assume that G has the minimum order, and among all such graphs, we may assume that the size of G is minimum. Let $C_1, C_2, \ldots, C_k$ be the k cycles of G. Let C_i be a leaf-cycle of G, where $i \in \{1, 2, \ldots, k\}$. Let $C_i = u_0u_1 \cdots u_lu_0$, where u_0 is a special cut-vertex of G. Assume that $\deg_G(u_j) = 2$ for each $j = 1, 2, \ldots, l$. Let $G' = G - u_1u_2$. Then by the choice of G, $fd_1(G') \leq \big(n(G') - 1\big)/2 + k - 1 = \big(n(G) - 1\big)/2 + k - 1$. Let S' be a $fd_1(G')$-set. Now if $\big|S' \cap \{u_1, u_2\}\big| \in \{0, 2\}$, then S' is a 1FD-set for G, a contradiction. Thus $\big|S' \cap \{u_1, u_2\}\big| = 1$. Assume that $u_1 \in S'$. Then $u_3 \in S'$, and so $\{u_2\} \cup S'$ is a 1FD-set in G of cardinality at most $\big(n(G) - 1\big)/2 + k$, a contradiction. If $u_2 \in S'$, then $u_0 \in S'$, and $\{u_1\} \cup S'$ is a 1FD-set in G of cardinality at most $\big(n(G)-1\big)/2+k$, a contradiction. We deduce that $\deg_G(u_i) \geq 3$ for some $i \in \{1, 2, \ldots, l\}$. Let v_d be a leaf of G such that $d(v_d, C_i - u_0)$ is as maximum as possible, and the shortest path from v_d to C_i does not contain u_0. Let $v_0v_1 \cdots v_d$ be the shortest path from v_d to C_i with $v_0 \in C_i$. Assume that $d \geq 2$. Assume that $\deg_G(v_{d-1}) = 2$. Let $G' = G - \{v_d, v_{d-1}\}$. By the choice of G, $fd_1(G') \leq \big(n(G') - 1\big)/2 + k$. Let S' be a $fd_1(G')$-set. If $v_{d-2} \in S'$, then $S' \cup \{v_{d-1}\}$ is a 1FD-set in G, and if $v_{d-2} \notin S'$, then $S' \cup \{v_d\}$ is a 1FD-set in G. Thus $fd_1(G) \leq (n - 1)/2 + k$, a contradiction. Thus assume that $\deg_G(v_{d-1}) \geq 3$. Clearly any vertex of $N_G(v_{d-1}) \setminus \{v_{d-2}\}$ is a leaf. Let G' be obtained from G by removing all leaves adjacent to v_{d-1}. By the choice of G, $fd_1(G') \leq \big(n(G') - 1\big)/2 + k$, since G has the minimum order among all graphs H with 1-fair domination number more than $\big(n(H) - 1\big)/2 + k$. Let S' be a $fd_1(G')$-set. If $v_{d-1} \in S'$, then S' is a 1FD-set in G, a contradiction. Thus assume that $v_{d-1} \notin S'$. Then $v_{d-2} \in S'$. Then $S' \cup \{v_{d-1}\}$ is a 1FD-set in G of cardinality at most $\big(n(G') - 1\big)/2 + k + 1 \leq \big(n(G) - 1\big)/2 + k$, a contradiction.

We thus assume that $d = 1$. Assume that u_i is a vertex of C_i such that $\deg_G(u_i) = 2$. Assume that $\deg_G(u_{i+1}) = 2$. Let $G' = G - u_iu_{i+1}$. By the choice of G, $fd_1(G') \le \big(n(G') - 1\big)/2 + k - 1 = \big(n(G) - 1\big)/2 + k - 1$. Let S' be a $fd_1(G')$-set. If $\big|S' \cap \{u_i, u_{i+1}\}\big| \in \{0, 2\}$, then S' is a 1FD-set for G, a contradiction. Then $\big|S' \cap \{u_i, u_{i+1}\}\big| = 1$. Assume that $u_i \in S'$. Then $u_{i+2} \in S'$ and so $\{u_{i+1}\} \cup S'$ is a 1FD-set in G of cardinality at most $\big(n(G) - 1\big)/2 + k$, a contradiction. Next assume that $u_{i+1} \in S'$. Then $u_{i-1} \in S'$ and so $\{u_i\} \cup S'$ is a 1FD-set in G of cardinality at most $\big(n(G) - 1\big)/2 + k$, a contradiction. Thus $\deg_G(u_{i+1}) \ge 3$, and similarly $\deg_G(u_{i-1}) \ge 3$. Since C_i is a leaf-cycle, it has precisely one special cut-vertex. Thus we may assume, without loss of generality, that u_{i+1} is a support vertex of G. Let $G' = G - u_{i-1}u_i$. By the choice of G, $fd_1(G') \le \big(n(G') - 1/2 + k - 1$. Let S' be a $fd_1(G')$-set. By Observation 1, $u_{i+1} \in S'$. If $u_{i-1} \notin S'$, then S' is a 1FD-set in G of cardinality at most $\big(n(G) - 1\big)/2 + k - 1$, a contradiction. Thus $u_{i-1} \in S'$. Then $S' \cup \{u_i\}$ is a 1FD-set in G of cardinality at most $\big(n(G) - 1\big)/2 + k$, a contradiction.

