ON THE CO-ROMAN DOMINATION IN GRAPHS

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Abstract

Let $G = (V, E)$ be a graph and let $f : V(G) \rightarrow \{0, 1, 2\}$ be a function. A vertex $v$ is said to be protected with respect to $f$, if $f(v) > 0$ or $f(v) = 0$ and $v$ is adjacent to a vertex of positive weight. The function $f$ is a co-Roman dominating function if (i) every vertex in $V$ is protected, and (ii) each $v \in V$ with positive weight has a neighbor $u \in V$ with $f(u) = 0$ such that the function $f_{uv} : V \rightarrow \{0, 1, 2\}$, defined by $f_{uv}(u) = 1$, $f_{uv}(v) = f(v) - 1$ and $f_{uv}(x) = f(x)$ for $x \in V \setminus \{v, u\}$, has no unprotected vertex. The weight of $f$ is $\omega(f) = \sum_{v \in V} f(v)$. The co-Roman domination number of a graph $G$, denoted by $\gamma_{cr}(G)$, is the minimum weight of a co-Roman dominating function on $G$. In this paper, we give a characterization of graphs of order $n$ for which co-Roman domination number is $\frac{2n}{3}$ or $n - 2$, which settles...

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For terminology and notation on graph theory not given here, the reader is referred to [9,10]. In this paper, $G$ is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of $G$ is denoted by $n = n(G)$. For every vertex $v \in V$, the open neighborhood $N(v)$ is the set $\{u \in V \mid uv \in E\}$ and the closed neighborhood of $v$ is the set $N[v] = N(v) \cup \{v\}$. The degree of a vertex $v \in V$ is $\deg_G(v) = \deg(v) = |N(v)|$. A universal vertex is a vertex that is adjacent to all other vertices of $G$. The open neighborhood of a set $S \subseteq V$ is the set $N(S) = \bigcup_{v \in S} N(v)$, and the closed neighborhood of $S$ is the set $N[S] = N(S) \cup S$.

For a set $S \subseteq V(G)$ and a vertex $v \in S$, the private neighborhood of $v$ with respect to $S$ is the set $pm(v; S) = \{u \in N(v), N(u) \cap S = \{v\}\}$. A leaf is a vertex of degree one, and a support vertex is a vertex adjacent to a leaf. We also denote by $L_v$ the set of all leaves adjacent to a support vertex $v$. For a vertex $v$ in a rooted tree $T$, let $D(v)$ denote the set of descendants of $v$ and $D[v] = D(v) \cup \{v\}$. The maximal subtree at $v$ is the subtree of $T$ induced by $D[v]$, and is denoted by $T_v$. A subdivision of an edge $uv$ is obtained by replacing the edge $uv$ with a path $uvw$, where $w$ is a new vertex. The subdivision graph $S(G)$ is the graph obtained from $G$ by subdividing each edge of $G$. The subdivision star $S(K_1,t)$ for $t \geq 2$, is called a healthy spider. We write $P_n$ for a path of length $n - 1$ and $K_{1,n}$ for a star. For integers $r \geq s \geq 1$, the double star $DS(r,s)$ is the tree obtained by connecting the centers of two stars $K_{1,r}$ and $K_{1,s}$ with an edge. The diameter of $G$, denoted by $\text{diam}(G)$, is the maximum value among minimum distances between all pairs of vertices of $G$. For a subset $S$ of vertices of $G$, we denote by $G[S]$ the subgraph induced by $S$. For a subset $S \subseteq V(G)$ of vertices of a graph $G$ and a function $f : V(G) \to R$, we define $f(S) = \sum_{x \in S} f(x)$. For a function $f : V(G) \rightarrow \{0,1,2\}$, let $V_i = \{v \in V \mid f(v) = i\}$ for $i = 0,1,2$. Since these three sets determine $f$, we can equivalently write $f = (V_0, V_1, V_2)$ (or $f = (V_0^f, V_1^f, V_2^f)$ to refer $f$). We note that $\omega(f) = |V_1| + 2|V_2|$.

A Roman dominating function on a graph $G$, abbreviated RD-function, is a function $f : V(G) \longrightarrow \{0,1,2\}$ satisfying the condition that every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. The weight, $\omega(f)$, of $f$ is defined as $\omega(f) = \sum_{v \in V} f(v)$. The Roman domination number, denoted by $\gamma_R(G)$. An RD-function with minimum weight $\gamma_R(G)$ in $G$ is called a $\gamma_R(G)$-function. The definition of the Roman dominating function
was given multiplicity by Steward [14] and ReVelle and Rosing [13]. Roman domination is now well studied in graph theory [1,3–7,15].

Let $f : V(G) \to \{0, 1, 2\}$ be a function. A vertex $v$ is said to be protected with respect to $f$, if $f(v) > 0$ or $f(v) = 0$ and $v$ is adjacent to a vertex of positive weight. The function $f$ is a weak Roman dominating function if for every vertex $u$ with $f(u) = 0$ there exists a vertex $v$ adjacent to $u$ such that $f(v) \in \{1, 2\}$ and the function $f_{uv} : V \to \{0, 1, 2\}$, defined by $f_{uv}(u) = 1$, $f_{uv}(v) = f(v) - 1$ and $f_{uv}(x) = f(x)$ for $x \in V \setminus \{v, u\}$, has no unprotected vertex. The weak Roman domination number of a graph $G$, denoted by $\gamma_r(G)$, is the minimum weight among all weak Roman dominating functions on $G$. The weak Roman domination number was introduced by Henning and Hedetniemi in [11].

The function $f : V(G) \to \{0, 1, 2\}$ is a co-Roman dominating function, abbreviated CRDF if (i) every vertex in $V$ is protected, and (ii) each $v \in V$ with positive weight has a neighbor $u \in V$ with $f(u) = 0$ such that the function $f_{uv} : V \to \{0, 1, 2\}$, defined by $f_{uv}(u) = 1$, $f_{uv}(v) = f(v) - 1$ and $f_{uv}(x) = f(x)$ for $x \in V \setminus \{v, u\}$, has no unprotected vertex. The weight of $f$ is $\omega(f) = \sum_{v \in V} f(v)$. The co-Roman domination number of a graph $G$, denoted by $\gamma_{cr}(G)$, is the minimum weight of a co-Roman dominating function on $G$. It follows from the definitions that for any connected graph $G$ of order $n \geq 2$,

$$\gamma_{cr}(G) \leq n - 1. \quad (1)$$

The co-Roman domination in graphs was investigated by Arumugam et al. in [2]. The proof of the next results can be found in [2].

**Proposition 1.** If $H$ is a spanning subgraph of a graph $H$, then $\gamma_{cr}(G) \leq \gamma_{cr}(H)$.

**Proposition 2.** For $n \geq 2$, $\gamma_{cr}(K_{1,n}) = 2$.

**Proposition 3.** For $n \geq 4$, $\gamma_{cr}(P_n) = \gamma_{cr}(C_n) = \lceil \frac{2n}{3} \rceil$.

**Proposition 4.** For every tree $T$ of order $n \geq 2$, $\gamma_{cr}(T) \leq \frac{2n}{3}$.

The next result is an immediate consequence of Propositions 1 and 4.

**Corollary 5.** For every connected graph $G$ of order $n \geq 2$, $\gamma_{cr}(G) \leq \frac{2n}{3}$.

**Observation 6.** Let $G$ be a graph of order $n \geq 2$. Then $\gamma_{cr}(G) = 1$ if and only if $G$ has two vertices of degree $n - 1$.

