Discussiones Mathematicae
Graph Theory 39 (2019) 439–453
doi:10.7151/dmgt.2083

ON SOME CHARACTERIZATIONS OF ANTIPODAL PARTIAL CUBES

Norbert Polat

I.A.E., Université Jean Moulin (Lyon 3)
6 cours Albert Thomas
69355 Lyon Cedex 08, France

e-mail: norbert.polat@univ-lyon3.fr

Abstract

We prove that any harmonic partial cube is antipodal, which was conjectured by Fukuda and K. Handa, *Antipodal graphs and oriented matroids*, Discrete Math. 111 (1993) 245–256. Then we prove that a partial cube G is antipodal if and only if the subgraphs induced by W_{ab} and W_{ba} are isomorphic for every edge ab of G. This gives a positive answer to a question of Klavžar and Kovše, *On even and harmonic-even partial cubes*, Ars Combin. 93 (2009) 77–86. Finally we prove that the distance-balanced partial cube that are antipodal are those whose pre-hull number is at most 1.

Keywords: diametrical graph, harmonic graph, antipodal graph, distance-balanced graph, partial cube, pre-hull number.

2010 Mathematics Subject Classification: 05C12, 05C75.

1. Introduction

If x, y are two vertices of a connected graph G, then y is said to be a *relative antipode* of x if $d_G(x, y) \geq d_G(x, z)$ for every neighbor z of x, where d_G denotes the usual distance in G; and it is said to be an *absolute antipode* of x if $d_G(x, y) = \mathrm{diam}(G)$ (the diameter of G). The graph G is said to be *antipodal* if every vertex x of G has exactly one relative antipode; it is *diametrical* if every vertex x of G has exactly one absolute antipode $\overline{x}$; and it is *harmonic* (or *automorphically diametrical* [27]) if it is diametrical and the *antipodal map* $x \mapsto \overline{x}$, $x \in V(G)$, is an automorphism of G, i.e., $\overline{x}\overline{y} \in E(G)$ whenever $xy \in E(G)$. Note that, if G is antipodal, then the unique relative antipode of a vertex x is an absolute antipode of x, and thus is denoted by $\overline{x}$.

Bipartite antipodal graphs were introduced by Kotzig [18] under the name of S-graphs. Later Glivjak, Kotzig and Plesník [10] proved in particular that *a graph G is antipodal if and only if for any $x \in V(G)$ there is an $\overline{x} \in V(G)$ such that*

$$d_G(x, y) + d_G(y, \overline{x}) = d_G(x, \overline{x}) \quad \text{for all} \quad y \in V(G), \tag{1}$$

where d_G denotes the usual distance in G. The definition was extended to the non-bipartite case by Kotzig and Laufer [19]. Several papers followed.

On the other hand diametrical graphs were introduced by Mulder [22] in the case of median graphs. They were later studied by Parthasarathy and Nandakumar [24] under the name of self-centered unique eccentric point graphs, then by Göbel and Veldman [11] under the name of even graphs, by Fukuda and Handa [9] who proved that the tope graphs of oriented matroids are harmonic partial cubes (i.e., isometric subgraphs of hypercubes), and more recently by Klavžar and Kovše [16] who gave a partial solution to a problem set in [9].

Any antipodal graph is clearly harmonic, and thus diametrical. Two results [9, Proposition 4.1 and Theorem 4.2] of Fukuda and Handa implicitely imply that any harmonic partial cube is antipodal. Partial cubes, i.e., isometric subgraphs of hypercubes, which were introduced by Firson [8] and characterized by Djoković [5] and Winkler [28], have been extensively studied, see [20, 3] for recent papers. Actually the aim of Fukuda and Handa in [9] was the characterization of the tope graph of an acycloid, and the fact that any harmonic partial cube is antipodal, which is clearly a consequence of their results, is not plainly expressed in their paper. This is why some had thought that this property was not proved in [9]. In Section 3, we give a direct proof of this property by using the fact that a diametrical partial cube is antipodal if and only if its diameter is equal to its isometric dimension, i.e., the least non-negative integer n such that this graph is an isometric subgraph of an n-cube (Lemma 3.2).

A graph G is said to be *distance-balanced* if $|W_{ab}| = |W_{ba}|$ for every edge ab of G, where W_{ab} denotes the set of vertices that are closer to a than to b. Since their introduction by Handa [13], distance-balanced graphs have played an important role, and given rise to several papers, see for example some recent ones [15, 14, 7]. Handa [13] observed that any harmonic graph is distance-balanced, but that there exist distance-balanced partial cubes that are not diametrical. In Section 6 we show that the distance-balanced partial cubes that are antipodal are those whose pre-hull number is at most 1 (see [26]).

Harmonic partial cubes have a property that is stronger than the one of being distance-balanced. Actually if a partial cube G is harmonic, and thus antipodal, then its antipodal map induces an isomorphism between the subgraphs induced by W_{ab} and W_{ba} for every edge ab of G. In Section 5 we prove that the converse is also true, i.e., that the above property characterizes antipodal partial cubes, which answers a question of Klavžar and Kovše [16, Section 5]. More generally, they asked [16, Problem 5.3] whether a partial cube G is harmonic if and only if the subgraphs induced by W_{ab} and W_{ba} are isomorphic for every edge ab of G. The above two results give several ways of tackling this problem.