We conclude that $\deg_G(u_i) \ge 3$ for $i = 0, 1, \ldots, l$. Furthermore, u_i is a support vertex for $i = 1, 2, \ldots, l$. Assume that u_i is a strong support vertex for some $i \in \{1, 2, \ldots, l\}$. Let G' be obtained from G by removal of all vertices in $\bigcup_{i=1}^{l}(N[u_i]) \setminus \{u_0, u_1, u_l\}$. Clearly u_0 is a strong support vertex of G'. By the choice of G, $fd_1(G') \le \big(n(G') - 1\big)/2 + k - 1 \le \big(n(G) - (2l + 1) + 2 - 1\big)/2 + k - 1$, since u_i is a strong support vertex of G. By Observation 1, $u_0 \in S'$, and so $S' \cup \{u_1, \ldots, u_l\}$ is a 1FD-set in G of cardinality at most $\big(n(G) - (2l + 1) + 2 - 1\big)/2 + k - 1 + l = n(G)/2 + k - 1$, a contradiction. Thus u_i is a weak support vertex, for each $i = 1, 2, \ldots, l$. Let G' be obtained from G by removal of any vertex in $\bigcup_{i=1}^{l}(N[u_i]) \setminus \{u_0\}$. By the choice of G, $fd_1(G') \le \big(n(G') - 1\big)/2 + k - 1$. Let S' be a $fd_1(G')$-set. If $u_0 \notin S'$, then $S' \cup \{w_1, \ldots, w_l\}$ is a 1FD-set in G of cardinality at most $\big(n(G) - 1\big)/2 + k - 1$, where w_i is the leaf adjacent to u_i, for $i = 1, 2, \ldots, l$. This is a contradiction. Thus $u_0 \in S'$. Then $S' \cup \{u_1, \ldots, u_l\}$ is a 1FD-set in G of cardinality at most $\big(n(G) - 1\big)/2 + k - 1$, a contradiction. ■

If G is a cactus graph of order n with $k \ge 1$ cycles and $fd_1(G) = (n-1)/2 + k$, then clearly $n \ge 3$ is odd, and since $fd_1(C_3) \ne 2$, we have $n \ge 5$. It is obvious that $fd_1(C_5) = 3 = (5 - 1)/2 + 1$.

Theorem 15. *If $G \ne C_5$ is a cactus graph of order $n \ge 5$ with $k \ge 1$ cycles, then $fd_1(G) = (n - 1)/2 + k$ if and only if $G \in \mathcal{G}_k$.*

Proof. We prove by an induction on k to show that any cactus graph G of order $n \ge 5$ with $k \ge 1$ cycles and $fd_1(G) = (n - 1)/2 + k$ belongs to $\mathcal{G}_k$. The base step of the induction follows by Theorem 5. Assume the result holds for all cactus graphs G' with $k' < k$ cycles. Now let G be a cactus graph of order n with $k \ge 2$ cycles and $fd_1(G) = (n - 1)/2 + k$. Clearly n is odd. Suppose to the contrary that $G \notin \mathcal{G}_k$. Assume that G has the minimum order, and among all such graphs, assume that the size of G is minimum. By Observation 8, G has at least two leaf-cycles. Let $C_1 = c_0c_1 \cdots c_rc_0$ and $C_2 = c'_0c'_1 \cdots c'_{r'}c'_0$, be two leaf-cycles of

G, where c_0 and c_0' are two special cut-vertices of G. Let G_1' be the component of $G - c_0c_1 - c_0c_r$ containing c_1, and G_1'' be the component of $G - c_0'c_1' - c_0'c_{r'}'$ containing c_1'.