**Theorem 7.** For every graph $G$, $\gamma_{cr}(G) \leq \gamma_r(G)$.

In [2], the authors posed the following open problems.

**Problem 1.** Characterize graphs $G$ of order $n$ such that $\gamma_{cr}(G) = n - 2$.

**Problem 2.** Characterize trees $T$ of order $n$ such that $\gamma_{cr}(T) = \frac{2n(T)}{3}$.

**Problem 3.** Characterize graphs $G$ such that $\gamma_{cr}(T) = \gamma(G)$.

In this paper, we settle the above open problems. Furthermore, we establish some sharp bounds on the co-Roman domination number.
1. Graphs $G$ with $\gamma_{cr}(G) = \gamma_r(G)$ or $\gamma_{cr}(G) = \gamma(G)$

In this section, we study the properties of graphs $G$ for which $\gamma_{cr}(G) = \gamma_r(G)$ or $\gamma_{cr}(G) = \gamma(G)$.

**Proposition 8.** Let $G$ be a connected graph of order at least two. Then $\gamma_{cr}(G) = \gamma_r(G)$ if and only if there exists a $\gamma_{cr}(G)$-function $f = (V_0, V_1, V_2)$ such that each vertex $x \in V_0$, either has a neighbor $x'$ in $V_2$ or has a neighbor $x'$ in $V_1$ for which $pn(x', V_1 \cup V_2) \subseteq N[x]$.

**Proof.** If there exists a $\gamma_{cr}(G)$-function $f = (V_0, V_1, V_2)$ such that each vertex $x \in V_0$, has a neighbor $x' \in V_1 \cup V_2$ with $pn(x', V_1 \cup V_2) \subseteq N[x]$, then clearly $f$ is a weak Roman dominating function of $G$ and so $\gamma_r(G) \leq \gamma_{cr}(G)$. It follows from Theorem 7 that $\gamma_r(G) = \gamma_{cr}(G)$.

Conversely, let $\gamma_r(G) = \gamma_{cr}(G)$. There exists a $\gamma_r(G)$-function $f = (V_0, V_1, V_2)$ such that $f$ is a co-Roman dominating function of $G$ (see Theorem 3.3 of [2]). By assumption, $f$ is a $\gamma_{cr}(G)$-function. Assume $x \in V_0$ is an arbitrary vertex. Since $f$ is a weak Roman dominating function, a vertex $x'$ in $V_1 \cup V_2$ such that the function $g : V(G) \rightarrow \{0, 1, 2\}$ defined by $g(x) = 1, g(x') = f(x') - 1$ and $g(u) = f(u)$ otherwise, is safe. If $x$ has a neighbor in $V_2$, then we are done. Assume $x$ has no neighbor in $V_2$. It follows that $x' \in V_1$. Since $f$ is safe, we must have $pn(x', V_1 \cup V_2) \subseteq N[x]$ and the proof is complete. ■

**Proposition 9.** Let $G$ be a connected graph of order at least two. Then $\gamma(G) = \gamma_{cr}(G)$ if and only if there exists a $\gamma(G)$-set $S$ such that each vertex $x \in S$ has a neighbor $x' \in V \setminus S$ with $pn(x, S) \subseteq N[x']$.

**Proof.** Let $\gamma(G) = \gamma_{cr}(G)$. Assume $f = (V_0, V_1, V_2)$ is a $\gamma_{cr}(G)$-function. Since $V_1 \cup V_2$ is a dominating set, we deduce from $\gamma(G) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = \gamma_{cr}(G)$ that $V_2 = \emptyset$ and $V_1$ is a $\gamma(G)$-set. Let $x \in V_1$ be an arbitrary vertex. Since $f$ is a co-Roman dominating function, there is a vertex $x' \in V_0 \cap N(x)$ such that $((V_0 \setminus \{x'\}) \cup \{x\}, (V_1 \setminus \{x\}) \cup \{x'\}, \emptyset)$ is a $\gamma_{cr}(G)$-function. It follows that $(V_1 \setminus \{x\}) \cup \{x'\}$ is a $\gamma(G)$-set and this implies that $pn(x, V_1) \subseteq N[x']$.

Conversely, let $S$ be a $\gamma(G)$-set such that each vertex $x \in S$ has a neighbor $x' \in V \setminus S$ with $pn(x, S) \subseteq N[x']$. Then the function $f = (V(G) \setminus S, S, \emptyset)$ is clearly a co-Roman dominating function of weight $\gamma(G)$ and so $\gamma_{cr}(G) \leq \gamma(G)$. It follows that $\gamma_{cr}(G) = \gamma(G)$. ■

**Corollary 10.** Let $G$ be a connected graph of order at least two with $\gamma(G) = \gamma_{cr}(G)$. Then for any $\gamma_{cr}(G)$-function $f = (V_0, V_1, V_2)$, $V_2 = \emptyset$.

**Corollary 11.** Let $G$ be a connected graph of order at least two. If $\gamma(G) = \gamma_{cr}(G)$, then $G$ has no strong support vertex.
For a tree $T$, let $M(T) = \{v \mid \text{there exists a } \gamma_{cr}(T)\text{-function } f \text{ such that } f(v) = 1\}$. In what follows, we present a constructive characterization of trees $T$ with $\gamma(T) = \gamma_{cr}(T)$. In order to do this, we define a family of trees as follows. Let $\mathcal{T}$ be the collection of trees $T$ that can be obtained from a sequence $T_1, T_2, \ldots, T_k$ of trees for some $k \geq 1$, where $T_1$ is a $P_2$ and $T = T_k$. If $k \geq 2$, $T_{i+1}$ can be obtained from $T_i$ by one of the following three operations. Let one vertex of $P_2$ be considered as a support vertex.

**Operation $O_1$.** If $v \in M(T_i)$, then the tree $T_{i+1}$ is obtained from $T_i$ by adding a pendant $P_3 = xyz$ and adding the edge $vx$ (see Figure 1(a)).

**Operation $O_2$.** If $v$ is a support vertex of $T_i$, then the tree $T_{i+1}$ is obtained from $T_i$ by adding a pendant $P_2 = xy$ and adding the edge $vx$ (see Figure 1(b)).

**Operation $O_3$.** If $v \in T_i$, then the tree $T_{i+1}$ is obtained from $T_i$ by adding a healthy spider with at least two feet headed at $x$ and adding the edge $vx$ (see Figure 1(c)).

\begin{figure}[h]
\begin{center}
\includegraphics[width=\textwidth]{figure1}
\caption{(a) Operation $O_1$. (b) Operation $O_2$. (c) Operation $O_3$.}
\end{center}
\end{figure}

**Lemma 12.** If $T_i$ is a tree with $\gamma(T_i) = \gamma_{cr}(T_i)$ and $T_{i+1}$ is a tree obtained from $T_i$ by Operation $O_1$, then $\gamma(T_{i+1}) = \gamma_{cr}(T_{i+1})$.

**Proof.** Let $f$ be a $\gamma_{cr}(T_i)$-function and $v$ a vertex of $T_i$ with $f(v) = 1$. Then the function $f' : V(T_{i+1}) \to \{0, 1, 2\}$ by $f'(y) = 1$, $f'(x) = f'(z) = 0$ and $f'(u) = f(u)$ for $u \in V(T_i)$, is a co-Roman dominating function on $T_{i+1}$ and so $\gamma_{cr}(T_{i+1}) \leq \gamma_{cr}(T_i) + 1$.

