ON THE CO-ROMAN DOMINATION IN GRAPHS

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Abstract

Let $G = (V, E)$ be a graph and let $f : V(G) \rightarrow \{0, 1, 2\}$ be a function. A vertex $v$ is said to be protected with respect to $f$, if $f(v) > 0$ or $f(v) = 0$ and $v$ is adjacent to a vertex of positive weight. The function $f$ is a co-Roman dominating function if (i) every vertex in $V$ is protected, and (ii) each $v \in V$ with positive weight has a neighbor $u \in V$ with $f(u) = 0$ such that the function $f_{uv} : V \rightarrow \{0, 1, 2\}$, defined by $f_{uv}(u) = 1$, $f_{uv}(v) = f(v) - 1$ and $f_{uv}(x) = f(x)$ for $x \in V \setminus \{v, u\}$, has no unprotected vertex. The weight of $f$ is $\omega(f) = \sum_{v \in V} f(v)$. The co-Roman domination number of a graph $G$, denoted by $\gamma_{cr}(G)$, is the minimum weight of a co-Roman dominating function on $G$. In this paper, we give a characterization of graphs of order $n$ for which co-Roman domination number is $\frac{2n}{3}$ or $n - 2$, which settles

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For terminology and notation on graph theory not given here, the reader is referred to [9, 10]. In this paper, $G$ is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of $G$ is denoted by $n = n(G)$. For every vertex $v \in V$, the open neighborhood $N(v)$ is the set $\{u \in V \mid uv \in E\}$ and the closed neighborhood of $v$ is the set $N[v] = N(v) \cup \{v\}$. The degree of a vertex $v \in V$ is $\deg_G(v) = \deg(v) = |N(v)|$. A universal vertex is a vertex that is adjacent to all other vertices of $G$. The open neighborhood of a set $S \subseteq V$ is the set $N(S) = \bigcup_{v \in S} N(v)$, and the closed neighborhood of $S$ is the set $N[S] = N(S) \cup S$.

For a set $S \subseteq V(G)$ and a vertex $v \in S$, the private neighborhood of $v$ with respect to $S$ is the set $pn(v; S) = \{u \in N(v), N(u) \cap S = \{v\}\}$. A leaf is a vertex of degree one, and a support vertex is a vertex adjacent to a leaf. We also denote by $L_v$ the set of all leaves adjacent to a support vertex $v$. For a vertex $v$ in a rooted tree $T$, let $D(v)$ denote the set of descendants of $v$ and $D[v] = D(v) \cup \{v\}$. The maximal subtree at $v$ is the subtree of $T$ induced by $D[v]$, and is denoted by $T_v$. A subdivision of an edge $uv$ is obtained by replacing the edge $uv$ with a path $uwv$, where $w$ is a new vertex. The subdivision graph $S(G)$ is the graph obtained from $G$ by subdividing each edge of $G$. The subdivision star $S(K_{1,t})$ for $t \geq 2$, is called a healthy spider. We write $P_n$ for a path of length $n - 1$ and $K_{1,n}$ for a star. For integers $r \geq s \geq 1$, the double star $DS(r,s)$ is the tree obtained by connecting the centers of two stars $K_{1,r}$ and $K_{1,s}$ with an edge. The diameter of $G$, denoted by $diam(G)$, is the maximum value among minimum distances between all pairs of vertices of $G$. For a subset $S$ of vertices of $G$, we denote by $G[S]$ the subgraph induced by $S$. For a subset $S \subseteq V(G)$ of vertices of a graph $G$ and a function $f : V(G) \to R$, we define $f(S) = \sum_{x \in S} f(x)$. For a function $f : V(G) \to \{0, 1, 2\}$, let $V_i = \{v \in V \mid f(v) = i\}$ for $i = 0, 1, 2$. Since these three sets determine $f$, we can equivalently write $f = (V_0, V_1, V_2)$ (or $f = (V_0^f, V_1^f, V_2^f)$) to refer $f$. We note that $\omega(f) = |V_1| + 2|V_2|$.

A Roman dominating function on a graph $G$, abbreviated RD-function, is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. The weight, $\omega(f)$, of $f$ is defined as $\omega(f) = \sum_{v \in V} f(v)$. The Roman domination number, denoted by $\gamma_R(G)$. An RD-function with minimum weight $\gamma_R(G)$ in $G$ is called a $\gamma_R(G)$-function. The definition of the Roman dominating function
was given multiplicity by Steward [14] and ReVelle and Rosing [13]. Roman domination is now well studied in graph theory [1,3–7,15].

Let \( f : V(G) \to \{0,1,2\} \) be a function. A vertex \( v \) is said to be protected with respect to \( f \), if \( f(v) > 0 \) or \( f(v) = 0 \) and \( v \) is adjacent to a vertex of positive weight. The function \( f \) is a weak Roman dominating function if for every vertex \( u \) with \( f(u) = 0 \) there exists a vertex \( v \) adjacent to \( u \) such that \( f(v) \in \{1,2\} \) and the function \( f_{uv} : V \to \{0,1,2\} \), defined by \( f_{uv}(u) = 1 \), \( f_{uv}(v) = f(v) - 1 \) and \( f_{uv}(x) = f(x) \) for \( x \in V \setminus \{v,u\} \), has no unprotected vertex. The weak Roman domination number of a graph \( G \), denoted by \( \gamma_{cr}(G) \), is the minimum weight among all weak Roman dominating functions on \( G \). The weak Roman domination number was introduced by Henning and Hedetniemi in [11].

The function \( f : V(G) \to \{0,1,2\} \) is a co-Roman dominating function, abbreviated CRDF if (i) every vertex in \( V \) is protected, and (ii) each \( v \in V \) with positive weight has a neighbor \( u \in V \) with \( f(u) = 0 \) such that the function \( f_{uv} : V \to \{0,1,2\} \), defined by \( f_{uv}(u) = 1 \), \( f_{uv}(v) = f(v) - 1 \) and \( f_{uv}(x) = f(x) \) for \( x \in V \setminus \{v,u\} \), has no unprotected vertex. The weight of \( f \) is \( \omega(f) = \sum_{v \in V} f(v) \). The co-Roman domination number of a graph \( G \), denoted by \( \gamma_{cr}(G) \), is the minimum weight of a co-Roman dominating function on \( G \). It follows from the definitions that for any connected graph \( G \) of order \( n \geq 2 \),

\[
\gamma_{cr}(G) \leq n - 1. \tag{1}
\]

The co-Roman domination in graphs was investigated by Arumugam et al. in [2]. The proof of the next results can be found in [2].

**Proposition 1.** If \( H \) is a spanning subgraph of a graph \( G \), then \( \gamma_{cr}(G) \leq \gamma_{cr}(H) \).

**Proposition 2.** For \( n \geq 2 \), \( \gamma_{cr}(K_{1,n}) = 2 \).

**Proposition 3.** For \( n \geq 4 \), \( \gamma_{cr}(P_n) = \gamma_{cr}(C_n) = \lceil \frac{2n}{3} \rceil \).

**Proposition 4.** For every tree \( T \) of order \( n \geq 2 \), \( \gamma_{cr}(T) \leq \frac{2n}{3} \).

The next result is an immediate consequence of Propositions 1 and 4.