Claim 1. *$V(G_1') \neq \{c_1, \ldots, c_r\}$, and $V(G_1'') \neq \{c_1', \ldots, c_{r'}'\}$.*

Proof. Suppose that $V(G_1') = \{c_1, \ldots, c_r\}$. Then $\deg_G(c_i) = 2$ for $i = 1, 2, \ldots, r$. Let $G^* = G - c_1c_2$, and S^* be a $fd_1(G^*)$-set. By Theorem 14, $fd_1(G^*) \leq \big(n(G^*) - 1\big)/2 + k - 1 = \big(n(G) - 1\big)/2 + k - 1$. Assume that $r = 2$. Then c_0 is a strong support vertex of G^*, and by Observation 1, $c_0 \in S^*$. Thus $\big|S^* \cap \{c_1, c_2\}\big| = 0$, and so S^* is a 1FD-set in G of cardinality at most $\big(n(G) - 1\big)/2 + k - 1 < \big(n(G) - 1\big)/2 + k$, a contradiction. Assume that $r = 3$. If $\big|S^* \cap \{c_1, c_2\}\big| \in \{0, 2\}$, then S^* is a 1FD-set in G of cardinality at most $\big(n(G) - 1\big)/2 + k - 1 < \big(n(G) - 1\big)/2 + k$, a contradiction. Thus $\big|S^* \cap \{c_1, c_2\}\big| = 1$. If $c_1 \in S^*$, then $c_3 \in S^*$, and so $c_0 \in S^*$. Then $S^* \setminus \{c_1\}$ is a 1FD-set in G^*, a contradiction. Thus $c_1 \notin S^*$, and so $c_2 \in S^*$. Since c_1 is dominated by S^*, we obtain that $c_0 \in S^*$, and so $c_3 \in S^*$. Then $S^* \setminus \{c_2\}$ is a 1FD-set in G^*, a contradiction. Assume that $r = 4$. Suppose that $fd_1(G^*) = \big(n(G^*) - 1\big)/2 + k - 1$. Let $G_1^* = G^* - \{c_2, c_3, c_4\}$. By Theorem 14, $fd_1(G_1^*) \leq \big(n(G_1^*) - 1\big)/2 + k - 1 = n/2 + k - 3$, and thus $fd_1(G_1^*) \leq (n-1)/2 + k - 3$, since n is odd. Let S_1^* be a $fd_1(G_1^*)$-set. If $c_0 \in S_1^*$, then $S_1^* \cup \{c_2\}$ is a 1FD-set for G^* and if $c_0 \notin S_1^*$, then $S_1^* \cup \{c_3\}$ is a 1FD-set for G^*. Thus $fd_1(G^*) \leq |S_2^*| + 1 \leq (n-1)/2 + k - 2$, a contradiction. Thus $fd_1(G^*) < \big(n(G^*) - 1\big)/2 + k - 1 = \big(n(G) - 1\big)/2 + k - 1$. If $\big|S^* \cap \{c_1, c_2\}\big| \in \{0, 2\}$, then S^* is a 1FD-set in G of cardinality at most $\big(n(G) - 1\big)/2 + k - 1 < \big(n(G) - 1\big)/2 + k$, a contradiction. Thus $|S^* \cap \{c_1, c_2\}| = 1$. Without loss of generality, assume that $c_1 \in S^*$. Then $S^* \cup \{c_2\}$ is a 1FD-set in G, and so $fd_1(G) \leq |S^*| + 1 < \big(n(G) - 1\big)/2 + k$, a contradiction. It remains to assume that $r \geq 5$. Suppose that $fd_1(G^*) = \big(n(G^*) - 1\big)/2 + k - 1$. Let $G_2^* = G^* - \{c_2, c_3, c_4\}$. By Theorem 14, $fd_1(G_2^*) \leq \big(n(G_2^*) - 1\big)/2 + k - 1 = n/2 + k - 3$, and thus $fd_1(G_2^*) \leq (n-1)/2 + k - 3$, since n is odd. Let S_2^* be a $fd_1(G_2^*)$-set. If $c_5 \in S_2^*$, then $S_2^* \cup \{c_2\}$ is a 1FD-set for G^* and if $c_5 \notin S_2^*$, then $S_2^* \cup \{c_3\}$ is a 1FD-set for G^*. Thus $fd_1(G^*) \leq |S_2^*| + 1 \leq (n-1)/2 + k - 2$, a contradiction. Thus $fd_1(G^*) < \big(n(G^*) - 1\big)/2 + k - 1 = \big(n(G) - 1\big)/2 + k - 1$. If $\big|S^* \cap \{c_1, c_2\}\big| \in \{0, 2\}$, then S^* is a 1FD-set in G of cardinality at most $\big(n(G) - 1\big)/2 + k - 1 < \big(n(G) - 1\big)/2 + k$, a contradiction. Thus $\big|S^* \cap \{c_1, c_2\}\big| = 1$. Without loss of generality, assume that $c_1 \in S^*$. Then $S^* \cup \{c_2\}$ is a 1FD-set in G, and so $fd_1(G) \leq |S^*| + 1 < \big(n(G) - 1\big)/2 + k$, a contradiction. We conclude that $V(G_1') \neq \{c_1, \ldots, c_r\}$. Similarly $V(G_1'') \neq \{c_1', \ldots, c_{r'}'\}$. $\square$

