FAIR DOMINATION NUMBER IN CACTUS GRAPHS

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Abstract

For $k \geq 1$, a $k$-fair dominating set (or just $k$FD-set) in a graph $G$ is a dominating set $S$ such that $|N(v) \cap S| = k$ for every vertex $v \in V \setminus S$. The $k$-fair domination number of $G$, denoted by $fd_k(G)$, is the minimum cardinality of a $k$FD-set. A fair dominating set, abbreviated FD-set, is a $k$FD-set for some integer $k \geq 1$. The fair domination number, denoted by $fd(G)$ of $G$ that is not the empty graph, is the minimum cardinality of an FD-set in $G$. In this paper, aiming to provide a particular answer to a problem posed in [Y. Caro, A. Hansberg and M.A. Henning, Fair domination in graphs, Discrete Math. 312 (2012) 2905–2914], we present a new upper bound for the fair domination number of a cactus graph, and characterize all cactus graphs $G$ achieving equality in the upper bound of $fd_1(G)$.

Keywords: fair domination, cactus graph, unicyclic graph.

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1. Introduction

For notation and graph theory terminology not given here, we follow [10]. Specifically, let $G$ be a graph with vertex set $V(G) = V$ of order $|V| = n$ and let $v$ be a vertex in $V$. The open neighborhood of $v$ is $N_G(v) = \{u \in V \mid uv \in E(G)\}$ and
the closed neighborhood of v is \( N_G[v] = \bigcup_{v \in S} N_G(v) \). If the graph G is clear from
the context, we simply write \( N(v) \) rather than \( N_G(v) \). The degree of a vertex
v, is \( \deg(v) = |N(v)| \). A vertex of degree one is called a leaf and its neighbor
a support vertex. We denote the set of leaves and support vertices of a graph
G by \( L(G) \) and \( S(G) \), respectively. A strong support vertex is a support vertex
adjacent to at least two leaves, and a weak support vertex is a support vertex
adjacent to precisely one leaf. For a set \( S \subseteq V \), its open neighborhood is the set
\( N(S) = \bigcup_{v \in S} N(v) \), and its closed neighborhood is the set \( N[S] = N(S) \cup S \).
The corona graph \( cor(G) \) of a graph G is a graph obtained by adding a leaf to
every vertex of G. We denote by \( P_n \) a path on n vertices. The distance \( d(u, v) \)
between two vertices u and v in a graph G is the minimum number of edges of
a path from u to v. The diameter \( diam(G) \) of G, is \( \max_{u,v \in V(G)} d(u,v) \). A path
of length \( diam(G) \) is called a diameterical path. A cactus graph is a connected
graph in which any two cycles have at most one vertex in common. For a subset
\( S \) of vertices of G, we denote by \( G[S] \) the subgraph of G induced by \( S \).

A subset \( S \subseteq V \) is a dominating set of G if every vertex not in \( S \) is adjacent
to a vertex in \( S \). The domination number of G, denoted by \( \gamma(G) \), is the minimum
cardinality of a dominating set of G. A vertex \( v \) is said to be dominated by a set
\( S \) if \( N(v) \cap S = \emptyset \).

Caro et al. [1] studied the concept of fair domination in graphs. For \( k \geq 1 \), a
\( k \)-fair dominating set, abbreviated kFD-set, in G is a dominating set \( S \) such that
\( |N(v) \cap D| = k \) for every vertex \( v \in V \setminus D \). The \( k \)-fair domination number of G,
denoted by \( fd_k(G) \), is the minimum cardinality of a kFD-set. A kFD-set of G
of cardinality \( fd_k(G) \) is called a \( fd_k(G) \)-set. A fair dominating set, abbreviated
FD-set, in G is a kFD-set for some integer \( k \geq 1 \). The fair domination number,
denoted by \( fd(G) \), of a graph G that is not the empty graph is the minimum
cardinality of an FD-set in G. An FD-set of G of cardinality \( fd(G) \) is called a
\( fd(G) \)-set.

A perfect dominating set in a graph G is a dominating set \( S \) such that every
vertex in \( V(G) \setminus S \) is adjacent to exactly one vertex in \( S \). Hence a 1FD-set
is precisely a perfect dominating set. The concept of perfect domination was
introduced by Cockayne et al. in [4], and Fellows et al. [7] with a different
terminology which they called semiperfect domination. This concept was further
studied, see for example, [2, 3, 5, 6, 9].

**Observation 1** (Caro et al. [1]). Every 1FD-set in a graph contains all its strong
support vertices.

The following is easily verified.

**Observation 2.** Let \( S \) be a 1FD-set in a graph G, \( v \) a support vertex of G and
\( v' \) a leaf adjacent to \( v \). If \( S \) contains a vertex \( u \in N_G(v) \setminus \{v'\} \), then \( v \in S \).
Among other results, Caro et al. [1] proved that $fd(G) \leq n - 2$ for any connected graph $G$ of order $n \geq 3$ with no isolated vertex, and constructed an infinite family of connected graphs achieving equality in this bound. They showed that $fd(G) < 17n/19$ for any maximal outerplanar graph $G$ of order $n$, and $fd(T) \leq n/2$ for any tree $T$ of order $n \geq 2$. They then showed that equality for the bound $fd(T) \leq n/2$ holds if and only if $T$ is the corona of a tree. Among open problems posed by Caro et al. [1], one asks to find $fd(G)$ for other families of graphs.

**Problem 3** (Caro et al. [1]). Find $fd(G)$ for other families of graphs.

In this paper, aiming to study Problem 3, we present a new upper bound for the 1-fair domination number of cactus graphs and characterize all cactus graphs achieving equality for the upper bound. We show that if $G$ is a cactus graph of order $n \geq 5$ with $k \geq 1$ cycles, then $fd_1(G) \leq (n - 1)/2 + k$. We also characterize all cactus graphs achieving equality for the upper bound.

## 2. Unicyclic Graphs

Fair domination in unicyclic graphs has been studied in [8]. A vertex $v$ of a cactus graph $G$ is a special vertex if $\deg_G(v) = 2$ and $v$ belongs to a cycle of $G$. Let $\mathcal{H}_1$ be the class of all graphs $G$ that can be obtained from the corona $\text{cor}(C)$ of a cycle $C$ by removing precisely one leaf of $\text{cor}(C)$. Let $\mathcal{G}_1$ be the class of all graphs $G$ that can be obtained from a sequence $G_1, G_2, \ldots, G_s = G$, where $G_1 \in \mathcal{H}_1$, and if $s \geq 2$, then $G_{j+1}$ is obtained from $G_j$ by one of the following Operations $O_1$ or $O_2$, for $j = 1, 2, \ldots, s - 1$.

**Operation $O_1$.** Let $v$ be a vertex of $G_j$ with $\deg(v) \geq 2$ such that $v$ is not a special vertex of $G_j$. Then $G_{j+1}$ is obtained from $G_j$ by adding a path $P_2$ and joining $v$ to a leaf of $P_2$.

**Operation $O_2$.** Let $v$ be a leaf of $G_j$. Then $G_{j+1}$ is obtained from $G_j$ by adding two leaves to $v$.

**Lemma 4** [8]. If $G \in \mathcal{G}_1$, then every 1FD-set in $G$ contains every vertex of $G$ of degree at least two.

**Theorem 5** [8]. If $G$ is a unicyclic graph of order $n$, then $fd_1(G) \leq (n + 1)/2$, with equality if and only if $G = C_5$ or $G \in \mathcal{G}_1$.