**Corollary 5.** For every connected graph \( G \) of order \( n \geq 2 \), \( \gamma_{cr}(G) \leq \frac{2n}{3} \).

**Observation 6.** Let \( G \) be a graph of order \( n \geq 2 \). Then \( \gamma_{cr}(G) = 1 \) if and only if \( G \) has two vertices of degree \( n - 1 \).

**Theorem 7.** For every graph \( G \), \( \gamma_{cr}(G) \leq \gamma_{r}(G) \).

In [2], the authors posed the following open problems.

**Problem 1.** Characterize graphs \( G \) of order \( n \) such that \( \gamma_{cr}(G) = n - 2 \).

**Problem 2.** Characterize trees \( T \) of order \( n \) such that \( \gamma_{cr}(T) = \frac{2n(T)}{3} \).

**Problem 3.** Characterize graphs \( G \) such that \( \gamma_{cr}(T) = \gamma(G) \).

In this paper, we settle the above open problems. Furthermore, we establish some sharp bounds on the co-Roman domination number.
1. Graphs $G$ with $\gamma_{cr}(G) = \gamma_r(G)$ or $\gamma_{cr}(G) = \gamma(G)$

In this section, we study the properties of graphs $G$ for which $\gamma_{cr}(G) = \gamma_r(G)$ or $\gamma_{cr}(G) = \gamma(G)$.

Proposition 8. Let $G$ be a connected graph of order at least two. Then $\gamma_{cr}(G) = \gamma_r(G)$ if and only if there exists a $\gamma_{cr}(G)$-function $f = (V_0, V_1, V_2)$ such that each vertex $x \in V_0$, either has a neighbor $x'$ in $V_2$ or has a neighbor $x'$ in $V_1$ for which $pn(x', V_1 \cup V_2) \subseteq N[x]$.

Proof. If there exists a $\gamma_{cr}(G)$-function $f = (V_0, V_1, V_2)$ such that each vertex $x \in V_0$, has a neighbor $x' \in V_1 \cup V_2$ with $pn(x', V_1 \cup V_2) \subseteq N[x]$, then clearly $f$ is a weak Roman dominating function of $G$ and so $\gamma_r(G) \leq \gamma_{cr}(G)$. It follows from Theorem 7 that $\gamma_r(G) = \gamma_{cr}(G)$.

Conversely, let $\gamma_r(G) = \gamma_{cr}(G)$. There exists a $\gamma_r(G)$-function $f = (V_0, V_1, V_2)$ such that $f$ is a co-Roman dominating function of $G$ (see Theorem 3.3 of [2]). By assumption, $f$ is a $\gamma_{cr}(G)$-function. Assume $x \in V_0$ is an arbitrary vertex. Since $f$ is a weak Roman dominating function, $x$ has a neighbor $x'$ in $V_1 \cup V_2$ such that the function $g : V(G) \rightarrow \{0, 1, 2\}$ defined by $g(x) = 1, g(x') = f(x') - 1$ and $g(u) = f(u)$ otherwise, is safe. If $x$ has a neighbor in $V_2$, then we are done. Assume $x$ has no neighbor in $V_2$. It follows that $x' \in V_1$. Since $f$ is safe, we must have $pn(x', V_1 \cup V_2) \subseteq N[x]$ and the proof is complete.

Proposition 9. Let $G$ be a connected graph of order at least two. Then $\gamma(G) = \gamma_{cr}(G)$ if and only if there exists a $\gamma(G)$-set $S$ such that each vertex $x \in S$ has a neighbor $x' \in V \setminus S$ with $pn(x, S) \subseteq N[x']$.

Proof. Let $\gamma(G) = \gamma_{cr}(G)$. Assume $f = (V_0, V_1, V_2)$ is a $\gamma_{cr}(G)$-function. Since $V_1 \cup V_2$ is a dominating set, we deduce from $\gamma(G) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = \gamma_{cr}(G)$ that $V_2 = \emptyset$ and $V_1$ is a $\gamma(G)$-set. Let $x \in V_1$ be an arbitrary vertex. Since $f$ is a co-Roman dominating function, there is a vertex $x' \in V_0 \cap N(x)$ such that $\{(V_0 \setminus \{x'\}) \cup \{x\}, (V_1 \setminus \{x\}) \cup \{x'\}$, $\emptyset\}$ is a $\gamma_{cr}(G)$-function. It follows that $\gamma(G)$-set and this implies that $pn(x, V_1) \subseteq N[x']$.

Conversely, let $S$ be a $\gamma(G)$-set such that each vertex $x \in S$ has a neighbor $x' \in V \setminus S$ with $pn(x, S) \subseteq N[x']$. Then the function $f = (V(G) \setminus S, S, \emptyset)$ is clearly a co-Roman dominating function of weight $\gamma(G)$ and so $\gamma_{cr}(G) \leq \gamma(G)$. It follows that $\gamma_{cr}(G) = \gamma(G)$.

Corollary 10. Let $G$ be a connected graph of order at least two with $\gamma(G) = \gamma_{cr}(G)$. Then for any $\gamma_{cr}(G)$-function $f = (V_0, V_1, V_2)$, $V_2 = \emptyset$.

Corollary 11. Let $G$ be a connected graph of order at least two. If $\gamma(G) = \gamma_{cr}(G)$, then $G$ has no strong support vertex.
For a tree $\mathbf{T}$, let $M(\mathbf{T}) = \{v \mid \text{there exists a } \gamma_{cr}(\mathbf{T}) \text{-function } f \text{ such that } f(v) = 1\}$. In what follows, we present a constructive characterization of trees $T$ with $\gamma(T) = \gamma_{cr}(T)$. In order to do this, we define a family of trees as follows. Let $\mathcal{T}$ be the collection of trees $T$ that can be obtained from a sequence $T_1, T_2, \ldots, T_k$ of trees for some $k \geq 1$, where $T_1$ is a $P_2$ and $T = T_k$. If $k \geq 2$, $T_{i+1}$ can be obtained from $T_i$ by one of the following three operations. Let one vertex of $P_2$ be considered as a support vertex.

Operation $\mathcal{O}_1$. If $v \in M(T_i)$, then the tree $T_{i+1}$ is obtained from $T_i$ by adding a pendant $P_3 = xyz$ and adding the edge $vx$ (see Figure 1(a)).

Operation $\mathcal{O}_2$. If $v$ is a support vertex of $T_i$, then the tree $T_{i+1}$ is obtained from $T_i$ by adding a pendant $P_2 = xy$ and adding the edge $vx$ (see Figure 1(b)).

Operation $\mathcal{O}_3$. If $v \in T_i$, then the tree $T_{i+1}$ is obtained from $T_i$ by adding a healthy spider with at least two feet headed at $x$ and adding the edge $vx$ (see Figure 1(c)).

Figure 1. (a) Operation $\mathcal{O}_1$. (b) Operation $\mathcal{O}_2$. (c) Operation $\mathcal{O}_3$.

**Lemma 12.** If $T_i$ is a tree with $\gamma(T_i) = \gamma_{cr}(T_i)$ and $T_{i+1}$ is a tree obtained from $T_i$ by Operation $\mathcal{O}_1$, then $\gamma(T_{i+1}) = \gamma_{cr}(T_{i+1})$.