2. Preliminaries

The graphs we consider are undirected, without loops or multiple edges, and are finite and connected. For a set S of vertices of a graph G we denote by $G[S]$ the subgraph of G induced by S, and $G - S := G[V(G) - S]$. A *path* P with $V(P) = \{x_0, \ldots, x_n\}$, $x_i \neq x_j$ if $i \neq j$, and $E(P) = \{x_i x_{i+1} : 0 \leq i < n\}$ is denoted by $\langle x_0, \ldots, x_n \rangle$ and is called an (x_0, x_n)*-path.* A *cycle* C with $V(C) = \{x_1, \ldots, x_n\}$, $x_i \neq x_j$ if $i \neq j$, and $E(C) = \{x_i x_{i+1} : 1 \leq i < n\} \cup \{x_n x_1\}$, is denoted by $\langle x_1, \ldots, x_n, x_1 \rangle$.

The usual *distance* between two vertices x and y of a graph G, that is, the length of any (x, y)*-geodesic* (= shortest (x, y)-path) in G, is denoted by $d_G(x, y)$. A connected subgraph H of G is *isometric* in G if $d_H(x, y) = d_G(x, y)$ for all vertices x and y of H. The (*geodesic*) *interval* $I_G(x, y)$ between two vertices x and y of G consists of the vertices of all (x, y)-geodesics in G.

In the *geodesic convexity*, that is, the convexity on the vertex set of a graph G which is induced by the *geodesic interval operator* I_G, a subset C of $V(G)$ is convex provided it contains the geodesic interval $I_G(x, y)$ for all $x, y \in C$. The *convex hull* $co_G(A)$ of a subset A of $V(G)$ is the smallest convex set which contains A. A subset H of $V(G)$ is a *half-space* if H and $V(G) - H$ are convex. We denote by $\mathcal{I}_G$ the *pre-hull operator* of the geodesic convex structure of G, i.e., the self-map of $\mathcal{P}(V(G))$ such that $\mathcal{I}_G(A) := \bigcup_{x,y \in A} I_G(x, y)$ for each $A \subseteq V(G)$. The convex hull of a set $A \subseteq V(G)$ is then $co_G(A) = \bigcup_{n \in \mathbb{N}} \mathcal{I}_G^n(A)$.

For an edge ab of a graph G, let

$$W_{ab} := \{x \in V(G) : d_G(a, x) < d_G(b, x)\}.$$

Note that the sets W_{ab} and W_{ba} are disjoint and that $V(G) = W_{ab} \cup W_{ab}$ if G is bipartite.

Two edges xy and uv are in the Djoković-Winkler relation Θ if

$$d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u).$$

The relation Θ is clearly reflexive and symmetric.

Remark 2.1. If G is bipartite, then, by [12, Lemma 11.2], *the notation can be chosen so that the edges xy and uv are in relation Θ if and only if*

$$d_G(x, u) = d_G(y, v) = d_G(x, v) - 1 = d_G(y, u) - 1,$$

or equivalently if and only if

$$y \in I_G(x, v) \text{ and } x \in I_G(y, u).$$

From now on, we will always use this way of defining the relation Θ. Note that, in this way, the edges xy and yx are not in relation Θ because $y \notin I_G(x, x)$ and $x \notin G(y, y)$. In other word, each time the relation Θ is used, the notation of an edge induces an orientation of this edge.

We recall that, by Djoković [5, Theorem 1] and Winkler [28], *a connected bipartite graph G is a partial cube, that is, an isometric subgraph of some hypercube, if it has the following equivalent properties:*

(Conv.) *For every edge ab of G, the sets W_{ab} and W_{ba} are convex.*

(Trans.) *The relation Θ is transitive, and thus is an equivalence relation.*

It follows in particular that *the non-trivial* (i.e., distinct from $\emptyset$ and $V(G)$) *half-spaces of a partial cube G are the sets W_{ab}, $ab \in E(G)$.* In the following lemma we recall two well-known properties of partial cubes that we will need later.

Lemma 2.2. *Let G be a partial cube. We have the following properties.*

- (i) *Let x, y be two vertices of G, P an (x, y)-geodesic and W an (x, y)-path of G. Then each edge of P is Θ-equivalent to some edge of W.*
- (ii) *A path P in G is a geodesic if and only if no two distinct edges of P are Θ-equivalent.*

3. Harmonicity Versus Antipodality

In this section we give an alternative proof of the following property.

Theorem 3.1. *Any harmonic partial cube is antipodal.*

Recall that the *isometric dimension* of a finite partial cube G, i.e., the least non-negative integer n such that G is an isometric subgraph of an n-cube, coincides with the number of Θ-classes of $E(G)$. We denote it by $\mathrm{idim}(G)$. By Lemma 2.2(ii) we clearly have $\mathrm{diam}(G) \leq \mathrm{idim}(G)$.

We need the following lemma which is an immediate consequence of Desharnais [4, Lemme 1.6.9]. However we give a short proof of it. Note that, by (1), a graph G is antipodal if and only if

$$I_G(x, \overline{x}) = V(G) \quad \text{for all} \quad x \in V(G). \tag{2}$$

Lemma 3.2. *Let G be a diametrical partial cube. Then G is antipodal if and only if* $\mathrm{diam}(G) = \mathrm{idim}(G)$.

Proof. *Necessity.* By Lemma 2.2(ii), $\operatorname{diam}(G) \leq \operatorname{idim}(G)$. Conversely, let uv be an edge of G, and P a $(u,\overline{u})$-geodesic. Then uv is Θ-equivalent to an edge of P, since otherwise $\langle v,u\rangle \cup P$ is a geodesic, contrary to the fact that P is a geodesic of maximal length. Therefore $\operatorname{idim}(G) \leq \operatorname{diam}(G)$. Whence the equality.

