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ON 3-COLORINGS OF DIRECT PRODUCTS OF GRAPHS

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Abstract

The k-independence number of a graph G, denoted as $\alpha_k(G)$, is the order of a largest induced k-colorable subgraph of G. In [S. Špacapan, *The k-independence number of direct products of graphs*, European J. Combin. 32 (2011) 1377–1383] the author conjectured that the direct product $G \times H$ of graphs G and H obeys the following bound

$$\alpha_k(G \times H) \leq \alpha_k(G)|V(H)| + \alpha_k(H)|V(G)| - \alpha_k(G)\alpha_k(H),$$

and proved the conjecture for $k = 1$ and $k = 2$. If true for $k = 3$ the conjecture strenghtens the result of El-Zahar and Sauer who proved that any direct product of 4-chromatic graphs is 4-chromatic [M. El-Zahar and N. Sauer, *The chromatic number of the product of two* 4*-chromatic graphs is* 4, Combinatorica 5 (1985) 121–126]. In this paper we prove that the above bound is true for $k = 3$ provided that G and H are graphs that have complete tripartite subgraphs of orders $\alpha_3(G)$ and $\alpha_3(H)$, respectively.

Keywords: independence number, direct product, Hedetniemi's conjecture.

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1. Introduction

The Hedetniemi's conjecture was raised in [6] where the author conjectured that

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}.$$

Since the upper bound is achived by canonical colorings (colorings of factors lifted to the product) the conjecture is equivalent to the following statement.

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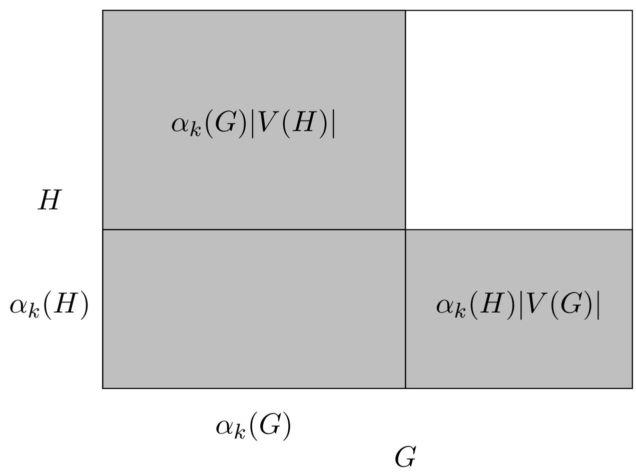
Conjecture 1 (Hedetniemi's conjecture)**.** *If G and H are not k-colorable, then $G \times H$ is not k-colorable.*

This is easy to prove for $k = 1, 2$, and in [2] the authors proved it for $k = 3$. A number of related conjectures, some stronger and some weaker, have been proposed by many authors (see [16] and [10]), in particular recently we have proposed in [9] a conjecture that bounds the maximum size of a subset of $V(G \times H)$ that induces a k-colorable subgraph. The *k-independence number* of G, denoted as $\alpha_k(G)$, is the order (number of vertices) of a largest induced k-colorable subgraph of G. The following conjecture is given in [9].

Conjecture 2. *For any graphs G and H,*

$$\alpha_k(G \times H) \leq \alpha_k(G)|V(H)| + \alpha_k(H)|V(G)| - \alpha_k(G)\alpha_k(H).$$

The above conjecture is true for $k = 1$ and $k = 2$, the case $k = 1$ gives an upper bound for the independence number of $G \times H$, and the case $k = 2$ gives an upper bound for the order of a largest induced bipartite subgraph of $G \times H$ (see [9]).



The conjecture suggests that the order of a maximum induced k-colorable subgraph of $G \times H$ is at most the size of the set colored by one canonical k-coloring $\alpha_k(G)|V(H)|$, plus the size of the set colored by another canonical k-coloring $\alpha_k(H)|V(G)|$, reduced by the size of the intersection of these two sets, see Figure 1. Observe that Conjecture 2 is stronger than Conjecture 1. To see this assume that G and H are not k-colorable, that is $\alpha_k(G) < |V(G)|$ and $\alpha_k(H) < |V(H)|$, as this is the case in Figure 1. Then according to Conjecture 2 we have $\alpha_k(G \times H) < |V(G \times H)|$, and hence $G \times H$ is not k-colorable. In particular note that proving Conjecture 2 for $k = 3$ improves the result by El-Zahar and Sauer who proved that if G and H are not 3-colorable, then $G \times H$ is not 3-colorable. In this paper we consider 3-colorings of $G \times H$ and prove that the bound from Conjecture 2 holds for $k = 3$ provided that G and H are graphs that have complete tripartite subgraphs of orders $\alpha_3(G)$ and $\alpha_3(H)$, respectively.

Several related results on independence number and the structure of maximum independent sets in direct products of graphs are given in articles [1, 3, 4, 7, 8, 13, 12, 15] and [14]. We also mention that the fractional version of Hedetniemi's conjecture was recently proved in [17].

We start by giving definitions and by setting the notation which we use in this and following sections. Let $G = (V(G), E(G))$ be a graph and

$$N(x) = \{x' \in V(G) \,|\, xx' \in E(G)\}$$

be the *neighborhood* of x in G. A *coloring* of a graph G is a function $f : V(G) \to \mathcal{C}$. We use the set of colors $\mathcal{C} = \{p, q, r\}$ for every 3-coloring of a graph. A coloring f is a *proper coloring* if $f(x) \neq f(y)$ whenever x is adjacent to y. We say that a coloring f is a *totally proper* coloring on a set $Y \subseteq V(G)$ if for every $y \in Y$ we have $f(y) \notin f(N(y))$ (note that a totally proper coloring on Y is not necessarily a proper coloring of G).

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be graphs. The *direct product* $G \times H$ of graphs G and H is the graph with vertex set $V(G \times H) = V(G) \times V(H)$ where vertices (x_1, y_1) and (x_2, y_2) are adjacent in $G \times H$ if $x_1x_2 \in E(G)$ and $y_1y_2 \in E(H)$. For a $y \in V(H)$ the *G-layer* G^y is the set of vertices in $V(G \times H)$ defined as follows

$$G^y = \{(x, y) \in V(G \times H) \,|\, x \in V(G)\}.$$

Analogously we define an H-layer

$$H^x = \{(x, y) \in V(G \times H) \,|\, y \in V(H)\}.$$

We use $p_G : V(G \times H) \to V(G)$ and $p_H : V(G \times H) \to V(H)$ to denote the projections from $V(G \times H)$ to $V(G)$ and $V(H)$, respectively. For a vertex $(x, y) \in V(G \times H)$ the *G-neighborhood* of (x, y) is defined as follows

$$N_G(x, y) = N(x) \times \{y\},$$

and similarly, the *H-neighborhood* of (x, y) is

$$N_H(x, y) = \{x\} \times N(y).$$

Note that we have reserved the notation $N(x)$ to denote the neighborhood of x in G (or H), whereas $N_G(x, y)$ and $N_H(x, y)$ denote two subsets of the product. If $a \in V(G), b \in V(H)$ and $X \subseteq V(G \times H)$ then we denote $X \cap H^a$ by X^a and $X \cap G^b$ by X^b.

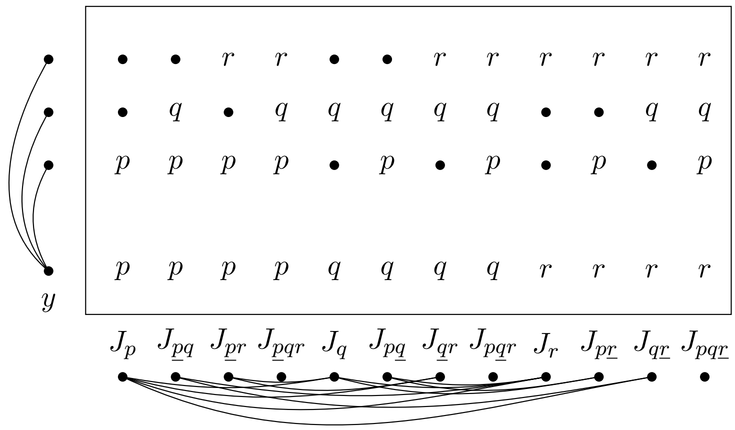


Figure 1. The partition of J^y. The graph G is on the x-axis and H is on the y-axis.

2. On 3-Colorable Subgraphs of $G \times H$

Let $I \subseteq V(G \times H)$ be a set that induces a 3-colorable subgraph of $G \times H$ and let f be a proper 3-coloring of I. Let $J \subseteq I$ be the set of vertices (x, y) that have an H-neighbor $(x, y') \in I$ such that $f(x, y) = f(x, y')$, and let $K \subseteq I$ be the set of vertices (x, y) that have a G-neighbor $(x', y) \in I$ such that $f(x, y) = f(x', y)$. That is

$$J = \left\{(x, y) \in I \,|\, \exists\, (x, y') \in I \text{ such that } yy' \in E(H) \text{ and } f(x, y) = f(x, y')\right\},$$

$$K = \left\{(x, y) \in I \,|\, \exists\, (x', y) \in I \text{ such that } xx' \in E(G) \text{ and } f(x, y) = f(x', y)\right\}.$$

Additionally, let $M = I \setminus (J \cup K)$. Note that, by the definition of J and K, we have $J \cap K = \emptyset$ (since $(x, y) \in J \cap K$ would imply that (x, y) has a G-neighbor and an H-neighbor colored by $f(x, y)$, which is a contradiction since these two neighbors are adjacent in $G \times H$). It also follows from the definition that for every $x \in V(G)$ and $y \in V(H)$ the projections of $K^x \cup M^x$ to H and $J^y \cup M^y$ to G induce 3-colorable subgraphs. Moreover the projection of $f_{/G^y}$ (here $f_{/G^y}$ denotes the restriction of f to G^y) to G, is a totally proper coloring on $p_G(J^y \cup M^y)$. This means that for every $x \in p_G(J^y \cup M^y)$ and every $x' \in N(x)$ we have $f(x, y) \neq f(x', y)$. Similarly, the projection of $f_{/H^x}$ to H is a totally proper coloring on $p_H(K^x \cup M^x)$. It follows from this discussion that

$$|I| + |M| = \sum_{y \in V(H)} |J^y \cup M^y| + \sum_{x \in V(G)} |K^x \cup M^x| \leq \alpha_3(G)|V(H)| + \alpha_3(H)|V(G)|$$

and hence

$$|I| \leq \alpha_3(G)|V(H)| + \alpha_3(H)|V(G)| - |M| \leq \alpha_3(G)|V(H)| + \alpha_3(H)|V(G)|.$$

The aim of this paper is to improve the above bound to

$$|I| \leq \alpha_3(G)|V(H)| + \alpha_3(H)|V(G)| - \alpha_3(G)\alpha_3(H).$$

This is done by making use of the fact that the projections of $f_{/G^y}$ and $f_{/H^x}$ are totally proper colorings on $p_G(J^y \cup M^y)$ and $p_H(K^x \cup M^x)$, and so we may add some vertices to $p_G(J^y \cup M^y)$ and $p_H(K^x \cup M^x)$ and obtain proper 3-colorings of some subgraphs that properly contain the sets $p_G(J^y \cup M^y)$ and $p_H(K^x \cup M^x)$. By doing this we get $\alpha_3(H) = |K^x \cup M^x| + \epsilon_x$ and $\alpha_3(G) = |J^y \cup M^y| + \epsilon_y$ where the sum of integers ϵ_x and ϵ_y is at least $\alpha_3(G)\alpha_3(H)$.