**Proof.** Let $f$ be a $\gamma_{cr}(T_i)$-function and $v$ a vertex of $T_i$ with $f(v) = 1$. Then the function $f' : V(T_{i+1}) \to \{0, 1, 2\}$ by $f'(y) = 1$, $f'(x) = f'(z) = 0$ and $f'(u) = f(u)$ for $u \in V(T_i)$, is a co-Roman dominating function on $T_{i+1}$ and so $\gamma_{cr}(T_{i+1}) \leq \gamma_{cr}(T_i) + 1$.

It is easy to see that $\gamma(T_{i+1}) = \gamma(T_i) + 1$. Now we have

$$\gamma(T_i) + 1 = \gamma(T_{i+1}) \leq \gamma_{cr}(T_{i+1}) \leq \gamma_{cr}(T_i) + 1 = \gamma(T_i) + 1$$

yielding $\gamma(T_{i+1}) = \gamma_{cr}(T_{i+1})$.

**Lemma 13.** If $T_i$ is a tree with $\gamma(T_i) = \gamma_{cr}(T_i)$ and $T_{i+1}$ is a tree obtained from $T_i$ by Operation $\mathcal{O}_2$, then $\gamma(T_{i+1}) = \gamma_{cr}(T_{i+1})$.
Proof. Clearly, any \( \gamma_{cr}(T_i) \)-function can be extended to a co-Roman dominating function by assigning 1 to \( x \) and 0 to \( y \) implying that \( \gamma_{cr}(T_{i+1}) \leq \gamma_{cr}(T_i) + 1 \).

Since \( v \) is a support vertex, one can easily check that \( \gamma(T_{i+1}) = \gamma(T_i) + 1 \). Now the result follows as in the proof of Lemma 12.

Lemma 14. If \( T_i \) is a tree with \( \gamma(T_i) = \gamma_{cr}(T_i) \) and \( T_{i+1} \) is a tree obtained from \( T_i \) by Operation \( O_3 \), then \( \gamma(T_{i+1}) = \gamma_{cr}(T_{i+1}) \).

Proof. Let the added spider has exactly \( k \) feet. Obviously, any \( \gamma_{cr}(T_i) \)-function can be extended to a co-Roman dominating function by assigning 1 to the support vertices of spider and 0 to the remaining vertices of spider and this implies that \( \gamma_{cr}(T_{i+1}) \leq \gamma_{cr}(T_i) + k \). Moreover, it is easy to verify that \( \gamma(T_{i+1}) = \gamma(T_i) + k \) and the result follows as in the proof of Lemma 12.

Lemma 15. If \( T \in \mathcal{T} \), then \( \gamma(T) = \gamma_{cr}(T) \).

Proof. Let \( T \in \mathcal{T} \). By definition, there exists a sequence of trees \( T_1, T_2, \ldots, T_k \) \( (k \geq 1) \) such that \( T_1 = K_2 \), and if \( k \geq 2 \), \( T_{i+1} \) can be obtained recursively from \( T_i \) by Operation \( O_1 \), \( O_2 \) or \( O_3 \) for \( i = 1, 2, \ldots, k-1 \). We proceed by induction on \( k \).

If \( T = K_2 \), then clearly \( \gamma(T) = \gamma_{cr}(T) = 1 \). Suppose \( k \geq 2 \) and the result holds for each tree \( T \in \mathcal{T} \) which can be obtained from a sequence of operations of length \( k-1 \) and let \( T' = T_{k-1} \). By the induction hypothesis, we have \( \gamma(T') = \gamma_{cr}(T') \).

Since \( T = T_k \) is obtained from \( T' \) by one of the Operations \( O_1 \), \( O_2 \) or \( O_3 \) from \( T' \), we have \( \gamma(T) = \gamma_{cr}(T) \) by Lemmas 12, 13 and 14.

Theorem 16. Let \( T \) be a tree of order \( n \geq 2 \). Then \( \gamma(T) = \gamma_{cr}(T) \) if and only if \( T \in \mathcal{T} \).

Proof. The sufficiency follows from Lemma 15. We use induction on \( n \) to prove the necessity. If \( n = 2 \), then \( T = P_2 \) that belongs to \( \mathcal{T} \). Assume \( n \geq 3 \) and that the result holds for any tree of order less than \( n \). Let \( T' \) be a tree of order \( n \) with \( \gamma(T) = \gamma_{cr}(T) \). Let \( P = v_1v_2 \cdots v_\ell \) be a diametral path in \( T \) and root \( T \) at \( v_\ell \).

By Corollary 11, we have \( d(v_2) = 2 \). Consider the following cases.

Case 1. \( v_3 \) is a support vertex. Let \( w \) be a leaf adjacent to \( v_3 \) and let \( T'' = T - \{v_1, v_2\} \). If \( f \) is a \( \gamma_{cr}(T) \)-function, then clearly \( f(v_1) + f(v_2) \geq 1 \) and \( f(v_3) + f(w) \geq 2 \). It is easy to verify that the function \( f \), restricted to \( T'' \) is a co-Roman dominating function implying that \( \gamma_{cr}(T) \geq \gamma_{cr}(T'') + 1 \). Clearly \( \gamma(T) = \gamma(T'') + 1 \), and we deduce from

\[
\gamma(T) = \gamma_{cr}(T) \geq \gamma_{cr}(T'') + 1 = \gamma(T'') + 1 = \gamma(T)
\]

that \( \gamma_{cr}(T'') = \gamma(T') \). By the induction hypothesis, we have \( T' \in \mathcal{T} \). Now \( T \) can be obtained from \( T' \) by Operation \( O_2 \).
Case 2. $d(v_3) = 2$. Let $T' = T - \{v_1, v_2, v_3\}$. By Proposition 3, $n \geq 4$. Clearly $\gamma(T) = \gamma(T') + 1$. Assume $f = (V_0, V_1, V_2)$ is a $\gamma_{cr}(T)$-function. By Corollary 10, $V_2 = \emptyset$. Clearly $f(v_1) + f(v_2) = 1$ and $f(v_3) + f(v_4) \geq 1$. If $f(v_3) = f(v_4) = 1$, then the function $g : V(T) \to \{0, 1, 2\}$ defined by $g(v_4) = g(v_2) = 1$, $g(v_1) = g(v_3) = 0$ and $g(x) = f(x)$ otherwise, is a co-Roman dominating function of $T$ of weight less than $\omega(f)$ which is a contradiction. Hence $f(v_3) = 0$ or $f(v_4) = 0$ and so $f(v_3) + f(v_4) = 1$. Consider the following.

- $f(v_3) = 1$ and $f(v_4) = 0$. If $f(x) = 1$ for some $x \in N_{T'}(v_4)$, then the function $g : V(G) \to \{0, 1\}$ defined by $g(v_2) = 1, g(v_1) = g(v_3) = 0$ and $g(x) = f(x)$ otherwise, is a dominating function of $T$ of weight less than $\omega(f)$ which contradicts the induction hypothesis, we have $T' \in \mathcal{T}$ and so $T$ can be obtained from $T'$ by Operation $O_1$. Thus $T \in \mathcal{T}$.
- $f(v_3) = 0$ and $f(v_4) = 1$. As above we have $f(x) = 0$ for some $x \in N_{T'}(v_4)$. Then the function $f$ restricted to $T'$ is a co-Roman dominating function of $T'$ and so $\gamma_{cr}(T) \geq \gamma_{cr}(T') + 1$. Using above argument, we obtain $T \in \mathcal{T}$.