Let $v_d \in V(G_1') \setminus \{c_1, \ldots, c_r\}$ be a leaf of G_1' at maximum distance from $\{c_1, \ldots, c_r\}$, and assume that $v_0v_1 \cdots v_d$ is the shortest path from v_d to $\{c_1, \ldots, c_r\}$, where $v_0 \in \{c_1, \ldots, c_r\}$. Likewise, let $v_{d'}' \in V(G_1'') \setminus \{c_1', \ldots, c_{r'}'\}$ be a leaf of G_1'' at maximum distance from $\{c_1', \ldots, c_{r'}'\}$, and assume that $v_0'v_1' \cdots v_{d'}'$ is the shortest path from $v'_{d'}$ to $\{c'_1, \ldots, c'_{r'}\}$, where $v'_0 \in \{c'_1, \ldots, c'_{r'}\}$. Without loss of generality, assume that $d' \leq d$.

Claim 2. *Every support vertex of G is adjacent to at most two leaves.*

Proof. Suppose that there is a support vertex $v \in S(G)$ such that v is adjacent to at least three leaves v_1, v_2 and v_3. Let $G' = G - \{v_1\}$, and let S' be a $fd_1(G')$-set. By Observation 1, $v \in S'$, and thus we may assume that $S' \cap \{v_2, v_3\} = \emptyset$. By Theorem 14, $|S'| \leq \big(n(G') - 1\big)/2 + k = (n-2)/2 + k$. Clearly S' is a 1FD-set for G, a contradiction. $\square$

Claim 3. *If $d \geq 2$, then $G \in \mathcal{G}_k$.*

Proof. Let $d \geq 2$. By Claim 2 , $2 \leq \deg_G(v_{d-1}) \leq 3$. Assume first that $\deg(v_{d-1}) = 3$. Let $x \neq v_d$ be a leaf adjacent to v_{d-1}. Let $G' = G - \{x, v_d\}$. By Theorem 14, $fd_1(G') \leq \big(n(G') - 1\big)/2 + k$. Suppose that $fd_1(G') < \big(n(G') - 1\big)/2 + k$. Let S' be a $fd_1(G')$-set. If $v_{d-1} \in S'$, then S' is a 1FD-set for G and if $v_{d-1} \notin S'$, then $S' \cup \{v_{d-1}\}$ is a 1FD-set for G. Thus $fd_1(G) \leq fd_1(G') + 1 < (n-1)/2 + k$, a contradiction. Hence, $fd_1(G') = \big(n(G') - 1\big)/2 + k$. By the choice of G, $G' \in \mathcal{G}_k$. Therefore G is obtained from G' by Operation $\mathcal{O}_2$. Consequently, $G \in \mathcal{G}_k$. Next assume that $\deg_G(v_{d-1}) = 2$. We consider the following cases.