Sufficiency. Assume that $\operatorname{diam}(G) = \operatorname{idim}(G) =: d$. Then G is an isometric subgraph of some d-cube H. Let u be some vertex of G, and $\overline{u}$ its antipode in G. Note that H is antipodal, and that $\overline{u}$ is the antipode of u in H, since G is an isometric subgraph of H and $\operatorname{diam}(H) = d = \operatorname{diam}(G)$. It follows that $I_H(u,\overline{u}) = V(H)$ by (2). Therefore $I_G(u,\overline{u}) = I_H(u,\overline{u}) \cap V(G) = V(H) \cap V(G) = V(G)$, which proves that G is antipodal. ∎

***Proof of Theorem* 3.1.** Let G be a harmonic partial cube. Denote by d its diameter, and by $\overline{u}$ the unique absolute antipode of any vertex u of G. Then the antipodal map $\alpha : u \to \overline{u}$ is an automorphism of G. Moreover denote by P_u some $(u,\overline{u})$-geodesic for every $u \in V(G)$. Finally, for an edge e of G and a subgraph F of G, let $\Theta[e]$ be the Θ-class of e, and

$$\Theta[F] := \{\Theta[e] : e \in E(F)\}.$$

Let $u \in V(G)$.

Claim 1. *Any neighbor v of u belongs to $I_G(u,\overline{u})$, and thus $\Theta[uv] \in \Theta[P_u]$.*

This is obvious if $v \in V(P_u)$. Suppose that $v \notin V(P_u)$. Then $d_G(v,\overline{u}) < d$ since $\overline{v} \neq \overline{u}$ because α is an automorphism. It follows that $v \in I_G(u,\overline{u})$. Hence the edge uv is Θ-equivalent to some edge of P_u by Lemma 2.2(i).

Claim 2. $\Theta[P_u] = \Theta[P_v]$ *for any neighbor v of u in G.*

Note that $\overline{v}$ is a neighbor of $\overline{u}$ since α is an automorphism. Let Q be some $(v,\overline{u})$-geodesic. Then $R := \langle u,v\rangle \cup Q$ and $R' := Q \cup \langle \overline{u},\overline{v}\rangle$ are a $(u,\overline{u})$-geodesic and a $(v,\overline{v})$-geodesic, respectively. On the other hand, the edges uv and $\overline{v}\overline{u}$ are Θ-equivalent.

It follows that $\Theta[P_u] = \Theta[R] = \Theta[R'] = \Theta[P_v]$ by Lemma 2.2(i).

Claim 3. $\Theta[P_u] = \Theta[P_v]$ *for any $v \in V(G)$.*

This is Claim 2 if v is a neighbor of u. Suppose that $d_G(u,v) = n > 1$, and let $\langle x_0,\dots,x_n\rangle$ be a (u,v)-geodesic with $x_0 = u$ and $x_n = v$. Then $\langle \overline{x_0},\dots,\overline{x_n}\rangle$ is a $(\overline{u},\overline{v})$-geodesic since α is an automorphism.

By a successive application of Claim 2, we obtain

$$\Theta[P_u] = \Theta[P_{x_0}] = \Theta[P_{x_1}] = \cdots = \Theta[P_{x_n}] = \Theta[P_v].$$

Now, each edge xy of G is such that $\Theta[xy] \in \Theta[P_x] = \Theta[P_u]$ by Claims 1 and 3, that is, each edge of G is Θ-equivalent to some edge of P_u. Therefore $\operatorname{idim}(G)$ is equal to the number of edges of P_u, because any two distinct edges of a geodesic are non-Θ-equivalent by Lemma 2.2(ii). It follows that $\operatorname{idim}(G) = d = \operatorname{diam}(G)$. Hence G is antipodal by Lemma 3.2. ∎

4. Expansion

In this section we recall some properties of expansions of a graph, a concept that we will need in the next section and which was introduced by Mulder [21] to characterize median graphs and which was later generalized by Chepoi [2].

Definition 4.1. A pair (V_0, V_1) of sets of vertices of a graph G is called a *proper cover* of G if it satisfies the following conditions

- $V_0 \cap V_1 \neq \emptyset$ and $V_0 \cup V_1 = V(G)$;
- there is no edge between a vertex in $V_0 - V_1$ and a vertex in $V_1 - V_0$;
- $G[V_0]$ and $G[V_1]$ are isometric subgraphs of G.

Recall that the *prism over* a graph G is the Cartesian product of G and K_2, i.e., the graph denoted by $G \Box K_2$ whose vertex set is $V(G) \times V(K_2)$, and such that, for all $x, y \in V(G)$ and $i, j \in V(K_2) = \{0, 1\}$, $(x, i)(y, j) \in E(G \Box K_2)$ if $xy \in E(G)$ and $i = j$, or $x = y$ and $i \neq j$.

Definition 4.2. An *expansion* of a graph G with respect to a proper cover (V_0, V_1) of G is the subgraph of the prism over G induced by the set $(V_0 \times \{0\}) \cup (V_1 \times \{1\})$.