Case 3. $v_3$ is not a support vertex and $d(v_3) \geq 3$. Let $T'$ be the component of $T - v_3v_4$ containing $v_3$. Then $T'$ is a spider with at least $k$ feet where $k = \deg(v_3) - 1$. Clearly $\gamma(T) = \gamma(T') + k$. Now we show that $\gamma_{cr}(T) \geq \gamma_{cr}(T') + k$. Let $u_1, \ldots, u_k$ be the children of $v_3$ and $w_i$ be the leaf adjacent to $u_i$ for $i = 1, \ldots, k$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{cr}(T)$-function. By Corollary 10, $V_2 = \emptyset$. Obviously $f(u_i) + f(w_i) = 1$ for each $i$. As Case 2, we can see that $f(v_4) = 0$ or $f(v_3) = 0$. If $f(v_4) = f(v_3) = 0$, then the function $f$ restricted to $T'$ is a co-Roman dominating function of weight $\gamma_{cr}(T) - k$ and so $\gamma_{cr}(T) \geq \gamma_{cr}(T') + k$. Consider the following subcases.

Subcase 3.1. $f(v_3) = 1$ and $f(v_4) = 0$. If $f(x) = 1$ for some $x \in N_{T'}(v_4)$, then the function $g : V(G) \to \{0, 1\}$ defined by $g(v_3) = g(w_i) = 0, g(u_i) = 1$ for $1 \leq i \leq k$ and $g(x) = f(x)$ otherwise, is a dominating function of $T$ of weight less than $\omega(f)$ contradicting $\gamma(T) = \gamma_{cr}(T)$. Thus $f(x) = 0$ for each $x \in N_{T'}(v_4)$. Now the function $h : V(T') \to \{0, 1\}$ defined by $h(v_4) = 1$ and $h(x) = f(x)$ otherwise, is a co-Roman dominating function of $T$ of weight $\omega(f) - k$ and hence $\gamma_{cr}(T) \geq \gamma_{cr}(T') + k$.

Subcase 3.2. $f(v_3) = 0$ and $f(v_4) = 1$. As above we have $f(x) = 0$ for some $x \in N_{T'}(v_4)$. Then the function $f$, restricted to $T'$ is a co-Roman dominating function of $T'$ and so $\gamma_{cr}(T) \geq \gamma_{cr}(T') + k$. 


Thus in all cases $\gamma_{cr}(T) \geq \gamma_{cr}(T') + k$. As Case 2, we deduce that $\gamma_{cr}(T') = \gamma(T')$ and so by the induction hypothesis we have $T' \in \mathcal{T}$. Now $T$ can be obtained from $T'$ by Operation $O_3$ and hence $T \in \mathcal{T}$. This completes the proof. 

2. Bounds on Co-Roman Domination

In this section, we present some sharp bounds on the co-Roman domination number. First we prove two upper bounds on the co-Roman domination number in terms of matching number.

**Theorem 17.** For any connected graph $G$ of order $n \geq 2$,

$$\gamma_{cr}(G) \leq n - \alpha'(G).$$

**Proof.** Let $M = \{u_1v_1, \ldots, u_{\alpha'}v_{\alpha'}\}$ be a maximum matching of $G$ and let $X$ be the independent set of $M$-unsaturated vertices. If $y$ and $z$ are vertices of $X$ and $yu_i \in E(G)$, then since the matching $M$ is maximum, $zv_i \notin E(G)$. Therefore, for all $i \in \{1, 2, \ldots, \alpha'\}$ there are at most two edges between the sets $\{u_i, v_i\}$ and $\{y, z\}$. Assume $S$ is the set of all vertices in $X$ which belongs to a triangle with an edge in $M$. Let $S = \{x_1, \ldots, x_s\}$ if $S \neq \emptyset$ and $X \setminus S = \{y_1, \ldots, y_k\}$ if $X \setminus S \neq \emptyset$.

First let $S = \emptyset$. Then $vu_i \notin E(G)$ or $v_iu_i \notin E(G)$ for each $v \in X$ and each $i \in \{1, \ldots, \alpha'\}$. We may assume $N(x) \subseteq \{u_1, \ldots, u_{\alpha'}\}$ for each $x \in X$. Define $f : V(G) \to \{0, 1, 2\}$ by $f(u_i) = 0$ for $1 \leq i \leq \alpha'$ and $f(x) = 1$ otherwise. Clearly, $f$ is a co-Roman dominating function of $G$ of weight $\alpha' + |X|$ and hence

$$\gamma_{cr}(G) \leq \alpha'(G) + |X| = \alpha'(G) + (n - 2\alpha'(G)) = n - \alpha'(G).$$

Now let $S \neq \emptyset$. We may assume, without loss of generality, that $x_iu_i, x_iv_i \in E(G)$ for $i = 1, \ldots, s$. As above, we can assume that $N(x) \subseteq \{u_1, \ldots, u_{\alpha'}\}$ for each $x \in X \setminus S$. Define $f : V(G) \to \{0, 1, 2\}$ by $f(x) = 0$ for $x \in S \cup \{u_1, \ldots, u_{\alpha'}\}$ and $f(x) = 1$ otherwise. Obviously, $f$ is a co-Roman dominating function of $G$ of weight $\alpha'(G) + |X| - |S|$ and hence

(2) $$\gamma_{cr}(G) \leq \alpha'(G) + |X| - |S| = \alpha'(G) + (n - 2\alpha') - |S| \leq n - \alpha'(G) - |S|.$$

This completes the proof. 

**Theorem 18.** For any connected graph $G$ of order $n \geq 2$ with $\delta(G) \geq 2$,

$$\gamma_{cr}(G) \leq \alpha'(G).$$
Proof. Let $M$, $X$ and $S$ be the sets defined in the proof of Theorem 17. Assume first that $S = \emptyset$. Then as above we may assume $N(x) \subseteq \{u_1, \ldots, u_\alpha\}$ for each $x \in X$. Define $f : V(G) \to \{0, 1, 2\}$ by $f(u_i) = 1$ for $1 \leq i \leq \alpha'$ and $f(x) = 0$ otherwise. Since $\delta(G) \geq 2$, the function $f_i : V(G) \to \{0, 1, 2\}$ defined by $f(u_i) = 0$, $f(v_i) = 1$ and $f_i(x) = f(x)$ otherwise, is safe for each $i$. Thus $f$ is a co-Roman dominating function of $G$ of weight $\alpha'(G)$ and so $\gamma_{cr}(G) \leq \alpha'(G)$.

Now let $S = \{x_1, \ldots, x_s\}$. We may assume, without loss of generality, that $x_iu_i, x_iv_i \in E(G)$ for $i = 1, \ldots, s$. As above, we can assume that $N(x) \subseteq \{u_1, \ldots, u_\alpha\}$ for each $x \in X \setminus S$. It is easy to see that the function $f$ defined above is a co-Roman dominating function of $G$. Thus $\gamma_{cr}(G) \leq \alpha'(G)$ and the proof is complete.