Case 1. $d \geq 3$. Suppose that $\deg_G(v_{d-2}) = 2$. Let $G' = G - \{v_{d-2}, v_{d-1}, v_d\}$. By Theorem 14, $fd_1(G') \leq \big(n(G') - 1\big)/2 + k = n/2 + k - 2$, and thus $fd_1(G') \leq (n-1)/2 + k - 2$, since n is odd. Let S' be a $fd_1(G')$-set. If $v_{d-3} \in S'$, then $S' \cup \{v_d\}$ is a 1FD-set for G and if $v_{d-3} \notin S'$, then $S' \cup \{v_{d-1}\}$ is a 1FD-set for G. Thus $fd_1(G) \leq |S'| + 1 \leq (n-1)/2 + k - 1$, a contradiction. Thus $\deg_G(v_{d-2}) \geq 3$. Let $G' = G - \{v_{d-1}, v_d\}$. By Theorem 14, $fd_1(G') \leq \big(n(G') - 1\big)/2 + k$. Suppose that $fd_1(G') < \big(n(G') - 1\big)/2 + k$. Let S' be a $fd_1(G')$-set. If $v_{d-2} \in S'$, then $S' \cup \{v_{d-1}\}$ is a 1FD-set for G and if $v_{d-2} \notin S'$, then $S' \cup \{v_d\}$ is a 1FD-set for G. Thus $fd_1(G) \leq |S'| + 1 \leq fd_1(G') + 1 < (n-1)/2 + k$, a contradiction. We deduce that $fd_1(G') = \big(n(G') - 1\big)/2 + k$. By the choice of G, $G' \in \mathcal{G}_k$. Since $d \geq 3$, v_{d-2} is not a special vertex of G'. Thus G is obtained from G' by Operation $\mathcal{O}_1$, and so $G \in \mathcal{G}_k$.

Case 2. $d = 2$. As noted, $\deg(v_1) = 2$. Clearly $\deg(v_0) \geq 3$. Assume first that $\deg(v_0) \geq 4$. Let $G' = G - \{v_2, v_1\}$. By Theorem 14 , $fd_1(G') \leq \big(n(G') - 1\big)/2 + k$. Suppose that $fd_1(G') < \big(n(G') - 1\big)/2 + k$. Let S' be a $fd_1(G')$-set. If $v_0 \in S'$, then $S' \cup \{v_1\}$ is a 1FD-set for G, and if $v_0 \notin S'$, then $S' \cup \{v_2\}$ is a 1FD-set for G. Thus $fd_1(G) \leq |S'| + 1 < (n-1)/2 + k$, a contradiction. Thus, $fd_1(G') = \big(n(G') - 1\big)/2 + k$. By the choice of G, $G' \in \mathcal{G}_k$. Since $\deg_{G'}(v_0) \geq 3$, v_0 is not a special vertex of G'. Hence G is obtained from G' by Operation $\mathcal{O}_1$. Consequently, $G \in \mathcal{G}_k$. Thus assume that $\deg(v_0) = 3$. Let $G' = G - \{v_2, v_1\}$. By Theorem 14, $fd_1(G') \leq \big(n(G') - 1\big)/2 + k$. Suppose that $fd_1(G') < \big(n(G') - 1\big)/2 + k$. Let S' be a $fd_1(G')$-set. If $v_0 \in S'$, then $S' \cup \{v_1\}$ is a 1FD-set for G, and if $v_0 \notin S'$, then $S' \cup \{v_2\}$ is a 1FD-set for G. Thus $fd_1(G) \leq |S'| + 1 \leq fd_1(G') + 1 < (n-1)/2 + k$, a contradiction. Thus we obtain that $fd_1(G') = \big(n(G') - 1\big)/2 + k$. By the choice of G, $G' \in \mathcal{G}_k$. Then v_0 is a special vertex of G'. From Observation 9(1), we obtain that $\deg_G(c_i) \geq 3$ for each $i \in \{1, \ldots, r\}$.