## 3. Main Result

Our aim in this paper is to give an upper bound for the fair domination number of a cactus graph $G$ in terms of the number of cycles of $G$, and then characterize
all cactus graphs achieving equality for the proposed bound. For this purpose we first introduce some families of graphs. Let $\mathcal{H}_1$ and $\mathcal{G}_1$ be the families of unicyclic graphs described in Section 2. For $i = 2, 3, \ldots, k$, we construct a family $\mathcal{H}_i$ from $\mathcal{G}_{i-1}$, and a family $\mathcal{G}_i$ from $\mathcal{H}_i$ as follows.

- **Family $\mathcal{H}_i$.** Let $\mathcal{H}_i$ be the family of all graphs $H_i$ such that $H_i$ can be obtained from a graph $H_1 \in \mathcal{H}_1$ and a graph $G \in \mathcal{G}_{i-1}$, by the following Procedure.

**Procedure A.** Let $w_0 \in V(H_1)$ be a support vertex of $H_1$, and $w \in V(G_{i-1})$ be a support vertex of $G_{i-1}$. We remove precisely one leaf adjacent to $w_0$ and precisely one leaf adjacent to $w$, and then identify the vertices $w_0$ and $w$.

- **Family $\mathcal{G}_i$.** Let $\mathcal{G}_i$ be the family of all graphs $G$ that can be obtained from a sequence $G_1, G_2, \ldots, G_s = G$, where $G_1 \in \mathcal{H}_i$, and if $s \geq 2$ then $G_{j+1}$ is obtained from $G_j$ by one of the Operations $O_1$ or $O_2$, described in Section 2, for $j = 1, 2, \ldots, s-1$.

Note that $\mathcal{H}_i \subseteq \mathcal{G}_i$, for $i = 1, 2, \ldots, k$. Figure 1 demonstrates the construction of the family $\mathcal{G}_k$.

We will prove the following.

**Theorem 6.** If $G$ is a cactus graph of order $n \geq 5$ with $k \geq 1$ cycles, then $fd_1(G) \leq (n-1)/2 + k$, with equality if and only if $G = C_5$ or $G \in \mathcal{G}_k$.

**Corollary 7.** If $G$ is a cactus graph of order $n \geq 5$ with $k \geq 1$ cycles, then $fd(G) \leq (n-1)/2 + k$.

4. Preliminary Results and Observations

4.1. Notation

We call a vertex $w$ in a cycle $C$ of a cactus graph $G$ a **special cut-vertex** if $w$ belongs to a shortest path from $C$ to a cycle $C' \neq C$. We call a cycle $C$ in a cactus graph $G$, a **leaf-cycle** if $C$ contains exactly one special cut-vertex. In the
Figure 2. $C_i$ is a leaf-cycle for $i = 1, 2, 3$ and $v_j$ is a special cut-vertex for $j = 1, 2, \ldots, 8$. Moreover, $C_j$ is a leaf-cycle for $j = 1, 2, 3$.

**Observation 8.** Every cactus graph with at least two cycles contains at least two leaf-cycles.

### 4.2. Properties of the family $\mathcal{G}_k$

The following observation can be proved by a simple induction on $k$.

**Observation 9.** If $G \in \mathcal{G}_k$ is a cactus graph of order $n$, then the following conditions are satisfied.

1. No cycle of $G$ contains a strong support vertex. Furthermore, any cycle of $G$ contains precisely one special vertex.
2. $n$ is odd.
3. $|L(G)| = (n + 1)/2 - k$.
4. If a vertex $v$ of $G$ belongs to at least two cycles of $G$, then $v$ is not a support vertex, and $v$ belongs to precisely two cycles of $G$.

**Observation 10.** Let $G \in \mathcal{G}_k$. Let $G$ be obtained from a sequence $G_1, G_2, \ldots, G_s = G$ ($s \geq 2$) such that $G_1 \in \mathcal{H}_1$ and $G_{j+1}$ is obtained from $G_j$ by one of the Operations $O_1$ or $O_2$ or procedure $A$, for $j = 1, 2, \ldots, s - 1$. If $v$ is a vertex of $G$ belonging to two cycles of $G$ then there is an integer $i \in \{2, 3, \ldots, s\}$ such that $G_i$ is obtained from $G_{i-1}$ by applying Procedure $A$ on the vertex $v$ using a graph $H \in \mathcal{H}_1$, such that $v$ belongs to a cycle of $G_{i-1}$.

**Observation 11.** Assume that $G \in \mathcal{G}_k$ and $v \in V(G)$ is a vertex of degree four belonging to two cycles. Let $D_1$ and $D_2$ be the components of $G - v$, $G_1^*$ be the
graph obtained from $G[D_1 \cup \{v\}]$ by adding a leaf $v^*_1$ to $v$, and $G^*_2$ be the graph obtained from $G[D_2 \cup \{v\}]$ by adding a leaf $v^*_2$ to $v$. Then there exists an integer $k' < k$ such that $G^*_1 \in G_{k'}$ or $G^*_2 \in G_{k'}$.

**Proof.** Let $G \in G_k$. Thus $G$ is obtained from a sequence $G_1, G_2, \ldots, G_s = G$ ($s \geq 2$) such that $G_1 \in H_1$ and $G_{j+1}$ is obtained from $G_j$ by one of the Operations $O_1$ or $O_2$ or Procedure $A$, for $j = 1, 2, \ldots, s - 1$. Note that $s \geq k$. We define the $j$-th Procedure-Operation or just $PO_j$ as one of the Operation $O_1$, Operation $O_2$, or Procedure $A$ that can be applied to obtain $G_{j+1}$ from $G_j$. Thus $G$ is obtained from $G_1$ by Procedure-Operations $PO_1, PO_2, \ldots, PO_{s-1}$.

Let $v$ be a vertex of $G$ of degree four belonging to two cycles of $G$, and $D_1$ and $D_2$ be the components of $G - v$. By Observation 10, there is an integer $i \in \{2, 3, \ldots, s\}$ such that $G_i$ is obtained from $G_{i-1}$ by applying Procedure $A$ on the vertex $v$ using a graph $H \in H_1$. Note that $v$ is a support vertex of $G_{i-1}$. Let $v^*$ be the leaf of $v$ in $G_{i-1}$ that is removed in Procedure $A$. Clearly, either $V(G_{i-1}) \cap D_1 \neq \emptyset$ or $V(G_{i-1}) \cap D_2 \neq \emptyset$. Without loss of generality, assume that $V(G_{i-1}) \cap D_1 \neq \emptyset$. Among $PO_1, PO_{i+1}, \ldots, PO_{s-1}$, let $PO_{r_1}, PO_{r_2}, \ldots, PO_{r_t}$, be the Procedure-Operations applied on a vertex of $D_1$, where $i \leq t \leq s - 1$. Let $G_{r_0} = G_{i-1}$ and $G_{r_{t+1}}$ be obtained from $G_{r_l}$ by $PO_{r_{l+1}}$, for $l = 0, 1, 2, \ldots, t - 1$. Clearly by an induction on $t$, we can deduce that there is an integer $k^* < k$ such that $G_{r_t} \in G_{k^*}$. Note that $G_{r_t} = G^*_1$.