Theorem 19. For any connected graph $G$ of order $n \geq 2$,

$$\gamma_{cr}(G) \leq 2\alpha'(G).$$

Proof. Let $M$, $X$ and $S$ be the sets defined in the proof of Theorem 17. As Theorem 17, we may assume that $x_iu_i, x_iv_i \in E(G)$ for $i = 1, \ldots, s$ if $S \neq \emptyset$ and $N(x) \subseteq \{u_1, \ldots, u_\alpha\}$ for each $x \in X \setminus S$. Then the function $f : V(G) \to \{0, 1, 2\}$ defined by $f(u_i) = 1$ if $u_i$ is adjacent to a vertex in $S$, $f(u_i) = 2$ if $u_i$ is adjacent to a vertex in $X \setminus S$ and $f(x) = 0$ otherwise, is a co-Roman dominating function of $G$ and so $\gamma_{cr}(G) \leq |S| + 2|X - S| = 2\alpha'(G) - |S| \leq 2\alpha'(G)$.

A set $X \subseteq V(G)$ is called a 2-packing if $d(u, v) > 2$ for any different vertices $u$ and $v$ of $X$. The 2-packing number $\rho(G)$ is the maximum cardinality of a 2-packing of $G$.

Theorem 20. For any connected graph $G$ of order $n \geq 2$ with $\delta(G) \geq 2$,

$$\gamma_{cr}(G) \leq n - \rho(G)(\delta(G) - 1).$$

Proof. Let $S$ be a 2-packing of $G$ of size $\rho(G)$. Define $f : V(G) \to \{0, 1, 2\}$ by $f(x) = 2$ for $x \in S$, $f(x) = 0$ for $x \in \bigcup_{v \in S} N(v)$ and $f(x) = 1$ otherwise. Clearly, $f$ is a co-Roman dominating function of $G$ and hence

$$\gamma_{cr}(G) \leq (n - |\bigcup_{v \in S} N(v)|) + 2|S| = n - \sum_{v \in S} |N[v]| + 2\rho(G) \leq n - \rho(G)(\delta(G) + 1) - 2\rho(G) = n - \rho(G)(\delta(G) - 1),$$

as desired.

Proposition 21. Let $G$ be a simple connected graph of order $n$ with $\delta(G) \geq 2$ and $g(G) \geq 5$. Then $\gamma_{cr}(G) \leq \frac{2(n - g(G))}{3} + \left\lceil \frac{2g(G)}{5} \right\rceil$. 
**Proof.** If $G$ is an $n$-cycle, then the result follows by Proposition 3. Assume $G$ is not a cycle and $C$ is a cycle of length $g(G)$ in $G$. Let $G'$ be the graph obtained from $G$ by removing the vertices of $V(C)$. Since $g(G) \geq 5$, each vertex of $G'$ can be adjacent to at most one vertex of $C$ which implies $\delta(G') \geq 1$. By Corollary 5, we have $\gamma_{cr}(G') \leq \frac{2(n-g(G))}{3}$. Let $g$ be a $\gamma_{cr}(G')$-function and $h$ be a $\gamma_{cr}(C)$-function. Define $f : V(G) \rightarrow \{0, 1, 2\}$ by $f(v) = g(v)$ for $v \in V(G')$ and $f(v) = h(v)$ for $v \in V(C)$. Obviously, $f$ is a co-Roman dominating function and so
\[
\gamma_{cr}(G) \leq \frac{2(n-g(G))}{3} + \left\lceil \frac{2g(G)}{5} \right\rceil.
\]
\[\square\]

3. Characterization of Graphs $G$ of Order $n$ with $\gamma_{cr}(G) = \frac{2n}{3}$

In this section, we characterize the graphs attaining the upper bound in Corollary 5. For any arbitrary tree $T$, let $T_{cr}$ be the tree obtained from $T$ by adding exactly two pendant edges at each vertex of $T$. Note that $n(T_{cr}) = 3n(T)$. Let $\mathcal{F}$ be the family of all trees $T_{cr}$. In fact, $\mathcal{F}$ is the family of trees $T$ such that $V(T)$ can be partitioned into sets inducing $P_3$ such that the subgraph induced by the central vertices of these paths is connected.

**Lemma 22.** If $T \in \mathcal{F}$, then $\gamma_{cr}(T) = \frac{2n(T)}{3}$.

**Proof.** Let $T \in \mathcal{F}$ and let $f$ be a $\gamma_{cr}$-function on $T$. Then $T$ is obtained from a tree $T'$ by adding exactly two pendant edges at each vertex of $T'$. For each non-leaf vertex $v \in V(T)$, let $L_v = \{v_1, v_2\}$. It is easy to see that for any non-leaf vertex $v \in V(T)$, $f(v) + f(v_1) + f(v_2) \geq 2$, otherwise we have an unprotected vertex in either $f$ or $f_{v_1}$ for some $i = 1, 2$. Hence, $\gamma_{cr}(T) = \omega(f) = \sum_{v \in V(T')} (f(v) + f(v_1) + f(v_2)) \geq 2n(T') = \frac{2n(T)}{3}$. Now the result follows from Proposition 4. \[\square\]

**Lemma 23.** Let $q \geq p \geq 1$ and let $T = DS(p, q)$. Then $\gamma_{cr}(T) = \frac{2n(T)}{3}$ if and only if $q = p = 2$.

**Proof.** If $q = p = 2$, then Lemma 22 implies $\gamma_{cr}(T) = \frac{2n(T)}{3}$. Conversely, let $\gamma_{cr}(T) = \frac{2n(T)}{3}$. It follows from Proposition 3 that $q \geq 2$. If $p = 1$, then clearly $\gamma_{cr}(T) = 3 < \frac{2n(T)}{3}$, a contradiction. Suppose that $p \geq 2$. If $q > 2$, then we have $\gamma_{cr}(T) \leq 4 < \frac{2n(T)}{3}$, a contradiction again. Thus $q = p = 2$ and the proof is complete. \[\square\]

**Theorem 24.** Let $T$ be a tree of order $n \geq 3$. Then $\gamma_{cr}(T) = \frac{2n}{3}$ if and only if $T \in \mathcal{F}$.
Proof. According to Lemma 22, we only need to prove the necessity. Let $T$ be a tree of order $n \geq 3$ with $\gamma_{cr}(T) = \frac{2n}{3}$. Note that $n$ is a multiple of 3. The proof is by induction on $n$. If $n = 3$, then the only tree $T$ of order 3 and $\gamma_{cr}(T) = 2$ is $P_3 \in \mathcal{F}$. Let $n \geq 4$ and let the statement hold for all trees of order less than $n$. Suppose that $T$ is a tree of order $n$ with $\gamma_{cr}(T) = \frac{2n}{3}$. If $\text{diam}(T) = 2$, then $T = K_{1,n}$ and we deduce from Proposition 2 that $T = P_3$ and so $T \in \mathcal{F}$. If $\text{diam}(T) = 3$, then we deduce from Lemma 23 that $T = DS(2,2)$ and so $T \in \mathcal{F}$. Henceforth we assume that $\text{diam}(T) \geq 4$. Let $v_1v_2 \cdots v_k$ ($k \geq 5$) be a diametral path in $T$ and root $T$ at $v_k$. We show that $\text{deg}_T(v_2) = 3$. Let $T' = T - T_{v_2}$ and $f$ be a $\gamma_{cr}(T')$-function. If $\text{deg}_T(v_2) \geq 4$, then the function $g : V(T) \to \{0,1,2\}$ defined by $g(v_2) = 2$, $g(x) = 0$ if $x \in L_{v_2}$ and $g(x) = f(x)$ for $x \in T'$, is a CRDF on $T$ of weight $\omega_f + 2$. By Proposition 4, we have $\gamma_{cr}(T) \leq \omega_f + 2 \leq \frac{2n(T')}{3} + 2 \leq \frac{2(n-4)}{3} + 2 < \frac{2n}{3}$, which is a contradiction. If $\text{deg}_T(v_2) = 2$, then the function $g : V(T) \to \{0,1,2\}$ defined by $g(v_2) = 1$, $g(v_1) = 0$ and $g(x) = f(x)$ for $x \in T'$, is a CRDF on $T$ of weight $\omega_f + 1$. By Proposition 4, we have $\gamma_{cr}(T) \leq \omega_f + 1 \leq \frac{2(n-2)}{3} + 1 < \frac{2n}{3}$, a contradiction again. Thus $\text{deg}_T(v_2) = 3$. Assume that $T' = T - T_{v_2}$. As above, we have

$$\frac{2n(T)}{3} = \gamma_{cr}(T) \leq \gamma_{cr}(T') + 2 \leq \frac{2n(T')}{3} + 2 = \frac{2(n-3)}{3} + 2 = \frac{2n}{3}.$$