It is easy to see that $\gamma(T_{i+1}) = \gamma(T_i) + 1$. Now we have

$$
\gamma(T_i) + 1 = \gamma(T_{i+1}) \leq \gamma_{cr}(T_{i+1}) \leq \gamma_{cr}(T_i) + 1 = \gamma(T_i) + 1
$$

yielding $\gamma(T_{i+1}) = \gamma_{cr}(T_{i+1})$.

**Lemma 13.** If $T_i$ is a tree with $\gamma(T_i) = \gamma_{cr}(T_i)$ and $T_{i+1}$ is a tree obtained from $T_i$ by Operation $O_2$, then $\gamma(T_{i+1}) = \gamma_{cr}(T_{i+1})$.
Proof. Clearly, any $\gamma_{cr}(T_i)$-function can be extended to a co-Roman dominating function by assigning 1 to $x$ and 0 to $y$ implying that $\gamma_{cr}(T_{i+1}) \leq \gamma_{cr}(T_i) + 1$.

Since $v$ is a support vertex, one can easily check that $\gamma(T_{i+1}) = \gamma(T_i) + 1$.

Now the result follows as in the proof of Lemma 12.

Lemma 14. If $T_i$ is a tree with $\gamma(T_i) = \gamma_{cr}(T_i)$ and $T_{i+1}$ is a tree obtained from $T_i$ by Operation $O_3$, then $\gamma(T_{i+1}) = \gamma_{cr}(T_{i+1})$.

Proof. Let the added spider has exactly $k$ feet. Obviously, any $\gamma_{cr}(T_i)$-function can be extended to a co-Roman dominating function by assigning 1 to the support vertices of spider and 0 to the remaining vertices of spider and this implies that $\gamma_{cr}(T_{i+1}) \leq \gamma_{cr}(T_i) + k$. Moreover, it is easy to verify that $\gamma(T_{i+1}) = \gamma(T_i) + k$ and the result follows as in the proof of Lemma 12.

Lemma 15. If $T \in \mathcal{T}$, then $\gamma(T) = \gamma_{cr}(T)$.

Proof. Let $T \in \mathcal{T}$. By definition, there exists a sequence of trees $T_1, T_2, \ldots, T_k$ ($k \geq 1$) such that $T_1 = K_2$, and if $k \geq 2$, $T_{i+1}$ can be obtained recursively from $T_i$ by Operation $O_1$, $O_2$ or $O_3$ for $i = 1, 2, \ldots, k-1$. We proceed by induction on $k$.

If $T = K_2$, then clearly $\gamma(T) = \gamma_{cr}(T) = 1$.

Suppose $k \geq 2$ and the result holds for each tree $T \in \mathcal{T}$ which can be obtained from a sequence of operations of length $k-1$ and let $T' = T_{k-1}$. By the induction hypothesis, we have $\gamma(T') = \gamma_{cr}(T')$.

Since $T = T_k$ is obtained from $T'$ by one of the Operations $O_1$, $O_2$ or $O_3$ from $T'$, we have $\gamma(T) = \gamma_{cr}(T)$ by Lemmas 12, 13 and 14.

Theorem 16. Let $T$ be a tree of order $n \geq 2$. Then $\gamma(T) = \gamma_{cr}(T)$ if and only if $T \in \mathcal{T}$.

Proof. The sufficiency follows from Lemma 15. We use induction on $n$ to prove the necessity. If $n = 2$, then $T = P_2$ that belongs to $\mathcal{T}$. Assume $n \geq 3$ and that the result holds for any tree of order less than $n$. Let $T'$ be a tree of order $n$ with $\gamma(T) = \gamma_{cr}(T)$. Let $P = v_1v_2\cdots v_\ell$ be a diametral path in $T$ and root $T$ at $v_\ell$. By Corollary 11, we have $d(v_2) = 2$. Consider the following cases.

Case 1. $v_3$ is a support vertex. Let $w$ be a leaf adjacent to $v_3$ and let $T' = T - \{v_1, v_2\}$. If $f$ is a $\gamma_{cr}(T)$-function, then clearly $f(v_1) + f(v_2) \geq 1$ and $f(v_3) + f(w) \geq 2$. It is easy to verify that the function $f$, restricted to $T'$ is a co-Roman dominating function implying that $\gamma_{cr}(T) \geq \gamma_{cr}(T') + 1$.

Clearly $\gamma(T) = \gamma(T') + 1$, and we deduce from $\gamma(T) = \gamma_{cr}(T) \geq \gamma_{cr}(T') + 1 = \gamma(T') + 1 = \gamma(T)$ that $\gamma_{cr}(T') = \gamma(T')$. By the induction hypothesis, we have $T' \in \mathcal{T}$. Now $T$ can be obtained from $T'$ by Operation $O_2$. 

Case 2. \(d(v_3) = 2\). Let \(T' = T - \{v_1, v_2, v_3\}\). By Proposition 3, \(n \geq 4\). Clearly \(\gamma(T) = \gamma(T') + 1\). Assume \(f = (V_0, V_1, V_2)\) is a \(\gamma_{cr}(T)\)-function. By Corollary 10, \(V_2 = \emptyset\). Clearly \(f(v_1) + f(v_2) = 1\) and \(f(v_3) + f(v_4) \geq 1\). If \(f(v_3) = f(v_4) = 1\), then the function \(g : V(T) \rightarrow \{0, 1, 2\}\) defined by \(g(v_4) = g(v_2) = 1, g(v_1) = g(v_3) = 0\) and \(g(x) = f(x)\) otherwise, is a co-Roman dominating function of \(T\) of weight less than \(\omega(f)\) which is a contradiction. Hence \(f(v_3) = 0\) or \(f(v_4) = 0\) and so \(f(v_3) + f(v_4) = 1\). Consider the following.

- \(f(v_3) = 1\) and \(f(v_4) = 0\). If \(f(x) = 1\) for some \(x \in N_{T'}(v_4)\), then the function \(g : V(G) \rightarrow \{0, 1\}\) defined by \(g(v_2) = 1, g(v_1) = g(v_3) = 0\) and \(g(x) = f(x)\) otherwise, is a dominating function of \(T\) of weight less than \(\omega(f)\) which contradicts \(\gamma(T) = \gamma_{cr}(T)\). Thus \(f(x) = 0\) for each \(x \in N_{T'}(v_4)\). Now the function \(h : V(T') \rightarrow \{0, 1\}\) defined by \(h(v_4) = 1\) and \(h(x) = f(x)\) otherwise, is a co-Roman dominating function of \(T\) of weight \(\omega(f) - 1\). It follows from

\[
\gamma(T) = \gamma_{cr}(T) \geq \gamma_{cr}(T') + 1 = \gamma(T') + 1 = \gamma(T)
\]

that \(\gamma_{cr}(T') = \gamma(T')\) and that \(h\) is a \(\gamma_{cr}(T')\)-function with \(h(v_4) = 1\). By the induction hypothesis, we have \(T' \in \mathcal{T}\) and so \(T\) can be obtained from \(T'\) by Operation \(O_1\). Thus \(T \in \mathcal{T}\).

