Discussiones Mathematicae
Graph Theory 39 (2019) 313–324
doi:10.7151/dmgt.2079

ON THE *b*-DOMATIC NUMBER OF GRAPHS

Mohammed Benatallah

UZA, Faculty of Sciences
RECITS Laboratory, Djelfa, Algeria

e-mail: m_benatallah@yahoo.fr

Noureddine Ikhlef-Eschouf

UYF, Faculty of Sciences
Medea, Algeria

e-mail: nour_echouf@yahoo.fr

and

Miloud Mihoubi

USTHB, Faculty of Mathematics
RECITS Laboratory, Algiers, Algeria

e-mail: mmihoubi@usthb.dz

Abstract

A set of vertices S in a graph $G = (V, E)$ is a *dominating set* if every vertex not in S is adjacent to at least one vertex in S. A *domatic partition* of graph G is a partition of its vertex-set V into dominating sets. A domatic partition $\mathcal{P}$ of G is called *b-domatic* if no larger domatic partition of G can be obtained from $\mathcal{P}$ by transferring some vertices of some classes of $\mathcal{P}$ to form a new class. The minimum cardinality of a b-domatic partition of G is called the *b-domatic number* and is denoted by $bd(G)$. In this paper, we explain some properties of b-domatic partitions, and we determine the b-domatic number of some families of graphs.

Keywords: domatic partition, domatic number, b-domatic partition, b-domatic number.

2010 Mathematics Subject Classification: 05C15, 05C69, 05C70.

1. Introduction

Let $G = (V, E)$ be a finite, simple and undirected graph with vertex-set V and edge-set E. We call $|V|$ the order of G and denote it by n. For any nonempty subset $S \subset V$, let $G[S]$ denote the subgraph of G induced by S. For any vertex v of G, the *open neighborhood* of v is the set $N_G(v) = \{u \in V(G) \mid (u, v) \in E\}$ and the *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. The *private neighborhood* of a vertex $v \in S$ with respect to S is the set $pn[v, S] = \{u \in V(G) : N_G[u] \cap S = \{v\}\}$. Each vertex in $pn\,[v, S]$ is called a *private neighbor* of v with respect to S. Remark that $pn\,[v, S]$ is a set contained in $\{v\} \cup (V \setminus S)$. Let $\Delta(G)$ (respectively, $\delta(G)$) be the *maximum* (respectively, *minimum*) degree in G. Through this paper, the notations P_n, C_n, and K_n always denote a path, a cycle, and a complete graph of order n, respectively, while $K_{p,q}$ $(p \geq q)$ denotes the complete bipartite graph with partite sets of sizes p, q. For further terminology on graphs we refer to the book by Berge [2].

Graph coloring and domination are two major areas in graph theory that have been extensively studied. These two concepts can be defined as a vertex partition into classes according to certain rules. By a *vertex partition* (partition for short), we will mean a partition of its vertex-set into disjoint subsets (classes). The cardinality of a partition is the number of its classes.

A set $S \subseteq V(G)$ is called *independent* if no two vertices in S are adjacent. A partition $\mathcal{P}$ in which each of its classes is an independent set is called a *proper coloring* of G. The smallest integer k such that G admits a proper coloring with k colors is called the *chromatic number* of G and is denoted by $\chi(G)$. In general, it is NP-complete to compute the chromatic number. For this reason, many heuristics have been developed for finding an approximate solution to this problem. One approach is to start with an arbitrary proper coloring and try to reduce the number of colors used by transferring all vertices from one color class to other classes. This technique is not possible if each color class contains a vertex having neighbors in all other color classes. A coloring satisfying such a property is called *b-coloring*. The *b-chromatic number* $b(G)$ of a graph G is the largest integer k such that G admits a b-coloring with k colors. This concept was introduced by Irving and Manlove [7, 8].