To realize the rough idea of the proof described in the previous paragraph we need to define and analyze vertices of different types as this is done in the sequel. For any $S \subseteq \mathcal{C}$ we define J_S and K_S as follows

$$J_S = \{(x, y) \in J \mid f(N_H(x, y)) = S\} \text{ and } K_S = \{(x, y) \in K \mid f(N_G(x, y)) = S\}.$$

We say that a vertex $(x, y) \in J$ is an *S-type vertex* if $(x, y) \in J_S$ or $(x, y) \in K_S$. (See Figure 1 where all possible types of vertices are shown, and where vertices in the product are marked by their colors p, q and r. Black vertices in the figure do not belong to I, and so they have not been assigned a color.) So if, for example, $(x, y) \in J$ is such that $f(N_H(x, y)) = \{p, q\}$ we say that (x, y) is a pq-type vertex and we denote the set of all such vertices by J_{pq}. If additionally $f(x, y) = p$ we say that (x, y) is a $\underline{p}q$-type vertex, where we underline the color of the vertex (x, y) (if $f(x, y) = q$ we say that (x, y) is a $p\underline{q}$-type vertex), and the set of all such vertices is denoted by $J_{\underline{p}q}$. So we have $J_{pq} = J_{\underline{p}q} \cup J_{p\underline{q}}$. Similar notation and terminology we use for the set K, where the G-neighborhood of a vertex defines its type.

Let $y \in V(H)$ and $J^y = J \cap G^y$. There is a partition of J^y into three sets such that the projection of each set to G is an independent set in G. The partition is (see Figure 1)

$$J^y = \left(J^y_p \cup J^y_{\underline{p}q} \cup J^y_{\underline{p}r} \cup J^y_{\underline{p}qr}\right) \cup \left(J^y_q \cup J^y_{p\underline{q}} \cup J^y_{\underline{q}r} \cup J^y_{p\underline{q}r}\right) \cup \left(J^y_r \cup J^y_{p\underline{r}} \cup J^y_{q\underline{r}} \cup J^y_{pq\underline{r}}\right).$$

Suppose that $A \subseteq V(G)$ and $B \subseteq V(H)$ are sets that induce maximal 3-colorable subgraphs of G and H, and let $A = \bigcup_{i=1}^3 A^i$ and $B = \bigcup_{i=1}^3 B^i$ be partitions of A and B. If A and B induce complete tripartite graphs and $f : I \to \mathcal{C}$ is a proper 3-coloring of $I \subseteq V(G \times H)$, then for every color $c \in \mathcal{C}$ there is an index $i \in [3]$ such that $f^{-1}(c) \cap (A \times B) \subseteq A^i \times B$, or an index $j \in [3]$ such that $f^{-1}(c) \cap (A \times B) \subseteq A \times B^j$ (note that both cases cannot appear simultaneously if $f^{-1}(c) \cap (A^i \times B^j) \neq \emptyset$ for at least two indices $i \in [3]$ or two indices $j \in [3]$). Therefore all possible colorings (up to a permutation of color classes and permutation of sets A^1, A^2 and A^3, and B^1, B^2 and B^3) of $(A \times B) \cap I$ are those given in Figure 2, where sets $A^i \times B^j$ are depicted by squares and colors

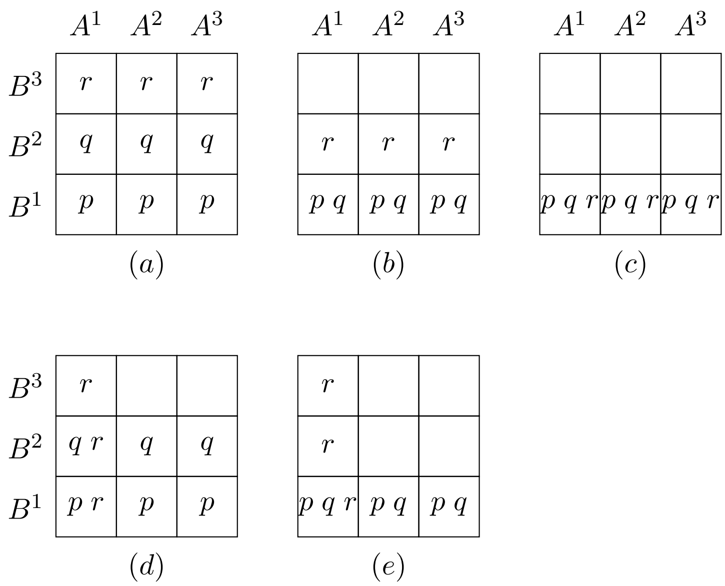


Figure 2. All possible colorings of $A \times B$.

used on $A^i \times B^j$ are written within the squares. In this paper we study colorings of the product shown in Figure 2 and prove that for every such coloring of $(A\times B)\cap I$ we have $|I| \leq \alpha_3(G)|V(H)| + \alpha_3(H)|V(G)| - \alpha_3(G)\alpha_3(H)$ (regardless of how f colors the vertices in the complement of $A \times B$). The motivation behind this study is that these are all possible colorings of the product of complete tripartite graphs A and B. However we drop the assumption that A and B induce complete tripartite graphs in the main theorem below.

Theorem 3. *Let G and H be graphs, $A \subseteq V(G)$ and $B \subseteq V(H)$ be sets that induce maximal 3-colorable subgraphs of G and H, and let A^i and B^i for $i \in [3]$ be their color classes, respectively. Let $I \subseteq V(G \times H)$ be a set that induces a 3-colorable subgraph of $G\times H$, and $f : I \to \mathcal{C}$ a proper 3-coloring of I. If f restricted to $A \times B$ is as in one of the cases from Figure 2, then $|I| \leq \alpha_3(G)|V(H)| + \alpha_3(H)|V(G)| - \alpha_3(G)\alpha_3(H)$.*

The following corollary follows straightforward from Theorem 3 and the fact that colorings from Figure 2 are the only colorings of products of complete tripartite graphs. Note however that a graph may have more than one maximal 3-colorable subgraph (which explains the formulation of the corollary).

Corollary 4. *Let G and H be graphs. If there exist maximal 3-colorable subgraphs of G and H that are complete tripartite graphs, then*

$$\alpha_3(G \times H) \leq \alpha_3(G)|V(H)| + \alpha_3(H)|V(G)| - \alpha_3(G)\alpha_3(H).$$

3. Preliminary Results

Before we prove our main theorem we shall prove few preparatory results and set the notation that we use in following claims. Let G be a graph and $D \subseteq V(G)$ a set that induces a maximal 3-colorable subgraph of G with parts (color classes) $D^i, i \in [3]$. Let $L \subseteq V(G)$ and $g : L \to \mathcal{C}$ be a coloring of L (note that we do not assume that g is a proper coloring). Additionaly let $Y \subseteq L$ be the maximum set such that g is a totally proper coloring on Y. Let $X = (L \setminus Y) \cap D$, and $X^i = X \cap D^i$ for $i \in [3]$ (see Figure 3). Let S be a subset of colors. We define

$$X_S = \{x \in X \mid g(N(x)) = S\}.$$

For example, if g is a 3-coloring and p, q, r are the colors, then X_{pq} denotes the set of $x \in X$ such that $g(N(x)) = \{p, q\}$, and we say that $x \in X_{pq}$ is a pq-type vertex. If also $g(x, y) = p$ then we say that (x, y) is a $\underline{p}q$-type vertex, and we denote the set of such vertices by $X_{\underline{p}q}$. Additionally let

$$R_i = \{x \in X : |g(N(x))| = i\}.$$

So R_1 contains p-type, q-type and r-type vertices, R_2 contains pq-type, pr-type and qr-type vertices, and finally R_3 contains pqr-type vertices, or equivalently $R_1 = X_p \cup X_q \cup X_r, R_2 = X_{pq} \cup X_{pr} \cup X_{qr}$ and $R_3 = X_{pqr}$. This notations are used in Claims 0, 1, 2 and 3.

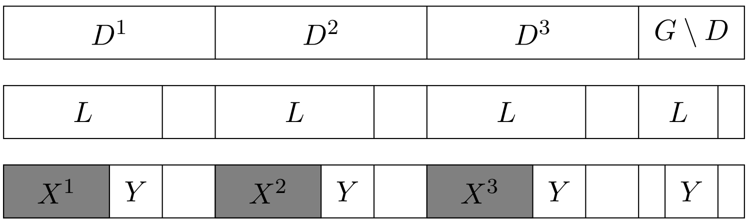


Figure 3. The definitions of D, L, X and Y.