- \(f(v_3) = 0\) and \(f(v_4) = 1\). As above we have \(f(x) = 0\) for some \(x \in N_{T'}(v_4)\). Then the function \(f\) restricted to \(T'\) is a co-Roman dominating function of \(T'\) and so \(\gamma_{cr}(T) \geq \gamma_{cr}(T') + 1\). Using above argument, we obtain \(T \in \mathcal{T}\).

Case 3. \(v_3\) is not a support vertex and \(d(v_3) \geq 3\). Let \(T'\) be the component of \(T - v_3 v_4\) containing \(v_3\). Then \(T'\) is a spider with at least \(k\) feet where \(k = \deg(v_3) - 1\). Clearly \(\gamma(T) = \gamma(T') + k\). Now we show that \(\gamma_{cr}(T) \geq \gamma_{cr}(T') + k\). Let \(u_1, \ldots, u_k\) be the children of \(v_3\) and \(w_i\) be the leaf adjacent to \(u_i\) for \(i = 1, \ldots, k\). Let \(f = (V_0, V_1, V_2)\) be a \(\gamma_{cr}(T)\)-function. By Corollary 10, \(V_2 = \emptyset\). Obviously \(f(u_i) + f(w_i) = 1\) for each \(i\). As Case 2, we can see that \(f(v_4) = 0\) or \(f(v_3) = 0\). If \(f(v_4) = f(v_3) = 0\), then the function \(f\) restricted to \(T'\) is a co-Roman dominating function of weight \(\gamma_{cr}(T) - k\) and so \(\gamma_{cr}(T) \geq \gamma_{cr}(T') + k\). Consider the following subcases.

Subcase 3.1. \(f(v_3) = 1\) and \(f(v_4) = 0\). If \(f(x) = 1\) for some \(x \in N_{T'}(v_4)\), then the function \(g : V(G) \rightarrow \{0, 1\}\) defined by \(g(v_4) = g(w_i) = 0, g(u_i) = 1\) for \(1 \leq i \leq k\) and \(g(x) = f(x)\) otherwise, is a dominating function of \(T\) of weight less than \(\omega(f)\) contradicting \(\gamma(T) = \gamma_{cr}(T)\). Thus \(f(x) = 0\) for each \(x \in N_{T'}(v_4)\). Now the function \(h : V(T') \rightarrow \{0, 1\}\) defined by \(h(v_4) = 1\) and \(h(x) = f(x)\) otherwise, is a co-Roman dominating function of \(T\) of weight \(\omega(f) - k\) and hence \(\gamma_{cr}(T) \geq \gamma_{cr}(T') + k\).

Subcase 3.2. \(f(v_3) = 0\) and \(f(v_4) = 1\). As above we have \(f(x) = 0\) for some \(x \in N_{T'}(v_4)\). Then the function \(f\) restricted to \(T'\) is a co-Roman dominating function of \(T'\) and so \(\gamma_{cr}(T) \geq \gamma_{cr}(T') + k\).
Thus in all cases $\gamma_{cr}(T) \geq \gamma_{cr}(T') + k$. As Case 2, we deduce that $\gamma_{cr}(T') = \gamma(T')$ and so by the induction hypothesis we have $T' \in \mathcal{T}$. Now $T$ can be obtained from $T'$ by Operation $O_3$ and hence $T \in \mathcal{T}$. This completes the proof.

2. Bounds on Co-Roman Domination

In this section, we present some sharp bounds on the co-Roman domination number. First we prove two upper bounds on the co-Roman domination number in terms of matching number.

**Theorem 17.** For any connected graph $G$ of order $n \geq 2$,

$$\gamma_{cr}(G) \leq n - \alpha'(G).$$

**Proof.** Let $M = \{u_1v_1, \ldots, u_{\alpha'}v_{\alpha'}\}$ be a maximum matching of $G$ and let $X$ be the independent set of $M$-unsaturated vertices. If $y$ and $z$ are vertices of $X$ and $yu_i \in E(G)$, then since the matching $M$ is maximum, $zv_i \notin E(G)$. Therefore, for all $i \in \{1, 2, \ldots, \alpha'\}$ there are at most two edges between the sets $\{u_i, v_i\}$ and $\{y, z\}$. Assume $S$ is the set of all vertices in $X$ which belongs to a triangle with an edge in $M$. Let $S = \{x_1, \ldots, x_s\}$ if $S \neq \emptyset$ and $X \setminus S = \{y_1, \ldots, y_k\}$ if $X \setminus S \neq \emptyset$.

First let $S = \emptyset$. Then $vu_i \notin E(G)$ or $vu_i \notin E(G)$ for each $v \in X$ and each $i \in \{1, \ldots, \alpha'\}$. We may assume $N(x) \subseteq \{u_1, \ldots, u_{\alpha'}\}$ for each $x \in X$. Define $f : V(G) \to \{0, 1, 2\}$ by $f(u_i) = 0$ for $1 \leq i \leq \alpha'$ and $f(x) = 1$ otherwise. Clearly, $f$ is a co-Roman dominating function of $G$ of weight $\alpha' + |X|$ and hence

$$\gamma_{cr}(G) \leq \alpha'(G) + |X| = \alpha'(G) + (n - 2\alpha'(G)) = n - \alpha'(G).$$

Now let $S \neq \emptyset$. We may assume, without loss of generality, that $x_iu_i, x_iv_i \in E(G)$ for $i = 1, \ldots, s$. As above, we can assume that $N(x) \subseteq \{u_1, \ldots, u_{\alpha'}\}$ for each $x \in X \setminus S$. Define $f : V(G) \to \{0, 1, 2\}$ by $f(x) = 0$ for $x \in S \cup \{u_1, \ldots, u_{\alpha'}\}$ and $f(x) = 1$ otherwise. Obviously, $f$ is a co-Roman dominating function of $G$ of weight $\alpha'(G) + |X| - |S|$ and hence

$$(2) \quad \gamma_{cr}(G) \leq \alpha'(G) + |X| - |S| = \alpha'(G) + (n - 2\alpha') - |S| \leq n - \alpha'(G) - |S|.$$

This completes the proof.

**Theorem 18.** For any connected graph $G$ of order $n \geq 2$ with $\delta(G) \geq 2$,

$$\gamma_{cr}(G) \leq \alpha'(G).$$
Proof. Let $M$, $X$ and $S$ be the sets defined in the proof of Theorem 17. Assume first that $S = \emptyset$. Then as above we may assume $N(x) \subseteq \{u_1, \ldots, u_\alpha\}$ for each $x \in X$. Define $f : V(G) \to \{0, 1, 2\}$ by $f(u_i) = 1$ for $1 \leq i \leq \alpha$ and $f(x) = 0$ otherwise. Since $\delta(G) \geq 2$, the function $f_i : V(G) \to \{0, 1, 2\}$ defined by $f(u_i) = 0$, $f(v_i) = 1$ and $f(x) = f(x)$ otherwise, is safe for each $i$. Thus $f$ is a co-Roman dominating function of $G$ of weight $\alpha'(G)$ and so $\gamma_{cr}(G) \leq \alpha'(G)$.

Now let $S = \{x_1, \ldots, x_s\}$. We may assume, without loss of generality, that $x_i u_i, x_i v_i \in E(G)$ for $i = 1, \ldots, s$. As above, we can assume that $N(x) \subseteq \{u_1, \ldots, u_\alpha\}$ for each $x \in X \setminus S$. It is easy to see that the function $f$ defined above is a co-Roman dominating function of $G$. Thus $\gamma_{cr}(G) \leq \alpha'(G)$ and the proof is complete.