An expansion of a bipartite graph (respectively, a partial cube) is a bipartite graph (respectively, a partial cube (see [2])). If G' is an expansion of a partial cube G, then we say that G is a Θ-*contraction* of G', because, as we can easily see, G is obtained from G' by contracting each element of some Θ-class of edges of G'. More precisely, let G be a partial cube different from K_1 and let uv be an edge of G. Let G/uv be the quotient graph of G whose vertex set $V(G/uv)$ is the partition of $V(G)$ such that x and y belong to the same block of this partition if and only if $x = y$ or xy is an edge which is Θ-equivalent to uv. The natural surjection γ_{uv} of $V(G)$ onto $V(G/uv)$ is a contraction (weak homomorphism in [12]) of G onto G/uv, that is, an application which maps any two adjacent vertices to adjacent vertices or to a single vertex. Then clearly the graph G/uv is a partial cube and $(\gamma_{uv}(W^G_{uv}), \gamma_{uv}(W^G_{vu}))$ is a proper cover of G/uv with respect to which G is an expansion of G/uv. We will say that G/uv is the Θ-*contraction of* G *with respect to the* Θ-*class of* uv.

Let G' be an expansion of a graph G with respect to a proper cover (V_0, V_1) of G. We will use the following notation.

- For $i = 0, 1$ denote by $\psi_i : V_i \to V(G')$ the natural injection $\psi_i : x \mapsto (x, i)$, $x \in V_i$, and let $V'_i := \psi_i(V_i)$. Note that V'_0 and V'_1 are complementary half-spaces of G'.
- For $A \subseteq V(G)$ put $\psi(A) := \psi_0(A \cap V_0) \cup \psi_1(A \cap V_1)$.

The following lemma is a restatement with more precisions of [25, Lemma 4.5] (also see [23, Lemma 8.1]).

Lemma 4.3. *Let G be a connected bipartite graph and G' an expansion of G with respect to a proper cover (V_0, V_1) of G, and let $P = \langle x_0, \ldots, x_n \rangle$ be a path in G. We have the following properties*

(i) *If $x_0, x_n \in V_i$ for some $i = 0$ or 1, then*
 – *if P is a geodesic in G, then there exists an (x_0, x_n)-geodesic R in $G[V_i]$ such that $V(P) \cap V_i \subseteq V(R)$;*
 – *P is a geodesic in $G[V_i]$ if and only if $P' = \langle \psi_i(x_0), \ldots, \psi_i(x_n) \rangle$ is a geodesic in G';*
 – *$d_{G'}(\psi_i(x_0), \psi_i(x_n)) = d_G(x_0, x_n)$;*
 – *$d_{G'}(\psi_i(x_0), \psi_i(x_n)) = d_G(x_0, x_n)$;*
 – *$I_{G'}(\psi_i(x_0), \psi_i(x_n)) = \psi_i(I_{G[V_i]}(x_0, x_n)) \subseteq \psi(I_G(x_0, x_n))$.*

(ii) *If $x_0 \in V_i$ and $x_n \in V_{1-i}$ for some $i = 0$ or 1, then*
 – *if there exists p such that $x_0, \ldots, x_p \in V_i$ and $x_p, \ldots, x_n \in V_{1-i}$, then P is a geodesic in G if and only if the path $P' = \langle \psi_i(x_0), \ldots, \psi_i(x_p), \psi_{1-i}(x_p), \ldots, \psi_{1-i}(x_n) \rangle$ is a geodesic in G';*
 – *$d_{G'}(\psi_i(x_0), \psi_{1-i}(x_n)) = d_G(x_0, x_n) + 1$;*
 – *$I_{G'}(\psi_i(x_0), \psi_{1-i}(x_n)) = \psi(I_G(x_0, x_n))$.*

Now we introduce a variety of expansions that are related to antipodal partial cubes. We need the following notation. If A is a set of vertices of an antipodal graph G, we write

$$\overline{A} := \{\bar{x} : x \in A\}.$$

Lemma 4.4. *If G is an antipodal partial cube, then $\overline{W_{ab}} = W_{ba}$ for every edge ab of G.*

Proof. Suppose both x and $\overline{x}$ belong to W_{ab} for some vertex x of G. Then $I_G(x, \overline{x}) \subseteq W_{ab}$ since W_{ab} is convex by (Conv.), hence $I_G(x, \overline{x}) \neq V(G)$, contrary to (2). ∎

Definition 4.5. A proper cover (V_0, V_1) of an antipodal partial cube G is said to *respect the antipodality*, or to be *antipodality-respectful*, if $\overline{V_0} = V_1$.

Clearly, if $\overline{V_0} = V_1$, then $\overline{V_1} = V_0$ and $\overline{V_0 \cap V_1} = V_0 \cap V_1$. For any antipodal partial cube G, there always exists a proper cover that respects the antipodality. For example, the proper cover (V_0, V_1) such that $V_0 = V_1 = V(G)$ respects the antipodality, and the expansion of G with respect to this proper cover is the prism over G.

Definition 4.6. An expansion of an antipodal partial cube G with respect to an antipodality-respectful proper cover of G is called an *antipodality-respectful expansion* of G.

These antipodality-respectful expansions were already defined in [9] under the name of acycloidal expansions.

