Discussiones Mathematicae
Graph Theory 39 (2019) 415–437
doi:10.7151/dmgt.2090

ARANKINGS OF TREES

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Abstract

For a graph $G = (V, E)$, a function $f : V(G) \to \{1, 2, \ldots, k\}$ is a k-ranking for G if $f(u) = f(v)$ implies that every $u - v$ path contains a vertex w such that $f(w) > f(u)$. A minimal k-ranking, f, of a graph, G, is a k-ranking with the property that decreasing the label of any vertex results in the ranking property being violated. The rank number $\chi_r(G)$ and the arank number $\psi_r(G)$ are, respectively, the minimum and maximum value of k such that G has a minimal k-ranking. This paper establishes an upper bound for ψ_r of a tree and shows the bound is sharp for perfect k-ary trees.

Keywords: minimal ranking, coloring, tree.

2010 Mathematics Subject Classification: 05C15, 05C05.

1. Introduction

In this paper the term graph refers to a simple graph; i.e., a graph with undirected edges, no loops, and no multiple edges between two vertices. Given a graph, G, a function $f : V(G) \to \{1, 2, \ldots, k\}$ is a *k-ranking* for G if $f(u) = f(v)$ implies that every $u - v$ path P contains a vertex w such that $f(w) > f(u)$. If the value of k is unimportant then f will be referred to as a *ranking* of G. The *rank number* of G, denoted $\chi_r(G)$, is the minimum value of k such that G has a k-ranking. By definition every ranking is a coloring, hence $\chi_r(G)$ is bounded below by the chromatic number, $\chi(G)$.

The rank number was initially studied from an algorithmic standpoint because it is related to the minimum height of an elimination tree [2] which has applications to the sparse factorization of matrices [15]. Some early papers on rankings include [3, 5, 6, 7].

A function $f : V \to \{1, 2, \ldots, k\}$ is a *minimal k-ranking* of G if

1. f is a k-ranking and

2. for all $x \in V$ such that $f(x) > 1$ the function g defined on V by $g(z) = f(z)$ for $z \neq x$ and $1 \leq g(x) < f(x)$ is not a ranking.

A minimal k-ranking may be referred to as a *minimal ranking* when the value of k is unimportant. Minimal rankings are introduced in [9] as a concept analagous to complete colorings. The second condition is the minimality criteria; when applied to rankings it allows for a worst case ranking, called the *arank number* of G, and denoted by $\psi_r(G)$. It is defined to be the maximum value of k for which G has a minimal k-ranking.

Paths are one of the few classes of graphs that have both ranking and aranking numbers determined. For P_n, the path on n vertices, both χ_r and ψ_r have been established.

Theorem 1 [11]**.** $\chi_r(P_n) = \lfloor \log_2(n) \rfloor + 1$.

Theorem 2 [16]**.** $\psi_r(P_n) = \lfloor \log_2(n+1) \rfloor + \lfloor \log_2 \left(n + 1 - 2^{\lfloor \log_2(n) \rfloor - 1} \right) \rfloor$.

If $n = 11$, then $\chi_r(P_{11}) = \lfloor \log_2(11) \rfloor + 1 = 3 + 1 = 4$ and $\psi_r(P_{11}) = \lfloor \log_2(12) \rfloor + \lfloor \log_2(12 - 2^{\lfloor \log_2(11) \rfloor - 1}) \rfloor = 3 + \lfloor \log_2 \left(12 - 2^{3-1}\right) \rfloor = 3 + \lfloor \log_2(8) \rfloor = 6$. With $\chi_r(P_{11})$ and $\psi_r(P_{11})$ known, consider the graphs in Figure 1.

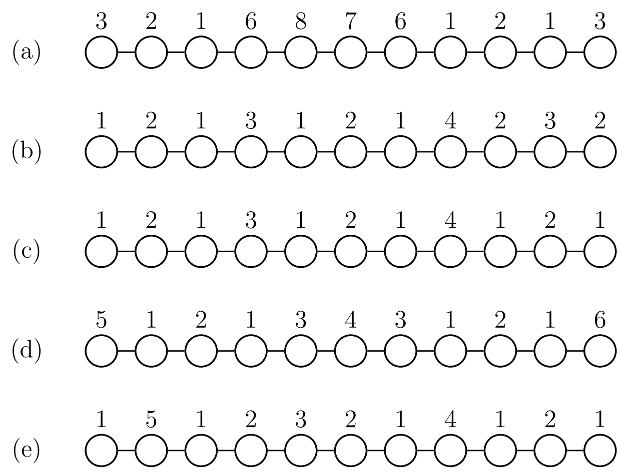

Figure 1. Different rankings of P_{11}.

The graph in (a) shows an 8-ranking since 8 is the largest label used. If the number, in this case 8, is unimportant, then the term *ranking* can be used. To establish that (a) is a ranking it is necessary to verify that every path between any two vertices with the same label contains a higher labeled vertex. This is true; for example, the vertices labeled 1 have a 2 or 6 on every path between them, the path between vertices labeled 2 contains a 7, the two vertices labeled 3 have an 8 between them, and the vertices labeled 6 have a 7 on the path between them. Note that this ranking contains no vertex labeled 4 or 5. This ranking is not minimal either; changing the label of any one of the vertices labeled 6 to a 4 still results in a ranking. The graph in part (b) can be verified as a ranking by the process explained for (a). Since the maximum label used is 4 and $\chi_r(P_{11}) = 4$ it can be called a χ_r-ranking. This is not a minimal ranking because reducing the label of the pendant vertex from 2 to a 1 results in a ranking. Reducing the labels from (b) eventually results in the minimal χ_r-ranking shown in (c). This ranking is minimal because no label can reduced and still result in a ranking. The authors of [9] observe that the process of reducing the labels of a χ_r-ranking which is not minimal, such as that in Figure 1(b), must eventually end with a minimal ranking, hence $\chi_r(G) = \min\{k : G \text{ has a minimal } k\text{-ranking}\}$. Next, it can be easily checked that (d) is a minimal ranking. Since $\psi_r(P_{11}) = 6$ and since ψ_r implies minimality it is common to call this a ψ_r-ranking. Finally, one can check that the graph in (e) has a minimal 5-ranking.

This paper relies on an extensive set of definitions. For definitions that are not given, consult a standard graph theory book such as [4]. Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. If x is a vertex of graph G the *neighborhood of* x, denoted $N(x)$, is the set of all vertices adjacent to x while $N[x] = N(x) \cup \{x\}$. A vertex x is *simplicial* if every vertex of $N(x)$ is adjacent to every other vertex of $N(x)$; that is, $N(x)$ is *complete.* If G is a graph with vertex set V and $S \subseteq V$ is nonempty then the *induced subgraph* of S, denoted $\langle S \rangle$, consists of all the vertices in S together with all the edges in E that are incident with two elements of S. If S is a subset of vertices from graph H which is a subgraph of G then $\langle S \rangle_H$ and $\langle S \rangle_G$ refer to the induced subgraph of S in H and the induced subgraph of S in G, respectively. If $G = (V, E)$ is a graph, a set $S \subseteq V$ is an *independent set* if whenever $x, y \in S$ then $(x, y) \notin E$. A set $S \subseteq V$ is a *dominating set* if for each vertex $y \in V - S$ there exists a vertex $x \in S$ such that $(x, y) \in E$. A set $S \subseteq V$ is an *independent dominating set* if S is an independent set and a dominating set. The *domination number* of G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set for G and the *independent domination number* of G, denoted $i(G)$, is the minimum cardinality of an independent dominating set for G. The *independence number* of G, denoted $\beta(G)$, is the maximum cardinality of an independent set for G. A set of edges $\{e_1, e_2, \ldots, e_m\}$ is a *strong matching* if no two edges share a common vertex and there is no edge in G between vertices e_i and e_j for $1 \leq i \neq j \leq m$. A vertex, x, is a *cut vertex* of graph G if $G - v$ has more components than G. A *clique* in a graph is a subgraph which is complete. Clique C is a *maximal clique* if there is no clique properly containing C.

In general, the distance between vertices x and y in a graph G will be denoted by $d(x, y)$ but if H is a subgraph of G then the distance between x and y could be different in H and G. The notation $d_G(x, y)$ and $d_H(x, y)$ will be used as needed to clarify whether the distance is in G or H, respectively. The notation $rad(G)$ refers to the *radius* of graph G.