Theorem 19. For any connected graph $G$ of order $n \geq 2$,

$$\gamma_{cr}(G) \leq 2\alpha'(G).$$

Proof. Let $M$, $X$ and $S$ be the sets defined in the proof of Theorem 17. As Theorem 17, we may assume that $x_i u_i, x_i v_i \in E(G)$ for $i = 1, \ldots, s$ if $S \neq \emptyset$ and $N(x) \subseteq \{u_1, \ldots, u_\alpha\}$ for each $x \in X \setminus S$. Then the function $f : V(G) \to \{0, 1, 2\}$ defined by $f(u_i) = 1$ if $u_i$ is adjacent to a vertex in $S$, $f(u_i) = 2$ if $u_i$ is adjacent to a vertex in $X \setminus S$ and $f(x) = 0$ otherwise, is a co-Roman dominating function of $G$ and so $\gamma_{cr}(G) \leq |S| + 2|X - S| = 2\alpha'(G) - |S| \leq 2\alpha'(G)$.

A set $X \subseteq V(G)$ is called a 2-packing if $d(u, v) > 2$ for any different vertices $u$ and $v$ of $X$. The 2-packing number $\rho(G)$ is the maximum cardinality of a 2-packing of $G$.

Theorem 20. For any connected graph $G$ of order $n \geq 2$ with $\delta(G) \geq 2$,

$$\gamma_{cr}(G) \leq n - \rho(G)(\delta(G) - 1).$$

Proof. Let $S$ be a 2-packing of $G$ of size $\rho(G)$. Define $f : V(G) \to \{0, 1, 2\}$ by $f(x) = 2$ for $x \in S$, $f(x) = 0$ for $x \in \bigcup_{v \in S} N(v)$ and $f(x) = 1$ otherwise. Clearly, $f$ is a co-Roman dominating function of $G$ and hence

$$\gamma_{cr}(G) \leq (n - |\bigcup_{v \in S} N[v]|) + 2|S| = n - \sum_{v \in S} |N[v]| + 2\rho(G) \leq n - \rho(G)(\delta(G) + 1) - 2\rho(G) = n - \rho(G)(\delta(G) - 1),$$

as desired.

Proposition 21. Let $G$ be a simple connected graph of order $n$ with $\delta(G) \geq 2$ and $g(G) \geq 5$. Then $\gamma_{cr}(G) \leq \frac{2(n - g(G))}{3} + \left\lceil \frac{2g(G)}{5} \right\rceil$. 

Proof. If $G$ is an $n$-cycle, then the result follows by Proposition 3. Assume $G$ is not a cycle and $C$ is a cycle of length $g(G)$ in $G$. Let $G'$ be the graph obtained from $G$ by removing the vertices of $V(C)$. Since $g(G) \geq 5$, each vertex of $G'$ can be adjacent to at most one vertex of $C$ which implies $\delta(G') \geq 1$. By Corollary 5, we have $\gamma_{cr}(G') \leq \frac{2(n-g(G))}{3}$. Let $g$ be a $\gamma_{cr}(G')$-function and $h$ be a $\gamma_{cr}(C)$-function. Define $f : V(G) \rightarrow \{0, 1, 2\}$ by $f(v) = g(v)$ for $v \in V(G')$ and $f(v) = h(v)$ for $v \in V(C)$. Obviously, $f$ is a co-Roman dominating function and so

$$\gamma_{cr}(G) \leq \frac{2(n-g(G))}{3} + \left\lceil \frac{2g(G)}{5} \right\rceil.$$ 

\[ \Box \]

3. Characterization of Graphs $G$ of Order $n$ with $\gamma_{cr}(G) = \frac{2n}{3}$

In this section, we characterize the graphs attaining the upper bound in Corollary 5. For any arbitrary tree $T$, let $T_{cr}$ be the tree obtained from $T$ by adding exactly two pendant edges at each vertex of $T$. Note that $n(T_{cr}) = 3n(T)$. Let $\mathcal{F}$ be the family of all trees $T_{cr}$. In fact, $\mathcal{F}$ is the family of trees $T$ such that $V(T)$ can be partitioned into sets inducing $P_3$ such that the subgraph induced by the central vertices of these paths is connected.

Lemma 22. If $T \in \mathcal{F}$, then $\gamma_{cr}(T) = \frac{2n(T)}{3}$.

Proof. Let $T \in \mathcal{F}$ and let $f$ be a $\gamma_{cr}$-function on $T$. Then $T$ is obtained from a tree $T'$ by adding exactly two pendant edges at each vertex of $T'$. For each non-leaf vertex $v \in V(T)$, let $L_v = \{v_1, v_2\}$. It is easy to see that for any non-leaf vertex $v \in V(T)$, $f(v) + f(v_1) + f(v_2) \geq 2$, otherwise we have an unprotected vertex in either $f$ or $f_{v_1}$ for some $i = 1, 2$. Hence, $\gamma_{cr}(T) = \omega(f) = \sum_{v \in V(T')} (f(v) + f(v_1) + f(v_2)) \geq 2n(T') = \frac{2n(T)}{3}$. Now the result follows from Proposition 4. \[ \Box \]

Lemma 23. Let $q \geq p \geq 1$ and let $T = DS(p, q)$. Then $\gamma_{cr}(T) = \frac{2n(T)}{3}$ if and only if $q = p = 2$.

Proof. If $q = p = 2$, then Lemma 22 implies $\gamma_{cr}(T) = \frac{2n(T)}{3}$. Conversely, let $\gamma_{cr}(T) = \frac{2n(T)}{3}$. It follows from Proposition 3 that $q \geq 2$. If $p = 1$, then clearly $\gamma_{cr}(T) = 3 < \frac{2n(T)}{3}$, a contradiction. Suppose that $p \geq 2$. If $q > 2$, then we have $\gamma_{cr}(T) \leq 4 < \frac{2n(T)}{3}$, a contradiction again. Thus $q = p = 2$ and the proof is complete. \[ \Box \]