The definitions from the above paragraph are applicaple to colorings of the product in the following sense. Suppose I is a 3-colorable subset of $V(G \times H)$, $f : I \to \mathcal{C}$ is a proper 3-coloring of I, and A, B induce maximum 3-colorable subgraphs of G and H, respectively. For $y \in V(H)$ we observe the G^y layer and the set of vertices I^y colored by f. Then we have the following (see the definitions in the previous section).

- A is a maximal 3-colorable subgraph of G, and, D is a maximal 3-colorable subgraph of G (as defined above).
- The projection of I^y to G is a subset of $V(G)$, and, L is a subset of $V(G)$ (as defined above).

- The projection of $f_{/G^y}$ to G is a totally proper coloring of $p_G(J^y \cup M^y)$, and $p_G(J^y \cup M^y)$ is a maximal such subset of $p_G(I^y)$, and, g is a totally proper coloring of Y, and Y is a maximal such subset of L (as defined above).
- $p_G(K^y \cap (A \times \{y\}))$ is the complement of $p_G(J^y \cup M^y)$ in $p_G(I^y \cap (A \times \{y\}))$, and, X is the complement of Y in $L \cap D$ (as defined above).

Thus we see that sets $A, I^y, J^y \cup M^y$ and $K^y \cap (A \times \{y\})$ take the role of D, L, Y and X, respectively. The coloring $f_{/G^y}$ takes the role of g from the previous paragraph. Alternatively, for an $x \in V(G)$, the sets $B, I^x, K^x \cup M^x$ and $J^x \cap (\{x\} \times B)$ take the role of D, L, Y and X, respectively (and here note that the projection of $f_{/H^x}$ to H is a totally proper coloring of $p_H(K^x \cup M^x)$).

Claim 0. *For every graph G we have $\alpha_3(G) \geq |Y| + |R_3|/3 + 2|R_2|/3 + |R_1|$.*

Proof. It follows from maximality of Y that every vertex $x \in X$ has a neighbor x' such that $g(x) = g(x')$. For every vertex $x \in X$ we define the list $\mathcal{L}(x)$ of admissible colors as follows

$$\mathcal{L}(x) = \{c \in \mathcal{C} \mid c = g(x) \text{ or } c \notin g(N(x))\}.$$

For example if $x \in X_p \cup X_q \cup X_r$ then $\mathcal{L}(x) = \{p, q, r\}$, and if $x \in X_{pq}$ then $\mathcal{L}(x) = \{p, r\}$. Note that for every color $c \in \mathcal{L}(x)$ we have $c \notin g(N(x) \cap Y)$ because g is a totally proper coloring on Y. Therefore any proper coloring g'' of vertices in $F \subseteq X$ from their lists yields a proper coloring g' of $F \cup Y$, this coloring is defined by $g'(u) = g(u)$ for $u \in Y$ and $g'(u) = g''(u)$ for $u \in F$ (note that lists for vertices in X are defined in such a way that they do not interfere with vertices in Y). Every vertex $x \in R_1$ has a list of size 3 because $g(N(x)) = \{g(x)\}$. Similarly vertices in R_2 and R_3 have lists of size 2 and 1, respectively. We define three proper 3-colorings of three subsets F_1, F_2 and F_3 of X as follows: color vertices in X^1, X^2 and X^3 from lists by p, q, and r respectively (if a vertex does not have the designated color in the list it remains uncolored). This coloring is depicted in Figure 4, where all types of vertices in X^1, X^2 and X^3 are listed and written in the middle of the squares, and the color used for a specific type of vertex is given in the upper right corner of the square. Call the set of vertices colored this way F_1. Then color X^1, X^2 and X^3 from lists by q, r, and p respectively, and call the set of colored vertices F_2 (see Figure 5). Finally color X^1, X^2 and X^3 from lists by r, p, and q respectively, and call the set of colored vertices F_3 (see Figure 6). Then we have

$$|F_1| + |F_2| + |F_3| \geq |R_3| + 2|R_2| + 3|R_1|,$$

so the size of at least one set, say F_1, is at least $|R_3|/3 + 2|R_2|/3 + |R_1|$. We conclude by $\alpha_3(G) \geq |Y| + |F_1| \geq |Y| + |R_3|/3 + 2|R_2|/3 + |R_1|$, which proves the claim. □

X^1	p $\underline{p}$	p $\underline{q}$	p $\underline{r}$	p $\underline{p}q$	$p\underline{q}$	p $\underline{p}r$	$p\underline{r}$	p $\underline{q}r$	p $q\underline{r}$	p $\underline{p}qr$	$p\underline{q}r$	$pq\underline{r}$
X^2	q $\underline{p}$	q $\underline{q}$	q $\underline{r}$	$\underline{p}q$	q $p\underline{q}$	q $\underline{p}r$	q $p\underline{r}$	q $\underline{q}r$	$q\underline{r}$	$\underline{p}qr$	q $p\underline{q}r$	$pq\underline{r}$
X^3	r $\underline{p}$	r $\underline{q}$	r $\underline{r}$	r $\underline{p}q$	r $p\underline{q}$	$\underline{p}r$	r $p\underline{r}$	$\underline{q}r$	r $q\underline{r}$	$\underline{p}qr$	$p\underline{q}r$	r $pq\underline{r}$

Figure 4. The set of colored vertices F_1.

X^1	q $\underline{p}$	q $\underline{q}$	q $\underline{r}$	$\underline{p}q$	q $p\underline{q}$	q $\underline{p}r$	q $p\underline{r}$	q $\underline{q}r$	$q\underline{r}$	$\underline{p}qr$	q $p\underline{q}r$	$pq\underline{r}$
X^2	r $\underline{p}$	r $\underline{q}$	r $\underline{r}$	r $\underline{p}q$	r $p\underline{q}$	$\underline{p}r$	r $p\underline{r}$	$\underline{q}r$	r $q\underline{r}$	$\underline{p}qr$	$p\underline{q}r$	r $pq\underline{r}$
X^3	p $\underline{p}$	p $\underline{q}$	p $\underline{r}$	p $\underline{p}q$	$p\underline{q}$	p $\underline{p}r$	$p\underline{r}$	p $\underline{q}r$	p $q\underline{r}$	p $\underline{p}qr$	$p\underline{q}r$	$pq\underline{r}$

Figure 5. The set of colored vertices F_2.

X^1	r $\underline{p}$	r $\underline{q}$	r $\underline{r}$	r $\underline{p}q$	r $p\underline{q}$	$\underline{p}r$	r $p\underline{r}$	$\underline{q}r$	r $q\underline{r}$	$\underline{p}qr$	$p\underline{q}r$	r $pq\underline{r}$
X^2	p $\underline{p}$	p $\underline{q}$	p $\underline{r}$	p $\underline{p}q$	$p\underline{q}$	p $\underline{p}r$	$p\underline{r}$	p $\underline{q}r$	p $q\underline{r}$	p $\underline{p}qr$	$p\underline{q}r$	$pq\underline{r}$
X^3	q $\underline{p}$	q $\underline{q}$	q $\underline{r}$	$\underline{p}q$	q $p\underline{q}$	q $\underline{p}r$	q $p\underline{r}$	q $\underline{q}r$	$q\underline{r}$	$\underline{p}qr$	q $p\underline{q}r$	$pq\underline{r}$

Figure 6. The set of colored vertices F_3.

We may apply the above claim to any 3-coloring of the G^y layer of $G \times H$ where sets $A, I^y, J^y \cup M^y$ and $K^y \cap (A \times \{y\})$ take the role of D, L, Y and X. Using the notation from the introduction gives us the following.

Corollary 5. *For every $y \in \bar{B}$ we have*

$$\begin{aligned}\alpha_3(G) \geq\ & |J^y \cup M^y| + \left|K_p^y \cap (A \times \bar{B})\right| + \left|K_q^y \cap (A \times \bar{B})\right| + \left|K_r^y \cap (A \times \bar{B})\right| \\ &+ \frac{2}{3}\left(\left|K_{pq}^y \cap (A \times \bar{B})\right| + \left|K_{pr}^y \cap (A \times \bar{B})\right| + \left|K_{qr}^y \cap (A \times \bar{B})\right|\right) \\ &+ \frac{1}{3}\left|K_{pqr}^y \cap (A \times \bar{B})\right|.\end{aligned}$$

Claim 1. *For every graph G we have*

(i) $\alpha_3(G) \geq |Y| + |R_1| + \left|R_2 \cap X^1\right| + \left|X_{pq} \cap X^2\right| - \left|X_r \cap X^3\right|$.

(ii) $\alpha_3(G) \geq |Y| + |R_1| - \frac{1}{3}\left|R_1 \cap X^3\right| + \left|R_2 \cap X^1\right| + \frac{1}{3}\left|R_2 \cap \left(X^2 \cup X^3\right)\right|$.