A set $S \subseteq V$ is called a *dominating set* if every vertex in $V \setminus S$ is adjacent to some vertex in S. The minimum cardinality of a dominating set of G is called the *domination number* of G and is denoted by $\gamma(G)$. By analogy to the concept of chromatic partition, Cockayne and Hedetniemi [3] introduced the concept of *domatic partition* of a graph. A partition $\mathcal{P}$ in which each of its classes is a dominating set is called a domatic partition of G. The *domatic number* $d(G)$ is defined as the largest number of sets in a domatic partition.

In [5], Favaron introduced the *b-domatic number* by considering a new type of domatic partition. As defined in [5], a domatic partition $\mathcal{P}$ of G is *b-domatic* if no larger domatic partition of G can be obtained from $\mathcal{P}$ by transferring some vertices of some classes of $\mathcal{P}$ to form a new class. Formally, a domatic partition $\mathcal{P} = \{U_1, \ldots, U_k\}$ is called *b-domatic* if there do not exist k non-empty subsets $\pi_i \subseteq U_i$, $i \in \{1, \ldots, k\}$ with $\bigcup_{i=1}^{k} (U_i \setminus \pi_i) \neq \emptyset$, such that $\{\pi_1, \ldots, \pi_k, \bigcup_{i=1}^{k} (U_i \setminus \pi_i)\}$ is a domatic partition of G. The minimum cardinality of a b-domatic partition of G is called the *b-domatic number* and is denoted by $bd(G)$.

It was observed in [5] that if $\delta(G) = 0$, then $\{V(G)\}$ is the unique domatic partition and so $bd(G) = d(G) = 1$. For this, all graphs considered in this paper are without isolated vertices. Many other properties of domatic and b-domatic partitions were given in [5]. In particular, it was observed that any graph G with minimum degree $\delta(G) \geq 1$, satisfies $2 \leq bd(G) \leq d(G)$. The same author [5] asked several questions, some of which we answer in this paper. We first investigate a new property of a b-domatic partition by giving a sufficient condition for which a given domatic partition of a graph G is b-domatic. We next present some classes of graphs for which $bd(G) = 2$ and $bd(G) = \delta(G) + 1$, and we determine the b-domatic number of some special bipartite graphs and block graphs. Other results are given for particular classes of graphs.

2. Known Results

The authors of [3] showed the following result.

Proposition 1 [3]. *Let G be a graph of order n and minimum degree $\delta(G)$. Then*

$$d(G) \leq \min\left\{\frac{n}{\gamma(G)}, \delta(G) + 1\right\}. \tag{1}$$

For some other results on domatic partitions see [1, 4, 10].

Proposition 2 [5]. *Let G be a graph of minimum degree $\delta(G)$. If $\delta(G) = 0$, then $bd(G) = d(G) = 1$, otherwise $2 \leq bd(G) \leq d(G)$.*

Hence, by Propositions 1 and 2, the next result follows immediately.

Proposition 3 [5]. *For any graph G of minimum degree $\delta(G)$, we have $bd(G) \leq \delta(G) + 1$.*

The following results are proved by Favaron in [5].

Theorem 4 [5]. *Let $G_1, \ldots, G_k$ be the components of a disconnected graph G without isolated vertices. Then $bd(G) = \min\{bd(G_i) : 1 \leq i \leq k\}$.*

Proposition 5 [5]. *Every domatic partition such that each class is a minimal dominating set of G is b-domatic.*

The same author has computed the b-domatic number for some particular classes of graphs.

Proposition 6 [5]**.** *$bd(K_n) = n$, $bd(C_3) = 3$, $bd(C_n) = 2$ for $n \geq 4$, and $bd(K_{p,q}) = 2$.*

3. Main Results

We start this section by giving a sufficient condition for which a given domatic partition of a graph G is b-domatic. Let $\mathcal{P}$ be a domatic partition of a graph G. For a vertex $v \in V(G)$, let U_v denote the class of $\mathcal{P}$ containing v.