Thus all inequalities in the above inequality chain must be equalities and so $\gamma_{cr}(T') = \frac{2n(T')}{3}$. By the induction hypothesis we have $T' \in \mathcal{F}$. Now we show that $v_3$ is not a leaf of $T'$. If $v_3$ is a leaf in $T'$, then let $T'' = T - T_{v_3}$ and let $h$ be a $\gamma_{cr}(T'')$-function. Define the function $g : V(T) \to \{0,1,2\}$ by $g(v_2) = 2$, $g(v) = 0$ if $v \in N_{T}(v_2)$ and $g(x) = h(x)$ for $x \in T''$. Clearly, $g$ is a CRDF on $T'$ of weight $\omega_f + 2$. By Proposition 4, we have $\gamma_{cr}(T) \leq \omega_f = \gamma_{cr}(T'') + 2 \leq 2 \frac{(n-4)}{3} + 2 < \frac{2n}{3}$, a contradiction. Thus $v_3$ is a non-leaf vertex of $T'$ and so $T \in \mathcal{F}$. This completes the proof.

**Theorem 25.** Let $G$ be a connected $n$-vertex graph with $n \geq 3$. Then $\gamma_{cr}(G) = \frac{2n}{3}$ if and only if $G$ is obtained from $\frac{n}{3}P_3$ by adding edges between the centers of the paths $P_3$ such that the resulting graph is connected.

**Proof.** If $G$ has the specified form, then clearly every CRDF puts weight at least 2 on the vertex set of each copy of $P_3$.

Now suppose that $\gamma_{cr}(G) = \frac{2n}{3}$. Since adding edges cannot increase $\gamma_{cr}(G)$, every spanning tree of $G$ belongs to $\mathcal{F}$. Given a spanning tree $T$, let $S_1, S_2, \ldots, S_n$ be the 3-sets in the special partition of $V(T)$. The assignment of weight 2 that guards $S_i$ can be chosen independently of any other $S_j$. If any edge of $G$ joins vertices of $S_i$ and $S_j$ that are not the centers of the paths they induce, then a CRDF with weight less than $\frac{2n}{3}$ can be built as in the proof of Theorem 24. This completes the proof.
4. Graphs with Large Co-Roman Domination Number

In this section, we characterize all graphs of order \( n \) with co-Roman domination number \( n-2 \) and \( n-3 \). The first result is an immediate consequence of Theorem 17.

**Corollary 26** (Theorem 4.2 in [2]). Let \( G \) be a connected graph on \( n \geq 2 \) vertices. Then \( \gamma_{cr}(G) = n-1 \) if and only if \( G = K_2 \) or \( K_{1,2} \).

Arunagam et al. [2] posed the following problem.

**Problem.** Characterize graphs \( G \) such that \( \gamma_{cr}(G) = n-2 \).

Next we solve this problem.

**Theorem 27.** Let \( G \) be a connected graph on \( n \geq 2 \) vertices. Then \( \gamma_{cr}(G) = n-2 \) if and only if \( G \) is a graph on four vertices different from \( K_4 \) and \( K_4 - e \), or \( G \cong DS(2,1) \), or \( G \cong DS(2,2) \).

**Proof.** By Theorem 17, we have \( \alpha'(G) \leq 2 \). If \( \alpha'(G) = 1 \), then \( G \) is the star \( K_{1,n-1} \) and we conclude from Proposition 2 that \( G = K_{1,3} \). Assume that \( \alpha'(G) = 2 \). Let \( M, X \) and \( S \) be the sets defined in the proof of Theorem 17. By (2), we have \( S = \emptyset \). As above, we may assume \( N(x) \subseteq \{u_1, \ldots, u_{\alpha'}\} \) for each \( x \in X \). If \( u_i \) has at least two neighbors in \( X \) for some \( i \), say \( i = 1 \), then the function \( f : V(G) \rightarrow \{0,1,2\} \) defined by \( f(u_1) = 2, f(u_i) = 0 \) for \( 2 \leq i \leq \alpha' \), \( f(x) = 0 \) if \( x = v_1 \) or \( x \in N(u_1) \cap X \) and \( f(x) = 1 \) otherwise, is clearly a co-Roman dominating function of \( G \) of weight \( n - \alpha'(G) - 1 \) which leads to a contradiction. Hence each \( u_i \) has at most one neighbor in \( X \) and this implies that \( |X| \leq 2 \). If \( |X| = 0 \), then \( n = 4 \) and obviously \( G \) is a connected graph on four vertices different from \( K_4 \) and \( K_4 - e \). Hence \( |X| \geq 1 \).

First let \( |X| = 2 \). Since \( X \) is independent and \( G \) is connected, we may assume that \( u_i y_i \in E(G) \) for \( i = 1, 2 \). Since each \( u_i \) has at most one neighbor in \( X \), we deduce that \( \deg(y_i) = 1 \) for \( i = 1, 2 \). Considering the matching \( M' = \{u_1 y_1, u_2 y_2\} \) instead of \( M \), we have \( \deg(v_1) = \deg(v_2) = 1 \). Since \( G \) is connected, we have \( u_1 u_2 \in E(G) \) and hence \( G = DS(2,2) \).

Now let \( |X| = 1 \). Since \( G \) is connected, we suppose that \( u_1 y_1 \in E(G) \). If \( u_2 y_1 \in E(G) \), then the function \( f_1 : V(G) \rightarrow \{0,1,2\} \) defined by \( f_1(u_1) = f_1(u_2) = 1 \) and \( f_1(x) = 0 \) otherwise, is clearly a co-Roman dominating function of \( G \) of weight 2, a contradiction. Thus \( \deg(y_1) = 1 \). Considering the matching \( M' = \{u_1 y_1, u_2 y_2\} \) instead of \( M \), we obtain \( \deg(v_1) = 1 \). Since \( G \) is connected, we may assume that \( u_1 u_2 \in E(G) \). If \( u_1 v_2 \in E(G) \), then clearly \( \gamma_{cr}(G) \leq 2 \) which is a contradiction. Thus \( G = DS(1,2) \) and the proof is complete. \( \blacksquare \)

The corona graph \( cor(H) \) of a graph \( H \) is the graph obtained from \( H \) by attaching a leaf to every vertex of \( H \). We recall the following result established by Payan and Xuong [12] (see also Fink et al. [8]).
Theorem 28. For a graph $G$ with even order $n$ and with no isolated vertices, \( \gamma(G) = \frac{n}{2} \) if and only if the components of $G$ are the cycle $C_4$ or the corona $\text{cor}(H)$ for any connected graph $H$.