Proof. We assume that $\left|R_2 \cap X^2\right| \geq \left|R_2 \cap X^3\right|$ (otherwise exchange the role of X^2 and X^3 below). To prove (i) consider the coloring g_1 given in Figure 7. In this figure rows 1, 2 and 3 correspond to vertices in X^1, X^2 and X^3, respectively. Each square represents a vertex type which is written in the center of the square, and the color that we assign to a particular type of vertices is given in the upper right corner of each square (if there is no color in the upper right corner, this means that we don't color this type of vertices). For every $y \in Y$ we define $g_1(y) = g(y)$, and note that to each vertex $x \in X$ the color $g_1(x) \notin g(N(x))$ or $g_1(x) = g(x)$ is assigned (we use the same lists $\mathcal{L}(x)$ to color x as in the previous claim). Hence if $y \in Y$ is adjacent to $x \in X$ we have $g_1(x) \neq g_1(y)$ (follows from the fact that g is a totally proper coloring on Y). It remains to prove that g_1 is a proper coloring when restricted to X. Since X^i is an independent set for $i \in [3]$ we have to check the $\underline{p}$-type and $\underline{q}$-type vertices in X^3 (since the restriction of g_1 to $X^1 \cup X^2$ is a proper coloring). The $\underline{p}$-type vertices in X^3 can only be adjacent to $\underline{p}, \underline{p}q, \underline{p}r$ and $\underline{p}qr$-type vertices in X^1. The color of the first is q and the color of the latter is p, so they are assigned different colors by g_1. Similarly we argue for $\underline{q}$-type vertices in X^3, they can only be adjacent to $\underline{q}, p\underline{q}, \underline{q}r$ and $p\underline{q}r$-type vertices in X^1, which receive the color q by g_1. We conclude that g_1 is a proper coloring and hence the bound (i) follows. In the sequel we use the notation $X_p^i = X_p \cap X^i$ (and similarly for other types of vertices). If $\left|X_p^3\right| + \left|X_q^3\right| + \left|X^2_{\underline{p}q}\right| + \left|X^2_{p\underline{q}}\right| + \left|X^2_{p\underline{r}}\right| + \left|X^2_{q\underline{r}}\right| \geq \frac{2}{3}\left(\left|R_1 \cap X^3\right| + \left|R_2 \cap X^2\right|\right)$ then the coloring g_1 in Figure 7, together with $\frac{2}{3}\left|R_2 \cap X^2\right| \geq \frac{1}{3}\left|R_2 \cap \left(X^2 \cup X^3\right)\right|$ which is obtained from the initial assumption, proves (ii). If $\left|X_q^3\right| + \left|X_r^3\right| + \left|X^2_{\underline{p}q}\right| + \left|X^2_{\underline{p}r}\right| + \left|X^2_{\underline{q}r}\right| + \left|X^2_{q\underline{r}}\right| \geq \frac{2}{3}\left(\left|R_1 \cap X^3\right| + \left|R_2 \cap X^2\right|\right)$ then the coloring g_2 in Figure 8 together with $\frac{2}{3}\left|R_2 \cap X^2\right| \geq \frac{1}{3}\left|R_2 \cap \left(X^2 \cup X^3\right)\right|$ proves (ii). If both the above inequalities fail to hold, then we obtain that $\left|X_q^3\right| + \left|X^2_{\underline{p}q}\right| + \left|X^2_{q\underline{r}}\right| < \frac{1}{3}\left(\left|R_1 \cap X^3\right| + \left|R_2 \cap X^2\right|\right)$ which together with $\frac{2}{3}\left|R_2 \cap X^2\right| \geq \frac{1}{3}\left|R_2 \cap \left(X^2 \cup X^3\right)\right|$ and the coloring g_3 in Figure 9 imply (ii). $\square$

We apply the above claim to the colorings of the product $G \times H$, where H takes the role of G in the above claim.

Corollary 6. *For every $x \in \bar{A}$ we have*

$$\begin{aligned}
\alpha_3(H) \geq\ & |K^x \cup M^x| + \left|J_p^x \cap (\bar{A} \times B)\right| + \left|J_q^x \cap (\bar{A} \times B)\right| \\
&+ \left|J_r^x \cap \left(\bar{A} \times \left(B^1 \cup B^2\right)\right)\right| + \left|J_{pq}^x \cap \left(\bar{A} \times B^1\right)\right| + \left|J_{pr}^x \cap \left(\bar{A} \times B^1\right)\right| \\
&+ \left|J_{qr}^x \cap \left(\bar{A} \times B^1\right)\right| + \left|J_{pq}^x \cap \left(\bar{A} \times B^2\right)\right|.
\end{aligned}$$

X^1	$\underline{p}^{\,p}$	$\underline{q}^{\,q}$	$\underline{r}^{\,p}$	$\underline{p}q^{\,p}$	$p\underline{q}^{\,q}$	$\underline{p}r^{\,p}$	$p\underline{r}^{\,q}$	$\underline{q}r^{\,q}$	$q\underline{r}^{\,p}$	$\underline{p}qr^{\,p}$	$p\underline{q}r^{\,q}$	$pq\underline{r}$
X^2	$\underline{p}^{\,r}$	$\underline{q}^{\,r}$	$\underline{r}^{\,r}$	$\underline{p}q^{\,r}$	$p\underline{q}^{\,r}$	$\underline{p}r$	$p\underline{r}^{\,r}$	$\underline{q}r$	$q\underline{r}^{\,r}$	$\underline{p}qr$	$p\underline{q}r$	$pq\underline{r}$
X^3	$\underline{p}^{\,q}$	$\underline{q}^{\,p}$	$\underline{r}$	$\underline{p}q$	$p\underline{q}$	$\underline{p}r$	$p\underline{r}$	$\underline{q}r$	$q\underline{r}$	$\underline{p}qr$	$p\underline{q}r$	$pq\underline{r}$

Figure 7. The coloring g_1 of X.

X^1	$\underline{p}^{\,q}$	$\underline{q}^{\,q}$	$\underline{r}^{\,r}$	$\underline{p}q^{\,r}$	$p\underline{q}^{\,q}$	$\underline{p}r^{\,q}$	$p\underline{r}^{\,r}$	$\underline{q}r^{\,q}$	$q\underline{r}^{\,r}$	$\underline{p}qr$	$p\underline{q}r^{\,q}$	$pq\underline{r}^{\,r}$
X^2	$\underline{p}^{\,p}$	$\underline{q}^{\,p}$	$\underline{r}^{\,p}$	$\underline{p}q^{\,p}$	$p\underline{q}$	$\underline{p}r^{\,p}$	$p\underline{r}$	$\underline{q}r^{\,p}$	$q\underline{r}^{\,p}$	$\underline{p}qr$	$p\underline{q}r$	$pq\underline{r}$
X^3	$\underline{p}$	$\underline{q}^{\,r}$	$\underline{r}^{\,q}$	$\underline{p}q$	$p\underline{q}$	$\underline{p}r$	$p\underline{r}$	$\underline{q}r$	$q\underline{r}$	$\underline{p}qr$	$p\underline{q}r$	$pq\underline{r}$

Figure 8. The coloring g_2 of X.

X^1	$\underline{p}^{\,p}$	$\underline{q}^{\,p}$	$\underline{r}^{\,r}$	$\underline{p}q^{\,p}$	$p\underline{q}^{\,r}$	$\underline{p}r^{\,p}$	$p\underline{r}^{\,r}$	$\underline{q}r^{\,p}$	$q\underline{r}^{\,r}$	$\underline{p}qr^{\,p}$	$p\underline{q}r$	$pq\underline{r}^{\,r}$
X^2	$\underline{p}^{\,q}$	$\underline{q}^{\,q}$	$\underline{r}^{\,q}$	$\underline{p}q$	$p\underline{q}^{\,q}$	$\underline{p}r^{\,q}$	$p\underline{r}^{\,q}$	$\underline{q}r^{\,q}$	$q\underline{r}$	$\underline{p}qr$	$p\underline{q}r$	$pq\underline{r}$
X^3	$\underline{p}^{\,r}$	$\underline{q}$	$\underline{r}^{\,p}$	$\underline{p}q$	$p\underline{q}$	$\underline{p}r$	$p\underline{r}$	$\underline{q}r$	$q\underline{r}$	$\underline{p}qr$	$p\underline{q}r$	$pq\underline{r}$

Figure 9. The coloring g_3 of X.

When we apply (ii) of Claim 1 to the coloring of the product we get the following corollary.

Corollary 7. *For every $x \in \bar{A}$ we have*

$$\begin{aligned}\alpha_3(H) \geq\ & |K^x \cup M^x| + \big|J_p^x \cap \big(\bar{A} \times \big(B^1 \cup B^2\big)\big)\big| + \big|J_q^x \cap \big(\bar{A} \times \big(B^1 \cup B^2\big)\big)\big| \\ & + \big|J_r^x \cap \big(\bar{A} \times \big(B^1 \cup B^2\big)\big)\big| + \frac{2}{3}\big(\big|J_p^x \cap \big(\bar{A} \times B^3\big)\big| + \big|J_q^x \cap \big(\bar{A} \times B^3\big)\big| \\ & + \big|J_r^x \cap \big(\bar{A} \times B^3\big)\big|\big) + \big|J_{pq}^x \cap \big(\bar{A} \times B^1\big)\big| + \big|J_{pr}^x \cap \big(\bar{A} \times B^1\big)\big| \\ & + \big|J_{qr}^x \cap \big(\bar{A} \times B^1\big)\big| + \frac{1}{3}\big(\big|J_{pq}^x \cap \big(\bar{A} \times \big(B^2 \cup B^3\big)\big)\big| \\ & + \big|J_{pr}^x \cap \big(\bar{A} \times \big(B^2 \cup B^3\big)\big)\big| + \big|J_{qr}^x \cap \big(\bar{A} \times \big(B^2 \cup B^3\big)\big)\big|\big).\end{aligned}$$

Claim 2. *For every graph G we have*

(i) $\alpha_3(G) \geq |Y| + |R_1| + \left|X_{qr} \cap X^1\right| + \left|X_{pr} \cap X^2\right| + \left|X_{pq} \cap X^3\right|.$

(ii) $\alpha_3(G) \geq |Y| + |R_1| + \left|X_{pq} \cap X^1\right| + \left|X_{qr} \cap X^2\right| + \left|X_{pr} \cap X^3\right|.$

Proof. The coloring that proves the bound (i) is given in Figure 4, and the coloring in Figure 6 proves (ii). □

When we apply (i) to the coloring f of the product $G \times H$ we get the following corollary.

Corollary 8. *For every $x \in \bar{A}$ we have*

$$\begin{aligned}\alpha_3(H) \geq\ & |K^x \cup M^x| + \left|J^x_p \cap (\bar{A} \times B)\right| + \left|J^x_q \cap (\bar{A} \times B)\right| + \left|J^x_r \cap (\bar{A} \times B)\right| \\ & + \left|J^x_{qr} \cap \left(\bar{A} \times B^1\right)\right| + \left|J^x_{pr} \cap \left(\bar{A} \times B^2\right)\right| + \left|J^x_{pq} \cap \left(\bar{A} \times B^3\right)\right|.\end{aligned}$$

When (ii) is applied we get the following.