The notation K_n will be used for the complete graph on n vertices. A *chordal graph* is a graph in which every cycle of length greater than 3 contains a chord; i.e., an edge which is not part of the cycle but connects two vertices of the cycle. A *perfect elimination ordering* of a graph is an ordering of the vertices $v_1, v_2, \ldots, v_n$ such that, for each v_i, $N(v_i)$ forms a clique in $\langle v_{i+1}, \ldots, v_n \rangle$.

In a rooted tree, all vertices below v that are adjacent to it are *children* of v and v is said to be their *parent.* All the vertices which have a path to v using vertices below v are *descendants* of v. The root is the only vertex with no parent. If the degree of a vertex v is 1 then v is a *pendant vertex* (or end vertex). An *internal vertex* of a tree is any vertex which is not pendant.

For $S \subset V$ the *reduction of G by S*, denoted by G^*_S, is the graph with vertex set $V - S$ and edge set given by $(x, y) \in E(G^*_S)$ if and only if $(x, y) \in E(G)$ or there exists a path $x - s_1 - s_2 - \cdots - s_k - y$ where $s_i \in S$ for $1 \leq i \leq k$. Figure 2 illustrates the reduction of the graph in (a) by the set $S = \{v_3, v_4\}$ which results in the graph (b).

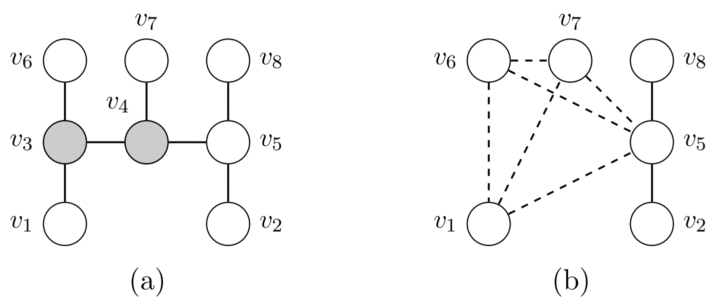

Figure 2. A graph and its reduction by $\{v_3, v_4\}$.

The edges (v_2, v_5) and (v_5, v_8) are in the original graph. The remaining edges, which are dashed, indicate the new edges between vertices of G that have a path between them containing internal vertices in $\{v_3, v_4\}$. For example, (v_1, v_6) is an edge of G^*_S because of the path $v_1 - v_3 - v_6$, where $v_3 \in S$. Similarly, (v_1, v_5) is an edge of G^*_S because the path $v_1 - v_3 - v_4 - v_5$ has internal vertices $v_3, v_4 \in S$.

2. Established Results

This paper also relies on many results that have already been established. Here are some results on rankings which will be needed.

Lemma 3 [9]**.** *If G is a graph on n vertices then $\psi_r(G) = n$ if and only if $\Delta(G) = n - 1$.*

Lemma 4 [9]**.** *The set $\mathcal{R}$ of vertices with repeated labels is a dominating set for G.*

Lemma 5 [13]**.** *Let f be a minimal ψ_r-ranking of a graph G and let t be the largest repeated label. Any permutation of the distinct labels which are greater than $t+1$ is a minimal ψ_r-ranking.*

Lemma 6 [9]**.** *If x is a pendant vertex of a graph G and y is the vertex adjacent to x, then in any minimal ranking f of G, either $f(x) = 1$ or $f(y) = 1$.*

Lemma 7 [9]**.** *If f is a minimal ψ_r-ranking of a connected graph G, then there exists a unique vertex x such that $f(x) = \psi_r$ and there exists a unique vertex y such that $f(y) = \psi_r - 1$.*

Lemma 8 [9]**.** *If G is a graph and $A_1, A_2 \subset V(G)$ such that $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 \subset V(G)$, then $G^*_{A_1 \cup A_2} = \left(G^*_{A_1}\right)^*_{A_2}$.*

Lemma 9 [9]**.** *Let G be a graph and let f be a minimal ψ_r-ranking of G. If $S_1 = \{x : f(x) = 1\}$, then $\psi_r(G^*_{S_1}) = \psi_r(G) - 1$.*

Lemma 6 makes it immediately obvious the ranking in Figure 1(b) is not minimal. Pendant vertices cannot be between two vertices with the same label, so reducing the pendant vertex labeled 2 to a 1 will result in a ranking unless it is adjacent to a vertex labeled 1. Since it is not, the ranking is not minimal. Lemma 9 is particularly important and is illustrated by the four graphs in Figure 3.

The graph, G, in (a) has a minimal 4-ranking. Reduction by S_1, the set of vertices labeled 1, creates the graph $G^*_{S_1}$ in (b). The dashed edges indicate edges that are in $G^*_{S_1}$ but are not in G. The remaining vertices of $G^*_{S_1}$ have been assigned a label one less than what it was in G. Note that this is a minimal 3-ranking. Reduction on the vertices labeled 1 in (b), which is equivalent to the reduction of the original graph by reduction on the vertices labeled 1 and 2, results in the graph in (c). This graph, $(G^*_{S_1})^*_{S_2}$, is equivalent to $G^*_{S_1 \cup S_2}$ by Lemma 8 which is a formal way of saying the reduction of a graph, G, by S_1 and then reduction of that graph by S_2 is equivalent to the reduction of G by $S_1 \cup S_2$. Observe the two resulting edges which do not appear in G are dashed. Once again, 1 has been subtracted from the each label of the vertices in (b), or 2 subtracted from the label the vertices had in G, and the result is a minimal 2-ranking. Reduction on the vertices labeled 1 in the graph of (c) results in an isolate with label 1. Lemma 9 asserts that given a $\psi_r(G)$-ranking, reduction of the graph by S_1 followed by subtracting 1 from each label results in a $\psi_r(G^*_{S_1})$-ranking. Indeed, even more can be said. The minimal 2-ranking in Figure 3 is

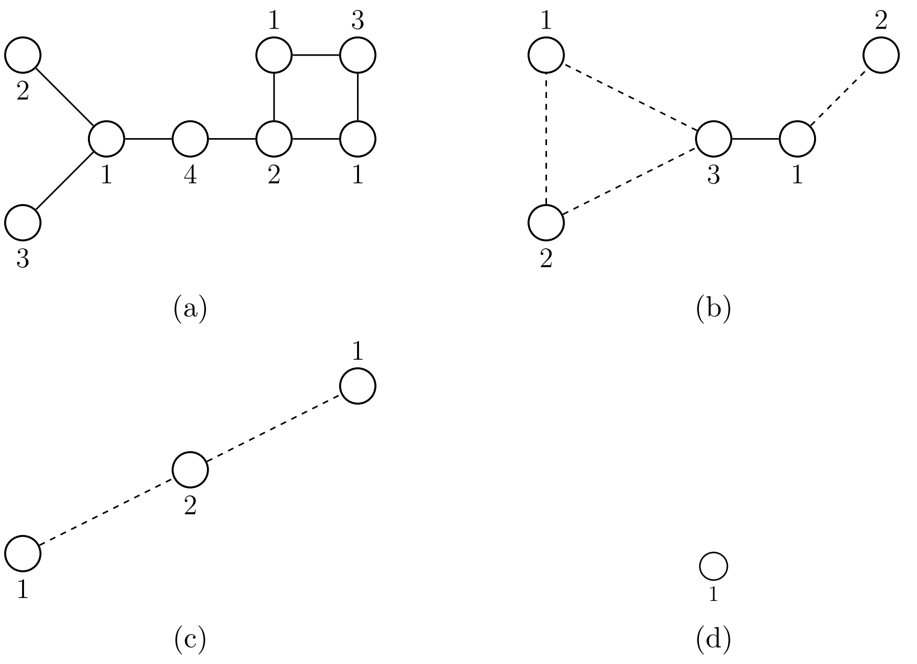

Figure 3. The reduction process for a ranking.

not a ψ_r-ranking. Lemma 3 asserts that the graph in (c) has $\psi_r = 3$ because the vertex labeled 2 has $\Delta(G) = 3 - 1 = 2$. This minimal ranking is achieved by labeling the vertices 2, 1, 3 from left to right results in a minimal 3-ranking. Since $\psi_r = 3$ for the graph in (c), Lemma 9 asserts the graph in (a) is not a ψ_r-ranking either. If it were, the labeling of the graph in (c) would be a ψ_r-ranking. Lemma 7 states that in a ψ_r-ranking the largest label and the second largest label are distinct labels. Therefore, the graph in (a) is not a ψ_r-ranking because although the largest label, $\psi_r = 4$ is distinct the second largest label $\psi_r - 1 = 3$ is a repeated label.