**Lemma 12.** If $G \in G_k$, then every 1FD-set in $G$ contains every vertex of $G$ of degree at least two.

**Proof.** Let $G \in G_k$, and $S$ be a 1FD-set in $G$. We prove by an induction on $k$, namely first-induction, to show that $S$ contains every vertex of $G$ of degree at least two. For the base step, if $k = 1$ then $G \in H_1$, and the result follows by Lemma 4. Assume the result holds for all graphs $G' \in G_{k'}$ with $k' < k$. Now consider the graph $G \in G_k$, where $k > 1$. Clearly, $G$ is obtained from a sequence $G_1, G_2, \ldots, G_l = G$, of cactus graphs such that $G_1 \in H_k$, and if $l \geq 2$, then $G_{i+1}$ is obtained from $G_i$ by one of the operations $O_1$ or $O_2$ for $i = 1, 2, \ldots, l - 1$.

We employ an induction on $l$, namely second-induction, to show that $S$ contains every vertex of $G$ of degree at least two.

For the base step of the second-induction, let $l = 1$. Thus $G \in H_k$. By the construction of graphs in the family $H_k$, there are graphs $H \in H_1$ and $G' \in G_{k-1}$ such that $G$ is obtained from $H$ and $G'$ by Procedure $A$. Clearly, $H$ is obtained from the corona $cor(C)$ of a cycle $C$, by removing precisely one leaf of $cor(C)$. Let $C = c_0c_1 \cdots c_rc_0$, where $c_0$ is the support vertex of $H$ that its leaf is removed according to Procedure $A$. Since $H$ has precisely one special vertex, let $c_1$ be the special vertex of $H$. Let $w \in V(G')$ be a support vertex of $G'$ that its leaf, say $w^*$, is removed to obtain $G$ according to Procedure $A$. First we show that $\{c_1, c_r\} \cap S \neq \emptyset$. Clearly $S \cap \{c_{l-1}, c_l, c_l+1\} \neq \emptyset$, since $deg_G(c_l) = 2$. Assume that
Since at least one of \( c_{t-1} \) or \( c_{t+1} \) is a support vertex, by Observation 2, \( \{c_{t-1}, c_{t+1}\} \cap S \neq \emptyset \). By applying Observation 2, we obtain that \( \{c_1, c_r\} \cap S \neq \emptyset \), since any vertex of \( \{c_1, \ldots, c_r\} \setminus \{c_t\} \) is a support vertex of \( G \). Thus assume that \( c_t \notin S \). Then \( \{c_{t-1}, c_{t+1}\} \cap S \neq \emptyset \), and so \( \{c_1, c_r\} \cap S \neq \emptyset \), since any vertex of \( \{c_1, \ldots, c_r\} \setminus \{c_t\} \) is a support vertex of \( G \). Hence, \( \{c_1, c_r\} \cap S \neq \emptyset \). If \( c_0 \notin S \), then \( (S \cap V(G')) \cup \{w'\} \) is a 1FD-set for \( G' \), and thus by the first-inductive hypothesis, \( S \) contains \( w = c_0 \), a contradiction. Thus \( c_0 \in S \). By Observation 2, \( V(C) \subseteq S \), since any vertex of \( \{c_1, \ldots, c_r\} \setminus \{c_t\} \) is a support vertex of \( G \). Thus \( S \cap V(G') \) is a 1FD-set for \( G' \). By the first-inductive hypothesis, \( (S \cap V(G')) \cup \{w\} \) contains every vertex of \( G' \) of degree at least two. Consequently, \( S \) contains every vertex of \( G \) of degree at least two. We conclude that the base step of the second-induction holds.

Assume that the result (for the second-induction) holds for \( 2 \leq l' < l \). Now let \( G = G_l \). Clearly \( G \) is obtained from \( G_{l-1} \) by applying one of the Operations \( O_1 \) or \( O_2 \).

Assume that \( G \) is obtained from \( G_{l-1} \) by applying Operation \( O_2 \). Let \( x \) be a leaf of \( G_{l-1} \) and \( G \) be obtained from \( G_{l-1} \) by adding two leaves \( x_1 \) and \( x_2 \) to \( x \). By Observation 1, \( x \in S \). Thus \( S \) is a 1FD-set for \( G_{l-1} \). By the second-inductive hypothesis \( S \) contains all vertices of \( G_{l-1} \) of degree at least two. Consequently, \( S \) contains every vertex of \( G \) of degree at least two.

Next assume that \( G \) is obtained from \( G_{l-1} \) by applying Operation \( O_1 \). Let \( x_1x_2 \) be a path and \( x_1 \) is joined to \( y \in V(G_{l-1}) \), where \( \deg_{G_{l-1}}(y) \geq 2 \) and \( y \) is not a special vertex of \( G_{l-1} \). Observe that \( \{x_1, x_2\} \cap S \neq \emptyset \). If \( x_1 \notin S \), then \( x_2 \in S \) and \( y \notin S \). Then \( S \setminus \{x_2\} \) is a 1FD-set for \( G_{l-1} \) that does not contain \( y \), a contradiction by the second-inductive hypothesis. Thus assume that \( x_1 \in S \). Suppose that \( y \notin S \). Clearly \( N_{G_{l-1}}(y) \cap S = \emptyset \).

Assume that there exists a component \( G'_l \) of \( G_{l-1} - y \) such that \( |V(G'_l) \cap N_{G_{l-1}}(y)| = 1 \). Then clearly \( S' = (S \cap V(G_{l-1})) \cup V(G'_l) \) is a 1FD-set for \( G_{l-1} \), and by the second-inductive hypothesis \( S' \) contains every vertex of \( G_{l-1} \) of degree at least two. Thus \( y \in S' \), and so \( y \in S \), a contradiction. Next assume that every component of \( G_{l-1} - y \) has at least two vertices in \( N_{G_{l-1}}(y) \). Since \( y \) is a non-special vertex of \( G_{l-1} \), \( y \) belongs to at least two cycles of \( G_{l-1} \). By Observation 9(4), \( y \) belongs to exactly two cycles of \( G_{l-1} \). Thus \( \deg_{G_{l-1}}(y) = 2 \). By Observation 11, \( G_{l-1} - y \) has exactly two components \( D_1 \) and \( D_2 \). Let \( G^* \) be a graph obtained from \( D_1 \cup \{v\} \) or \( D_2 \cup \{v\} \), by adding a leaf \( v^* \) to \( y \). Then there exists \( k' \leq k \) such that \( G^* \in G_{k'} \). Evidently, \( S^* = (S \cap V(G^*)) \cup \{v^*\} \) is a 1FD-set for \( G^* \), and so by the first-inductive hypothesis, \( S^* \) contains every vertex of \( G^* \) of degree at least two (since \( G^* \in G_{k'} \)). Thus \( y \in S^* \), and so \( y \in S \), a contradiction. We conclude that \( y \in S \). Observe that \( S \cap V(G_{l-1}) \) is a 1FD-set for \( G_{l-1} \), and so by the second-inductive hypothesis, \( S \cap V(G_{l-1}) \) contains every vertex of \( G_{l-1} \) of degree at least two. Consequently \( S \) contains every vertex of \( G \).
of degree at least two.

As a consequence of Observation 9(3) and Lemma 12, we obtain the following.

**Corollary 13.** If $G \in G_k$ is a cactus graph of order $n$, then $V(G) \setminus L(G)$ is the unique $fd_1(G)$-set.

5. **Proof of Theorem 6**

**Theorem 14.** If $G$ is a cactus graph of order $n$ with $k \geq 1$ cycles, then $fd_1(G) \leq \frac{(n(G)-1)}{2} + k$.