Suppose that $N_G(c_j) \setminus V(C_1)$ contains no strong support vertex for each $j \in \{1, \ldots, r\}$. Observation 9(1) implies that c_j is not a strong support vertex of G, since $G' \in \mathcal{G}_k$. Assume that there is a vertex $c_j \in \{c_1, \ldots, c_r\}$ such that c_j has a neighbor a which is a support vertex. By assumption, a is a weak support vertex. If a' is the leaf adjacent to a, then a' plays the role of v_d. Since $\deg(v_0) = 3$, we may assume that $\deg(c_j) = 3$. Thus by Observation 9(1), we may assume that $\deg_G(c_i) = 3$ for each $c_i \in \{c_1, \ldots, c_r\}$. Let $F = \bigcup_{i=1}^{r}(N[c_i]) \setminus \{c_0, \ldots, c_r\}$. Clearly $|F| = r$, since $\deg_G(c_i) = 3$ for each $c_i \in \{c_1, \ldots, c_r\}$. Let $F = \{u_1, u_2, \ldots, u_r\}$, $F' = \{u_i \in F \mid \deg_G(u_i) = 1\}$, and $F'' = F \setminus F'$. Then every vertex of F'' is a weak support vertex. Since $v_1 \in F''$, $|F''| \geq 1$. Now let $G^* = G - c_0c_1 - c_0c_r$, and G_1^* and G_2^* be the components of G^*, where $c_1 \in V(G_1^*)$. By Theorem 14, $fd_1(G_2^*) \leq \big(n(G_2^*) - 1\big)/2 + k - 1$. Clearly $n(G_2^*) = n(G) - 2r - |F''|$. Let S_2^* be a $fd_1(G_2^*)$-set. If $c_0 \notin S_2^*$, then $S_2^* \cup F$ is a 1FD-set for G, and so $fd_1(G) \leq \big(n(G) - 2r - |F''| - 1\big)/2 + k - 1 + r < (n-1)/2 + k$, a contradiction. Thus $c_0 \in S_2^*$. If $|F''| = 1$, then $S_2^* \cup C_1 \cup \{v_1\}$ is a 1FD-set for G and thus $fd_1(G) \leq fd_1(G_2^*) + r + 1 \leq (n-2)/2 + k$, a contradiction. Thus assume that $|F''| \geq 2$. Let $\{u_t, u_{t'}\} \subseteq F''$ (assume without loss of generality that $t < t'$) such that $\deg_G(u_i) = 1$ for $1 \leq i < t$ and $t' < i \leq r$. Let u'_t and $u'_{t'}$ be the leaves of u_t and $u_{t'}$, respectively. Clearly $S_2^* \cup \{c_1, \ldots, c_{t-1}\} \cup \{c_{t'+1}, \ldots, c_r\} \cup \{u_{t+1}, \ldots, u_{t'-1}\} \cup \{u'_t, u'_{t'}\}$ is a 1FD-set for G and thus $fd_1(G) \leq fd_1(G_2^*) + r < (n-1)/2 + k - 1$, a contradiction.

Thus we may assume that $N(c_j) \setminus C_1$ contains at least one strong support vertex for some $c_j \in \{c_1, \ldots, c_r\}$. Let u_j be a strong support vertex in $N(c_j) \setminus C_1$. By Claim 2, there are precisely two leaves adjacent to u_j . Let u' and u'' be the leaves adjacent to u_j, and $G^* = G - \{u', u''\}$. By Theorem 14, $fd_1(G^*) \leq \big(n(G^*) - 1\big)/2 + k$. Assume that $fd_1(G^*) < \big(n(G^*) - 1\big)/2 + k$. Let S' be a $fd_1(G^*)$-set. If $u_j \in S'$, then S' is a 1FD-set for G, and if $u_j \notin S'$, then $S' \cup \{u'_j\}$ is a 1FD-set for G. Thus $fd_1(G) \leq fd_1(G^*) + 1 < (n-1)/2 + k$, a contradiction. We deduce that $fd_1(G^*) = \big(n(G^*) - 1\big)/2 + k$. By the choice of G, $G^* \in \mathcal{G}_k$. Thus G is obtained from G^* by Operation $\mathcal{O}_2$. Consequently, $G \in \mathcal{G}_k$. □

By Claim 3, we assume that $d = d' = 1$.