Theorem 7. *Let $\mathcal{P}$ be a domatic partition of a graph $G = (V, E)$. If G has a vertex v such that for each $u \in N_G[v]$ the set $pn[u, U_u]$ is not empty, then $\mathcal{P}$ is b-domatic.*

Proof. Let $\mathcal{P} = \{U_1, \ldots, U_k\}$ be a domatic partition of a graph G and let $v \in V(G)$ such that for each vertex $u \in N_G[v]$, $pn[u, U_u] \neq \emptyset$. Suppose, to the contrary, that $\mathcal{P}$ is not b-domatic. Then, there exist k non-empty subsets $\pi_i \subseteq U_i$, $i \in \{1, \ldots, k\}$ with $\bigcup_{i=1}^{k}(U_i \setminus \pi_i) \neq \emptyset$ for which $\pi = \left\{\pi_1, \ldots, \pi_k, \bigcup_{i=1}^{k}(U_i \setminus \pi_i)\right\}$ is a domatic partition of G. Let $\pi_{k+1} = \bigcup_{i=1}^{k}(U_i \setminus \pi_i)$. We claim that any vertex u in $N_G[v]$ cannot be in π_{k+1}. Suppose, to the contrary, that $u \in N_G[v] \cap \pi_{k+1}$. Then, there is a class $\pi_p \subset U_u$ that does not contain u for a some $p \in \{1, \ldots, k\}$. Therefore, either u is isolated in U_u in which case no vertex of π_p can dominate u for the partition π, or there exists a vertex $z \in pn[u, U_u]$ in which case no vertex of π_p can dominate z for the partition π. In either case, we have a contradiction with the fact that π is a domatic partition of G. This means that neither v nor its neighbors are in π_{k+1}, so v is not dominated by π_{k+1}, which contradicts again that π is a domatic partition of G. ■

Note that the converse is not true in general. For example, the domatic partition $\mathcal{P}_0$=$\{\{v_1, v_2, v_3\}, \{v_4, v_5, v_6\}\}$ of the graph H_0 in Figure 1 is b-domatic but there is no vertex of H_0 which satisfies the sufficient condition of Theorem 7 for $\mathcal{P}_0$. Remark that, as $\mathcal{P}_0$ is a b-domatic partition of H_0 of cardinality 2, the lower bound of Proposition 2 implies that $bd(H_0) = 2$.

We next show that for any integer $k \geq 6$, there exists a graph G_k of order k that contains H_0 as an induced subgraph and has b-domatic number equal to 2. Recall that, as proved in [9], if $G = (V, E)$ is a graph with no isolated vertices, then the complement $V \setminus S$ of every minimal dominating set S is a dominating set.

Theorem 8. *For any integer $k \geq 6$, there exists a graph G_k of order k containing H_0 as an induced subgraph, such that $bd(G_k) = 2$.*

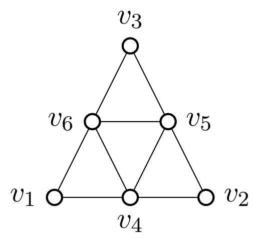

Figure 1. Graph H_0.