Corollary 9. *For every $y \in \bar{B}$ we have*

$$\begin{aligned}\alpha_3(G) \geq\ & |J^y \cup M^y| + \left|K^y_p \cap (A \times \bar{B})\right| + \left|K^y_q \cap (A \times \bar{B})\right| + \left|K^y_r \cap (A \times \bar{B})\right| \\ & + \left|K^y_{pq} \cap \left(A^1 \times \bar{B}\right)\right| + \left|K^y_{qr} \cap \left(A^2 \times \bar{B}\right)\right| + \left|K^y_{pr} \cap \left(A^3 \times \bar{B}\right)\right|.\end{aligned}$$

Claim 3. *For every graph G, any colors $c, d \in \mathcal{C}$ and any $i \in [3]$ we have*

$$\alpha_3(G) \geq |Y| + |R_1| + \frac{1}{2}|R_2| + \frac{1}{2}\left|X^i_{cd}\right|.$$

Proof. Without loss of generality assume $i = 1, c = p$ and $d = q$. We shall prove that $\alpha_3(G) \geq |Y| + |R_1| + \frac{1}{2}|R_2| + \frac{1}{2}\left|X^1_{pq}\right|$. We distinguish two cases.

Case 1. $\left|X^1_{\underline{p}r}\right| + \left|X^1_{\underline{q}r}\right| \geq \left|X^1_{q\underline{r}}\right| + \left|X^1_{p\underline{r}}\right| + \left|X^2_{\underline{p}r}\right| + \left|X^2_{\underline{q}r}\right| + \left|X^3_{\underline{p}r}\right| + \left|X^3_{\underline{q}r}\right|$. In this case consider the coloring g_1 given in Figure 10 (here only the colors of vertices in X are given, for vertices $y \in Y$ we define $g_1(y) = g(y)$). Assume without loss of generality that $\left|X^1_{p\underline{r}}\right| \leq \left|X^1_{q\underline{r}}\right|$ and $\left|X^2_{\underline{p}q}\right| + \left|X^2_{p\underline{q}}\right| + \left|X^2_{p\underline{r}}\right| + \left|X^2_{q\underline{r}}\right| \geq \left|X^3_{\underline{p}q}\right| + \left|X^3_{p\underline{q}}\right| + \left|X^3_{p\underline{r}}\right| + \left|X^3_{q\underline{r}}\right|$. (If any of this two inequalities is not true we may alter the coloring g_1 given in Figure 10 and proceed the same way. If $\left|X^1_{p\underline{r}}\right| \geq \left|X^1_{q\underline{r}}\right|$, then color $X^1_{p\underline{r}}$ and X^1_r by q, do not color $X^1_{q\underline{r}}$, and color X^3_r by p. The color of all other vertices remains the same as in Figure 10. If the second inequality is not true just exchange the role of X^2 and X^3 and proceed the same way.) The size of the set colored by g_1 is at least

$$\begin{aligned}& |Y| + |X_p| + |X_q| + |X_r| + \left|X^1_{\underline{p}q}\right| + \left|X^1_{p\underline{q}}\right| + \left|X^1_{\underline{p}r}\right| + \left|X^1_{\underline{q}r}\right| + \left|X^1_{q\underline{r}}\right| \\ & + \frac{1}{2}\left(\left|X^2_{\underline{p}q}\right| + \left|X^2_{p\underline{q}}\right| + \left|X^2_{p\underline{r}}\right| + \left|X^2_{q\underline{r}}\right| + \left|X^3_{\underline{p}q}\right| + \left|X^3_{p\underline{q}}\right| + \left|X^3_{p\underline{r}}\right| + \left|X^3_{q\underline{r}}\right|\right).\end{aligned}$$

Since $\left|X^1_{\underline{p}r}\right| + \left|X^1_{\underline{q}r}\right| \geq \left|X^1_{q\underline{r}}\right| + \left|X^1_{p\underline{r}}\right| + \left|X^2_{\underline{p}r}\right| + \left|X^2_{\underline{q}r}\right| + \left|X^3_{\underline{p}r}\right| + \left|X^3_{\underline{q}r}\right|$ and $\left|X^1_{p\underline{r}}\right| \leq \left|X^1_{q\underline{r}}\right|$, we find that

$$
\begin{aligned}
&\left|X^1_{\underline{p}r}\right| + \left|X^1_{\underline{q}r}\right| \\
&\geq \frac{1}{2}\left(\left|X^1_{\underline{p}r}\right| + \left|X^1_{\underline{q}r}\right| + \left|X^1_{q\underline{r}}\right| + \left|X^1_{p\underline{r}}\right| + \left|X^2_{\underline{p}r}\right| + \left|X^2_{\underline{q}r}\right| + \left|X^3_{\underline{p}r}\right| + \left|X^3_{\underline{q}r}\right|\right)
\end{aligned}
$$

and $\left|X^1_{q\underline{r}}\right| \geq \frac{1}{2}\left(\left|X^1_{p\underline{r}}\right| + \left|X^1_{q\underline{r}}\right|\right)$. Combining these two inequalities with the above bound on the size of the set colored by g_1, we find that this size is at least $|Y| + |R_1| + \frac{1}{2}|R_2| + \frac{1}{2}\left|X^1_{pq}\right|$ which completes this case.

Case 2. $\left|X^1_{\underline{p}r}\right| + \left|X^1_{\underline{q}r}\right| < \left|X^1_{q\underline{r}}\right| + \left|X^1_{p\underline{r}}\right| + \left|X^2_{\underline{p}r}\right| + \left|X^2_{\underline{q}r}\right| + \left|X^3_{\underline{p}r}\right| + \left|X^3_{\underline{q}r}\right|$. In this case consider the coloring g_2 given in Figure 11. Assume that

$$
\begin{aligned}
&\left|X^2_{\underline{p}q}\right| + \left|X^2_{\underline{p}r}\right| + \left|X^2_{\underline{q}r}\right| + \left|X^2_{q\underline{r}}\right| + \left|X^3_{p\underline{q}}\right| + \left|X^3_{\underline{p}r}\right| + \left|X^3_{p\underline{r}}\right| + \left|X^3_{\underline{q}r}\right| \\
&\geq \left|X^3_{\underline{p}q}\right| + \left|X^3_{\underline{p}r}\right| + \left|X^3_{\underline{q}r}\right| + \left|X^3_{q\underline{r}}\right| + \left|X^2_{p\underline{q}}\right| + \left|X^2_{\underline{p}r}\right| + \left|X^2_{p\underline{r}}\right| + \left|X^2_{\underline{q}r}\right|
\end{aligned}
$$

(if not exchange the role of X^2 and X^3 in Figure 11, and color X^2 as X^3 is colored in Figure 11, and vice versa). So the size of the set colored by g_2 is

$$
\begin{aligned}
&|Y| + |X_p| + |X_q| + |X_r| + \left|X^1_{\underline{p}q}\right| + \left|X^1_{p\underline{q}}\right| + \left|X^1_{p\underline{r}}\right| + \left|X^1_{q\underline{r}}\right| + \left|X^2_{\underline{p}q}\right| \\
&+ \left|X^2_{\underline{p}r}\right| + \left|X^2_{\underline{q}r}\right| + \left|X^2_{q\underline{r}}\right| + \left|X^3_{p\underline{q}}\right| + \left|X^3_{\underline{p}r}\right| + \left|X^3_{p\underline{r}}\right| + \left|X^3_{\underline{q}r}\right|.
\end{aligned}
$$

It follows (with an analogous calculation as in the previous case) from the above assumptions that the size of the set colored by g_2 is at least $|Y| + |R_1| + \frac{1}{2}|R_2| + \frac{1}{2}\left|X^1_{pq}\right|$, which completes the proof. $\square$

X^1	p $\underline{p}$	q $\underline{q}$	p $\underline{r}$	p $\underline{p}q$	q $p\underline{q}$	p $\underline{p}r$	$p\underline{r}$	q $\underline{q}r$	p $q\underline{r}$	$\underline{p}qr$	$p\underline{q}r$	$pq\underline{r}$
X^2	r $\underline{p}$	r $\underline{q}$	r $\underline{r}$	r $\underline{p}q$	r $p\underline{q}$	$\underline{p}r$	r $p\underline{r}$	$\underline{q}r$	r $q\underline{r}$	$\underline{p}qr$	$p\underline{q}r$	$pq\underline{r}$
X^3	q $\underline{p}$	p $\underline{q}$	q $\underline{r}$	$\underline{p}q$	$p\underline{q}$	$\underline{p}r$	$p\underline{r}$	$\underline{q}r$	$q\underline{r}$	$\underline{p}qr$	$p\underline{q}r$	$pq\underline{r}$

Figure 10. The coloring g_1 of X.

Setting $i = 1, c = p$ and $d = q$, we arrive at the following corollary.

X^1	$\underline{p}^{\,r}$	$\underline{q}^{\,r}$	$\underline{r}^{\,r}$	$\underline{p}q^{\,r}$	$p\underline{q}^{\,r}$	$\underline{p}r$	$p\underline{r}^{\,r}$	$\underline{q}r$	$q\underline{r}^{\,r}$	$\underline{p}qr$	$p\underline{q}r$	$pq\underline{r}$
X^2	$\underline{p}^{\,p}$	$\underline{q}^{\,p}$	$\underline{r}^{\,p}$	$\underline{p}q^{\,p}$	$p\underline{q}$	$\underline{p}r^{\,p}$	$p\underline{r}$	$\underline{q}r^{\,p}$	$q\underline{r}^{\,p}$	$\underline{p}qr$	$p\underline{q}r$	$pq\underline{r}$
X^3	$\underline{p}^{\,q}$	$\underline{q}^{\,q}$	$\underline{r}^{\,q}$	$\underline{p}q$	$p\underline{q}^{\,q}$	$\underline{p}r^{\,q}$	$p\underline{r}^{\,q}$	$\underline{q}r^{\,q}$	$q\underline{r}$	$\underline{p}qr$	$p\underline{q}r$	$pq\underline{r}$

Figure 11. The coloring g_2 of X.