Figure 3 can illustrate some of the other lemmas as well. Lemma 4 asserts that the repeated label vertices form a dominating set. The graph in (a) has repeated labels $1, 2, 3$ and every other vertex, in this case just the vertex labeled 4, is adjacent to some vertex with a repeated label.

Finally, the dashed edges are known as *implicit edges* of the graph for that specific ranking. The concept of an implicit edge is introduced in [12] where the precise formulation is: Let f be a k-ranking of a graph $G = (V, E)$ and let $S_i = \{x : f(x) = i\}$ for $1 \leq i \leq k$. Define $Im(G, f) = (V, E')$ where $(u, v) \in E'$ if and only if $(u, v) \in E(G) \cup E(G^*_{S_1}) \cup \cdots \cup E(G^*_{S_1 \cup \cdots \cup S_{k-1}})$. We call $Im(G, f)$ the *implied graph* of G with respect to f. An edge which is in the implied graph but not in the graph itself is an *implicit edge*.

Figure 4 shows the implied graph for the first graph, G, of Figure 3 under the minimal 4-ranking it started with. The resulting graph is still minimally ranked. The implied edges are dashed to distinguish them from the original edges of G.

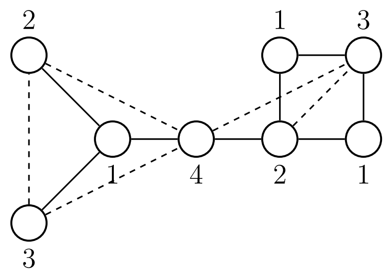

Figure 4. The implied graph of G under the ranking in Figure 3.

Notice the implied graph is chordal. This is always true, for any graph G and any ranking as has been established in Theorem 10.

Theorem 10 [12]**.** *If G is a graph and f is a k-ranking of G, then $Im(G, f)$ is chordal.*

Not every ranking will produce implicit edges; one trivial example is any ranking of K_n. Since a chordal graphs is hiding under every minimal ranking of a graph, the next well known characterization of chordal graphs is useful for studying minimal rankings.

Theorem 11 [8]**.** *A graph is chordal if and only if it has a perfect elimination ordering.*

3. Preliminary Results

This section begins with some results on how parameters change under the reduction process. Theorem 12 asserts that the distance between any two vertices of G^*_S cannot be greater than in G.

Theorem 12. *Let G be a connected graph, $S \subset V(G)$, and suppose $u, v \in V(G^*_S)$. The graph $H = G^*_S$ is connected and $d_H(u, v) \leq d_G(u, v)$.*

Proof. Let $u, v \in V(G^*_S)$; since G is connected there is a shortest path $u = p_1 - p_2 - \cdots - p_k = v$. If all edges (p_i, p_{i+1}) from G are in H then $d_H(u, v) \leq d_G(u, v)$ is clearly true. If some of those edges are not in H then some p_i are in S. Now

$u, v \in V(H)$ implies $u = p_1$ and $v = p_k$ are in $V - S$. Removing the vertices of the path that are in S results in a subsequence of $p_1, \ldots, p_k$ which will be represented by $u = a_1, a_2, \ldots, a_m = v$. Each a_i and a_{i+1} which are not consecutive vertices of $p_1, \ldots, p_k$ are separated by consectutive vertices $p_t - p_{t+1} - \cdots - p_{t+r}$ from S. Therefore (a_i, a_{i+1}) is always an implicit edge of H and so $u = a_1 - a_2 - \cdots - a_m = v$ is a path in H which is shorter than the path in G. As u and v are arbitrary, it follows that H is connected and $d_H(u, v) \leq d_G(u, v)$. ∎

Corollary 13 follows immediately.

Corollary 13. *If G is a graph and $S \subset V$, then $rad(G) \geq rad(G^*_S)$ and $diam(G) \geq diam(G^*_S)$.*

Corollary 14. *If G is a graph and $S \subset V$, then $\beta(G) \geq \beta(G^*_S)$.*

Proof. If A is an independent set in G^*_S with $\beta(G^*_S)$ elements then the distance in G^*_S between any two vertices of A is at least 2. By Theorem 12, the distance in G between any two vertices of A is at least 2. It follows that A is independent in G and $\beta(G) \geq |A| = \beta(G^*_S)$. ∎

For any connnected graph G, let $\mathcal{T}$ denote the set of spanning trees of G and let $\varepsilon(T)$ be the number of pendant vertices of tree $T \in \mathcal{T}$. Finally, let $\varepsilon(G) = \max\{\varepsilon(T) : T \in \mathcal{T}\}$.

For the graph G in Figure 3(a) it is easy to calculate the maximum number of pendant vertices in a spanning tree is 4. Likewise, the maximum number of pendant vertices in a spanning tree of (b) $\varepsilon(G^*_{S_1}) = 3$, and $\varepsilon(G^*_{S_1 \cup S_2}) = 2$ for the graph in (c). Theorem 15 establishes that the maximum number of pendant vertices in a spanning tree cannot increase after a reduction.

Theorem 15. *If G is a connected graph and $S \subset V$, then $\varepsilon(G) \geq \varepsilon(G^*_S)$.*

Proof. Let T^* be a spanning tree of G^* with $\varepsilon(T^*)$ pendant vertices. A spanning tree T of G will be constructed with at least $\varepsilon(T^*)$ pendant vertices in 3 steps. First, any edges of T^* that are in G will be used in T. Second, edges of T^* that are not in G are implicit edges of G^*_S. Each implicit edge corresponds to a path in G using internal vertices which are only in S. The edges of this path are added to T and then any cycle edges that may have resulted are removed. At this point a tree has been created, not necessarily spanning, with $\varepsilon(T^*)$ pendant vertices using edges of G. Third, if the tree is not a spanning tree then it will be extended to create a spanning tree while not decreasing the number of pendant vertices.

The first step is easily accomplished. To complete the second step, observe that any implicit edge in T^*_S between vertices x and y is the result of a path between x and y with internal vertices in S. Add those vertices and edges to T. Note that if either x or y is a pendant vertex of T^* then it is still a pendant vertex in T. Continue the process of adding the vertices and edges to T for each implicit edge of T^* until all implicit edges have been replaced. That resulting graph, call it T', might not be a tree since a cycle could have been created along the way. One by one, remove any edges from T and T' that are on a cycle. Observe that removing a cycle edge cannot decrease the number of pendant vertices so the process ends with a tree having at least as many pendant vertices as T^*. However, T' might not be a spanning tree. If T' is a spanning tree let $T = T'$. Otherwise T is created by finding a vertex $s_1 \in S$ which is not in T' and is adjacent to some vertex $x \in T'$; this is possible since G is connected. Add the vertex s_1 and edge (s_1, x) to T. If x is a pendant vertex of T' then x is no longer pendant but s_1 is now pendant. If x is an internal vertex of T' then x is not a pendant vertex but s_1 is. Therefore, T has at least as many pendant vertices as T'. Continue the process of finding a vertex in S which is adjacent to T and adding it along with the edge to connect it to T. Eventually, since S is finite, the result is a spanning tree T which has at least $\varepsilon(G^*_S)$ pendant vertices. It follows that $\varepsilon(G) \geq \varepsilon(T) \geq \varepsilon(G^*_S)$. ∎

Lemma 4, proven in [9], established $\mathcal{R}$ is a dominating set hence $|\mathcal{R}| \geq \gamma(G)$. The same paper noted $|\mathcal{D}| \leq n - \gamma(G)$, and then used that to prove the upper bound for ψ_r in Lemma 5.

Theorem 16 [9]**.** *If G is a graph with n vertices such that $i(G) > 1$, then $\psi_r(G) \leq n - \frac{\gamma(G)}{2}$.*

The parameter $\varepsilon(G)$ can be used to establish another upper bound for $\psi_r(G)$. First, observe that a minimal ranking, f, of a graph G partitions the vertices into a set $\mathcal{R}$ of vertices that have a repeated label and a set $\mathcal{D}$ of vertices that have a distinct label. If G is a ranking of a graph then $\mathcal{R}$, the set of vertices with a repeated label and $\mathcal{D}$, the set of vertices with a distinct label, partition the vertex set. For any graph, G, which is minimally ranked, reduction by $\mathcal{R}$ will result in a graph which has a minimal ranking if the remaining vertices have their label reduced by the number of different repeated labels. Take, for example, the minimal 6-ranking of the graph, G, shown in Figure 5(a).