Proof. Let $v_1, v_2, v_3, v_4, v_5, v_6$ be the vertices of H_0 as shown in Figure 1. Let $V(H_0) = A_1 \cup A_2$ such that $A_1 = \{v_1, v_2, v_3\}$ and $A_2 = \{v_4, v_5, v_6\}$. It is not difficult to see that $\{A_1, A_2\}$ is a domatic partition of H_0. Let G_k ($k \geq 6$) be a graph of order k, having no isolated vertices and with vertex-set $V(G_k) = A_1 \cup A_2 \cup A_3$ where $G[A_1 \cup A_2] = H_0$ and A_3 is a set of extra vertices (may be empty) such that there is no edge between A_1 and A_3. Note that if $k = 6$, then A_3 is an empty set and therefore $G_6 = H_0$. By the remark before the Theorem 8, $bd(G_6) = 2$. Assume now that $k \geq 7$, so $|A_3| \geq 1$. Let $H_1 = G[A_3]$, and let S be a minimal dominating set of H_1. Then, as proved in [9], $A_3 \setminus S$ is a dominating set of H_1 implying that $\{S, A_3 \setminus S\}$ is a domatic partition of H_1. Let $U_1 = A_1 \cup S$ and $U_2 = A_2 \cup (A_3 \setminus S)$. Clearly $V(G_k) = U_1 \cup U_2$, further $\{U_1, U_2\}$ is a domatic partition of G_k. In addition, it is a routine exercise to show that U_1 is a minimal dominating set of G_k. We shall show that $\{U_1, U_2\}$ is a b-domatic partition of G_k. Suppose not. Then there exist two subsets $\pi_i \subseteq U_i$ ($i = 1, 2$) such that $(U_1 \setminus \pi_1) \cup (U_2 \setminus \pi_2) \neq \emptyset$ and $\pi = \{\pi_1, \pi_2, (U_1 \setminus \pi_1) \cup (U_2 \setminus \pi_2)\}$ is a domatic partition of G_k. Let $\pi_3 = (U_1 \setminus \pi_1) \cup (U_2 \setminus \pi_2)$. Since U_1 is a minimal dominating set of G_k, then no vertex of U_1 can be in π_3, so $\pi_1 = U_1$. Likewise, no vertex of A_2 belongs to π_3 because if not, there is a vertex in A_1 ($\subset \pi_1$) that is not dominated by π_3 (or by π_2), a contradiction. For example, if v_4 is the only vertex of A_2 in π_3, then v_3 will have no neighbors in π_3, and if A_2 has at least two vertices, say v_4, v_5 in π_3, then v_2 will have no neighbors in π_2; in each case, we have a contradiction with the fact that π is a domatic partition of G_k. Thus, no vertex of A_2 can be in π_3, which means that $A_2 \subset \pi_2$. Therefore, since there is no edge between A_1 and A_3, no vertex of A_1 is dominated by π_3, this contradicts again that π is a domatic partition of G_k. Thus $\{U_1, U_2\}$ is a b-domatic partition of G_k, and so $bd(G_k) \leq 2$. The lower bound of Proposition 2 implies that $bd(G_k) = 2$. ■

We now give other classes of graphs for which the b-domatic number is equal to 2.

Theorem 9. *If G has a vertex such that its neighbors form an independent set, then $bd(G) = 2$.*

Proof. Let H be a connected component of G (possibly $H = G$). Let v be a vertex of H such that $N_H(v)$ is an independent set. Let U_1 be the set of vertices of H whose distance from v is even, and let U_2 be the set of vertices of H whose distance from v is odd. Observe that $v \in U_1$ and each neighbor of v is in U_2. Clearly, $\{U_1, U_2\}$ is a domatic partition of H. In view of Theorem 7, $\{U_1, U_2\}$ is a b-domatic partition of H because each neighbor of v is isolated in U_2 and v is isolated in U_1. Therefore $bd(H) \leq 2$, and so Theorem 4 and Proposition 2 yield $bd(G) = 2$. ■

Corollary 10. *If G is triangle-free, then $bd(G) = 2$.*

Consider a graph H with vertex-set $V(H)$. For any permutation π of $V(H)$, the *prism* of H with respect to π is the graph obtained by taking two disjoint copies of H, denoted by H_1 and H_2, and joining every $u \in V(H_1)$ with $\pi(u) \in V(H_2)$. The complementary prism of H is the graph formed from the disjoint union of H and its complementary graph $\overline{H}$ by adding the edges of a perfect matching between the corresponding vertices of H and $\overline{H}$.

Proposition 11. *Let H be a graph. If G is a prism of H or a complementary prism of H, then $bd(G) = 2$.*

Proof. Let H_1, H_2 be two disjoint copies of H and $\mathcal{P} = \{V(H_1), V(H_2)\}$ be a partition of $V(G)$. It is not difficult to see that $\mathcal{P}$ is a domatic partition of G, further, for $i = 1, 2$, each vertex of $V(H_i)$ has a private neighbor with respect to $V(H_i)$. Therefore, in view of Theorem 7, $\mathcal{P}$ is b-domatic of G, and hence by Proposition 2, $bd(G) = 2$. The same proof still holds if G is the complementary prism of H, by substituting H_2 with the complementary graph $\overline{H}$. ■