Corollary 10. *For every $y \in \bar{B}$ we have*

$$\begin{aligned}\alpha_3(G) \geq\ & |J^y \cup M^y| + \left|K_p^y \cap (A \times \bar{B})\right| + \left|K_q^y \cap (A \times \bar{B})\right| + \left|K_r^y \cap (A \times \bar{B})\right| \\ & + \frac{1}{2}\left(\left|K_{pq}^y \cap (A \times \bar{B})\right| + \left|K_{pr}^y \cap (A \times \bar{B})\right| + \left|K_{qr}^y \cap (A \times \bar{B})\right|\right) \\ & + \frac{1}{2}\left|K_{pq}^y \cap \left(A^1 \times \bar{B}\right)\right|.\end{aligned}$$

4. Proof of the Main Theorem

We begin by deriving an upper bound for the size of I, and we use the notation from the introduction.

$$\begin{aligned}|I| + |M| &= \sum_{x \in V(G)} |K^x \cup M^x| + \sum_{y \in V(H)} |J^y \cup M^y| \\ &= \sum_{x \in A} |K^x \cup M^x| + \sum_{x \in \bar{A}} |K^x \cup M^x| + \sum_{y \in B} |J^y \cup M^y| + \sum_{y \in \bar{B}} |J^y \cup M^y| \\ &= |I \cap (A \times B)| + |M \cap (A \times B)| + \sum_{x \in A} \left|(K^x \cup M^x) \cap (A \times \bar{B})\right| \\ &\quad + \sum_{y \in B} \left|(J^y \cup M^y) \cap (\bar{A} \times B)\right| + \sum_{x \in \bar{A}} |K^x \cup M^x| + \sum_{y \in \bar{B}} |J^y \cup M^y|\end{aligned}$$

and therefore

$$\begin{aligned}|I| &= |I \cap (A \times B)| + \sum_{x \in A} |K^x \cap (A \times \bar{B})| + \sum_{y \in B} |J^y \cap (\bar{A} \times B)| \\ &\quad + \sum_{x \in \bar{A}} |K^x \cup M^x| + \sum_{y \in \bar{B}} |J^y \cup M^y| - \left|M \cap (\bar{A} \times \bar{B})\right| \\ &\leq |I \cap (A \times B)| + \sum_{y \in \bar{B}} |K^y \cap (A \times \bar{B})| + \sum_{x \in \bar{A}} |J^x \cap (\bar{A} \times B)|\end{aligned}$$

$$
\begin{aligned}
&\le |I \cap (A \times B)| + \sum_{y \in \bar{B}} |K^y \cap (A \times \bar{B})| + \sum_{x \in \bar{A}} |J^x \cap (\bar{A} \times B)| \\
&+ \sum_{x \in \bar{A}} |K^x \cup M^x| + \sum_{y \in \bar{B}} |J^y \cup M^y| \\
&= |I \cap (A \times B)| + \sum_{y \in \bar{B}} \left(\left| K^y \cap (A \times \bar{B}) \right| + |J^y \cup M^y| \right) \\
&+ \sum_{x \in \bar{A}} \left(\left| J^x \cap (\bar{A} \times B) \right| + |K^x \cup M^x| \right).
\end{aligned}
$$

To prove the theorem it remains to prove that

$$
\begin{aligned}
&|I \cap (A \times B)| + \sum_{y \in \bar{B}} \left(|K^y \cap (A \times \bar{B})| + |J^y \cup M^y| \right) \\
&+ \sum_{x \in \bar{A}} \left(\left| J^x \cap (\bar{A} \times B) \right| + |K^x \cup M^x| \right) \\
&\le \alpha_3(G)|V(H)| + \alpha_3(H)|V(G)| - \alpha_3(G)\alpha_3(H).
\end{aligned} \tag{1}
$$

In order to prove (1) we give in each of the five cases, (a), (b), (c), (d) and (e) shown in Figure 2, two upper bounds for $|I \cap (A \times B)|$ of the following form

$$
|I \cap (A \times B)| \le \alpha_3(G)\alpha_3(H) - \sum_{j=1}^{3} \sum_{S \subseteq \mathcal{C}} \beta_S^j \left| J_S \cap \left(\bar{A} \times B^j \right) \right| \tag{2}
$$

and

$$
|I \cap (A \times B)| \le \alpha_3(G)\alpha_3(H) - \sum_{i=1}^{3} \sum_{S \subseteq \mathcal{C}} \gamma_S^i \left| K_S \cap \left(A^i \times \bar{B} \right) \right| \tag{3}
$$

where the coefficients β_S^j, γ_S^i for $S \subseteq \mathcal{C}$ and $i, j \in [3]$ are declared and explained later for each case individually. Combining both inequalities (2) and (3) we can (upper) bound $|I \cap (A \times B)|$ by

$$
\begin{aligned}
\alpha_3(G)\alpha_3(H) \; &- \Lambda_1 \sum_{j=1}^{3} \sum_{S \subseteq \mathcal{C}} \beta_S^j \left| J_S \cap \left(\bar{A} \times B^j \right) \right| \\
&- \Lambda_2 \sum_{i=1}^{3} \sum_{S \subseteq \mathcal{C}} \gamma_S^i \left| K_S \cap \left(A^i \times \bar{B} \right) \right|
\end{aligned} \tag{4}
$$

where Λ_1 and Λ_2 are two constants such that $\Lambda_1 + \Lambda_2 \le 1$, and these constants will also be declared later. Then we also bound the other two terms that appear on the left side of (1) as follows

$$\begin{aligned}(5)\qquad &\sum_{x\in\bar{A}}\left(\left|J^x\cap(\bar{A}\times B)\right|+|K^x\cup M^x|\right)\\ &\le(|V(G)|-\alpha_3(G))\alpha_3(H)+\Lambda_1\sum_{j=1}^{3}\sum_{S\subseteq\mathcal{C}}\beta_S^j\left|J_S\cap\left(\bar{A}\times B^j\right)\right|\end{aligned}$$

and

$$\begin{aligned}(6)\qquad &\sum_{y\in\bar{B}}\left(|K^y\cap(A\times\bar{B})|+|J^y\cup M^y|\right)\\ &\le(|V(H)|-\alpha_3(H))\alpha_3(G)+\Lambda_2\sum_{i=1}^{3}\sum_{S\subseteq\mathcal{C}}\gamma_S^i\left|K_S\cap\left(A^i\times\bar{B}\right)\right|\end{aligned}$$

which gives (when combining (4), (5) and (6)) the desired result (1). Note that

$$\left|J^x\cap(\bar{A}\times B)\right|=\sum_{j=1}^{3}\sum_{S\subseteq\mathcal{C}}\left|J_S^x\cap\left(\bar{A}\times B^j\right)\right|.$$

To prove (5) it suffices to show that for every $x\in\bar{A}$ we have

$$\left|J^x\cap(\bar{A}\times B)\right|+|K^x\cup M^x|\le\alpha_3(H)+\Lambda_1\sum_{j=1}^{3}\sum_{S\subseteq\mathcal{C}}\beta_S^j\left|J_S^x\cap\left(\bar{A}\times B^j\right)\right|,$$

which is equivalent to

$$(7)\qquad \alpha_3(H)\ge|K^x\cup M^x|+\sum_{j=1}^{3}\sum_{S\subseteq\mathcal{C}}(1-\Lambda_1\beta_S^j)\left|J_S^x\cap(\bar{A}\times B^j)\right|.$$

Similarly, to prove (6) it suffices to show that for every $y\in\bar{B}$ we have

$$(8)\qquad \alpha_3(G)\ge|J^y\cup M^y|+\sum_{i=1}^{3}\sum_{S\subseteq\mathcal{C}}\left(1-\Lambda_2\gamma_S^i\right)\left|K_S^y\cap\left(A^i\times\bar{B}\right)\right|.$$

Now we are ready to give the values of the coefficients β_S^j and γ_S^i for $S\subseteq\mathcal{C}$ such that (2), (3), (7) and (8) are fulfilled. The values are given in Table 1.

Case (a). We first consider the coloring from Case (a) (see Figure 2 where in each case sets A^1,A^2,A^3 are arranged horizontally from left to right, and B^1,B^2,B^3 vertically from bottom to top), and prove the bounds (2), (3), (7) and (8) for this case. To prove (2) note that for every $y\in B^1$ the projections of $U=(J_{pq}^y\cup J_{pqr}^y)\cap(\bar{A}\times B^1)$ and $V=(J_{pr}^y\cup J_{pqr}^y)\cap(\bar{A}\times B^1)$ to G are independent sets in G, and for every $(x,y)\in U\cup V$ we have $p\in f(N_H(x,y))$. Since $f(A\times B^1)=p$ we find that $N_G(U)\cap(A\times B^1)$ and $N_G(V)\cap(A\times B^1)$ have empty intersections with I. Since A is a maximum tripartite subgraph of G we find that

$Case:$	(a)	(b)	(c)	(d)	(e)
$\beta_p^j, j \in [2]$	0	0	0	0	0
$\beta_q^j, j \in [2]$	0	0	0	0	0
$\beta_r^j, j \in [2]$	0	0	0	0	0
β_p^3	0	0	1	0	3/4
β_q^3	0	0	1	0	3/4
β_r^3	0	3/2	1	0	3/4
β_{pq}^1	3/2	0	0	3/2	0
β_{pr}^1	3/2	0	0	3/2	0
β_{qr}^1	0	0	0	0	0
β_{pq}^2	3/2	0	3/2	3/2	3/2
β_{pr}^2	0	3/2	3/2	0	3/2
β_{qr}^2	3/2	3/2	3/2	3/2	3/2
β_{pq}^3	0	3/2	3/2	0	3/2
β_{pr}^3	3/2	3/2	3/2	3/2	3/2
β_{qr}^3	3/2	3/2	3/2	3/2	3/2
$\beta_{pqr}^j, j \in [3]$	3	3	3	3	3
$\gamma_p^i, i \in [3]$	0	0	0	0	0
$\gamma_q^i, i \in [3]$	0	0	0	0	0
$\gamma_r^i, i \in [3]$	0	0	0	0	0
γ_{pq}^1	1	1	1	0	0
γ_{pr}^1	1	1	1	3/2	1
γ_{qr}^1	1	1	1	3/2	1
γ_{pq}^2	1	1	1	3/2	1
γ_{pr}^2	1	1	1	3/2	1
γ_{qr}^2	1	1	1	3/2	1
γ_{pq}^3	1	1	1	3/2	1
γ_{pr}^3	1	1	1	3/2	1
γ_{qr}^3	1	1	1	3/2	1
$\gamma_{pqr}^i, i \in [3]$	3	3	3	3	3

Table 1. Values of β_S^j and γ_S^i for $S \subseteq \mathcal{C}$.