Claim 4. *C_i has precisely one special vertex, for $i = 1, 2$.*

Proof. We first show that C_i has at least one special vertex, for $i = 1, 2$. Suppose that C_1 has no special vertex. Thus $\deg_G(c_i) \geq 3$ for $i = 1, \ldots, r$. Clearly, c_i is a support vertex for $i = 1, 2, \ldots, r$. Suppose that c_j is a strong support vertex for some $j \in \{1, 2, \ldots, r\}$. Let G' be obtained from G by removal of all vertices in $\bigcup_{i=1}^{r} \big(N[c_i]\big) \setminus \{c_0, c_1, c_r\}$. Clearly, c_0 is a strong support vertex of G'. By Theorem 14, $fd_1(G') \leq \big(n(G') - 1\big)/2 + k - 1$. Since c_j is a strong support vertex of G, we have $n(G') \leq n(G) - (2r+1) + 2$. Thus, $fd_1(G') \leq \big(n(G) - (2r+1) + 2 - 1\big)/2 + k - 1$. By Observation 1, $c_0 \in S'$, and so $S' \cup \{c_1, \ldots, c_r\}$ is a 1FD-set in G of cardinality at most $\big(n(G) - (2r+1) + 2 - 1\big)/2 + k - 1 + r = n(G)/2 + k - 1 < \big(n(G) - 1\big)/2 + k$, a contradiction. Thus c_i is a weak support vertex for each $i = 1, 2, \ldots, r$. Let G' be obtained from G by removal of any vertex in $\bigcup_{i=1}^{r} \big(N[c_i]\big) \setminus \{c_0\}$. By Theorem 14, $fd_1(G') \leq \big(n(G') - 1\big)/2 + k - 1$. Let S' be a $fd_1(G')$-set. If $c_0 \notin S'$, then $S' \cup \{u_1, \ldots, u_r\}$ is a 1FD-set in G of cardinality at most $\big(n(G) - 1\big)/2 + k - 1 < \big(n(G) - 1\big)/2 + k$, where u_i is the leaf adjacent to c_i for $i = 1, 2, \ldots, r$. This is a contradiction. Thus $c_0 \in S'$. Then $S' \cup \{c_1, \ldots, c_r\}$ is a 1FD-set in G of cardinality at most $\big(n(G) - 1\big)/2 + k - 1 < \big(n(G) - 1\big)/2 + k$, a contradiction. Thus C_1 has at least one special vertex. Similarly, C_2 has at least one special vertex. Let c_t be a special vertex of C_1 and c'_h be a special vertex of C_2.

We show that c_t is the unique special vertex of C_1. Suppose to the contrary that C_1 has at least two special vertices. Assume that $\deg_G(c'_{h+1}) = 2$. Let $G' = G - c'_h c'_{h+1}$, and S' be a $fd_1(G')$-set. By Theorem 14, $fd_1(G') \leq \big(n(G') - 1\big)/2 + k - 1$. If $fd_1(G') = \big(n(G') - 1\big)/2 + k - 1$, then by the inductive hypothesis, $G' \in \mathcal{G}_{k-1}$. This is a contradiction by Observation 9(1), since C_1 has at least two special vertices. Thus $fd_1(G') < \big(n(G') - 1\big)/2 + k - 1$. If $\big|S' \cap \{c'_h, c'_{h+1}\}\big| \in \{0, 2\}$, then S' is a 1FD-set in G of cardinality at most $\big(n(G) - 1\big)/2 + k - 1$, a contradiction. Thus $\big|S' \cap \{c'_h, c'_{h+1}\}\big| = 1$. Without loss of generality, assume that $c'_h \in S'$. Then $\{c'_{h+1}\} \cup S'$ is a 1FD-set in G, and so $fd_1(G) < \big(n(G) - 1\big)/2 + k$, a contradiction. We thus assume that $\deg_G(c'_{h+1}) \geq 3$. Likewise, we may assume that $\deg_G(c'_{h-1}) \geq 3$. Since C_2 is a leaf-cycle, c'_0 is its unique special cut-vertex. Thus we may assume, without loss of generality, that $c'_{h+1} \neq c'_0$. Clearly, c'_{h+1} is a support vertex of G. Let $G' = G - c'_h c'_{h-1}$, and S' be a $fd_1(G')$-set. Clearly c'_{h+1} is a strong support vertex of G'. By Theorem 14, $fd_1(G') \leq \big(n(G') - 1\big)/2 + k - 1$. If $fd_1(G') = \big(n(G') - 1\big)/2 + k - 1$, then by the inductive hypothesis $G' \in \mathcal{G}_{k-1}$. This is a contradiction by Observation 9(1), since C_1 has at least two special vertices. Thus $fd_1(G') < \big(n(G') - 1\big)/2 + k - 1$. By Observation 1, $c'_{h+1} \in S'$. If $c'_{h-1} \notin S'$, then S' is a 1FD-set in G of cardinality at most $\big(n(G) - 1\big)/2 + k - 1$, a contradiction. Thus $c'_{h-1} \in S'$. Now, $S' \cup \{c'_h\}$ is a 1FD-set in G, and thus $fd_1(G) \leq |S'| + 1 < \big(n(G) - 1\big)/2 + k$, a contradiction. Thus c_t is the unique special vertex of C_1. Similarly, c'_h is the unique special vertex of C_2. □

Let c_t be the unique special vertex of C_1, and c'_h be the unique special vertex of C_2, and note that Claim 4 guarantees the existence of c_t and c'_h.