Theorem 12. *Let $G = (V, E)$ be an r-regular graph and $\mu = \max\{|S_v| : v \in V(G)$ and S_v is a maximum independent set in $G[N(v)]\}$. If $d(G) = r + 1$, then $bd(G) \leq r - \mu + 2$.*

Proof. Let $\mathcal{P} = \{U_1, \ldots, U_{r+1}\}$ be a domatic partition of G of cardinality $r + 1$. We can easily observe that for $i \in \{1, \ldots, r + 1\}$,

(2) U_i is an independent set of G, and each vertex in U_i has exactly one neighbor in each other class $U_j, j \neq i$.

Let v be a vertex of G such that $\mu = |S_v|$. Clearly $r \geq \mu \geq 1$. Let $v_1, \ldots, v_r$ be the neighbors of v. By (2), we may assume that $v \in U_1$ and $v_i \in U_{i+1}$ for each $i \in \{1, \ldots, r\}$. Without loss of generality, assume also that $S_v = \{v_1, \ldots, v_\mu\}$. Set $q = r - \mu + 2$ and let $\pi = \{\pi_1, \ldots, \pi_q\}$ be a partition of G of cardinality q obtained from $\mathcal{P}$ as follows. $\pi_1 = \{v\} \cup \big(\big(\bigcup_{i=1}^{\mu} U_{i+1} \big) \setminus S_v \big)$, $\pi_2 = S_v \cup (U_1 \setminus \{v\})$ and $\pi_i = U_{i+\mu-1}$ for $i \in \{3, \ldots, q\}$. Now, we shall show that π is a domatic partition of

G. For $i \in \{3, \ldots, q\}, \pi_i$ is a dominating set of G because $U_{i+\mu-1}$ is a dominating set of G. So, it suffices to show that π_1 and π_2 are dominating sets of G. Observe first that (2) implies that each vertex in $U_{i+1} \setminus \{v_i\}$, $i = 1, \ldots, \mu$, has at least one neighbor in $U_1 \setminus \{v\}$ and vice versa. Therefore, each vertex of π_2 is dominated by π_1 and vice versa. On the other hand, as S_v is a maximum independent set of $G[N_G(v)]$, each vertex in $\{v_{\mu+1}, \ldots, v_r\}$ has at least one neighbor in S_v and so in π_2. Notice that v has no neighbor in $\bigcup_{i=1}^{r}(U_{i+1} \setminus \{v_i\})$. Therefore, since U_1 is a dominating set of G, each vertex in $U_{i+1} \setminus \{v_i\}$ $(i \geq 1)$ has at least one neighbor in $U_1 \setminus \{v\}$ and so in π_2. Hence, each vertex in π_i, $3 \leq i \leq q$ has a neighbor in π_2. This means that π_2 is a dominating set of G. Thus, it remains to show that π_1 is a dominating set of G. To this end, we show that each vertex $u \in U_{j+1}$, $(j \geq \mu + 1)$ is dominated by π_1. Remember that v_j is the neighbor of v in U_{j+1}. Clearly, if $u = v_j$ $(j \geq \mu + 1)$, then u is adjacent to v and so u is dominated by π_1. Suppose now that $u \neq v_j$. Then u cannot be adjacent to all vertices of S_v, otherwise, since u and v_j are in the same class U_{j+1}, the second part of Observation 2 implies that v_j cannot be adjacent to any vertex of S_v and so $S_v \cup \{v_j\}$ is an independent set of $G[N_G(v)]$, a contradiction. Hence u is non-adjacent to at least one vertex in S_v. Therefore, u must be adjacent to at least one vertex of $\bigcup_{i=1}^{\mu}(U_{i+1} \setminus \{v_i\})$ and hence u is dominated by π_1. Thus each vertex of $\bigcup_{i=3}^{q} \pi_i$ is dominated by π_1. Hence, π_1 is a dominating set of G. Consequently, π is a domatic partition of cardinality q for which each vertex of $N_G[v]$ is isolated in its class. Therefore, Theorem 7 implies that π is b-domatic of G, which means that $bd(G) \leq q = r - \mu + 2$. ■

This bound is achieved, for example, by a complete bipartite graph minus a perfect matching G of order $2p$, with partite sets of the same size p, in which $\delta(G) = \mu = p - 1$ and $d(G) = p$, while, by Theorem 9, $bd(G) = 2$.