Reduction by the set of vertices with a repeated label, $\mathcal{R} = S_1$, results in $G^*_{\mathcal{R}}$, shown in Figure 5(b). It is vertex set is $\mathcal{D}$ and it can be minimally ranked by reducing the label of each vertex by the number of different repeated labels, which is 1. Once again, the dashed edges represent the edges that are not in the graph in (a); that is, the implied edges.

Notice for the graph in Figure 5(b) that all vertices have a path to each other through the vertex labeled 1. Lemma 5 asserts that the labels of vertices in $G^*_{\mathcal{R}}$ bigger than 1 can be permuted; so labels bigger than 2 in G can be permuted. Next, Lemma 3 says $\Delta(G^*_{\mathcal{R}}) = |\mathcal{D}| - 1$. A spanning tree T^* of $G^*_{\mathcal{R}}$ can be

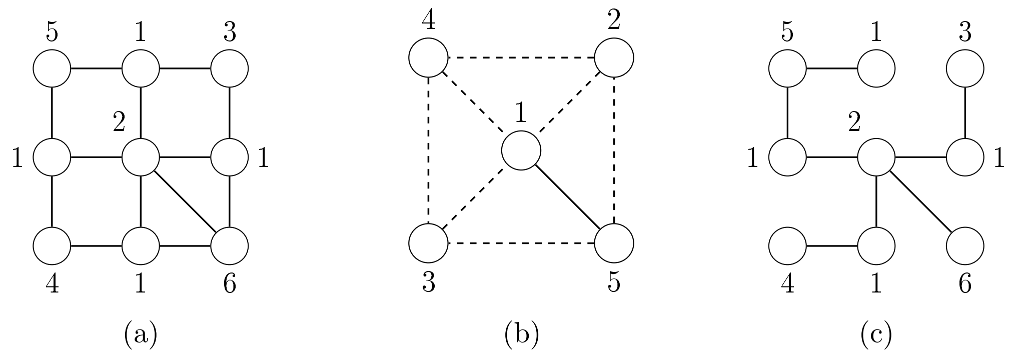

Figure 5. A spanning tree of G with $|\mathcal{D}| - 1$ pendant vertices exists.

constructed with $\varepsilon(G^*_{\mathcal{R}}) = |\mathcal{D}| - 1$ provided $|\mathcal{D}| - 2 > 0$; T^* would consist of all the vertices in Figure 5(b) with edges from the vertex labeled 1 to vertices labeled $2, 3, 4, 5$. A spanning tree, T, of G in Figure 5(a) is achieved by first taking all the vertices and the edges of T^* in G (the edge from vertices labeled 1 to 5 which corresponds ot the edge from 2 to 6 in G) along with edges in G that create implicit edges. For example, there is an implicit edge from vertices labeled 1 to 4 in T^* so there is a path from the vertex labeled 2 to the vertex labeled 5 in (a). This process continues for each implicit edge. After removing any cycle edges that may have resulted from crossing paths and connecting up any vertices of $\mathcal{R}$ that have not been used, a spanning tree T for G is formed with $|\mathcal{D}| - 1$ edges. Figure 5(c) is one such possibility. This is the basic idea behind the next result, Theorem 17, which more formally establishes a second upper bound for $|\mathcal{D}|$.

Theorem 17. *If f is a ψ_r-ranking of a connected graph G on n vertices, then* $|\mathcal{D}| \leq \varepsilon(G) + 1$.

Proof. If $|V(G)| = 1$ then G is a K_1 and $1 = |\mathcal{D}| \leq \varepsilon(G) + 1$, likewise if $|V(G)| = 2$ then G is a K_2 and $2 = |\mathcal{D}| \leq \varepsilon(G) + 1$. Therefore, assume G is a connected graph on at least three vertices. Since G is connected it contains either P_3 or K_3 as an induced subgraph hence $\psi_r(G) \geq 3$ and by Lemma 7, $|\mathcal{D}| \geq 2$. Let $\mathcal{R}$ be the set of vertices that have a repeated label. If $|\mathcal{R}| = 0$ then by Lemma 3 there is one vertex, v, adjacent to every other vertex. The spanning tree consisting of every edge incident with v has $n - 1$ pendant vertices and since every spanning tree has more than one edge, $\varepsilon(G) \leq n - 1$, hence $n = |\mathcal{D}| \leq n - 1 + 1$.

So suppose $\mathcal{R}$ is nonempty; form the reduction of G by the vertices labeled 1 then by the vertices labeled 2, and so on until all the repeated labels have been used. The resulting graph $G^*_{\mathcal{R}}$ has $|\mathcal{D}|$ vertices and $\psi_r(G^*_{\mathcal{R}}) = |\mathcal{D}|$. Once again, Lemma 3 implies there is a vertex v adjacent to every other vertex of $G^*_{\mathcal{R}}$. Since $|\mathcal{D}| \geq 2$, for every vertex $u \in \mathcal{D} - \{v\}$ is either adjacent to v in G or has a path $u - r_1 - r_2 - \cdots - r_k - v$ in G, where each r_i is in the nonempty set $\mathcal{R}$.

A spanning tree for G will be formed which has pendant vertices $\mathcal{D}-\{v\}$. The vertex v, which has the smallest distinct label, will be the root of the spanning tree and the remaining vertices with distinct labels will be the pendant vertices of the spanning tree. Since the spanning tree has more than two edges v will not be pendant. Start by including all the edges from vertices in $\mathcal{D} - \{v\}$ to v. If there exist vertices in $\mathcal{D} - \{v\}$ not adjacent to v in G then, since v is adjacent to it in $G^*_{\mathcal{R}}$, there exists a path from that vertex, call it x, to v through vertices of $\mathcal{R}$. Add all the edges to the tree and the number of pendant vertices increases by one. Continue the process of connecting up the remaining vertices of $\mathcal{D} - \{v\}$. Remove any cycle edge that is created; this can only increase the number of pendant vertices in the tree. The result is a tree T with at least $|\mathcal{D}| - 1$ pendant vertices.

Expand the tree one vertex at a time by finding a vertex not in T that is adjacent to a vertex of T. Add any edge of G that connects it to T. The edge either connects to a pendant vertex, in which case the number of pendant vertices stays the same, or it connects with a vertex which is not pendant, in which case the number of pendant vertices increases. Continue the process until all vertices are used and the result is a spanning tree of T with at least $|\mathcal{D}| - 1$ pendant vertices. Therefore, $|\mathcal{D}| - 1 \leq \varepsilon(T) \leq \varepsilon(G)$; that is, $|\mathcal{D}| \leq \varepsilon(G) + 1$. ∎

This bound on the cardinality of $\mathcal{D}$ means that $|\mathcal{R}| \leq n - (\varepsilon(G) + 1)$ which results in another bound on ψ_r.

Corollary 18. *If G has n vertices, then $\psi_r(G) \leq \frac{n+\varepsilon(T)+1}{2}$.*

Proof. There can be at most $\varepsilon(G) + 1$ distinct labels and $\frac{n-(\varepsilon(G)+1)}{2}$ different repeated labels. Therefore, $\psi_r(G) \leq \frac{n-(\varepsilon(G)+1)}{2} + \varepsilon(G) + 1 = \frac{n+\varepsilon(G)+1}{2}$. ∎

If $n \geq 3$ then $\varepsilon(K_n) = n - 1$, hence $\psi_r(K_n) \leq \frac{n+\varepsilon(G)+1}{2} = \frac{n+(n-1)+1}{2} = n$ shows the bound is sharp.

4. Reductions on Chordal Graphs and Reduction Trees

Theorem 19 will be used to show the reduction of a chordal graph is a chordal graph. This will allow the possibility of using induction on any chordal graph.