Theorem 24. Let $T$ be a tree of order $n \geq 3$. Then $\gamma_{cr}(T) = \frac{2n}{3}$ if and only if $T \in \mathcal{F}$.
Proof. According to Lemma 22, we only need to prove the necessity. Let $T$ be a tree of order $n \geq 3$ with $\gamma_{cr}(T) = \frac{2n}{3}$. Note that $n$ is a multiple of 3. The proof is by induction on $n$. If $n = 3$, then the only tree $T$ of order 3 and $\gamma_{cr}(T) = 2$ is $P_3 \in \mathcal{F}$. Let $n \geq 4$ and let the statement hold for all trees of order less than $n$. Suppose that $T$ is a tree of order $n$ with $\gamma_{cr}(T) = \frac{2n}{3}$. If diam$(T) = 2$, then $T = K_{1,n}$ and we deduce from Proposition 2 that $T = P_3$ and so $T \in \mathcal{F}$. If diam$(T) = 3$, then we deduce from Lemma 23 that $T = DS(2,2)$ and so $T \in \mathcal{F}$. Henceforth we assume that diam$(T) \geq 4$. Let $v_1v_2 \cdots v_k$ ($k \geq 5$) be a diametral path in $T$ and root $T$ at $v_k$. We show that deg$_T(v_2) = 3$. Let $T' = T - T_{v_2}$ and $f$ be a $\gamma_{cr}(T')$-function. If deg$_T(v_2) \geq 4$, then the function $g : V(T) \to \{0,1,2\}$ defined by $g(v_2) = 2$, $g(x) = 0$ if $x \in L_{v_2}$ and $g(x) = f(x)$ for $x \in T'$, is a CRDF on $T$ of weight $\omega(f) + 2$. By Proposition 4, we have $\gamma_{cr}(T) \leq \omega(g) \leq \gamma_{cr}(T') + 2 \leq \frac{2n(T')}{3} + 2 \leq \frac{2(n-4)}{3} + 2 < \frac{2n}{3}$, which is a contradiction. If deg$_T(v_2) = 2$, then the function $g : V(T) \to \{0,1,2\}$ defined by $g(v_2) = 1$, $g(v_1) = 0$ and $g(x) = f(x)$ for $x \in T'$, is a CRDF on $T$ of weight $\omega(f) + 1$. By Proposition 4, we have $\gamma_{cr}(T) \leq \omega(g) \leq \gamma_{cr}(T') + 1 \leq \frac{2(n-2)}{3} + 1 < \frac{2n}{3}$, a contradiction again. Thus deg$_T(v_2) = 3$. Assume that $T' = T - T_{v_2}$. As above, we have

$$\frac{2n(T)}{3} = \gamma_{cr}(T) \leq \gamma_{cr}(T') + 2 \leq \frac{2n(T')}{3} + 2 = \frac{2(n-3)}{3} + 2 = \frac{2n}{3}. $$

Thus all inequalities in the above inequality chain must be equalities and so $\gamma_{cr}(T') = \frac{2n(T')}{3}$. By the induction hypothesis we have $T' \in \mathcal{F}$. Now we show that $v_3$ is a leaf of $T'$. If $v_3$ is a leaf in $T'$, then let $T'' = T - T_{v_3}$ and let $h$ be a $\gamma_{cr}(T'')$-function. Define the function $g : V(T) \to \{0,1,2\}$ by $g(v_2) = 2$, $g(v) = 0$ if $v \in N_T(v_2)$ and $g(x) = h(x)$ for $x \in T''$. Clearly, $g$ is a CRDF on $T$ of weight $\omega(f) + 2$. By Proposition 4, we have $\gamma_{cr}(T) \leq \omega(g) = \gamma_{cr}(T') + 2 \leq \frac{2(n-4)}{3} + 2 < \frac{2n}{3}$, a contradiction. Thus $v_3$ is a non-leaf vertex of $T'$ and so $T \in \mathcal{F}$. This completes the proof.

Theorem 24. Let $G$ be a connected $n$-vertex graph with $n \geq 3$. Then $\gamma_{cr}(G) = \frac{2n}{3}$ if and only if $G$ is obtained from $\frac{n}{3}P_3$ by adding edges between the centers of the paths $P_3$ such that the resulting graph is connected.

Proof. If $G$ has the specified form, then clearly every CRDF puts weight at least 2 on the vertex set of each copy of $P_3$.

Now suppose that $\gamma_{cr}(G) = \frac{2n}{3}$. Since adding edges cannot increase $\gamma_{cr}(G)$, every spanning tree of $G$ belongs to $\mathcal{F}$. Given a spanning tree $T$, let $S_1, S_2, \ldots, S_\frac{n}{3}$ be the 3-sets in the special partition of $V(T)$. The assignment of weight 2 that guards $S_i$ can be chosen independently of any other $S_j$. If any edge of $G$ joins vertices of $S_i$ and $S_j$ that are not the centers of the paths they induce, then a CRDF with weight less than $\frac{2n}{3}$ can be built as in the proof of Theorem 24. This completes the proof.
4. Graphs with Large Co-Roman Domination Number

In this section, we characterize all graphs of order $n$ with co-Roman domination number $n-2$ and $n-3$. The first result is an immediate consequence of Theorem 17.

**Corollary 26** (Theorem 4.2 in [2]). Let $G$ be a connected graph on $n \geq 2$ vertices. Then $\gamma_{cr}(G) = n - 1$ if and only if $G = K_2$ or $K_{1,2}$.

Arumugam et al. [2] posed the following problem.

**Problem.** Characterize graphs $G$ such that $\gamma_{cr}(G) = n - 2$.

Next we solve this problem.

**Theorem 27.** Let $G$ be a connected graph on $n \geq 2$ vertices. Then $\gamma_{cr}(G) = n - 2$ if and only if $G$ is a graph on four vertices different from $K_4$ and $K_4 - e$, or $G \cong DS(2,1)$, or $G \cong DS(2,2)$.

**Proof.** By Theorem 17, we have $\alpha'(G) \leq 2$. If $\alpha'(G) = 1$, then $G$ is the star $K_{1,n-1}$ and we conclude from Proposition 2 that $G = K_{1,3}$. Assume that $\alpha'(G) = 2$. Let $M$, $X$ and $S$ be the sets defined in the proof of Theorem 17. By (2), we have $S = \emptyset$. As above, we may assume $N(x) \subseteq \{u_1, \ldots, u_{\alpha'}\}$ for each $x \in X$.

If $u_i$ has at least two neighbors in $X$ for some $i$, say $i = 1$, then the function $f : V(G) \rightarrow \{0, 1, 2\}$ defined by $f(u_1) = 2, f(u_i) = 0$ for $2 \leq i \leq \alpha'$, $f(x) = 0$ if $x = v_1$ or $x \in N(u_1) \cap X$ and $f(x) = 1$ otherwise, is clearly a co-Roman dominating function of $G$ of weight $n - \alpha'(G) - 1$ which leads to a contradiction. Hence each $u_i$ has at most one neighbor in $X$ and this implies that $|X| \leq 2$. If $|X| = 0$, then $n = 4$ and obviously $G$ is a connected graph on four vertices different from $K_4$ and $K_4 - e$. Hence $|X| \geq 1$.

First let $|X| = 2$. Since $X$ is independent and $G$ is connected, we may assume that $u_iy_i \in E(G)$ for $i = 1, 2$. Since each $u_i$ has at most one neighbor in $X$, we deduce that $\deg(y_i) = 1$ for $i = 1, 2$. Considering the matching $M' = \{u_1y_1, u_2y_2\}$ instead of $M$, we have $\deg(v_1) = \deg(v_2) = 1$. Since $G$ is connected, we have $u_1u_2 \in E(G)$ and hence $G = DS(2,2)$.

Now let $|X| = 1$. Since $G$ is connected, we suppose that $u_1y_1 \in E(G)$. If $u_2y_1 \in E(G)$, then the function $f_1 : V(G) \rightarrow \{0, 1, 2\}$ defined by $f_1(u_1) = f_1(u_2) = 1$ and $f_1(x) = 0$ otherwise, is clearly a co-Roman dominating function of $G$ of weight 2, a contradiction. Thus $\deg(y_1) = 1$. Considering the matching $M' = \{u_1y_1, u_2v_2\}$ instead of $M$, we obtain $\deg(v_1) = 1$. Since $G$ is connected, we may assume that $u_1u_2 \in E(G)$. If $u_1v_2 \in E(G)$, then clearly $\gamma_{cr}(G) \leq 2$ which is a contradiction. Thus $G = DS(1,2)$ and the proof is complete.