Theorem 13. *Let r be a positive integer and let G be a r-regular graph. Then $bd(G) = r + 1$ if and only if $G = pK_{r+1}$ for some positive integer p.*

Proof. Using Theorem 4 and Proposition 6, we can easily verify that the statement is true when $G = pK_{r+1}$. So, let us prove the converse. As $bd(G) = r + 1 = d(G)$, Theorem 12 implies that $r + 1 \leq r - \mu + 2$. So, since $\mu \geq 1$, it follows that $r + 1 \leq r - \mu + 2 \leq r + 1$. Hence $\mu = 1$ implying that the neighborhood of any vertex of G induces a complete subgraph. This means that G is the union of $p \geq 1$ copies of complete graphs of order $r + 1$. ■

A vertex v in a graph G is *universal* if it is adjacent to every other vertex in G. Recall that if G has no universal vertex, then $\gamma(G) \geq 2$. So, the next result follows immediately by applying Propositions 1 and 2.

Observation 14. *If G is a graph of order n without universal vertices, then $bd(G) \leq \frac{n}{2}$.*

This bound is achieved, for example, by $(n-2)$-regular graphs of order n.

Proposition 15. *If G is an $(n-2)$-regular graph of order n, then $bd(G)=\frac{n}{2}$.*

Proof. Let G be an r-regular graph of order $n=r+2$. Obviously, n is even, every vertex of G has exactly $n-2$ neighbors and one non-neighbor, and G is without isolated vertices. Hence, according to the Observation 14, we have $bd(G)\leq\frac{n}{2}$. Suppose to the contrary that $bd(G)=k<\frac{n}{2}$, and consider a b-domatic partition $\mathcal{P}=\{U_1,\ldots,U_k\}$ of G of cardinality k. Therefore, since any set of two vertices of G dominates G, there are two classes U_i, U_j $(i\neq j)$ of $\mathcal{P}$ such that both together contain at least 6 vertices. In this case, we can split $U_i\cup U_j$ into three dominating sets U_i',U_j',U_{k+1} each of them of size at least two such that $U_i'\subseteq U_i$, $U_j'\subseteq U_j$ and $U_{k+1}\subseteq U_i\cup U_j$. It is easy to check that $\left(\mathcal{P}\setminus\{U_i,U_j\}\right)\cup\left(\{U_i',U_j',U_{k+1}\}\right)$ is a domatic partition of G of cardinality $k+1$, a contradiction. So $bd(G)=\frac{n}{2}$. ■

It was shown in [4] that if v is a universal vertex, then $d(G)=d(G\setminus v)+1$. We give here a similar result for the b-domatic number.

Proposition 16. *If v is a universal vertex in G, then $bd(G)=bd(G\setminus v)+1$.*

Proof. Let v be a universal vertex in G. Set $k=bd(G\setminus v)$ and let $\{U_1,\ldots,U_k\}$ be a b-domatic partition of $G\setminus v$. Clearly, $\{U_1,\ldots,U_k,\{v\}\}$ is a b-domatic partition of G, so $bd(G)\leq bd(G\setminus v)+1$. Now set $t=bd(G)$ and let $\{\pi_1,\ldots,\pi_t\}$ be a b-domatic partition of G. Assume that $v\in\pi_1$. Observe that $\{(\pi_1\cup\pi_2)\setminus\{v\},\pi_3,\ldots,\pi_t\}$ is a b-domatic partition of $G\setminus v$. Thus $bd(G\setminus v)\leq bd(G)-1$ which gives the desired result. ■

A *threshold graph* is a graph that can be constructed from the one-vertex graph by repeatedly adding an isolated vertex or a universal vertex. As a consequence of Proposition 16, we determine the b-domatic number of threshold graph and its complementary graph.