Theorem 19. *If G is a chordal graph on $n \geq 2$ vertices and $v \in V(G)$, then G^*_v is a chordal graph.*

Proof. The proof is by induction on the number of vertices. If G is a chordal graph on $n = 2$ vertices then G is either two isolated vertices or G is K_2. In either case, $G_v^* \cong K_1$ which is chordal. Suppose the statement is true for any chordal graph on n vertices and consider a chordal graph on $n + 1$ vertices. If v is a simplicial vertex then G_v^* is equivalent to $\langle V - v \rangle$, the graph formed by removing v, which is chordal. If v is not simplicial then, since G is chordal, there exists a perfect elimination ordering for G, say $p_1, p_2, \ldots, p_k = v, p_{k+1}, \ldots, p_n$. It will be shown that $p_1, p_2, \ldots, p_{k-1}, p_{k+1}, \ldots, p_n$ is a perfect elimination ordering for $H = G_v^*$. Now $N_G(p_i)$ is already complete in the induced subgraph $\langle \{p_{i+1}, \ldots, p_n\} \rangle$ of G and that does not change in H if $v \notin (N_H(p_i) \cap \{p_{i+1}, \ldots, p_n\})$. If $v \in (N_H(p_i) \cap \{p_{i+1}, \ldots, p_n\})$ then v is adjacent to every other vertex in $N_H(p_i)$ contained in $\{p_{i+1}, \ldots, p_n\}$. Therefore, for each $x \in (N_H(p_i) \cap \{p_{i+1}, \ldots, p_n\})$ and for each $y \in (N_H(v) \cap \{p_{i+1}, \ldots, p_n\})$ there is always an $x - v - y$ path hence (x, y) is an edge of H. Moreover, if $y_1, y_2 \in (N(v) \cap \{p_{i+1}, \ldots, p_n\})$ then there is a $y_1 - v - y_2$ path in G hence y_1 and y_2 are adjacent in H. Therefore $N_H(p_i)$ is simplicial in the graph $\langle \{p_{i+1}, \ldots, p_n\} \rangle_H$. It follows that this is a perfect elimination ordering and G_v^* is a chordal graph. ■

Apply Theorem 19 for each s in a proper subset $S \subset V$ to get Corollary 20.

Corollary 20. *If G is a chordal graph on $n \geq 2$ vertices and $S \subset V$, then G_S^* is a chordal graph.*

In order to prove a bound for the arank number of a tree, a new concept is introduced. A graph, G, is defined to be a *reduction tree* if there exists a tree T and subset $A \subset V(T)$ such that $G = T_A^*$. This is illustrated in Figure 6.

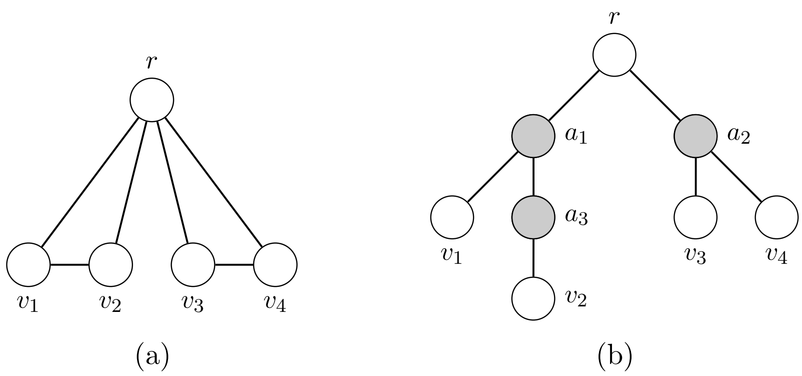

Figure 6. The graph, G, in (a) is a reduction tree of the tree, T in (b).

Graph, G, in (a) is a reduction tree because the tree T in (b) contains a set of vertices $A = \{a_1, a_2, a_3\}$ with the property that $G = T_A^*$. Observe that

T is not unique; adding another vertex $a_4 \in A$ and making it adjacent to any vertex of the tree in Figure 6 results in the same reduction tree. Of course, any vertices of A which are pendant are not needed to create a reduction tree. Also notice, by Theorem 12, the distance between two vertices in G is less than the distance between them in T. For example, the distance from $d_G(r, v_2) = 1$ but $d_T(r, v_2) = 3$.

Theorem 21. *The reduction of a reduction tree is a reduction tree.*

Proof. If G is a reduction tree then there exists a tree T and a set of vertices A such that $G = T_A^*$. The reduction G_B^* is equivalent to $(T_A^*)_B^* = T_{A\cup B}^*$ by Lemma 8. ■

Since a tree is a chordal graph and the reduction of a chordal graph is a chordal graph by Corollary 20, reduction trees are chordal graphs. In order to characterize reduction trees, consider Figure 7, which is chordal with perfect elimination order c, b, a, d.

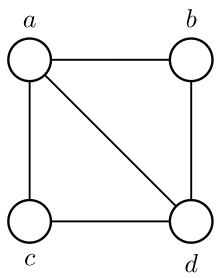

Figure 7. K_4 minus an edge.

Theorem 22 asserts that an induced subgraph of K_4 minus an edge is precisely what keeps a chordal graph from being a reduction tree. Therefore, K_4 minus an edge is an example of a chordal graph which is not a reduction tree.

Theorem 22. *Let G be a chordal graph. G is a reduction tree if and only if it contains no induced subgraph isomorphic to K_4 minus an edge.*

Proof. Let F denote the forbidden induced subgraph K_4 minus an edge shown in Figure 7. Let T be a tree rooted at r whose reduction by a set of vertices, A, results in the reduction tree G containing F as an induced subgraph. Consider the vertices a, c, d; there is only one path between these vertices in T. If one of the vertices was on the path between the other, such as a path from a to d which contained c, then the vertices would not from a clique in G because there is no path from a to d with internal vertices in A. The only way that the clique can form is if there is some vertex x in A such that the paths from x to a, b, c whose internal vertices are entirely within A. This makes it impossible for there (a, b) and (b, d) to be edges of G. Imagine that the tree is rooted at x; of course b cannot be on any path from x to a (or b or c) because that would imply that x and a (or b or c) are adjacent after reduction by A. Since T is a tree, this would imply that there is a path from b to a as well as d whose internal vertices are in A. But this would mean that there is a path from b to c with internal vertices in A, hence a, b, c, d induce a K_4. It follows that F cannot be an induced subgraph of G as all cases result in a contradiction.

Conversely, given a connected chordal graph, G, without F as an induced subgraph, identify the m maximal cliques of G and assign a one vertex a_i, such that $1 \leq i \leq m$, to the vertices in maximal clique i. Let $A = \{a_1, \ldots, a_m\}$ and T be the graph with vertex set $V(T) = V(G) \cup A$. The edges of T will consist of ordered pairs $\{(a_i, x) : a_i \in A$ and x is in maximal clique $i\}$. If T were to have a cycle then it would require an even number of vertices which alternate between being in A and being in G. Let $a_1 - v_1 - a_2 - v_2 - \cdots - a_k, v_k$ be such a cycle where $a_1, \ldots, a_k$ are elements of A and $v_1, \ldots, v_k$ are vertices of T. Since the vertices of A only connect to vertices of their respective maximal cliques, $v_1, \ldots, v_k$ is a cycle of chordal graph. Therefore, there is a chord for each cycle greater than 3 hence assume the 3 consecutive vertices v_1, v_2, v_3 form a K_3 in G. Now a_2 is adjacent to v_1 and v_2 because (v_1, v_2) is part of a maximal clique C_1 containing v_1 and v_2. Similarly, there is a maximal clique C_2 for the cycle containing the K_3 formed by v_1, v_2, v_3. Since the C_1 and C_2 are different there exists $x \in C_1$ and $y \in C_2$ which are not adjacent, hence $\langle\{x, y, v_1, v_2\}\rangle$ forms a K_4 minus an edge.

■

Figure 8 illustrates the process of constructing the tree, T and the set, A, of vertices such that $T_A^* = G$.

Graph G in (a) is a chordal graph with simplicial vertices $\{s_1, s_2, s_3, s_4, s_5, s_6\}$ and cut vertices $\{c_1, c_2\}$. The proof assumes that K_4 minus an edge is not an induced subgraph, and that forbidden subgraph is not part of G. The graph in (b) adds vertices a_1, a_2, a_3, a_4; one vertex to associate with each of the four maximal cliques of G. That is, a_1 is associated with the clique containing vertices s_1, s_2, c_1, vertex a_2 is associated with the clique containing vertices c_1, s_6, vertex a_3 is associated with the clique containing vertices c_1, c_2, and a_4 is associated with the clique containing vertices c_2, s_3, s_4, s_5. In graph (c) all the edges of G are removed. In (d), edges are added from each a_i to the vertices which are in the clique it is associated with. This creates a tree, T, along with a set, $A = \{a_1, a_2, a_3, a_4\}$ such that $T_A^* = G$ so the graph in (a) is a reduction tree.

Theorem 22 gives another way to show that a tree on $n > 1$ vertices is a reduction tree: replace the i^{th} edge, (x, y), of the tree with two edges (x, a_i) and (a_i, y), where $a_1 \in A$ for $1 \leq i \leq n - 1$.