**Proof.** The result follows by Theorem 5 if $k = 1$. Thus assume that $k \geq 2$. Suppose to the contrary that $fd_1(G) > \frac{(n(G)-1)}{2} + k$. Assume that $G$ has the minimum order, and among all such graphs, we may assume that the size of $G$ is minimum. Let $C_1, C_2, \ldots, C_k$ be the $k$ cycles of $G$. Let $C_i$ be a leaf-cycle of $G$, where $i \in \{1, 2, \ldots, k\}$. Let $C_i = u_0u_1 \cdots u_lu_0$, where $u_0$ is a special cut-vertex of $G$. Assume that $\deg_G(u_j) = 2$ for each $j = 1, 2, \ldots, l$. Let $G' = G - u_1u_2$. Then by the choice of $G$, $fd_1(G') \leq (n(G') - 1)/2 + k - 1 = (n(G) - 1)/2 + k - 1$. Let $S'$ be a $fd_1(G')$-set. Now if $|S' \cap \{u_1, u_2\}| \in \{0, 2\}$, then $S'$ is a $1FD$-set for $G$, a contradiction. Thus assume that $|S' \cap \{u_1, u_2\}| = 1$. Assume that $u_1 \in S'$. Then $u_3 \in S'$, and so $\{u_2\} \cup S'$ is a $1FD$-set in $G$ of cardinality at most $(n(G) - 1)/2 + k$, a contradiction. If $u_2 \in S'$, then $u_0 \in S'$, and $\{u_1\} \cup S'$ is a $1FD$-set in $G$ of cardinality at most $(n(G) - 1)/2 + k$, a contradiction. We deduce that $\deg_G(u_i) \geq 3$ for some $i \in \{1, 2, \ldots, l\}$. Let $v_d$ be a leaf of $G$ such that $d(v_d, C_i - u_0)$ is as maximum as possible, and the shortest path from $v_d$ to $C_i$ does not contain $u_0$. Let $v_0v_1 \cdots v_d$ be the shortest path from $v_d$ to $C_i$ with $v_0 \in C_i$. Assume that $d \geq 2$. Assume that $\deg_G(v_{d-1}) = 2$. Let $G' = G - \{v_d, v_{d-1}\}$. By the choice of $G$, $fd_1(G') \leq (n(G') - 1)/2 + k$. Let $S'$ be a $fd_1(G')$-set. If $v_{d-2} \in S'$, then $S' \cup \{v_{d-1}\}$ is a $1FD$-set in $G$, and if $v_{d-2} \notin S'$, then $S' \cup \{v_d\}$ is a $1FD$-set in $G$. Thus $fd_1(G) \leq (n-1)/2 + k$, a contradiction. Thus assume that $\deg_G(v_{d-1}) \geq 3$. Clearly any vertex of $N_G(v_{d-1}) \setminus \{v_{d-2}\}$ is a leaf. Let $G'$ be obtained from $G$ by removing all leaves adjacent to $v_{d-1}$. By the choice of $G$, $fd_1(G') \leq (n(G') - 1)/2 + k$, since $G$ has the minimum order among all graphs $H$ with 1-fair domination number more than $(n(H) - 1)/2 + k$. Let $S'$ be a $fd_1(G')$-set. If $v_{d-1} \in S'$, then $S'$ is a $1FD$-set in $G$, a contradiction. Thus assume that $v_{d-1} \notin S'$. Then $v_{d-2} \in S'$. Then $S' \cup \{v_{d-1}\}$ is a $1FD$-set in $G$ of cardinality at most $(n(G') - 1)/2 + k + 1 \leq (n(G) - 1)/2 + k$, a contradiction.

We thus assume that $d = 1$. Assume that $u_i$ is a vertex of $C_i$ such that $\deg_G(u_i) = 2$. Assume that $\deg_G(u_{i+1}) = 2$. Let $G' = G - u_iu_{i+1}$. By the
We prove by induction on $n \geq n$. Let $S'$ be a $f(G')$-set. If $|S' \cap \{u_i, u_{i+1}\}| \in \{0, 2\}$, then $S'$ is a 1FD-set for $G$, a contradiction. Then $|S' \cap \{u_i, u_{i+1}\}| = 1$. Assume that $u_i \in S'$. Then $u_{i+2} \in S'$ and so $\{u_{i+1}\} \cup S'$ is a 1FD-set in $G$ of cardinality at most $(n(G) - 1)/2 + k$, a contradiction. Next assume that $u_{i+1} \in S'$. Then $u_{i-1} \in S'$ and so $\{u_i\} \cup S'$ is a 1FD-set in $G$ of cardinality at most $(n(G) - 1)/2 + k$, a contradiction. Thus deg$_G(u_{i+1}) \geq 3$, and similarly deg$_G(u_{i-1}) \geq 3$. Since $C_i$ is a leaf-cycle, it has precisely one special cut-vertex. Thus we may assume, without loss of generality, that $u_{i+1}$ is a support vertex of $G$. Let $G' = G - u_{i-1}u_i$. By the choice of $G$, $f(G') \leq (n(G') - 1)/2 + k - 1$. Let $S'$ be a $f(G')$-set. By Observation 1, $u_{i+1} \in S'$. If $u_{i-1} \notin S'$, then $S'$ is a 1FD-set in $G$ of cardinality at most $(n(G) - 1)/2 + k - 1$, a contradiction. Thus $u_{i-1} \in S'$. Then $S' \cup \{u_i\}$ is a 1FD-set in $G$ of cardinality at most $(n(G) - 1)/2 + k$, a contradiction.

We conclude that deg$_G(u_i) \geq 3$ for $i = 0, 1, \ldots, l$. Furthermore, $u_i$ is a support vertex for $i = 1, 2, \ldots, l$. Assume that $u_i$ is a strong support vertex for some $i \in \{1, 2, \ldots, l\}$. Let $G'$ be obtained from $G$ by removal of all vertices in $\bigcup_{i=1}^l (N[u_i]) \setminus \{u_0, u_1, u_l\}$. Clearly $u_0$ is a strong support vertex of $G'$. By the choice of $G$, $f(G') \leq (n(G') - 1)/2 + k - 1 \leq (n(G) - (2l + 1) + 2 - 1)/2 + k - 1$, since $u_i$ is a strong support vertex of $G$. By Observation 1, $u_0 \in S'$, and so $S' \cup \{u_1, \ldots, u_l\}$ is a 1FD-set in $G$ of cardinality at most $(n(G) - (2l + 1) + 2 - 1)/2 + k - 1 + l = n(G)/2 + k - 1$, a contradiction. Thus $u_i$ is a weak support vertex, for each $i = 1, 2, \ldots, l$. Let $G'$ be obtained from $G$ by removal of any vertex in $\bigcup_{i=1}^l (N[u_i]) \setminus \{u_0\}$. By the choice of $G$, $f(G') \leq (n(G') - 1)/2 + k - 1$. Let $S'$ be a $f(G')$-set. If $u_0 \notin S'$, then $S' \cup \{u_1, \ldots, u_l\}$ is a 1FD-set in $G$ of cardinality at most $(n(G) - 1)/2 + k - 1$, where $w_i$ is the leaf adjacent to $u_i$, for $i = 1, 2, \ldots, l$. This is a contradiction. Thus $u_0 \in S'$. Then $S' \cup \{u_1, \ldots, u_l\}$ is a 1FD-set in $G$ of cardinality at most $(n(G) - 1)/2 + k - 1$, a contradiction.