Lemma 4.7. *Any antipodality-respectful expansion of an antipodal partial cube is an antipodal partial cube.*

Proof. Let G' be an expansion of an antipodal partial cube G with respect to an antipodality-respectful proper cover (V_0, V_1) of G. Clearly G' is a bipartite partial cube such that $\mathrm{diam}(G') = \mathrm{diam}(G) + 1$. Denote by pr the projection of G' onto G. Let $x \in V(G')$. Then $x \in V'_i$ for some $i = 0$ or 1, and thus $\mathrm{pr}(x) \in V_i$ and $\overline{\mathrm{pr}(x)} \in V_{1-i}$ because (V_0, V_1) respects the antipodality. Let $y \in V(G')$. Then $\mathrm{pr}(y) \in I_G(\mathrm{pr}(x), \overline{\mathrm{pr}(x)})$ since G is antipodal. Hence $y \in I_{G'}(x, \psi_{1-i}(\overline{\mathrm{pr}(x)}))$ by Lemma 4.3. It follows, by (2), that G' is antipodal with $\overline{x} = \psi_{1-i}(\overline{\mathrm{pr}(x)})$. ■

Lemma 4.8. *Let G' be an expansion of a partial cube G with respect to a proper cover (V_0, V_1). If G' is antipodal, then so is G and moreover (V_0, V_1) is an antipodality-respectful proper cover of G.*

Proof. Assume that G' is antipodal. We use the notations introduced above.

Claim 1. *G is antipodal.*

Because V'_0 and V'_1 are complementary half-spaces of G', it follows that $\overline{V'_i} = V'_{1-i}$ for $i = 0, 1$ by Lemma 4.4.

Let $x \in V(G)$. Without loss of generality, we can suppose that $x \in V_0$. Then $\psi_0(x) \in V'_0$ and $\overline{\psi_0(x)} \in V'_1$. Hence, by Lemma 4.3,

$$\psi(I_G(x, pr(\overline{\psi_0(x)}))) = I'_G(\psi_0(x), \psi_1(\overline{\psi_0(x)}) = V(G').$$

It follows that $I_G(x, pr(\overline{\psi_0(x)}) = V(G)$, which proves that $pr(\overline{\psi_0(x)})$ is the antipode of x in G. Therefore G is antipodal.

Claim 2. (V_0, V_1) *respects the antipodality.*

By Lemma 4.4, $\overline{V'_i} = V'_{1-i}$ for $i = 0, 1$ since G is antipodal by Claim 1. Hence, by Lemma 4.3, $\overline{V_i} = \overline{\mathrm{pr}(V'_i)} = \mathrm{pr}(\overline{V'_i}) = \mathrm{pr}(V'_{1-i}) = V_{1-i}$ for $i = 0, 1$, and thus (V_0, V_1) respects the antipodality, or in other words, G' is an antipodality-respectful expansion of G. ■

The following theorem, which is similar to a characterization of median graphs by Mulder [21] and of partial cubes by Chepoi [2] (also see [9, Theorem 4.6]), is easily proved by induction on the isometric dimension by using the above two lemmas.

Theorem 4.9. *A finite graph is an antipodal partial cube if and only if it can be obtained from K_1 by a sequence of antipodality-respectful expansions.*

The number of iterations to obtain some antipodal partial cube G from K_1 is equal to the isometric dimension of G.

5. Special Automorphisms

By Lemma 4.4, if G is an antipodal partial cube, then the antipodal map $x \to \overline{x}$, $x \in V(G)$, is an isomorphism of $G[W_{ab}]$ onto $G[W_{ba}]$. We will show that such a property characterizes antipodal partial cubes. Let G be a partial cube. Recall that the subgraphs $G[W_{ab}]$, $ab \in E(G)$, are called *semicubes* by Eppstein [6], and that the semicubes $G[W_{ab}]$ and $G[W_{ba}]$ are said to be *opposite*.

Definition 5.1. Let α be an automorphism of a partial cube G. We say that α is

(i) *semicube-switching* if it induces an isomorphism between the subgraphs $G[W_{ab}]$ and $G[W_{ba}]$ for each edge ab of G.

(ii) Θ*-faithful* if the edges uv and $\alpha(v)\alpha(u)$ are Θ-equivalent for each $uv \in E(G)$.

In Remark 2.1 we observed that the edges xy and yx are not Θ-equivalent. It follows that the identity automorphism of a partial cube distinct from K_1 is not Θ-faithful. Also note that, if, for example, G is a 4-cycle $\langle x_1, x_2, x_3, x_4, x_1\rangle$, then the only Θ-faithful automorphism, and also the only semicube-switching automorphism, is the antipodal map. Indeed, the involution β mapping x_1 to x_2, and x_3 to x_4 is not Θ-faithful because the edges x_1x_4 and $\beta(x_4)\beta(x_1) = x_3x_2$ are not Θ-equivalent since $x_4 \notin I_G(x_1, x_2)$.

Theorem 5.2. *Let G be a partial cube. The following assertions are equivalent.*

(i) *G is antipodal.*

(ii) *There exits an automorphism of G that is Θ-faithful.*

(iii) *There exits an automorphism of G that is semicube-switching.*

Proof. (i) $\Rightarrow$ (ii): Assume that G is antipodal, and let $uv \in E(G)$. Then, by (2), $v \in I_G(u, \overline{u})$ and $u \in I_G(v, \overline{v})$. Hence the edges uv and $\overline{v}\overline{u}$ are Θ-equivalent. Therefore the antipodal map α is Θ-faithful.

(ii) $\Rightarrow$ (iii): Assume that there exists a Θ-faithful automorphism α of G. Let $uv \in E(G)$, and let $x \in W_{uv}$. Then $u \in I_G(x, v)$, and thus no edge of any (u, x)-geodesic is Θ-equivalent to uv. Because α is Θ-faithful, it follows that no edge of any $(\alpha(u), \alpha(x))$-geodesic is Θ-equivalent to uv, and thus to $\alpha(v)\alpha(u)$. Hence $\alpha(u) \in I_G(\alpha(x), \alpha(v))$, and thus $\alpha(x) \in W_{\alpha(u)\alpha(v)} = W_{vu}$. Therefore α is an isomorphism between $G[W_{ab}]$ and $G[W_{ba}]$, and thus α is semicube-switching.