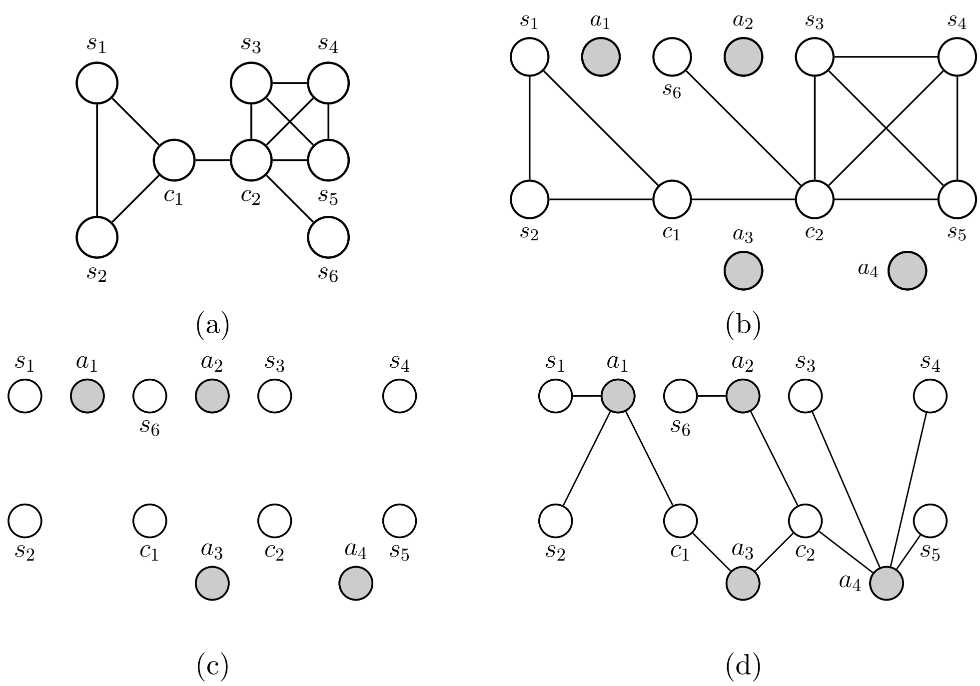

Figure 8. A graph is shown to be a reduction tree using Theorem 22.

Reduction trees can be distinguished from chordal graphs in another way. Borrowing from tree vocabulary, Lemma 23 asserts that the parent of a vertex in a reduction tree is unique.

Lemma 23. *If G is a reduction tree and x is a vertex distance k from a central vertex r, then there is a unique vertex y in G that is distance $k-1$ from r such that (x, y) is an edge of G.*

Proof. Let r be a central vertex of the reduction tree, G. The vertices distance 1 from r will be on the first level below r, the vertices distance 2 from r will be on the second level below r, and so until the vertices which are distance $rad(G)$ from r are on the final level. Corresponding to the reduction tree there is a tree T with a subset A of vertices such that $G = T_A^*$. If y_1 and y_2 are adjacent to x in G and both y_1 and y_2 are distance $k-1$ from the root r in G then there exist paths $y_1 - a_1 - a_2 - \cdots - a_m - x$ and $y_2 - b_1 - b_2 - \cdots - b_t - x$ in T such that $a_1, a_2, \ldots, a_m \in A$ and $b_1, b_2, \ldots, b_t \in A$. Since T is a tree there exists a smallest i such that $a_i = b_j$. Likewise, since there is a unique path from the root, r, of T to both y_1 and y_2 there is a vertex c (which may be r) in common to the $r - y_1$ path and the $r - y_2$ path. This implies c, y_1, x, and y_2 are on a common cycle, contradicting that T is a tree. It follows that result is true. ∎

The graph in Figure 9 is chordal because every cycle of length greater than 3 contains a chord. One perfect elimination ordering is $s_1, s_2, x, y, r, z, s_3$.

By Lemma 23 the graph is not a reduction tree since s_1 (and s_2) has two vertices, x and y above it that it is adjacent to. Likewise, the graph is chordal and the vertices r, x, y, s_1 induce a K_4 minus an edge so Theorem 22 could also be used to establish the graph is not a reduction tree.

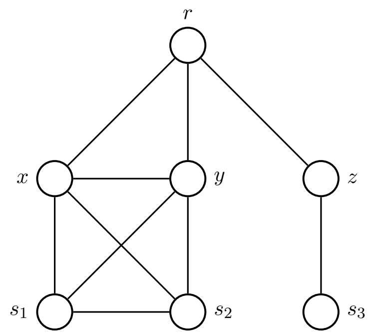

Figure 9. This chordal graph contains the forbidden subgraph of Figure 7.

Lemma 6 can be generalized to any simplicial vertex. It asserts that a simplicial vertex does not really separate vertices, hence the only reason its label could not be decreased to 1 is because it is adjacent to some vertex labeled 1.

Lemma 24. *If G is a graph with simplicial vertex s and f is a minimal ranking of G, then either $f(s) = 1$ or there exists a $y \in N(s)$ such that $f(y) = 1$.*

Proof. Let f be any minimal ranking of G with simplicial vertex s. By Lemma 6 the degree of s can be assumed to be at least 2. If either $f(s) = 1$ or $f(y) = 1$ then there is nothing to prove so assume that both $f(s)$ and $f(y)$ are greater than 1. Consider any path, P, containing s between two vertices u and v such that $f(u) = f(v)$. Let P be represented by $u = a_1 - a_2 - \cdots - a_m - s - b_1 - b_2 - \cdots - b_n - v$. Since s is simplicial, (a_m, b_1) is an edge of G hence P contains a shorter path P' given by $u = a_1 - a_2 - \cdots - a_m - b_1 - b_2 - \cdots - b_n - v$. Since $f(u) = f(v)$ and f is a ranking there exists an x on P' such that $f(x) > f(u)$. This means that the label of s can be decreased to 1 because x is a vertex of P such that $f(x) > f(u)$. This contradicts the fact that f is a minimal ranking, which establishes the result. ■

Lemma 25. *If f is a minimal ranking of a reduction tree G and S_1 is the set of vertices labeled* 1, *then $rad(G) - rad(G^*_{S_1}) + \varepsilon(G) - \varepsilon(G^*_{S_1}) > 0$.*

Proof. Since $rad(G) \geq rad(G^*_{S_1})$ by Corollary 13 and $\varepsilon(G) \geq \varepsilon(G^*_{S_1})$ by Theorem 15, it suffices to show that $rad(G) > rad(G^*_{S_1})$ or $\varepsilon(G) > \varepsilon(G^*_{S_1})$. Now G is a reduction tree so by Lemma 23 every vertex which is $rad(G)$ away from a central vertex, r, has a unique parent, p. It follows that the parent is an internal vertex of any spanning tree G. By Lemma 31 those vertices $rad(G)$ from r are simplicial so by Lemma 24 each vertex is either labeled 1 or is adjacent to a vertex labeled 1. There are three cases to consider.

Case 1. If the parent is labeled 1 then all vertices below it are distance $rad(G)-1$ after reduction.

Case 2. If a simplicial vertex which is not pendant is labeled 1 then, since a unique parent implies all its children are pendant in any spanning tree of G with $\varepsilon(G)$ vertices, it follows $\varepsilon(G) > \varepsilon(G^*_{S_1})$.

Case 3. If a simplicial vertex is pendant and labeled 1 then $\varepsilon(G) > \varepsilon(G^*_{S_1})$ unless the parent, p, becomes pendant. In this case, p is now distance $rad(G)-1$ from r. Therefore, if Case 2 occurs then the result is true since $\varepsilon(G) > \varepsilon(G^*_{S_1})$ while if only Case 1 and Case 3 occur then $rad(G) > rad(G^*_{S_1})$. It follows $rad(G) - rad(G^*_{S_1}) + \varepsilon(G) - \varepsilon(G^*_{S_1}) > 0$. ■

5. MAIN RESULTS

With the preliminary results complete, the main results follow easily.