If $G$ is a cactus graph of order $n$ with $k \geq 1$ cycles and $f_1(G) = (n-1)/2 + k$, then clearly $n \geq 3$ is odd, and since $f_1(C_3) \neq 2$, we have $n \geq 5$. It is obvious that $f_1(C_5) = 3 = (5 - 1)/2 + 1$.

**Theorem 15.** If $G \neq C_5$ is a cactus graph of order $n \geq 5$ with $k \geq 1$ cycles, then $f_1(G) = (n-1)/2 + k$ if and only if $G \in \mathcal{G}_k$.

**Proof.** We prove by an induction on $k$ to show that any cactus graph $G$ of order $n \geq 5$ with $k \geq 1$ cycles and $f_1(G) = (n-1)/2 + k$ belongs to $\mathcal{G}_k$. The base step of the induction follows by Theorem 5. Assume the result holds for all cactus graphs $G'$ with $k' < k$ cycles. Now let $G$ be a cactus graph of order $n$ with $k \geq 2$ cycles and $f_1(G) = (n-1)/2 + k$. Clearly $n$ is odd. Suppose to the contrary that $G \notin \mathcal{G}_k$. Assume that $G$ has the minimum order, and among all such graphs, assume that the size of $G$ is minimum. By Observation 8, $G$ has at least two leaf-cycles. Let $C_1 = c_0c_1 \cdots c_r c_0$ and $C_2 = c'_0 c'_1 \cdots c'_r c'_0$, be two leaf-cycles of
$G$, where $c_0$ and $c'_0$ are two special cut-vertices of $G$. Let $G'_1$ be the component of $G - c_0c_1c_0c_r$ containing $c_1$, and $G''_r$ be the component of $G - c'_0c'_1c'_0c'_r$ containing $c'_1$.

Claim 1. $V(G'_1) \neq \{c_1, \ldots, c_r\}$, and $V(G''_r) \neq \{c'_1, \ldots, c'_r\}$.

**Proof.** Suppose that $V(G'_1) = \{c_1, \ldots, c_r\}$. Then $\deg_{G}(c_i) = 2$ for $i = 1, 2, \ldots, r$. Let $G^* = G - c_1c_2$, and $S^*$ be a $fd_1(G^*)$-set. By Theorem 14, $fd_1(G^*) \leq (n(G^*) - 1)/2 + k - 1 = (n(G) - 1)/2 + k - 1$. Assume that $r = 2$. Then $c_0$ is a strong support vertex of $G^*$, and by Observation 1, $c_0 \in S^*$. Thus $|S^* \cap \{c_1, c_2\}| = 0$, and so $S^*$ is a 1FD-set in $G$ of cardinality at most $(n(G) - 1)/2 + k - 1 < (n(G) - 1)/2 + k$, a contradiction. Assume that $r = 3$. If $|S^* \cap \{c_1, c_2\}| \in \{0, 2\}$, then $S^*$ is a 1FD-set in $G$ of cardinality at most $(n(G) - 1)/2 + k - 1 < (n(G) - 1)/2 + k$, a contradiction. Thus $|S^* \cap \{c_1, c_2\}| = 1$. If $c_1 \in S^*$, then $c_3 \in S^*$, and so $c_0 \in S^*$. Then $S^* \setminus \{c_1\}$ is a 1FD-set in $G^*$, a contradiction. Thus $c_1 \notin S^*$, and so $c_2 \in S^*$. Since $c_1$ is dominated by $S^*$, we obtain that $c_0 \in S^*$, and so $c_3 \in S^*$. Then $S^* \setminus \{c_2\}$ is a 1FD-set in $G^*$, a contradiction. Assume that $r = 4$. Suppose that $fd_1(G^*) = (n(G^*) - 1)/2 + k - 1$. Let $G'_2 = G^* - \{c_2, c_3, c_4\}$. By Theorem 14, $fd_1(G'_2) \leq (n(G'_2) - 1)/2 + k - 1 = n/2 + k - 3$, and thus $fd_1(G'_2) \leq (n(G) - 1)/2 + k - 3$, since $n$ is odd. Let $S'_3$ be a $fd_1(G'_2)$-set. If $c_0 \notin S'_3$, then $S'_3 \cup \{c_3\}$ is a 1FD-set for $G^*$. Thus $fd_1(G^*) \leq |S'_3| + 1 \leq (n - 1)/2 + k - 2$, a contradiction. Thus $fd_1(G^*) < (n(G^*) - 1)/2 + k - 1 = (n(G) - 1)/2 + k - 1$. If $|S^* \cap \{c_1, c_2\}| \in \{0, 2\}$, then $S^*$ is a 1FD-set in $G$ of cardinality at most $(n(G) - 1)/2 + k - 1 < (n(G) - 1)/2 + k$, a contradiction. Thus $|S^* \cap \{c_1, c_2\}| = 1$. Without loss of generality, assume that $c_1 \in S^*$. Then $S^* \cup \{c_2\}$ is a 1FD-set in $G$, and so $fd_1(G) \leq |S^*| + 1 < (n(G) - 1)/2 + k$, a contradiction. It remains to assume that $r \geq 5$. Suppose that $fd_1(G^*) = (n(G^*) - 1)/2 + k - 1$. Let $G^*_2 = G^* - \{c_2, c_3, c_4\}$. By Theorem 14, $fd_1(G^*_2) \leq (n(G^*_2) - 1)/2 + k - 1 = n/2 + k - 3$, and thus $fd_1(G^*_2) \leq (n(G) - 1)/2 + k - 3$, since $n$ is odd. Let $S'_3$ be a $fd_1(G'_2)$-set. If $c_3 \notin S'_3$, then $S'_3 \cup \{c_3\}$ is a 1FD-set for $G^*$ and if $c_3 \notin S'_3$, then $S'_3 \setminus \{c_3\}$ is a 1FD-set for $G^*$. Thus $fd_1(G^*) \leq |S'_3| + 1 \leq (n - 1)/2 + k - 2$, a contradiction. Thus $fd_1(G^*) < (n(G^*) - 1)/2 + k - 1 = (n(G) - 1)/2 + k - 1$. If $|S^* \cap \{c_1, c_2\}| \in \{0, 2\}$, then $S^*$ is a 1FD-set in $G$ of cardinality at most $(n(G) - 1)/2 + k - 1 < (n(G) - 1)/2 + k$, a contradiction. Thus $|S^* \cap \{c_1, c_2\}| = 1$. Without loss of generality, assume that $c_1 \in S^*$. Then $S^* \cup \{c_2\}$ is a 1FD-set in $G$, and so $fd_1(G) \leq |S^*| + 1 < (n(G) - 1)/2 + k$, a contradiction. We conclude that $V(G'_1) \neq \{c_1, \ldots, c_r\}$. Similarly $V(G''_r) \neq \{c'_1, \ldots, c'_r\}$. □

Let $v_d \in V(G'_1) \setminus \{c_1, \ldots, c_r\}$ be a leaf of $G'_1$ at maximum distance from $\{c_1, \ldots, c_r\}$, and assume that $v_0v_1 \cdots v_d$ is the shortest path from $v_d$ to $\{c_1, \ldots, c_r\}$, where $v_0 \in \{c_1, \ldots, c_r\}$. Likewise, let $v'_d \in V(G''_r) \setminus \{c'_1, \ldots, c'_r\}$ be a leaf of $G''_r$ at maximum distance from $\{c'_1, \ldots, c'_r\}$, and assume that $v'_0v'_1 \cdots v'_d$ is the shortest
path from $v_{d'}$ to $\{c'_1, \ldots, c'_r\}$, where $v_{d'} \in \{c'_1, \ldots, c'_r\}$. Without loss of generality, assume that $d' \leq d$.