Now we characterize all connected graphs $G$ of order $n \geq 4$ with $\gamma_{cr}(G) = n - 3$. To do this, we introduced some families of graphs.

Let

- $G_1 = \{K_4, K_4 - e, K_{1,4}\}$,
- $G_2$ be the family of connected graphs $G$ obtained from a triangle and a path $P_2$ by adding some edges between them so that the resulting graph has at most one universal vertex,
• $G_3$ be the family of connected graphs $G$ obtained from a path $P_3$ and a path $P_2$ by adding some edges between them such that the resulting graph is different from $DS(1, 2)$ and has at most one universal vertex,

• $G_4$ be the family of connected graphs $G \not\cong DS(2, 2)$ of order 6 consisting of $cor(P_3), cor(C_3)$ and all graphs $G$ with $\Delta(G) \leq 4$, for which every $\gamma(G)$-set $S$ has a vertex $x$ such that $x$ has no neighbor $x' \in V \setminus S$ with $pn(x, S) \subseteq N[x']$,

• $G_5 = \{G_1, G_2, \ldots, G_{13}\}$,

• $G_6$ be the family of connected graphs $G$ obtained from three paths $v_1u_1y_1, v_2u_2y_2$ and $v_3u_3$ by adding edges between $u_1, u_2, u_3$ such that the resulting graph is connected,

• $G_7$ be the family of connected graphs $G$ obtained from $3P_3$ by adding edges between the centers of the paths $P_3$ such that the resulting graph is connected.

![Diagram of two graphs](image)

Figure 3. Two graphs $G$ of order 6 with $\gamma_{cr}(G) = 3$.

Theorem 29. Let $G$ be a connected graph on $n \geq 4$ vertices, then $\gamma_{cr}(G) = n - 3$ if and only if $G \in \bigcup_{i=1}^{7} G_i$.

Proof. Let $G \in \bigcup_{i=1}^{7} G_i$. We deduce from (1), Corollary 26 and Theorem 27 that $\gamma_{cr}(G) \leq n - 3$. If $G = K_{1, 4}$, then $\gamma_{cr}(G) = 2 = n - 3$ by Proposition 2, and if $G \in G_1 \setminus \{K_{1, 4}\}$ then $\gamma_{cr}(G) = 1 = n - 3$ by Observation 6. If $G \in G_2 \cup G_3$, then we conclude from Observation 6 that $\gamma_{cr}(G) \geq 2 = n - 3$ and so $\gamma_{cr}(G) = n - 3$. If $G \in \{cor(P_3), cor(C_3)\}$, then by Proposition 9 and Theorem 28 we have $\gamma_{cr}(G) = \gamma(G) = 3$, and if $G \in G_4 \setminus \{cor(P_3), cor(C_3)\}$, then clearly $\gamma(G) = 2$ and Proposition 9 implies that $\gamma_{cr}(G) \geq \gamma(G) + 1 = 3 = n - 3$ and so $\gamma_{cr}(G) = n - 3$. If $G \in G_5 \cup G_6$, then it is easy to see that $\gamma_{cr}(G) = n - 3$. Finally, if $G \in G_7$, then by Theorem 25, we have $\gamma_{cr}(G) = 6 = n - 3$. 
Conversely, let \( \gamma_{cr}(G) = n - 3 \). By Corollary 5 and Theorem 17, we obtain \( n \leq 9 \) and \( \alpha'(G) \leq 3 \). If \( \alpha'(G) = 1 \), then \( G \) is the star \( K_{1,n-1} \) and we conclude from Proposition 2 that \( G = K_{1,4} \in \mathcal{G}_1 \). Assume that \( \alpha'(G) \geq 2 \). Suppose \( M, X \) and \( S \) are the sets defined in the proof of Theorem 17. We consider the following cases.

Case 1. \( \alpha'(G) = 3 \). Since \( n \leq 9 \), we must have \( |X| \leq 3 \). If \( |X| = 3 \), then \( n = 9 \) and we conclude from Theorem 25 that \( G \in \mathcal{G}_7 \). Let \( |X| \leq 2 \). By (2), we have \( S = \emptyset \). As above, we may assume \( N(x) \subseteq \{u_1, u_2, u_3\} \) for each \( x \in X \). Consider the following subcases.

Subcase 1.1. \( |X| = 2 \). Assume \( u_iy_1, u_iy_2 \in E(G) \) for some \( i \), say \( i = 1 \), then the function \( f_1 : V(G) \to \{0, 1, 2\} \) defined by \( f_1(u_1) = 2, f_1(u_2) = f_1(u_3) = 1 \) and \( f_1(x) = 0 \) otherwise, is clearly a co-Roman dominating function of \( G \) of weight 4 which is a contradiction. Thus each \( u_i \) has at most one neighbor in \( X \). Assume without loss of generality that \( u_1y_1, u_2y_2 \in E(G) \). If \( y_1u_3 \in E(G) \) (the case \( y_1u_2 \in E(G) \) is similar), then the function \( f_2 : V(G) \to \{0, 1, 2\} \) defined by \( f_2(u_1) = f_2(u_3) = 1, f_2(u_2) = 2 \) and \( f_2(x) = 0 \) otherwise, is clearly a co-Roman dominating function of \( G \) of weight 4 which is a contradiction again. Hence \( y_1u_3, y_2u_3 \not\in E(G) \). It follows that \( \deg(y_1) = \deg(y_2) = 1 \). Considering the matching \( M' = \{u_1y_1, u_2y_2, u_3v_3\} \) instead of \( M \), we obtain \( \deg(v_1) = \deg(v_2) = 1 \). Since \( G \) is connected, we may assume, without loss of generality, that \( u_1v_3 \in E(G) \). If \( u_1v_3 \in E(G) \) or \( u_2v_3 \in E(G) \), then the function \( f_3 : V(G) \to \{0, 1, 2\} \) defined by \( f_3(u_1) = f_3(u_2) = 2 \) and \( f_3(x) = 0 \) otherwise, is clearly a co-Roman dominating function of \( G \) of weight 4, a contradiction. Therefore, \( \deg(v_3) = 1 \). Since \( G \) is connected, we conclude that \( G \) is a graph obtained from three paths \( v_1u_1y_1, v_2u_2y_2 \) and \( v_3u_3 \) by adding edges between \( u_1, u_2, u_3 \) such that the resulting graph is connected. Hence \( G \in \mathcal{G}_6 \).