Theorem 26. *If G is a reduction tree on more than one vertex, then $\psi_r(G) \leq rad(G) + \varepsilon(G)$.*

Proof. Observe that $\psi_r(K_2) = 2 \leq 3 = rad(K_2)+\varepsilon(K_2)$. Let G be the reduction tree on the smallest number of vertices for which $\psi_r(G) > rad(G) + \varepsilon(G)$. By Lemma 9, $\psi_r(G) - 1 = \psi_r(G^*_{S_1})$ and since $G^*_{S_1}$ is a reduction tree by Theorem 21 it follows that $rad(G) + \varepsilon(G) - 1 < \psi_r(G) - 1 = \psi_r(G^*_{S_1}) \leq rad(G^*_{S_1}) + \varepsilon(G^*_{S_1})$. Therefore, $rad(G) - rad(G^*_{S_1}) + \varepsilon(G) - \varepsilon(G^*_{S_1}) < 1$. Now every quantity is an integer so this is equivalent to $rad(G) - rad(G^*_{S_1}) + \varepsilon(G) - \varepsilon(G^*_{S_1}) \leq 0$ which contradicts Lemma 25. Therefore, there does not exist a graph G for which $\psi_r(G) > rad(G) + \varepsilon(G)$, which establishes the result. ■

Corollary 27. *If G is a tree on more than one vertex, then $\psi_r(G) \leq rad(G) + \varepsilon(G)$.*

Proof. By Theorem 22, every tree is a reduction tree, hence Theorem 26 applies to trees as well. ■

The next theorem gives a sufficient condition to achieve the bound.

Theorem 28. *Let T be a tree with $rad(T) \geq 3$ and c the unique central vertex of T. If 1) every pendant vertex is $rad(T)$ from the center, and 2) $d(x, c) \leq rad(T) - 3$ implies x has at least two children, then $\psi_r(T) = rad(T) + \varepsilon(T)$.*

Proof. Root the tree at c and label the vertices distance k $(0 \leq k \leq rad(T) - 1)$ from c with $rad(T) - k$. The $\varepsilon(T)$ pendant vertices which are $rad(T)$ away from c are labeled $rad(T) + 1$ to $rad(T) + \varepsilon(T)$. This labeling is a ranking because whenever two vertices are the same label, the only path between them is through their parent, which has a higher label. The ranking is minimal as well. The vertices labeled 2 are $rad(T) - 2$ away from c and since every pendant vertex is $rad(T)$ away from the center, they all have a child labeled 1. Vertices labeled 3 are distance $rad(T) - 3$ from c; it has two children labeled 2, so it is label cannot be reduced. The same argument applies to every label up to $rad(T)$; it has at least two children with labeled $rad(T) - 1$ below it preventing it from being lowered. The remaining $\varepsilon(T)$ pendant vertices are all, by hypothesis, $rad(T)$ from c. The vertex labeled $rad(T)+1$ is cannot be changed to k $(1 \leq k \leq rad(T))$ because the vertices above it form a path with labels $k - 1 - 2 - \cdots - k$ path which would not be a ranking. The same argument applies to the vertex labeled $rad(T) + 2$ and above. These vertices cannot be lowered to k for $rad(T) \leq k \leq rad(T) + \varepsilon(T) - 1$ either because then there is a path with labels $k - 1 - 2 - \cdots - rad(T) - (rad(T) - 1) - \cdots - 1 - k$. Since this is a minimal ranking uses $rad(T) + \varepsilon(T)$ labels it is a ψ_r-ranking. ∎

The ranking scheme using Theorem 28 is demonstrated in Figure 10. Notice, for example, that the vertex labeled 11 cannot have its label reduced to 1 because it is adjacent to 1 and cannot be reduced to 2 or 3 because that would create a path with labels $2 - 1 - 2$ and $3 - 1 - 2 - 3$, respectively. Reducing the label from 11 to 4 creates a path with labels $4 - 1 - 2 - 3 - 4 - 1 - 4$. Similarly, reducing the label from 11 to any distinct label results in a path where two vertices have the same label and are connected by a path containing no higher labeled vertex.

Theorem 28 applies to a specific class of trees but first some more definitions are needed. A *k-ary tree* is a rooted tree in which each node has no more than k children. A *full k-ary tree* is a k-ary tree where within each level every node has either 0 or k children. A *perfect k-ary tree* is a full k-ary tree in which all pendant vertices are the same distance from the root. Figure 10 shows a ψ_r-ranking of a perfect k-ary tree. Lemma 5 tells us that this is not the only ψ_r-ranking; the labels greater than 3 can be permuted to get a different ψ_r-ranking.

Corollary 29. *If G is a perfect k-ary tree where $k \geq 2$, then $\psi_r(G) = rad(G) + \varepsilon(G)$.*

Proof. By Corollary 27, $\psi_r(T) \leq rad(T) + \varepsilon(T)$ and the ranking scheme of Theorem 28 shows $\psi_r(T) = rad(T) + \varepsilon(T)$. ∎

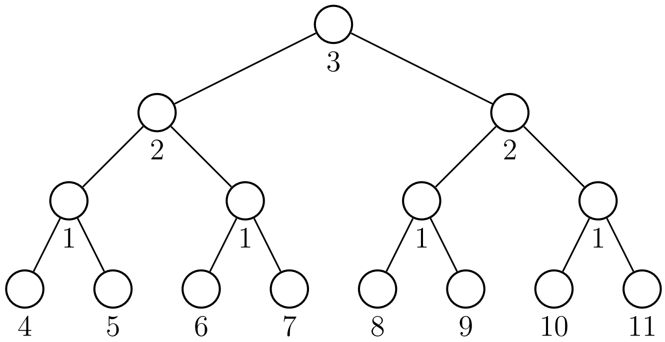

Figure 10. A ψ_r-ranking for a perfect 2-ary tree with radius 3.

Corollary 30. *If T is a perfect k-ary tree ($k \geq 2$), then $\psi_r(T) = rad(T) + k^{rad(T)}$.*

Lemma 31. *If G is a reduction tree, then every vertex $rad(G)$ from a central vertex is simplicial.*

Proof. Let G be a reduction tree. There exists a tree T and set of vertices A in T such that $T_A^* = G$. Color the vertices of A with blue and the rest of the vertices of tree T with white. The white vertices are the vertices that form G after T is reduced by the blue vertices of A. Root T at r, a (white) central vertex of G. Any vertex $rad(G)$ away from r in G is at least $rad(G)$ away from r in T by Theorem 12. Moreover, that vertex can have no descendants in T which are white vertices as they would be at least $rad(G) + 1$ from the center of G. Consider any white vertex, x, which is $rad(G)$ away from r. If x has a parent in T which is a white vertex then x is pendant in G, hence x is simplicial in G. If x has a parent in T which is blue then consider any white vertex y for which $x - b_1 - b_2 - \cdots - b_k - y$ is a path in which $b_1, b_2, \ldots, b_k, y \in A$. The vertices x and y are adjacent in G. Moreover, if there exist white vertices $y_1, y_2 \in T$ such that $x - b_1 - b_2 - \cdots - b_k - y_1$ and $x - a_1 - a_2 - \cdots - a_m - y_2$ are paths with $b_1, b_2, \ldots, b_k \in A$ and $a_1, a_2, \ldots, a_m \in A$ then $a_1 = b_1$ since T is a tree. It follows that $y_1 - b_k - \cdots - b_2 - b_1 = a_1 - a_2 - \cdots - a_m - y_2$ is a path from y_1 to y_2 with internal vertices in A. This implies (y_1, y_2) is an edge of G and since y_1 and y_2 are arbitrary, $N_G(x)$ is complete. That is, x is a simplicial vertex of G. ∎

Theorem 32. *If G is a reduction tree such that $\psi_r(G) = rad(G) + \varepsilon(G)$ and S_1 is the set of vertices labeled 1 in a $\psi_r(G)$-ranking, then $\psi_r(G^*_{S_1}) = rad(G^*_{S_1}) + \varepsilon(G^*_{S_1})$.*

Proof. The proof is by induction on $rad(G)$. If G is a graph on $n \geq 4$ vertices such that $rad(G) = 1$ then G contains a vertex, x such that $\Delta(G) = n - 1$.