$\left|N_G(U) \cap (A^i \times B^1)\right| \geq |U|$ and $|N_G(V) \cap (A^i \times B^1)| \geq |V|$ for $i \in [3]$. Assume that $|U| \geq |V|$. Since $|N_G(U) \cap (A \times B^1)| \geq 3|U|$ we find that $|N_G(U) \cap (A \times B^1)| \geq \frac{3}{2}(|U| + |V|)$ and therefore

$$\begin{aligned} \left|N_G(U) \cap \left(A \times B^1\right)\right| \geq \frac{3}{2}\left(\left|J_{pq}^y \cap \left(\bar{A} \times B^1\right)\right| + \left|J_{pr}^y \cap \left(\bar{A} \times B^1\right)\right|\right) \\ + 3\left|J_{pqr}^y \cap \left(\bar{A} \times B^1\right)\right|. \end{aligned}$$

Since $N_G(U) \cap (A \times B^1) \cap I = \emptyset$ we find that for every $y \in B^1$ we have

$$\begin{aligned}|I^y \cap (A \times B)| \leq \alpha_3(G) - \frac{3}{2}\left(\left|J^y_{pq} \cap \left(\bar{A} \times B^1\right)\right| + \left|J^y_{pr} \cap \left(\bar{A} \times B^1\right)\right|\right) \\ - 3\left|J^y_{pqr} \cap (\bar{A} \times B^1)\right|.\end{aligned}$$

Note that the same bound is true (and with the same proof) for the coloring in Case (d). Similarly (by observing sets $U' = \left(J^y_{pq} \cup J^y_{pqr}\right) \cap \left(\bar{A} \times B^2\right)$ and $V' = \left(J^y_{qr} \cup J^y_{pqr}\right) \cap \left(\bar{A} \times B^2\right)$ and deducing similar claims as above) we prove that for every $y \in B^2$ we have

$$\begin{aligned}|I^y \cap (A \times B)| \leq \alpha_3(G) - \frac{3}{2}\left(\left|J^y_{pq} \cap (\bar{A} \times B^2)\right| + \left|J^y_{qr} \cap \left(\bar{A} \times B^2\right)\right|\right) \\ - 3\left|J^y_{pqr} \cap \left(\bar{A} \times B^2\right)\right|.\end{aligned}$$

and (by observing sets $U'' = \left(J^y_{qr} \cup J^y_{pqr}\right) \cap \left(\bar{A} \times B^2\right)$ and $V'' = \left(J^y_{pr} \cup J^y_{pqr}\right) \cap \left(\bar{A} \times B^2\right)$ we find that) for every $y \in B^3$

$$\begin{aligned}|I^y \cap (A \times B)| \leq \alpha_3(G) - \frac{3}{2}\left(\left|J^y_{pr} \cap \left(\bar{A} \times B^3\right)\right| + \left|J^y_{qr} \cap \left(\bar{A} \times B^3\right)\right|\right) \\ - 3\left|J^y_{pqr}\left(\bar{A} \times B^3\right)\right|.\end{aligned}$$

Combining all three bounds we get

$$\begin{aligned}|I \cap (A \times B)| = & \sum_{y \in B} |I^y \cap (A \times B)| \leq \alpha_3(G)\alpha_3(H) \\ & - \frac{3}{2}\left(\left|J_{pq} \cap \left(\bar{A} \times B^1\right)\right| + \left|J_{pr} \cap \left(\bar{A} \times B^1\right)\right|\right) - 3\left|J_{pqr} \cap \left(\bar{A} \times B^1\right)\right| \\ & - \frac{3}{2}\left(\left|J_{pq} \cap \left(\bar{A} \times B^2\right)\right| + \left|J_{qr} \cap \left(\bar{A} \times B^2\right)\right|\right) - 3\left|J_{pqr} \cap \left(\bar{A} \times B^2\right)\right| \\ & - \frac{3}{2}\left(\left|J_{pr} \cap \left(\bar{A} \times B^3\right)\right| + \left|J_{qr} \cap \left(\bar{A} \times B^3\right)\right|\right) - 3\left|J_{pqr} \cap \left(\bar{A} \times B^3\right)\right|.\end{aligned}$$

This is exactly the bound (2) with coefficients β^j_S as given in the first column of Table 1 (again note that this bound applies also for the coloring in Case (d), so this also proves inequality (2) for Case (d)).

Now we prove (3). Let $a \in A$ and consider the sets $U = \left(K^x_{pq} \cup K^x_{pr} \cup K^x_{pqr}\right) \cap (A \times \bar{B})$ and $V = \left(K^x_{pq} \cup K^x_{qr} \cup K^x_{pqr}\right) \cap (A \times \bar{B})$. Note that for every $(x, y) \in U$ we have $p \in f(N_G(x, y))$ and for every $(x, y) \in V$ we have $q \in f(N_G(x, y))$. Moreover the projections of both sets to H are independent sets in H. Therefore $|N_H(U) \cap (A \times B^1)| \geq |U|$ and since $p \in f(N_G(x, y))$ for all $(x, y) \in U$ we find that $(N_H(U) \cap (A \times B^1)) \cap I = \emptyset$. Similarly $(N_H(V) \cap (A \times B^2)) \cap I = \emptyset$ and

$|N_H(V)\cap(A\times B^2)|\geq|V|$. Finally let $Z=K^x_{pqr}$. We have $(N_H(Z)\cap(A\times B^3))\cap I=\emptyset$ and $|N_H(Z)\cap(A\times B^3)|\geq|Z|$. Since

$$|U|+|V|+|Z| = \left|K^x_{pq}\cap(A\times\bar{B})\right|+\left|K^x_{pr}\cap(A\times\bar{B})\right| + \left|K^x_{qr}\cap(A\times\bar{B})\right|+3\left|K^x_{pqr}\cap(A\times\bar{B})\right|$$

we find that

$$|I\cap(A\times B)| = \sum_{x\in A}|I^x\cap(A\times B)|\leq\alpha_3(G)\alpha_3(H)-\left|K_{pq}\cap(A\times\bar{B})\right| - \left|K_{pr}\cap(A\times\bar{B})\right|-\left|K_{qr}\cap(A\times\bar{B})\right|-3\left|K_{pqr}\cap(A\times\bar{B})\right|.$$

This is exactly the bound (3) with coefficients γ^i_S as given in the first column of Table 1.

It remains to prove (7) and (8). If we choose $\Lambda_1=2/3$ and $\Lambda_2=1/3$ then (7) and (8) follow from Corollaries 8 and 5.

Case (b). The bound (3) is the same as in Case (a), and we use sets $U=K^x_{\underline{pr}}\cup K^x_{qr}\cup K^x_{pqr}$, $V=K^x_{\underline{pr}}\cup K^x_{pq}\cup K^x_{pqr}$ and $Z=K^x_{pqr}$ to prove (3) (an analogous arumentation as in Case (a) works). Next we prove (2). To prove (2) note that for every $y\in B^3$ the projections of $U=\left(J^y_{\underline{pr}}\cup J^y_{pq}\cup J^y_{pqr}\right)\cap(\bar{A}\times B^3)$ and $V=(J^y_r\cup J^y_{\underline{pr}}\cup J^y_{qr}\cup J^y_{pqr})\cap(\bar{A}\times B^3)$ to G are independent sets in G. Since A is a maximum tripartite subgraph of G and the projections of U and V induce independent sets in $\bar{A}$, we find that $|A|\geq 3|U|$ and $|A|\geq 3|U|$, and so $|A|\geq\frac{3}{2}(|U|+|V|)$. Therefore

$$|A|\geq\frac{3}{2}\left|\left(J^y_{pq}\cup J^y_{pr}\cup J^y_{qr}\cup J^y_r\right)\cap\left(\bar{A}\times B^3\right)\right|+3\left|J^y_{pqr}\cap\left(\bar{A}\times B^3\right)\right|.$$

Since

$$0=\left|I^y\cap\left(A\times B^3\right)\right|=\alpha_3(G)-|A|$$

we find that for every $y\in B^3$ we have

$$|I^y\cap(A\times B)|\leq\alpha_3(G)-\frac{3}{2}\left|\left(J^y_{pq}\cup J^y_{pr}\cup J^y_{qr}\cup J^y_r\right)\cap\left(\bar{A}\times B^3\right)\right| - 3\left|J^y_{pqr}\cap\left(\bar{A}\times B^3\right)\right|.$$

This inequaity explains the coefficients $\beta^3_{pq},\beta^3_{pr},\beta^3_{qr},\beta^3_r$ and β^3_{pqr}. We prove, analogously as in Case (a), that for every $y\in B^2$ we have

$$|I^y\cap(A\times B)|\leq\alpha_3(G)-\frac{3}{2}\left(\left|J^y_{pr}\cap\left(\bar{A}\times B^2\right)\right|+\left|J^y_{qr}\cap\left(\bar{A}\times B^2\right)\right|\right) - 3\left|J^y_{pqr}\cap\left(\bar{A}\times B^2\right)\right|,$$

which explains coefficients $\beta^2_{pr}, \beta^2_{qr}$ and β^2_{pqr}. Also for every $y \in B^1$ we have

$$|I^y \cap (A \times B)| \le \alpha_3(G) - 3\left|J^y_{pqr} \cap \left(\bar{A} \times B^1\right)\right|,$$

explaining the coeficient β^1_{pqr}. If we set $\Lambda_1 = 2/3$ and $\Lambda_2 = 1/3$ then (7) follows from Corollary 6 and (8) follows from Corollary 5.