Subcase 1.2. \( |X| = 1 \). Assume that \( u_1y_1 \in E(G) \). If \( y_1u_3 \in E(G) \) (the case \( y_1u_2 \in E(G) \) is similar), then the function \( f_4 : V(G) \to \{0, 1, 2\} \) defined by \( f_4(u_1) = f_4(u_2) = f_4(u_3) = 1 \) and \( f_4(x) = 0 \) otherwise, is clearly a co-Roman dominating function of \( G \) of weight 3 which is a contradiction. Hence \( y_1u_3, y_1v_1 \not\in E(G) \). Hence \( \deg(y_1) = 1 \). Regarding the matching \( M' = \{u_1y_1, u_2v_2, u_3v_3\} \) instead of \( M \), we have \( \deg(v_1) = 1 \). Since \( G \) is connected, we may assume that \( u_1v_3 \in E(G) \). If \( u_1v_3 \in E(G) \), then the function \( h_1 : V(G) \to \{0, 1, 2\} \) defined by \( h_1(u_1) = 2, h_1(u_2) = 1 \) and \( h_1(x) = 0 \) otherwise, is clearly a co-Roman dominating function of \( G \) of weight 3, a contradiction. Therefore \( u_1v_3 \not\in E(G) \). Consider the following.

- \( u_1v_2 \in E(G) \) (the case \( u_1v_2 \in E(G) \) is similar). Then as above \( u_1v_2 \not\in E(G) \). If \( v_2v_3 \in E(G) \), then the function \( h_2 : V(G) \to \{0, 1, 2\} \) defined by \( h_2(u_1) = 2, h_2(v_2) = 1 \) and \( h_2(x) = 0 \) otherwise, is clearly a co-Roman dominating function of \( G \) of weight 3, a contradiction. Hence \( v_2v_3 \not\in E(G) \). If
\{u_2v_3, u_3v_2\} \subseteq E(G), then the function \(h_3 : V(G) \to \{0, 1, 2\}\) defined by 
\(h_3(u_1) = 2, h_3(u_2) = 1\) and \(h_3(x) = 0\) otherwise, is clearly a co-Roman dominating function of \(G\) of weight 3, a contradiction. Thus \(\{u_2v_3, u_3v_2\} \not\subseteq E(G)\).

It follows that \(G \in \{G_1, G_2, G_3, G_4, G_5\}\) and so \(G \in G_5\).

- \(u_1u_2, u_1v_2 \not\in E(G)\). If \(\{u_2, v_2, v_3\}\) induces a triangle, then the function \(h_4 : V(G) \to \{0, 1, 2\}\) defined by 
\(h_4(u_1) = 2, h_4(u_2) = 1\) and \(h_4(x) = 0\) otherwise, is clearly a co-Roman dominating function of \(G\) of weight 3, a contradiction. Thus \(\{u_2, v_2, v_3\}\) does not induce a triangle. As above we have \(\{u_2v_3, u_3v_2\} \not\subseteq E(G)\).

Since \(G\) is connected, the graph induced by \(u_2, v_2, u_3, v_3\) is connected. This implies that \(G \in \{G_6, G_7, G_8, G_9, G_{10}\}\) and so \(G \in G_5\).

Subcase 1.3. \(|X| = 0\). Then \(n = 6\). Since \(\gamma_{cr}(G) = 3\), we have \(\Delta(G) \leq 4\) by Propositions 1 and 2. Hence \(\gamma(G) \geq 2\). If \(\gamma(G) = 3\), then we deduce from Theorem 28 that \(G\) is the corona \(cor(P_3)\) or \(cor(C_3)\) and so \(G \in G_4\). Assume \(\gamma(G) = 2\). Then we conclude from Proposition 9 that every \(\gamma(G)\)-set \(S\) contains a vertex \(x\) such that \(x\) has no neighbor \(x' \in V \setminus S\) with \(pm(x, S) \subseteq N[x']\). It follows that \(G \in G_4\).

Case 2. \(\alpha'(G) = 2\). First let \(S \neq \emptyset\). We deduce from (2) that \(|S| = 1\) and so \(S = \{x_1\}\). Let \(x_1u_1, x_1v_1 \in E(G)\). Then we assume that each other vertex of \(X\) is adjacent only to \(u_2\). It follows that \(\deg(x) = 1\) for each \(x \in X \setminus \{x_1\}\). Since the function \(g : V(G) \to \{0, 1, 2\}\) defined by \(g(u_1) = 1, g(u_2) = 2\) and \(g(x) = 0\) otherwise, is an co-Roman dominating function of \(G\), we deduce that \(n - 3 \leq 3\) and so \(n \leq 6\). If \(n = 6\), then clearly \(X = \{x_1, y_1\}\). By considering the matching \(M' = \{u_1v_1, u_2y_1\}\) instead of \(M\), we have \(\deg(v_2) = 1\). Since \(G\) is connected and \(\gamma_{cr}(G) = 3\), \(u_2\) must be adjacent to at least one vertex and at most two vertices in \(\{u_1, v_1, x_1\}\). Thus \(G\) is a graph obtained from a triangle by adding a path \(P_3\) and joining its center to at least one and at most two vertices of triangle and so \(G \simeq H_1\) or \(H_2\). Hence \(G \in G_4\). Assume that \(n = 5\). Since \(G\) is connected, \(G\) is a graph obtained from a triangle and a path \(P_2\) by adding some edges between them so that the resulting graph has at most one universal vertex. Thus \(G \in G_4\).

Now let \(S = \emptyset\). As above, we may assume \(N(x) \subseteq \{u_1, u_2\}\) for each \(x \in X\).

By Theorem 19, we have \(\gamma_{cr}(G) \leq 4\) and this implies that \(n \leq 7\). Thus \(|X| \leq 3\).

If \(n = 4\), then we have \(\gamma_{cr}(G) = 1\) yielding \(G \in \{K_4, K_4-e\}\) \(\subseteq G_1\) by Observation 6. If \(n = 5\), then \(G\) is a graph obtained from a path \(P_3\) and a path \(P_2\) by adding some edges between them such that the resulting graph is different from \(DS(1, 2)\) and has at most one universal vertex. Thus \(G \in G_4\). Let \(n \geq 6\). Since \(\gamma_{cr}(G) \geq 3\), \(G\) has no vertex of degree \(n - 1\) and so \(\gamma(G) \geq 2\). Since \(\{u_1, u_2\}\) is a dominating set, we have \(\gamma(G) = 2\). If \(n = 6\), then clearly \(G \in G_4\). Suppose \(n = 7\). Then \(X = \{y_1, y_2, y_3\}\). If \(u_i\) is adjacent to all vertices of \(X\) for some \(i\), say \(i = 1\), then the function \(g : V(G) \to \{0, 1, 2\}\) defined by \(g(u_1) = 2, g(u_2) = 1\) and \(g(x) = 0\) otherwise, is an co-Roman dominating function of \(G\) of weight 3 which leads to a contradiction. Hence, each \(u_i\) is adjacent to at most two vertices in \(X\). We may
assume without loss of generality that \( u_1 y_1, u_1 y_2, u_2 y_3 \in E(G) \) and \( u_1 y_3 \notin E(G) \). Since \( \{y_1, y_2, y_3, v_1\} \) is independent, we deduce that \( \deg(y_3) = 1 \). Considering the matching \( M' = \{u_1 v_1, u_2 y_3\} \) instead of \( M \), we obtain \( \deg(v_2) = 1 \). Since \( \gamma_{cr}(G) = 4 \), \( u_2 \) is adjacent to at most one vertex in \( \{y_1, y_2, v_1\} \). Thus \( G \) is a connected graph obtained from \( K_{1,3} \) and a path \( P_3 \) by joining the center of \( P_3 \) to the center or at most one leaf of \( K_{1,3} \). This implies that \( G \in \{G_{11}, G_{12}, G_{13}\} \) and so \( G \in \mathcal{G}_5 \). This completes the proof.

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**References**


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