The corona graph $cor(H)$ of a graph $H$ is the graph obtained from $H$ by attaching a leaf to every vertex of $H$. We recall the following result established by Payan and Xuong [12] (see also Fink et al. [8]).
Theorem 28. For a graph $G$ with even order $n$ and with no isolated vertices, 
\[ \gamma(G) = \frac{n}{2} \] if and only if the components of $G$ are the cycle $C_4$ or the corona $\text{cor}(H)$ for any connected graph $H$.

Now we characterize all connected graphs $G$ of order $n \geq 4$ with $\gamma_{cr}(G) = n - 3$. To do this, we introduced some families of graphs.

Let

- $\mathcal{G}_1 = \{K_4, K_4 - e, K_{1,4}\}$,
- $\mathcal{G}_2$ be the family of connected graphs $G$ obtained from a triangle and a path $P_2$ by adding some edges between them so that the resulting graph has at most one universal vertex,
• $G_3$ be the family of connected graphs $G$ obtained from a path $P_3$ and a path $P_2$ by adding some edges between them such that the resulting graph is different from $DS(1, 2)$ and has at most one universal vertex,

• $G_4$ be the family of connected graphs $G \not\cong DS(2, 2)$ of order 6 consisting of $cor(P_3), cor(C_3)$ and all graphs $G$ with $\Delta(G) \leq 4$, for which every $\gamma(G)$-set $S$ has a vertex $x$ such that $x$ has no neighbor $x' \in V \setminus S$ with $pn(x, S) \subseteq N[x']$.

• $G_5 = \{G_1, G_2, \ldots, G_{13}\}$,

• $G_6$ be the family of connected graphs $G$ obtained from three paths $v_1u_1y_1, v_2u_2y_2$ and $v_3u_3$ by adding edges between $u_1, u_2, u_3$ such that the resulting graph is connected,

• $G_7$ be the family of connected graphs $G$ obtained from $3P_3$ by adding edges between the centers of the paths $P_3$ such that the resulting graph is connected.

![Graphs H1 and H2](image)

Figure 3. Two graphs $G$ of order 6 with $\gamma_{cr}(G) = 3$.

**Theorem 29.** Let $G$ be a connected graph on $n \geq 4$ vertices, then $\gamma_{cr}(G) = n - 3$ if and only if $G \in \bigcup_{i=1}^{7} G_i$.

**Proof.** Let $G \in \bigcup_{i=1}^{7} G_i$. We deduce from (1), Corollary 26 and Theorem 27 that $\gamma_{cr}(G) \leq n - 3$. If $G = K_{1,4}$, then $\gamma_{cr}(G) = 2 = n - 3$ by Proposition 2, and if $G \in G_1 \setminus \{K_{1,4}\}$ then $\gamma_{cr}(G) = 1 = n - 3$ by Observation 6. If $G \in G_2 \cup G_3$, then we conclude from Observation 6 that $\gamma_{cr}(G) \geq 2 = n - 3$ and so $\gamma_{cr}(G) = n - 3$. If $G \in \{cor(P_3), cor(C_3)\}$, then by Proposition 9 and Theorem 28 we have $\gamma_{cr}(G) = \gamma(G) = 3$, and if $G \in G_4 - \{cor(P_3), cor(C_3)\}$, then clearly $\gamma(G) = 2$ and Proposition 9 implies that $\gamma_{cr}(G) \geq \gamma(G) + 1 = 3 = n - 3$ and so $\gamma_{cr}(G) = n - 3$. If $G \in G_5 \cup G_6$, then it is easy to see that $\gamma_{cr}(G) = n - 3$. Finally, if $G \in G_7$, then by Theorem 25, we have $\gamma_{cr}(G) = 6 = n - 3$. 


Conversely, let \( \gamma_{cr}(G) = n - 3 \). By Corollary 5 and Theorem 17, we obtain \( n \leq 9 \) and \( \alpha'(G) \leq 3 \). If \( \alpha'(G) = 1 \), then \( G \) is the star \( K_{1,n-1} \) and we conclude from Proposition 2 that \( G = K_{1,4} \in \mathcal{G}_1 \). Assume that \( \alpha'(G) \geq 2 \). Suppose \( M, X \) and \( S \) are the sets defined in the proof of Theorem 17. We consider the following cases.

**Case 1.** \( \alpha'(G) = 3 \). Since \( n \leq 9 \), we must have \( |X| \leq 3 \). If \( |X| = 3 \), then \( n = 9 \) and we conclude from Theorem 25 that \( G \in \mathcal{G}_7 \). Let \( |X| \leq 2 \). By (2), we have \( S = \emptyset \). As above, we may assume \( N(x) \subseteq \{u_1, u_2, u_3\} \) for each \( x \in X \). Consider the following subcases.

**Subcase 1.1.** \( |X| = 2 \). If \( u_iy_1, u_iy_2 \in E(G) \) for some \( i \), say \( i = 1 \), then the function \( f_1 : V(G) \to \{0, 1, 2\} \) defined by \( f_1(u_1) = 2, f_1(u_2) = f_1(u_3) = 1 \) and \( f_1(x) = 0 \) otherwise, is clearly a co-Roman dominating function of \( G \) of weight 4 which is a contradiction. Thus each \( u_i \) has at most one neighbor in \( X \). Assume without loss of generality that \( u_1y_1, u_2y_2 \in E(G) \). If \( y_1u_3 \in E(G) \) (the case \( y_2u_3 \in E(G) \) is similar), then the function \( f_2 : V(G) \to \{0, 1, 2\} \) defined by \( f_2(u_1) = f_2(u_3) = 1, f_2(u_2) = 2 \) and \( f_2(x) = 0 \) otherwise, is clearly a co-Roman dominating function of \( G \) of weight 4 which is a contradiction again. Hence \( y_1u_3, y_2u_3 \notin E(G) \). It follows that \( \deg(y_1) = \deg(y_2) = 1 \). Considering the matching \( M' = \{u_1y_1, u_2y_2, u_3v_3\} \) instead of \( M \), we obtain \( \deg(v_1) = \deg(v_2) = 1 \). Since \( G \) is connected, we may assume, without loss of generality, that \( u_1v_3 \in E(G) \). If \( u_1v_3 \in E(G) \) or \( u_2v_3 \in E(G) \), then the function \( f_3 : V(G) \to \{0, 1, 2\} \) defined by \( f_3(u_1) = f_3(u_2) = 2 \) and \( f_3(x) = 0 \) otherwise, is clearly a co-Roman dominating function of \( G \) of weight 4, a contradiction. Therefore, \( \deg(v_3) = 1 \). Since \( G \) is connected, we conclude that \( G \) is a graph obtained from three paths \( v_1u_1y_1, v_2u_2y_2 \) and \( v_3u_3 \) by adding edges between \( u_1, u_2, u_3 \) such that the resulting graph is connected. Hence \( G \in \mathcal{G}_6 \).