(iii) $\Rightarrow$ (i): We prove by induction on the isometric dimension that any partial cube G that has a semicube-switching automorphism α is antipodal and that α is its antipodal map. This is obvious if $\mathrm{idim}(G) \leq 2$, i.e., if G is K_1, K_2 or a 4-cycle. Note that a path of length 2, which is also a partial cube of isometric dimension 2, has no semicube-switching automorphism. Suppose that this holds for any partial cube of isometric dimension n for some $n \geq 2$. Let G be a partial cube with $\mathrm{idim}(G) = n + 1$ that has a semicube-switching automorphism α. Let uv be an edge of G, $F := G/uv$ the Θ-contraction of G with respect to the Θ-class of uv, and γ the natural surjection of $V(G)$ onto $V(F)$. Then F is a partial cube with $\mathrm{idim}(F) = n$.

Note that, if xy is an edge of G that is Θ-equivalent to uv, then so is the edge $\alpha(v)\alpha(u)$, because $\alpha(u) \in W_{vu}$, $\alpha(v) \in W_{uv}$ and $\alpha(u)$ and $\alpha(v)$ are adjacent since α is a semicube-switching automorphism.

Let $\beta : V(F) \to V(F)$ be such that $\beta \circ \gamma = \gamma \circ \alpha$. Because α is an automorphism of G, it is sufficient to prove that β preserves the edges to show that it is an automorphism of F. Let x and y be two adjacent vertices of G. If xy is not Θ-equivalent to uv, then so is $\alpha(y)\alpha(x)$, and thus $\gamma(x)$ and $\gamma(y)$ are adjacent, and so are $\gamma(\alpha(x))$ and $\gamma((\alpha(y))$, and hence so also are $\beta(\gamma(x))$ and $\beta(\gamma(y))$. If xy is Θ-equivalent to uv, then so is $\alpha(y)\alpha(x)$, and thus $\gamma(x) = \gamma(y)$ and $\gamma(\alpha(x)) = \gamma((\alpha(y))$. Therefore β is an automorphism of F.

We now show that β is semicube-switching. Note that each edge of F is the image by γ of somme edge of G that is not Θ-equivalent to uv. Let ab be an edge of G that is not Θ-equivalent to uv, and x a vertex of G. Without loss of generality we can suppose that $x \in W_{ab}$. Then $a \in I_G(x, b)$, and thus each (x, a)-geodesic contains an edge that is Θ-equivalent to uv if and only if so does each (x, b)-geodesic. It follows, by Lemma 4.3, that $\gamma(a) \in I_F(\gamma(x), \gamma(b))$, and thus $\gamma(x) \in W^F_{\gamma(a)\gamma(b)}$. On the other hand, $\alpha(x) \in W_{ba}$ since α is semicube-switching. Hence, as above, $\gamma(\alpha(x)) \in W^F_{\gamma(b)\gamma(a)}$, that is, $\beta(\gamma(x)) \in W^F_{\gamma(b)\gamma(a)}$, which implies that β is semicube-switching.

It follows, by the induction hypothesis and since $\mathrm{idim}(F) = n$, that the partial cube F is antipodal and that β is its antipodal map. Note that $\beta(\gamma(W_{uv})) = \gamma(\alpha(W_{uv})) = \gamma(W_{vu})$. Hence $(\gamma(W_{uv}), \gamma(W_{vu}))$ is an antipodality-respectful proper cover of F, and G is the expansion of F with respect to this proper cover. Consequently G is antipodal by Lemma 4.7. ∎

6. Pre-Hull Number and Distance-Balanced Graphs

We now give a characterization of antipodal partial cubes that uses the concept of pre-hull number, a concept which was introduced in [26] and that we first recall.

A *copoint* at a vertex x of a graph G is a convex set C which is maximal with respect to the property that $x \notin C$; x is an *attaching point* of C. Note that $co_G(C \cup \{x\}) = co_G(C \cup \{y\})$ for any two attaching points x, y of C. We denote by $\mathrm{Att}(C)$ the set of all attaching points of C, i.e.,

$$\mathrm{Att}(C) := co_G(C \cup \{x\}) - C. \tag{3}$$

By [26, Proposition 5.6], the copoints of a partial cube G are precisely the sets W_{ab}, $ab \in E(G)$, and thus are the non-trivial half-spaces of G.

Definition 6.1. Let G be a graph. The least non-negative integer n (if it exists) such that $co_G(C \cup \{x\}) = \mathcal{I}^n_G(C \cup \{x\})$ for each vertex x of G and each copoint C at x, is called the *pre-hull number* of G and is denoted by $ph(G)$. If there is no such n we put $ph(G) := \infty$.

Recall that by [26, Corollary 3.8], the pre-hull number of a connected bipartite graph G is zero if and only if G is a tree. For a graph G, $ph(G) \leq 1$ if $co_G(C \cup \{x\}) = \mathcal{I}_G(C \cup \{x\})$ for each vertex x of G and each copoint C at x, i.e., for all $x, y \in Att(C)$ there exists some $z \in C$ such that $y \in I_G(x, z)$.