Claim 5. *No vertex of C_i is a strong support vertex, for $i = 1, 2$.*

Proof. Suppose that $c_j \in C_1$ is a strong support vertex. Since C_2 is a leaf-cycle, c'_0 is its unique special cut-vertex. Thus, we may assume, without loss of generality, that c'_{h+1} is a support vertex of G. Let $G' = G - c'_h c'_{h-1}$, and S' be a $fd_1(G')$-set. Clearly c'_{h+1} is a strong support vertex of G'. By Theorem 14, $fd_1(G') \le \big(n(G') - 1\big)/2 + k - 1$. If $fd_1(G') = \big(n(G') - 1\big)/2 + k - 1$, then by the inductive hypothesis $G' \in \mathcal{G}_{k-1}$. This is a contradiction by Observation 9(1), since C_1 has a strong support vertex. Thus $fd_1(G') < \big(n(G') - 1\big)/2 + k - 1$. By Observation 1, $c'_{h+1} \in S'$. If $c'_{h-1} \notin S'$, then S' is a 1FD-set in G of cardinality at most $\big(n(G) - 1\big)/2 + k - 1$, a contradiction. Thus $c'_{h-1} \in S'$. Then $S' \cup \{c'_h\}$ is a 1FD-set in G, and so $fd_1(G) \le |S'| + 1 < \big(n(G) - 1\big)/2 + k$, a contradiction. We deduce that C_1 has no strong support vertex. Similarly, C_2 has no strong support vertex. □

We deduce that c_i is a weak support vertex for each $i \in \{1, 2, \ldots, r\} \setminus \{t\}$, and similarly c'_i is a weak support vertex for each $i \in \{1, 2, \ldots, r'\} \setminus \{h\}$. For each $i \in \{1, 2, \ldots, r\} \setminus \{t\}$, let u_i be the leaf adjacent to c_i.

Let G'_2 be the component of $G - c_0c_1 - c_0c_r$ that contains c_0, and G^* be a graph obtained from G'_2 by adding a leaf v^* to c_0. Clearly $n(G^*) = n(G) - 2r + 2$. By Theorem 14, $fd_1(G^*) \le \big(n(G^*) - 1\big)/2 + k - 1$. Suppose that $fd_1(G^*) < \big(n(G^*) - 1\big)/2 + k - 1$. Let S^* be a $fd_1(G^*)$-set. If $c_0 \in S^*$, then $S^* \cup \{c_1, c_2, \ldots, c_r\}$ is a 1FD-set in G, so we obtain that $fd_1(G) < (n-1)/2 + k$, a contradiction. Thus $c_0 \notin S^*$. Then $v^* \in S^*$. If $t > 1$, then $S^* \cup \{c_1, \ldots, c_{t-1}\} \cup \{u_{t+1}, \ldots, u_r\} \setminus \{v^*\}$ is a 1FD-set in G of cardinality at most $\big(n(G^*) - 1\big)/2 + k - 1 - 1 + r - 1 = \big(n(G) - 2r + 2 - 1\big)/2 + k - 1 - 1 + r - 1 = \big(n(G) - 1\big)/2 + k - 2$, a contradiction. Thus assume that $t = 1$. Then $S^* \cup \{c_2, \ldots, c_r\} \setminus \{v^*\}$, is a 1FD-set in G of cardinality at most $\big(n(G) - 1\big)/2 + k - 2$, a contradiction. Thus $fd_1(G^*) = \big(n(G^*) - 1\big)/2 + k - 1$. By the inductive hypothesis, $G^* \in \mathcal{G}_{k-1}$. Let H^* be the graph obtained from $G[\{c_0, c_1, \ldots, c_r, u_1, \ldots, u_{t-1}, u_{t+1}, \ldots, u_r\}]$ by adding a leaf to c_0. Clearly $H^* \in \mathcal{H}_1$. Thus G is obtained from $G^* \in \mathcal{G}_{k-1}$ and $H^* \in \mathcal{H}_1$ by Procedure A. Consequently, $G \in \mathcal{H}_k \subseteq \mathcal{G}_k$.

For the converse, by Corollary 13, $V(G) \setminus L(G)$ is the unique $fd_1(G)$-set. Now Observation 9 implies that $fd_1(G) = (n - 1)/2 + k$. ■

Acknowledgments

We would like to thank both referees for their careful review and useful comments.

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Received 4 May 2017
Revised 5 September 2017
Accepted 19 September 2017