**Claim 2.** Every support vertex of $G$ is adjacent to at most two leaves.

**Proof.** Suppose that there is a support vertex $v \in S(G)$ such that $v$ is adjacent to at least three leaves $v_1, v_2$, and $v_3$. Let $G' = G - \{v_1\}$, and let $S'$ be a $fd_1(G')$-set. By Observation 1, $v \in S'$, and thus we may assume that $S' \cap \{v_2, v_3\} = \emptyset$. By Theorem 14, $|S'| \leq \left(\frac{n(G') - 1}{2}\right)/2 + k = (n - 2)/2 + k$. Clearly $S'$ is a 1FD-set for $G$, a contradiction.

**Claim 3.** If $d \geq 2$, then $G \in \mathcal{G}_k$.

**Proof.** Let $d \geq 2$. By Claim 2, $2 \leq \deg_G(v_{d-1}) \leq 3$. Assume first that $\deg_G(v_{d-1}) = 3$. Let $x \neq v_d$ be a leaf adjacent to $v_{d-1}$. Let $G' = G - \{x, v_d\}$. By Theorem 14, $\deg_{G'}(G) \leq \left(\frac{n(G') - 1}{2}\right)/2 + k$. Suppose that $\deg_{G'}(G') < \left(\frac{n(G') - 1}{2}\right)/2 + k$. Let $S'$ be a $\deg_{G'}(G')$-set. If $v_{d-1} \in S'$, then $S'$ is a 1FD-set for $G$ and if $v_{d-1} \notin S'$, then $S' \cup \{v_{d-1}\}$ is a 1FD-set for $G$. Thus $\deg_{G'}(G) \leq \deg_{G'}(G') + 1 < (n - 1)/2 + k$, a contradiction. Hence, $\deg_{G'}(G') = \left(\frac{n(G') - 1}{2}\right)/2 + k$. By the choice of $G$, $G' \in \mathcal{G}_k$. Therefore $G$ is obtained from $G'$ by Operation $\mathcal{O}_2$. Consequently, $G \in \mathcal{G}_k$. Next assume that $\deg_G(v_{d-1}) = 2$. We consider the following cases.

**Case 1.** $d \geq 3$. Suppose that $\deg_G(v_{d-2}) = 2$. Let $G' = G - \{v_{d-2}, v_{d-1}, v_d\}$. By Theorem 14, $\deg_{G'}(G') \leq \left(\frac{n(G') - 1}{2}\right)/2 + k = n/2 + k - 2$, and thus $\deg_{G'}(G') \leq (n - 1)/2 + k - 2$, since $n$ is odd. Let $S'$ be a $\deg_{G'}(G')$-set. If $v_{d-3} \in S'$, then $S' \cup \{v_d\}$ is a 1FD-set for $G$ and if $v_{d-3} \notin S'$, then $S' \cup \{v_{d-1}\}$ is a 1FD-set for $G$. Thus $\deg_{G'}(G) \leq |S'| + 1 \leq (n - 1)/2 + k - 1$, a contradiction. Hence $\deg_G(v_{d-2}) \geq 3$. Let $G' = G - \{v_{d-1}, v_d\}$. By Theorem 14, $\deg_{G'}(G') \leq \left(\frac{n(G') - 1}{2}\right)/2 + k$. Suppose that $\deg_{G'}(G') \leq \left(\frac{n(G') - 1}{2}\right)/2 + k$. Let $S'$ be a $\deg_{G'}(G')$-set. If $v_{d-2} \in S'$, then $S' \cup \{v_{d-1}\}$ is a 1FD-set for $G$ and if $v_{d-2} \notin S'$, then $S' \cup \{v_d\}$ is a 1FD-set for $G$. Thus $\deg_{G'}(G) \leq |S'| + 1 \leq \deg_{G'}(G') + 1 < (n - 1)/2 + k$, a contradiction. We deduce that $\deg_{G'}(G') = \left(\frac{n(G') - 1}{2}\right)/2 + k$. By the choice of $G$, $G' \in \mathcal{G}_k$. Since $d \geq 3$, $v_{d-2}$ is not a special vertex of $G'$. Thus $G$ is obtained from $G'$ by Operation $\mathcal{O}_1$, and so $G \in \mathcal{G}_k$.

**Case 2.** $d = 2$. As noted, $\deg(v_1) = 2$. Clearly $\deg(v_0) \geq 3$. Assume first that $\deg(v_0) \geq 4$. Let $G' = G - \{v_2, v_1\}$. By Theorem 14, $\deg_{G'}(G') \leq \left(\frac{n(G') - 1}{2}\right)/2 + k$. Suppose that $\deg_{G'}(G') < \left(\frac{n(G') - 1}{2}\right)/2 + k$. Let $S'$ be a $\deg_{G'}(G')$-set. If $v_0 \in S'$, then $S' \cup \{v_1\}$ is a 1FD-set for $G$, and if $v_0 \notin S'$, then $S' \cup \{v_2\}$ is a 1FD-set for $G$. Thus $\deg_{G'}(G) \leq |S'| + 1 < (n - 1)/2 + k$, a contradiction. Hence, $\deg_{G'}(v_0) = \left(\frac{n(G') - 1}{2}\right)/2 + k$. By the choice of $G$, $G' \in \mathcal{G}_k$. Since $\deg_{G'}(v_0) \geq 3$, $v_0$ is not a special vertex of $G'$. Hence $G$ is obtained from $G'$ by Operation $\mathcal{O}_1$. Consequently, $G \in \mathcal{G}_k$. Thus assume that $\deg(v_0) = 3$. Let $G' = G - \{v_2, v_1\}$. By Theorem 14, $\deg_{G'}(G') \leq \left(\frac{n(G') - 1}{2}\right)/2 + k$. Suppose
that $fd_1(G') < \frac{(n(G') - 1)}{2} + k$. Let $S'$ be a $fd_1(G')$-set. If $v_0 \in S'$, then $S' \cup \{v_1\}$ is a 1FD-set for $G$, and if $v_0 \notin S'$, then $S' \cup \{v_2\}$ is a 1FD-set for $G$. Thus $fd_1(G) \leq |S'| + 1 \leq fd_1(G') + 1 < (n-1)/2 + k$, a contradiction. Thus we obtain that $fd_1(G') = \frac{(n(G') - 1)}{2} + k$. By the choice of $G$, $G' \in \mathcal{G}_k$. Then $v_0$ is a special vertex of $G'$. From Observation 9(1), we obtain that $deg_G(c_i) \geq 3$ for each $i \in \{1, \ldots, r\}$.