Lemma 6.2. *If a partial cube G is distance-balanced, then every non-trivial half-space of G is maximal.*

Proof. Assume that G is distance-balanced, and let H be a non-trivial half-space of G. Then $H = W_{uv}$ for some $uv \in E(G)$ (see Section 2). Then $|W_{uv}| = |W_{vu}|$ since G is distance-balanced. Suppose that H is not maximal. Then there exists a non-trivial half-space H' that contains $H \cup \{x\}$ for some $x \in W_{vu}$. Hence $H' = W_{ab}$ for some $ab \in E(G)$, and thus $|W_{ba}| < |W_{vu}| = |W_{uv}| < |W_{ab}|$, contrary to the assumption. Therefore H is maximal. ■

Theorem 6.3. *Let G be a partial cube. The following assertions are equivalent.*

(i) *G is antipodal.*

(ii) *$ph(G) \leq 1$ and G is distance-balanced.*

(iii) *$ph(G) \leq 1$ and every non-trivial half-space of G is maximal.*

Proof. (i) ⇒ (ii): Assume that G is antipodal. Then $ph(G) \leq 1$ by [26, Section 8]. Because the antipodal map of G is semicube-switching by Theorem 5.2, it follows that $|W_{uv}| = |W_{vu}|$ for every $uv \in E(G)$.

(ii) ⇒ (iii) is a consequence of Lemma 6.2.

(iii) ⇒ (i): Assume that G satisfies (iii). Let $uv \in E(G)$. Because the half-space W_{vu} is maximal, it follows that $Att(W_{vu}) = W_{uv}$. Indeed, if some $x \in W_{uv}$ is not an attaching point of W_{vu}, then there exists a copoint at x, and thus a half-space by what we saw above, which strictly contains W_{vu}, contrary to the maximality of W_{vu}.

Let $x, y \in W_{uv}$. Because $ph(G) \leq 1$, there is some $z \in W_{vu}$ such that $y \in I_G(x, z)$. This implies that no geodesic in the subgraph $G[W_{uv}]$ is maximal in G. Hence, for every $x \in W_{uv}$, any relative antipode of x belongs to W_{vu}.

It follows that, if x has several antipodes, then, for any edge uv of G, if $x \in W_{uv}$, then all antipodes of x belong to W_{vu}, contrary to the fact that any partial cube has the Separation Property S_2, i.e., any two vertices can be separated by a half-space. Therefore any vertex of G has exactly one relative antipode, and thus G is antipodal. ■

In other words, *the distance-balanced partial cubes that are antipodal are those whose pre-hull number is at most* 1.

7. Crossing Graph

Let G be a partial cube. We say that two Θ-classes A, B of edges of G *cross* if, for $a_0a_1 \in A$ and $b_0b_1 \in B$,

$$W_{a_i a_{1-i}} \cap W_{b_j b_{1-j}} \neq \emptyset \quad \text{for all} \quad i, j \in \{0, 1\}.$$

Note that this definition is independent of the choice of the edges in A and B.

The *crossing graph* of a partial cube G is the graph $G^{\#}$ whose vertices are the Θ-classes of G, and where two vertices are adjacent if they cross. The concept of crossing graph was introduced by Bandelt and Dress [1] under the name of incompatibility graph, and extensively studied by Klavžar and Mulder [17].

Proposition 7.1. *The crossing graph $G^{\#}$ of a partial cube G is a complete graph if and only if every non-trivial half-space of G is maximal.*

Proof. Suppose that some non-trivial half-space H of G is not maximal. Then there exist two edges uv and ab of G such that $H = W_{uv}$ and $W_{uv} \subset W_{ab}$. It follows that $W_{uv} \cap W_{ba} = \emptyset$, and thus that the Θ-classes of uv and ab do not cross. Therefore $G^{\#}$ is not complete.

Conversely suppose that two Θ-classes A and B of G do not cross. Then there exist $a_0a_1 \in A$ and $b_0b_1 \in B$ such that $W_{a_0a_1} \subset W_{b_0b_1}$. It follows that the half-space $W_{a_0a_1}$ is not maximal. ∎

From Theorem 6.3 we then deduce immediately.

Theorem 7.2. *A partial cube G is antipodal if and only if $ph(G) \leq 1$ and $G^{\#}$ is complete.*

Median graphs are particular partial cubes whose pre-hull number is at most 1 ([26, Theorem 4.4]). We recall the following result.

Proposition 7.3. (Klavžar and Mulder [17, Proposition 4.1] and Mulder [22, Corollary 5]) *Let G be a median graph. The following assertions are equivalent.*

(i) *G is antipodal.*

(ii) *$G^{\#}$ is complete.*

(iii) *G is a hypercube.*

In [17], Klavžar and Mulder defined an *all-color expansion* of a partial cube G as an expansion with respect to a proper cover (V_0, V_1) such that each Θ-classes of G has a representative occurring in $E(G[V_i])$ for $i = 0, 1$. They prove [17, Proposition 4.4] that *the crossing graph of a partial cube G is complete if and only if G can be obtained from K_1 by a sequence of all-color expansions.* By comparing this result with Theorem 4.9, we see that any antipodality-respectful expansion is an all-color expansion, but that the converse is false.

8. On Two Problems of Klavžar and Kovše

At the end of their paper [16], after having noticed that, for any harmonic (and thus antipodal by Theorem 3.1) partial cube G, the antipodal map induces an isomorphism between each opposite semicubes, Klavžar and Kovše asked if the converse is true. Theorem 5.2 asserts that so it is. However they asked the more general question [16, Problem 5.3] whether *a partial cube G is harmonic if and only if $G[W_{ab}]$ is isomorphic to $G[W_{ba}]$ for every $ab \in E(G)$.* We still have no definitive answer to this problem, but Theorems 5.2 and 6.3 (because a partial cube all of whose opposite semicubes are isomorphic is obviously distance-balanced) give new approaches of dealing with this question. More precisely, if a partial cube G is such that the opposite semicubes $G[W_{ab}]$ and $G[W_{ba}]$ are isomorphic for every $ab \in E(G)$, then we can tackle the above problem by asking one of the following question:

(a) Does there exist an automorphism of G that is Θ-faithful?