This implies $\psi_r(G) = n$ by labeling x with 1 and the rest of the vertices labeled 2 through n, in any order. For this graph $\varepsilon(G) = n - 1$ by taking the G is a K_2 and has $\psi_r(G)$-ranking consisting of one vertex labeled 1 and the other labeled 2. Suppose $\psi_r(G^*_{S_1}) = rad(G^*_{S_1}) + \varepsilon(G^*_{S_1})$ for any reduction tree such that $rad(G) = k$ and let G be a reduction tree with $\psi_r(G^*_{S_1}) = rad(G^*_{S_1}) + \varepsilon(G^*_{S_1})$ and $rad(G) > 1$. Since every vertex $rad(G)$ from the center is simplicial by Lemma 31 either the simplicial vertex, s is labeled 1 or a vertex in $N(s)$ is labeled 1. Since G is a reduction tree, there exists a tree T and a subset A of vertices such that $T^*_A = G$. If the tree is ranked at vertex r then x is a pendant vertex of T and looking at the vertices that x is descended from means that there is some vertex y closest to x which is not in A. This implies that y is a cut vertex since the only implied edge are between vertices that are connected through a path whose internal vertices are in A. This implies that all the simplicial vertices that every spanning tree must contain y hence all the vertices would be pendant in a tree having $\varepsilon(G)$ pendant vertices. Making a simplicial vertex 1 will decrease $\varepsilon(G)$ by one while choosing labeling the cut vertex adjacent to s will decrease the distance of the simplicial vertices from the root.

Now $\psi_r(G) = rad(G) + \varepsilon(G)$ by hypothesis, $\psi_r(G^*_{S_1}) \leq rad(G^*_{S_1}) + \varepsilon(G^*_{S_1})$ by the Theorem and $\psi_r(G^*_{S_1}) = \psi_r(G) - 1$ by Theorem . Combining the results gives $rad(G) + \varepsilon(G) - 1 = \psi_r(G) - 1 = \psi_r(G^*_{S_1}) \leq rad(G^*_{S_1}) + \varepsilon(G^*_{S_1})$ which simplifies to $rad(G) + \varepsilon(G) - 1 \leq rad(G^*_{S_1}) + \varepsilon(G^*_{S_1})$. By Theorem 15, $\varepsilon(G) \geq \varepsilon(G^*_S)$.

Case 1. $\varepsilon(G) = \varepsilon(G^*_S)$. This gives $rad(G) - 1 \leq rad(G^*_{S_1})$ so by Corollary 13, $rad(G) \geq rad(G^*_{S_1})$ and therefore $rad(G) - 1 \leq rad(G^*_{S_1})$. That is, if the number of pendant vertices does not change then the radius must decrease by 1.

Case 2. $\varepsilon(G) > \varepsilon(G^*_S)$. It follows that $\varepsilon(G) - 1 \geq \varepsilon(G^*_S)$ and so $rad(G) + \varepsilon(G^*_{S_1}) \leq rad(G^*_{S_1}) + \varepsilon(G^*_{S_1})$. This simplifies to $rad(G) \leq rad(G^*_{S_1})$. By Corollary 13 $rad(G) = rad(G^*_{S_1})$; that is, if the number of pendant vertices decreases then the radius does not change. This implies that all the distinct label vertices must be simplicial. ∎

6. Conclusion

Although a sharp upper bound for the the arank number of a tree has been found, it is natural to try to extend Theorem 32 to other classes of graphs. The graph in Figure 11 will show the bound can fail for chordal graphs which are not reduction trees. The graph is chordal since every cycle with more 4 vertices contains a chord. A possible perfect elimination ordering is $t_1, t_3, t_2, s_1, s_2, r, t_4, t_6, t_5, s_3, s_4$. This graph is not a reduction tree because $\langle\{t_1, t_2, s_1, s_2\}\rangle$ induces K_4 minus an edge. However, it will be shown that $\psi_r(G) > rad(G) + \varepsilon(G)$ for this graph.

It is easy to see that $\Delta(G) = 4$, $|V(G)| = 11$, and $\gamma(G) \geq 3$. The set of

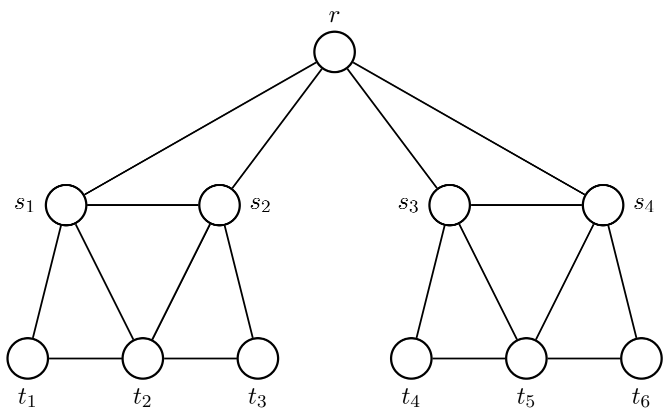

Figure 11. This graph will be a counterexample.

vertices $\{t_2, t_5, r\}$ is a dominating set for G, hence $\gamma(G) = 3$. Likewise, r is the only central vertex and $rad(G) = 2$. To establish $\varepsilon(G)$, note that a spanning tree must contain either the edge (s_1, r) or (s_2, r). Likewise, a spanning tree must contain (s_3, r) or (s_4, r). The symmetry of the graph means you can assume that the spanning tree must contain edges (s_1, r) and (s_3, r). In order for t_1, t_3, t_4 and t_6 to be part of the spanning tree 4 other vertices will need to be internal vertices. This implies $\varepsilon(G) \leq 6$ and the spanning tree consisting of edges (s_1, r), (s_3, r), (s_1, t_1), (s_1, t_2), (s_2, r), (s_2, t_3), (s_3, t_4), (s_3, t_5), (s_3, s_4), (s_4, t_6) establishes that $\varepsilon(G) = 6$.

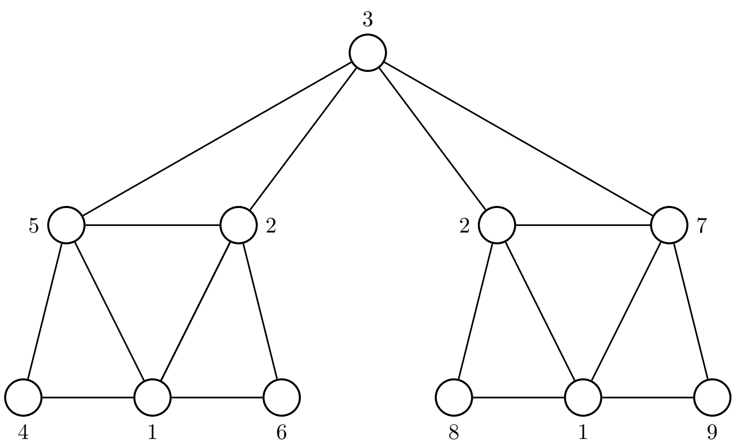

Figure 12. This chordal graph has $\psi_r(G) = 9 > 8 = rad(G) + \varepsilon(G)$.

To establish this as a counterexample, it must be shown that $\psi_r(G) > rad(G) + \varepsilon(G) = 2 + 6$. Since $\gamma(G) = 3$ it follows that $\psi_r(G) \leq 11 - \frac{3}{2} = 9.5$ by Theorem 16. Now consider the labeling shown in Figure 12.

This is a ranking because the only vertices labeled 1 are not adjacent and the only vertices labeled 2 are separated by a vertex labeled 3. It is straightforward to check that this is a minimal ranking as well. For example the vertex 9 cannot be labeled 1 because it is adjacent to a vertex labeled 1. Making it 2 creates a $2-1-2$ path and making it 3 creates a $3-1-2-3$ path. Likewise, making the label 4 creates a $4-1-2-3-2-1-4$ path and changing the label to 5 creates a $5-1-2-3-5$ path. Changing it to 6 creates a $6-1-2-3-2-6$ path, making it 7 creates a $7-7$ path and making it 8 creates an $8-1-8$ path. The reader can easily confirm that no other label can be reduced.

Here are some other questions to consider.

1. Is the converse of Theorem 28 true? That is, if T is a tree with $rad(T) \geq 3$ such that $\psi_r(T) = rad(T) + \varepsilon(T)$ then is it true that T has a unique central vertex c such that 1) every pendant vertex is $rad(T)$ from the center, and 2) $d(x, c) \leq rad(T) - 3$ implies x has at least two children?

2. Are there other classes of graphs, besides reduction trees, for which $\psi_r(G) = rad(G) + \varepsilon(G)$?

3. It was shown that reduction of a chordal graph is chordal and certainly the reduction of a path is a path. What other classes of graphs have this property?

4. Since the reduction of a chordal graph is chordal an inductive argument could be used in finding a bound. What is a good bound for the arank number of a chordal graph?

Acknowledgment

The author would like to thank his referees for helpful suggestions.

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Received 9 September 2016
Revised 13 September 2017
Accepted 13 September 2017