Suppose that $N_G(c_j) \setminus V(C_1)$ contains no strong support vertex for each $j \in \{1, \ldots, r\}$. Observation 9(1) implies that $c_j$ is not a strong support vertex of $G$, since $G' \in \mathcal{G}_k$. Assume that there is a vertex $c_j \in \{c_1, \ldots, c_r\}$ such that $c_j$ has a neighbor $a$ which is a support vertex. By assumption, $a$ is a weak support vertex. If $a'$ is the leaf adjacent to $a$, then $a'$ plays the role of $v_d$. Since $deg(v_0) = 3$, we may assume that $deg(c_j) = 3$. Thus by Observation 9(1), we may assume that $deg_G(c_i) = 3$ for each $c_i \in \{c_1, \ldots, c_r\}$. Let $F = \bigcup_{i=1}^{r} (N[c_i]) \setminus \{c_0, \ldots, c_r\}$. Clearly $|F| = r$, since $deg_G(c_i) = 3$ for each $c_i \in \{c_1, \ldots, c_r\}$. Let $F = \{u_1, u_2, \ldots, u_r\}$, $F' = \{u_i \in F | \deg_G(u_i) = 1\}$, and $F'' = F \setminus F'$. Then every vertex of $F''$ is a weak support vertex. Since $v_1 \in F''$, $|F''| \geq 1$. Now let $G^* = G - c_0c_1 - c_0c_r$, and $G'^*_1$ and $G'^*_2$ be the components of $G^*$, where $c_1 \in V(G'^*_1)$. By Theorem 14, $fd_1(G'^*_2) \leq \frac{(n(G'^*_2) - 1)}{2} + k - 1$. Clearly $n(G'^*_2) = n(G) - 2r - |F''|$. Let $S'^*_2$ be a $fd_1(G'^*_2)$-set. If $c_0 \notin S'^*_2$, then $S'^*_2 \cup F$ is a 1FD-set for $G$, and so $fd_1(G) \leq \frac{(n(G) - 2r - |F''| - 1)}{2} + k - 1 < (n-1)/2 + k$, a contradiction. Thus $c_0 \notin S'^*_2$. If $|F''| = 1$, then $S'^*_2 \cup C_1 \cup \{v_1\}$ is a 1FD-set for $G$ and thus $fd_1(G) \leq fd_1(G'^*_2) + r + 1 \leq \frac{(n-2)}{2} + k$, a contradiction. Thus assume that $|F''| \geq 2$. Let $\{u_t, u_t'\} \subseteq F''$ (assume without loss of generality that $t < t'$) such that $deg_G(u_i) = 1$ for $1 \leq i < t$ and $t' < i < r$. Let $u_t'$ and $u_t''$ be the leaves of $u_t$ and $u_t'$, respectively. Clearly $S'^*_2 \cup \{c_1, \ldots, c_t-1\} \cup \{c_{t+1}, \ldots, c_r\} \cup \{u_{t+1}, \ldots, u_{t-1}\} \cup \{u_t', u_t''\}$ is a 1FD-set for $G$ and thus $fd_1(G) \leq fd_1(G'^*_2) + r < (n-1)/2 + k - 1$, a contradiction.

Thus we may assume that $N(c_j) \setminus C_1$ contains at least one strong support vertex for some $c_j \in \{c_1, \ldots, c_r\}$. Let $u_j$ be a strong support vertex in $N(c_j) \setminus C_1$. By Claim 2, there are precisely two leaves adjacent to $u_j$. Let $u'$ and $u''$ be the leaves adjacent to $u_j$, and $G^* = G - \{u', u''\}$. By Theorem 14, $fd_1(G^*) \leq \frac{(n(G^*) - 1)}{2} + k$. Assume that $fd_1(G^*) < \frac{(n(G^*) - 1)}{2} + k$. Let $S'$ be a $fd_1(G^*)$-set. If $u_j \in S'$, then $S'$ is a 1FD-set for $G$, and if $u_j \notin S'$, then $S' \cup \{u_j\}$ is a 1FD-set for $G$. Thus $fd_1(G) \leq fd_1(G^*) + 1 < (n-1)/2 + k$, a contradiction. We deduce that $fd_1(G^*) = \frac{(n(G^*) - 1)}{2} + k$. By the choice of $G$, $G^* \in \mathcal{G}_k$. Thus $G$ is obtained from $G^*$ by Operation $O_2$. Consequently, $G \in \mathcal{G}_k$.

By Claim 3, we assume that $d = d' = 1$.

Claim 4. $C_i$ has precisely one special vertex, for $i = 1, 2$.

**Proof.** We first show that $C_i$ has at least one special vertex, for $i = 1, 2$. Suppose that $C_1$ has no special vertex. Thus $deg_G(c_i) \geq 3$ for $i = 1, \ldots, r$. Clearly, $c_i$ is a
support vertex for \( i = 1, 2, \ldots, r \). Suppose that \( c_j \) is a strong support vertex for some \( j \in \{1, 2, \ldots, r\} \). Let \( G' \) be obtained from \( G \) by removal of all vertices in \( \bigcup_{i=1}^{r} (N[c_i]) \setminus \{c_0, c_1, \ldots, c_r\} \). Clearly, \( c_0 \) is a strong support vertex of \( G' \). By Theorem 14, \( fd_1(G') \leq (n(G') - 1)/2 + k - 1 \). Since \( c_j \) is a strong support vertex of \( G \), we have \( n(G') \leq n(G) - (2r + 1) + 2 \). Thus, \( fd_1(G') \leq (n(G) - (2r + 1) + 2 - 1)/2 + k - 1 \).

By Observation 1, \( c_0 \in S' \), and so \( S' \cup \{c_1, \ldots, c_r\} \) is a 1FD-set in \( G \) of cardinality at most \( (n(G) - (2r + 1) + 2 - 1)/2 + k - 1 + r = (n(G) - k - 1) < (n(G) - 1)/2 + k \), a contradiction. Thus \( c_i \) is a weak support vertex for each \( i = 1, 2, \ldots, r \). Let \( G' \) be obtained from \( G \) by removal of any vertex in \( \bigcup_{i=1}^{r} (N[c_i]) \setminus \{c_0\} \). By Theorem 14, \( fd_1(G') \leq (n(G') - 1)/2 + k - 1 \). Let \( S' \) be a \( fd_1(G') \)-set. If \( c_0 \notin S' \), then \( S' \cup \{u_1, \ldots, u_r\} \) is a 1FD-set in \( G \) of cardinality at most \( (n(G) - 1)/2 + k - 1 < (n(G) - 1)/2 + k \), where \( u_i \) is the leaf adjacent to \( c_i \) for \( i = 1, 2, \ldots, r \). This is a contradiction. Thus \( c_0 \in S' \). Then \( S' \cup \{c_1, \ldots, c_r\} \) is a 1FD-set in \( G \) of cardinality at most \( (n(G) - 1)/2 + k - 1 < (n(G) - 1)/2 + k \), a contradiction.

Thus \( C_1 \) has at least one special vertex. Similarly, \( C_2 \) has at least one special vertex. Let \( c_1 \) be a special vertex of \( C_1 \) and \( c'_h \) be a special vertex of \( C_2 \).