(b) Does there exist an automorphism of G that is semicube-switching?

(c) Do we have $ph(G) \leq 1$?

Acknowledgments

I would like to thank one of the referees for his careful reading of the manuscript and helpful suggestions.

References

[1] H.-J. Bandelt and A.W.M. Dress, *A canonical decomposition theory for metrics on a finite set*, Adv. Math. **92** (1992) 47–105.
doi:10.1016/0001-8708(92)90061-O

[2] V. Chepoi, *Isometric subgraphs of Hamming graphs and d-convexity*, Cybernetics **24** (1988) 6–11.
doi:10.1007/BF01069520

[3] V. Chepoi, K. Knauer and T. Marc, *Partial cubes without Q_3^- minors*, CoRR abs/1606.02154 (2016) (submitted).

[4] J. Desharnais, *Maille et Plongements de Graphes Antipodaux*, Mémoire de Maitrise (Université de Montréal, 1993).

[5] D. Djoković, *Distance preserving subgraphs of hypercubes*, J. Combin. Theory Ser. B **14** (1973) 263–267.
doi:10.1016/0095-8956(73)90010-5

[6] D. Eppstein, *The lattice dimension of a graph*, European J. Combin. **26** (2005) 585–592.
doi:10.1016/j.ejc.2004.05.001

[7] M. Faghani, E. Pourhadi and H. Kharazi, *On the new extension of distance-balanced graphs*, Trans. Comb. **5** (2016) 23–34.

[8] V.V. Firsov, *Isometric embedding of a graph into a Boolean cube*, Cybernetics **1** (1965) 112–113.
doi:10.1007/BF01074705

[9] K. Fukuda and K. Handa, *Antipodal graphs and oriented matroids*, Discrete Math. **111** (1993) 245–256.
doi:10.1016/0012-365X(93)90159-Q

[10] F. Glivjak, A. Kotzig and J. Plesník, *Remarks on the graphs with a central symmetry*, Monatsh. Math. **74** (1970) 302–307.
doi:10.1007/BF01302697

[11] F. Göbel and H.J. Veldman, *Even graphs*, J. Graph Theory **10** (1986) 225–239.
doi:10.1002/jgt.3190100212

[12] R. Hammack, W. Imrich and S. Klavžar, Handbook of Product Graphs, Second Edition (CRC Press, 2011).

[13] K. Handa, *Bipartite graphs with balanced (a, b)-partitions*, Ars Combin. **51** (1999) 113–119.

[14] A. Ilíc, S. Klavžar and M. Milanovíc, *On distance-balanced graphs*, European J. Combin. **31** (2010) 732–737.
doi:10.1016/j.ejc.2009.10.006

[15] J. Jerebic, S. Klavžar and D.F. Rall, *Distance-balanced graphs*, Ann. Comb. **12** (2008) 71–79.
doi:10.1007/s00026-008-0337-2

[16] S. Klavžar and M. Kovše, *On even and harmonic-even partial cubes*, Ars Combin. **93** (2009) 77–86.

[17] S. Klavžar and H.M. Mulder, *Partial cubes and crossing graphs*, SIAM J. Discrete Math. **15** (2002)) 235–251.
doi:10.1137/S0895480101383202

[18] A. Kotzig, *Centrally symmetric graphs*, Czechoslovak Math. J. **93** (1968) 605–615.

[19] A. Kotzig and P. Laufer, Generalized S-Graphs, preprint CRM-779, Centre de recherches mathématiques (Université de Montréal, 1978).

[20] T. Marc, *There are no finite partial cubes of girth more than* 6 *and minimum degree at least* 3, European J. Combin. **55** (2016) 62–72.
doi:10.1016/j.ejc.2016.01.005

[21] H.M. Mulder, *The structure of median graphs*, Discrete Math. **24** (1978) 197–204. doi:10.1016/0012-365X(78)90199-1

[22] H.M. Mulder, *n-cubes and median graphs*, J. Graph Theory **4** (1980) 107–110. doi:10.1002/jgt.3190040112

[23] S. Ovchinnikov, *Partial cubes: structures, characterizations, and constructions*, Discrete Math. **308** (2008) 5597–5621. doi:10.1016/j.disc.2007.10.025

[24] K.R. Parthasarathy and R. Nandakumar, *Unique eccentric point graphs*, Discrete Math. **46** (1983) 69–74. doi:10.1016/0012-365X(83)90271-6

[25] N. Polat, *Netlike partial cubes* I. *General properties*, Discrete Math. **307** (2007) 2704–2722. doi:10.1016/j.disc.2007.01.018

[26] N. Polat and G. Sabidussi, *On the geodesic pre-hull number of a graph*, European J. Combin. **30** (2009) 1205–1220. doi:10.1016/j.ejc.2008.09.022

[27] G. Sabidussi, *Graphs without dead ends*, European J. Combin. **17** (1996) 69–87. doi:10.1006/eujc.1996.0006

[28] P. Winkler, *Isometric embeddings in products of complete graphs*, Discrete Appl. Math. **7** (1984) 221–225. doi:10.1016/0166-218X(84)90069-6

Received 28 April 2017
Revised 13 September 2017
Accepted 13 September 2017