Case (c). In this case (3) is proved as in Case (a). When we set $\Lambda_1 = 2/3$ and $\Lambda_2 = 1/3$ we find that (8) follows from Corollary 5 and (7) follows from Corollary 7, so it remains to prove (2) to complete this case. Let $y \in B^3$ and assume, without loss of generality, that $\left|J^y_p \cap \left(\bar{A} \times B^3\right)\right| + \left|J^y_q \cap \left(\bar{A} \times B^3\right)\right| \ge \frac{2}{3}\left(\left|J^y_p \cap \left(\bar{A} \times B^3\right)\right| + \left|J^y_q \cap \left(\bar{A} \times B^3\right)\right| + \left|J^y_r \cap \left(\bar{A} \times B^3\right)\right|\right)$. Let $U = \left(J^y_p \cup J^y_{pq} \cup J^y_{pr} \cup J^y_{pqr}\right) \cap \left(\bar{A} \times B^3\right)$ and $V = \left(J^y_q \cup J^y_{pq} \cup J^y_{qr} \cup J^y_{pqr}\right) \cap \left(\bar{A} \times B^3\right)$. The projections of U and V to G induce independent sets in $\bar{A}$, so it follows from maximality of A that $|A| \ge 3|U|$ and $|A| \ge 3|V|$. Therefore $|A| \ge \frac{3}{2}(|U| + |V|)$ and thus

$$0 = \left|I^y \cap \left(A \times B^3\right)\right| \le \alpha_3(G) - \frac{3}{2}(|U| + |V|).$$

Now we use the initial asuumption $\left|J^y_p \cap \left(\bar{A} \times B^3\right)\right| + \left|J^y_q \cap \left(\bar{A} \times B^3\right)\right| \ge \frac{2}{3}\left(\left|J^y_p \cap \left(\bar{A} \times B^3\right)\right| + \left|J^y_q \cap \left(\bar{A} \times B^3\right)\right| + \left|J^y_r \cap \left(\bar{A} \times B^3\right)\right|\right)$ to get

$$\begin{aligned}\left|I^y \cap \left(A \times B^3\right)\right| \le\ & \alpha_3(G) - \Big(\left|J^y_p \cap \left(\bar{A} \times B^3\right)\right| + \left|J^y_q \cap \left(\bar{A} \times B^3\right)\right| \\ & + \left|J^y_r \cap \left(\bar{A} \times B^3\right)\right| - \frac{3}{2}\left(\left|J^y_{pq} \cap \left(\bar{A} \times B^3\right)\right| + \left|J^y_{pr} \cap \left(\bar{A} \times B^3\right)\right|\right) \\ & + \left|J^y_{qr} \cap \left(\bar{A} \times B^3\right)\right| - 3\left|J^y_{pqr} \cap \left(\bar{A} \times B^3\right)\right|.\end{aligned}$$

Since also for every $y \in B^2$ we have

$$\begin{aligned}\left|I^y \cap \left(A \times B^2\right)\right| \le\ & \alpha_3(G) - \frac{3}{2}\Big(\left|J^y_{pq} \cap \left(\bar{A} \times B^2\right)\right| + \left|J^y_{pr} \cap \left(\bar{A} \times B^2\right)\right| \\ & + \left|J^y_{qr} \cap \left(\bar{A} \times B^2\right)\right|\Big) - 3\left|J^y_{pqr} \cap \left(\bar{A} \times B^2\right)\right|.\end{aligned}$$

and for every $y \in B^1$ we have

$$\left|I^y \cap \left(A \times B^1\right)\right| \le \alpha_3(G) - 3\left|J^y_{pqr} \cap \left(\bar{A} \times B^1\right)\right|,$$

we find that (2) with coefficients β_S as given in Table 1 is fulfilled.

Case (d). The bound (2) is the same as in Case (a) and it was already proved, so it remains to prove (3). Let $x \in A^1$ and consider the sets $U = (K^x_{pr} \cup K^x_{pqr}) \cap (A \times \bar{B})$ and $V = (K^x_{qr} \cup K^x_{pqr}) \cap (A \times \bar{B})$. Note that for every $(x, y) \in U$ we have $p, r \in f(N_G(x, y))$ and for every $(x, y) \in V$ we have $q, r \in f(N_G(x, y))$. Moreover the projections of both sets to H are independent sets in H. Thus we have $|N_H(U) \cap (A^1 \times B^1)| \ge |U|$ and $|N_H(U) \cap (A^1 \times B^3)| \ge |U|$, and since

$p, r \in f(N_G(x,y))$ for all $(x,y) \in U$ we find that $(N_H(U) \cap (A^1 \times B^1)) \cap I = (N_H(U) \cap (A^1 \times B^3)) \cap I = \emptyset$. Similarly $(N_H(V) \cap (A^1 \times B^2)) \cap I = (N_H(V) \cap (A^1 \times B^3)) \cap I = \emptyset$, and $|N_H(V) \cap (A^1 \times B^2)| \geq |V|$ and $|N_H(V) \cap (A^1 \times B^3)| \geq |V|$. Putting all together we get

$$\begin{aligned}|I \cap (A^1 \times B)| &= \sum_{x \in A^1} \left|I^x \cap (A^1 \times B)\right| \leq |A^1|\alpha_3(H) - \frac{3}{2}(|K_{pr} \cap (A^1 \times \bar{B})| \\ &- |K_{qr} \cap (A^1 \times \bar{B})|) - 3|K_{pqr} \cap (A^1 \times \bar{B})|.\end{aligned}$$

If $x \in A^2 \cup A^2$ then the we set $U = \left(K^x_{\underline{pq}} \cup K^x_{pr} \cup K^x_{pqr}\right) \cap (A \times \bar{B})$ and $V = \left(K^x_{p\underline{q}} \cup K^x_{qr} \cup K^x_{pqr}\right) \cap (A \times \bar{B})$ and derive similar claims as in Case (a) to obtain

$$\begin{aligned}\left|I \cap \left(\left(A^2 \cup A^3\right) \times B\right)\right| &= \sum_{x \in A^2 \cup A^3} \left|I^x \cap \left(\left(A^2 \cup A^3\right) \times B\right)\right| \leq \left|A^2 \cup A^3\right|\alpha_3(H) \\ &\quad -\frac{3}{2}\left(\left|K_{pq} \cap \left(\left(A^2 \cup A^3\right) \times \bar{B}\right)\right| + \left|K_{pr} \cap \left(\left(A^2 \cup A^3\right) \times \bar{B}\right)\right|\right. \\ &\quad +\left|K_{qr} \cap \left(\left(A^2 \cup A^3\right) \times \bar{B}\right)\right| - 3\left|K_{pqr} \cap \left(\left(A^2 \cup A^3\right) \times \bar{B}\right)\right|.\end{aligned}$$

Adding both inequalities we get (3) with coefficients γ^i_S as declared in the fourth column of Table 1. Finally when we set $\Lambda_1 = 2/3$ and $\Lambda_2 = 1/3$ we find that (8) follows from Corollary 9 and (7) follows from Corollary 8.

Case (e). We first prove (2). Let $y \in B^2 \cup B^3$ and assume without loss of generality that $\left|J^y_r \cap \left(\bar{A} \times B^3\right)\right| \geq \left|J^y_q \cap \left(\bar{A} \times B^3\right)\right|$. Let $U = \left(J^y_p \cup J^y_{\underline{p}r} \cup J^y_{pq} \cup J^y_{pqr}\right) \cap \left(\bar{A} \times B^3\right)$ and $V = \left(J^y_r \cup J^y_{p\underline{r}} \cup J^y_{qr} \cup J^y_{pqr}\right) \cap \left(\bar{A} \times B^3\right)$. Without loss of generality assume that $|U| \geq |V|$. Since the projections of U and V induce independent sets in $\bar{A}$ we find that $|U| \geq |V| \geq |A^i|$ for $i \in [3]$. Note that for every vertex $(x,y) \in V$ we have $r \in N_H(x,y)$ and so $N_G(V) \cap (A^1 \times B^3) \cap I = \emptyset$. Since $\left|N_G(V) \cap \left(A^1 \times B^3\right)\right| \geq |V|$ we find that

$$\left|I^y \cap \left(A \times B^3\right)\right| \leq \alpha_3(G) - \frac{3}{2}(|U| + |V|),$$

and since $\left|\left(J^y_r \cup J^y_p\right) \cap \left(\bar{A} \times B^3\right)\right| \geq \frac{1}{2}\left|\left(J^y_r \cup J^y_p \cup J^y_q\right) \cap \left(\bar{A} \times B^3\right)\right|$ we find that

$$\begin{aligned}\left|I^y \cap \left(A \times B^3\right)\right| &\leq \alpha_3(G) - \frac{3}{2}\left(\left|J^y_{pq} \cap \left(\bar{A} \times B^3\right)\right| + \left|J^y_{pr} \cap \left(\bar{A} \times B^3\right)\right|\right. \\ &\quad + \left.\left|J^y_{qr} \cap \left(\bar{A} \times B^3\right)\right|\right) - \frac{3}{4}\left(\left|J^y_p \cap \left(\bar{A} \times B^3\right)\right| + \left|J^y_q \cap \left(\bar{A} \times B^3\right)\right|\right. \\ &\quad + \left.\left|J^y_r \cap \left(\bar{A} \times B^3\right)\right|\right) - 3\left|J^y_{pqr} \cap \left(\bar{A} \times B^3\right)\right|.\end{aligned}$$

which proves (2).

To prove (3) let $x \in A^1$ and define $U = (K^x_{pr} \cup K^x_{pqr}) \cap (A \times \bar{B})$, $V = (K^x_{qr} \cup K^x_{pqr}) \cap (A \times \bar{B})$ and $Z = K^x_{pqr} \cap (A \times \bar{B})$. Every vertex in $U \cup V \cup Z$ has in its G-neighborhood a vertex colored by r. So we have

$$\begin{aligned}\left|I^x \cap \left(A^1 \times B\right)\right| \leq \alpha_3(H) &- \left(\left|K^x_{pr} \cap \left(A^1 \times \bar{B}\right)\right| + \left|K^x_{qr} \cap \left(A^1 \times \bar{B}\right)\right|\right)\\ &- 3\left|K^x_{pqr} \cap \left(A^1 \times \bar{B}\right)\right|.\end{aligned}$$

For $x \in A^2 \cup A^3$ the proof is analogous as in Case (a). In this case we set $\Lambda_1 = \Lambda_2 = 1/2$, and so (8) follows from Corollary 10 and (7) follows from Corollary 7. This completes the proof of the last case.

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