We show that \( c_1 \) is the unique special vertex of \( C_1 \). Suppose to the contrary that \( C_1 \) has at least two special vertices. Assume that \( \deg_G(c'_{h+1}) \geq 2 \). Let \( G' = G - c'_{h+1}, \) and \( S' \) be a \( fd_1(G') \)-set. By Theorem 14, \( fd_1(G') \leq (n(G') - 1)/2 + k - 1 \). If \( fd_1(G') = (n(G') - 1)/2 + k - 1 \), then by the inductive hypothesis, \( G' \in \mathcal{G}_{k-1} \). This is a contradiction by Observation 9(1), since \( C_1 \) has at least two special vertices. Thus \( fd_1(G') < (n(G') - 1)/2 + k - 1 \). If \( |S' \cap \{c'_h, c'_{h+1}\}| \in \{0, 2\} \), then \( S' \) is a 1FD-set in \( G \) of cardinality at most \( (n(G) - 1)/2 + k - 1 \), a contradiction. Thus \( |S' \setminus \{c'_h, c'_{h+1}\}| = 1 \). Without loss of generality, assume that \( c'_h \in S' \). Then \( \{c'_{h+1}\} \cup S' \) is a 1FD-set in \( G \), and so \( fd_1(G) < (n(G) - 1)/2 + k \), a contradiction. We thus assume that \( \deg_G(c'_{h+1}) \geq 3 \). Likewise, we may assume that \( \deg_G(c'_{h-1}) \geq 3 \). Since \( C_2 \) is a leaf-cycle, \( c'_h \) is its unique special cut-vertex. Thus we may assume, without loss of generality, that \( c'_{h-1} \neq c'_h \). Clearly, \( c'_{h+1} \) is a support vertex of \( G \). Let \( G' = G - c'_{h-1}, \) and \( S' \) be a \( fd_1(G') \)-set. Clearly \( c'_{h+1} \) is a strong support vertex of \( G' \). By Theorem 14, \( fd_1(G') \leq (n(G') - 1)/2 + k - 1 \). If \( fd_1(G') = (n(G') - 1)/2 + k - 1 \), then by the inductive hypothesis \( G' \in \mathcal{G}_{k-1} \). This is a contradiction by Observation 9(1), since \( C_1 \) has at least two special vertices. Thus \( fd_1(G') < (n(G') - 1)/2 + k - 1 \). By Observation 1, \( c'_{h+1} \in S' \). If \( c'_{h-1} \notin S' \), then \( S' \) is a 1FD-set in \( G \) of cardinality at most \( (n(G) - 1)/2 + k - 1 \), a contradiction. Thus \( c'_{h-1} \in S' \). Now, \( S' \setminus \{c'_h\} \) is a 1FD-set in \( G \), and thus \( fd_1(G) \leq |S'| + 1 < (n(G) - 1)/2 + k \), a contradiction. Thus \( c_1 \) is the unique special vertex of \( C_1 \). Similarly, \( c'_h \) is the unique special vertex of \( C_2 \).

Let \( c_1 \) be the unique special vertex of \( C_1 \), and \( c'_h \) be the unique special vertex of \( C_2 \), and note that Claim 4 guarantees the existence of \( c_1 \) and \( c'_h \).

**Claim 5.** No vertex of \( C_i \) is a strong support vertex, for \( i = 1, 2 \).
\textbf{Proof.} Suppose that \( c_j \in C_1 \) is a strong support vertex. Since \( C_2 \) is a leaf-cycle, \( c_0^{'} \) is its unique special cut-vertex. Thus, we may assume, without loss of generality, that \( c_{h+1}^{'} \) is a support vertex of \( G \). Let \( G' = G - c_{h+1}^{'} c_{h-1}^{'} \), and \( S' \) be a \( fd(G') \)-set. Clearly \( c_{h+1}^{'} \) is a strong support vertex of \( G' \). By Theorem 14, \( fd(G') \leq (n(G') - 1)/2 + k - 1 \). If \( fd(G') = (n(G') - 1)/2 + k - 1 \), then by the inductive hypothesis \( G' \in \mathcal{G}_{k-1} \). This is a contradiction by Observation 9(1), since \( C_1 \) has a strong support vertex. Thus \( fd(G') < (n(G') - 1)/2 + k - 1 \). By Observation 1, \( c_{h+1}^{'} \in S' \). If \( c_{h+1}^{'} \notin S' \), then \( S' \) is a \( 1FD \)-set in \( G \) at most \( n(G) - 1 \)/2 + k - 1, a contradiction. Thus \( c_{h-1}^{'} \in S' \). Then \( S' \cup \{ c_h^{'} \} \) is a \( 1FD \)-set in \( G \), and so \( fd(G) \leq |S'| + 1 < (n(G) - 1)/2 + k \), a contradiction.

We deduce that \( C_1 \) has no strong support vertex. Similarly, \( C_2 \) has no strong support vertex.

\[ \square \]

We deduce that \( c_i \) is a weak support vertex for each \( i \in \{1, 2, \ldots, r\} \setminus \{t\} \), and similarly \( c_i^{'} \) is a weak support vertex for each \( i \in \{1, 2, \ldots, r\} \setminus \{h\} \). For each \( i \in \{1, 2, \ldots, r\} \setminus \{t, h\} \), let \( u_i \) be the leaf adjacent to \( c_i \).

Let \( G_2^{'} \) be the component of \( G - c_0 c_1 - c_0 c_r \) that contains \( c_0 \), and \( G^* \) be a graph obtained from \( G'_2 \) by adding a leaf \( v^* \) to \( c_0 \). Clearly \( n(G^*) = n(G) - 2r + 2 \). By Theorem 14, \( fd(G^*) \leq (n(G^*) - 1)/2 + k - 1 \). Suppose that \( fd(G^*) < (n(G^*) - 1)/2 + k - 1 \). Let \( S' \) be a \( fd(G^*) \)-set. If \( c_0 \in S' \), then \( S' \cup \{c_1, c_2, \ldots, c_r\} \) is a \( 1FD \)-set in \( G \), so we obtain that \( fd(G) < (n(G) - 1)/2 + k \), a contradiction. Thus \( c_0 \notin S' \). Then \( v^* \in S^* \). If \( t > 1 \), then \( S^* \cup \{c_1, \ldots, c_{t-1}\} \cup \{u_{t+1}, \ldots, u_r\} \setminus \{v^*\} \) is a \( 1FD \)-set in \( G \) of cardinality at most \( (n(G^*) - 1)/2 + k - 1 + r - 1 = (n(G) - 2r + 2 - 1)/2 + k - 1 + r - 1 = (n(G) - 1)/2 + k - 2 \), a contradiction. Thus assume that \( t = 1 \). Then \( S^* \cup \{c_2, \ldots, c_r\} \setminus \{v^*\} \), is a \( 1FD \)-set in \( G \) of cardinality at most \( (n(G^*) - 1)/2 + k - 2 \), a contradiction. Thus \( fd(G^*) = (n(G^*) - 1)/2 + k - 1 \). By the inductive hypothesis, \( G^* \in \mathcal{G}_{k-1} \). Let \( H^* \) be the graph obtained from \( G'[\{c_0, c_1, \ldots, c_r, u_1, \ldots, u_{t-1}, u_{t+1}, \ldots, u_r\}] \) by adding a leaf to \( c_0 \). Clearly \( H^* \in \mathcal{H}_1 \). Thus \( G \) is obtained from \( G^* \in \mathcal{G}_{k-1} \) and \( H^* \in \mathcal{H}_1 \) by Procedure A. Consequently, \( G \in \mathcal{H}_k \subseteq \mathcal{G}_k \).

For the converse, by Corollary 13, \( V(G) \setminus L(G) \) is the unique \( fd(G) \)-set. Now Observation 9 implies that \( fd(G) = (n-1)/2 + k \).

\[ \blacksquare \]

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References


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