**Subcase 1.2.** \( |X| = 1 \). Assume that \( u_1y_1 \in E(G) \). If \( y_1u_3 \in E(G) \) (the case \( y_2u_3 \in E(G) \) is similar), then the function \( f_4 : V(G) \to \{0, 1, 2\} \) defined by \( f_4(u_1) = f_4(u_2) = f_4(u_3) = 1 \) and \( f_4(x) = 0 \) otherwise, is clearly a co-Roman dominating function of \( G \) of weight 3 which is a contradiction. Hence \( y_1u_3, y_2u_3 \notin E(G) \). Hence \( \deg(y_1) = 1 \). Regarding the matching \( M' = \{u_1y_1, u_2v_2, u_3v_3\} \) instead of \( M \), we have \( \deg(v_1) = 1 \). Since \( G \) is connected, we may assume that \( u_1v_3 \in E(G) \). If \( u_1v_3 \in E(G) \), then the function \( h_1 : V(G) \to \{0, 1, 2\} \) defined by \( h_1(u_1) = 2, h_1(u_2) = 1 \) and \( h_1(x) = 0 \) otherwise, is clearly a co-Roman dominating function of \( G \) of weight 3, a contradiction. Therefore \( u_1v_3 \notin E(G) \). Consider the following.

- \( u_1v_2 \in E(G) \) (the case \( u_1v_2 \in E(G) \) is similar). Then as above \( u_1v_2 \notin E(G) \). If \( v_2v_3 \in E(G) \), then the function \( h_2 : V(G) \to \{0, 1, 2\} \) defined by \( h_2(u_1) = 2, h_2(v_2) = 1 \) and \( h_2(x) = 0 \) otherwise, is clearly a co-Roman dominating function of \( G \) of weight 3, a contradiction. Hence \( v_2v_3 \notin E(G) \). If
\{u_2v_3, u_3v_2\} \subseteq E(G)$, then the function $h_3 : V(G) \rightarrow \{0, 1, 2\}$ defined by $h_3(u_1) = 2, h_3(u_2) = 1$ and $h_3(x) = 0$ otherwise, is clearly a co-Roman dominating function of $G$ of weight 3, a contradiction. Thus $\{u_2v_3, u_3v_2\} \not\subseteq E(G)$.

It follows that $G \in \{G_1, G_2, G_3, G_4, G_5\}$ and so $G \in \mathcal{G}_5$.

- $u_1u_2, u_1v_2 \not\in E(G)$. If $\{u_2, v_2, v_3\}$ induces a triangle, then the function $h_4 : V(G) \rightarrow \{0, 1, 2\}$ defined by $h_4(u_1) = 2, h_4(u_2) = 1$ and $h_4(x) = 0$ otherwise, is clearly a co-Roman dominating function of $G$ of weight 3, a contradiction. Thus $\{u_2, v_2, v_3\}$ does not induce a triangle. As above we have $\{u_2v_3, u_3v_2\} \not\subseteq E(G)$. Since $G$ is connected, the graph induced by $u_2, v_2, u_3, v_3$ is connected. This implies that $G \in \{G_6, G_7, G_8, G_9, G_{10}\}$ and so $G \in \mathcal{G}_5$.

Subcase 1.3. $|X| = 0$. Then $n = 6$. Since $\gamma_{cr}(G) = 3$, we have $\Delta(G) \leq 4$ by Propositions 1 and 2. Hence $\gamma(G) \geq 2$. If $\gamma(G) = 3$, then we deduce from Theorem 28 that $G$ is the corona $cor(P_3)$ or $cor(C_3)$ and so $G \in \mathcal{G}_4$. Assume $\gamma(G) = 2$. Then we conclude from Proposition 9 that every $\gamma(G)$-set $S$ contains a vertex $x$ such that $x$ has no neighbor $x' \in V \setminus S$ with $pm(x, S) \subseteq N[x']$. It follows that $G \in \mathcal{G}_4$.

Case 2. $\alpha'(G) = 2$. First let $S \neq \emptyset$. We deduce from (2) that $|S| = 1$ and so $S = \{x_1\}$. Let $x_1u_1, x_1v_1 \in E(G)$. Then we assume that each other vertex of $X$ is adjacent only to $u_2$. It follows that $\deg(x) = 1$ for each $x \in X \setminus \{x_1\}$. Since the function $g : V(G) \rightarrow \{0, 1, 2\}$ defined by $g(u_1) = 1, g(u_2) = 2$ and $g(x) = 0$ otherwise, is an co-Roman dominating function of $G$, we deduce that $n - 3 \leq 3$ and so $n \leq 6$. If $n = 6$, then clearly $X = \{x_1, y_1\}$. By considering the matching $M' = \{u_3v_1, u_2y_1\}$ instead of $M$, we have $\deg(v_2) = 1$. Since $G$ is connected and $\gamma_{cr}(G) = 3$, $u_2$ must be adjacent to at least one vertex and at most two vertices in $\{u_1, v_1, x_1\}$. Thus $G$ is a graph obtained from a triangle by adding a path $P_3$ and joining its center to at least one and at most two vertices of triangle and so $G \simeq H_1$ or $H_2$. Hence $G \in \mathcal{G}_4$. Assume that $n = 5$. Since $G$ is connected, $G$ is a graph obtained from a triangle and a path $P_2$ by adding some edges between them so that the resulting graph has at most one universal vertex. Thus $G \in \mathcal{G}_2$.

Now let $S = \emptyset$. As above, we may assume $N(x) \subseteq \{u_1, u_2\}$ for each $x \in X$. By Theorem 19, we have $\gamma_{cr}(G) \leq 4$ and this implies that $n \leq 7$. Thus $|X| \leq 3$. If $n = 4$, then we have $\gamma_{cr}(G) = 1$ yielding $G \in \{K_4, K_4-e\} \subseteq \mathcal{G}_1$ by Observation 6. If $n = 5$, then $G$ is a graph obtained from a path $P_3$ and a path $P_2$ by adding some edges between them such that the resulting graph is different from $DS(1,2)$ and has at most one universal vertex. Thus $G \in \mathcal{G}_3$. Let $n \geq 6$. Since $\gamma_{cr}(G) \geq 3$, $G$ has no vertex of degree $n - 1$ and so $\gamma(G) \geq 2$. Since $\{u_1, u_2\}$ is a dominating set, we have $\gamma(G) = 2$. If $n = 6$, then clearly $G \in \mathcal{G}_4$. Suppose $n = 7$. Then $X = \{y_1, y_2, y_3\}$. If $u_i$ is adjacent to all vertices of $X$ for some $i$, say $i = 1$, then the function $g : V(G) \rightarrow \{0, 1, 2\}$ defined by $g(u_1) = 2, g(u_2) = 1$ and $g(x) = 0$ otherwise, is a co-Roman dominating function of $G$ of weight 3 which leads to a contradiction. Hence, each $u_i$ is adjacent to at most two vertices in $X$. We may
assume without loss of generality that $u_1y_1, u_1y_2, u_2y_3 \in E(G)$ and $u_1y_3 \notin E(G)$. Since $\{y_1, y_2, y_3, v_1\}$ is independent, we deduce that $\deg(y_3) = 1$. Considering the matching $M' = \{u_1v_1, u_2y_3\}$ instead of $M$, we obtain $\deg(v_2) = 1$. Since $\gamma_{cr}(G) = 4$, $u_2$ is adjacent to at most one vertex in $\{y_1, y_2, v_1\}$. Thus $G$ is a connected graph obtained from $K_{1,3}$ and a path $P_3$ by joining the center of $P_3$ to the center or at most one leaf of $K_{1,3}$. This implies that $G \in \{G_{11}, G_{12}, G_{13}\}$ and so $G \in \mathcal{G}_5$. This completes the proof.

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