Corollary 17. *Let G_n be a threshold graph of order n. Then*

$$bd(G_1)=1 \text{ and for } n\geq 2,\ bd(G_n)=1+\sum_{j=2}^{n}\alpha_n\cdots\alpha_j,$$

$$\text{where } \alpha_j=\begin{cases}1 & \text{if the added vertex to } G_{j-1} \text{ is an universal vertex,}\\ 0 & \text{if the added vertex to } G_{j-1} \text{ is an isolated vertex.}\end{cases}$$

Proof. Proposition 2 and Proposition 16 imply that $bd(G_1)=1$ and $bd(G_n)=\alpha_n\cdot bd(G_{n-1})+1$ for $n\geq 2$. This recurrence relation admits the unique solution $bd(G_n)=1+\sum_{j=2}^{n}\alpha_n\cdots\alpha_j$. ■

Corollary 18. *If $\overline{G_n}$ is the complementary graph of a threshold graph G_n, then*

$$bd(\overline{G_n}) = 1 + \sum_{j=2}^{n}(1-\alpha_n)\cdots(1-\alpha_j).$$

A *block* of a graph is a maximal connected subgraph that has no cut-vertex. A block is trivial if it has only one edge. A *block graph* is a connected graph in which each block induces a complete subgraph. In the next theorem, we determine the b-domatic number of any block graph G such that each of its blocks contains at least one vertex that is non-cut for G.

Theorem 19. *Let G be a block graph and $B_1, \ldots, B_r$ $(r \geq 2)$ be the blocks of G. For $i \in \{1, \ldots, r\}$, let $|V(B_i)| = n_i$, and let k_i denote the number of cut vertices in B_i. If $l = \min\{n_i - k_i : 1 \leq i \leq r\} \geq 1$, then $bd(G) = l + 1$.*

Proof. Let $r \geq 2$. For $i \in \{1, \ldots, r\}$, denote by δ_i the minimum degree in B_i and let $l_i = n_i - k_i$. As $l = \min\{l_i : 1 \leq i \leq r\} \geq 1$, it follows that $\delta = \min \delta_i \geq 1$ and $1 \leq l_i \leq \delta_i$. Thus, $l \leq \delta_i$ and in particular, we have

$$l \leq \delta. \tag{3}$$

If $r = 2$, then G has exactly one cut vertex, say w. Hence,

$$l_i = n_i - 1 = \delta_i \text{ for each } i \text{ in } \{1, 2\}. \tag{4}$$

Observe that w is a universal vertex in G, and $G \setminus w = K_{\delta_1} \cup K_{\delta_2}$ is the union of two complete subgraphs of G. Hence, by Theorem 4, $bd(G \setminus w) = \min\{bd(K_{\delta_1}), bd(K_{\delta_2})\} = \min\{\delta_1, \delta_2\}$, and by (4), we get $bd(G \setminus w) = \min\{l_1, l_2\} = l$. Therefore, Proposition 16 implies that $bd(G) = bd(G \setminus w) + 1 = l + 1$. Hence, the statement is true. Assume now that $r \geq 3$. Denote by

$$V(B_i) = \left\{v_1^i, v_2^i, \ldots, v_{l_i}^i, u_1^i, u_2^i, \ldots, u_{k_i}^i\right\},$$

the set of the vertices of the block B_i such that $v_1^i, v_2^i, \ldots, v_{l_i}^i$ and $u_1^i, u_2^i, \ldots, u_{k_i}^i$ are respectively the non-cut vertices and the cut vertices of G in B_i.

We first show that $bd(G) \geq l+1$. Let $k = bd(G)$ and suppose to the contrary that $k \leq l$. Hence, by (3), we obtain

$$k \leq l \leq \delta. \tag{5}$$

Let $\mathcal{P} = \{U_1, U_2, \ldots, U_k\}$ be a b-domatic partition of G of cardinality k. Notice that $|B_i| = \delta_i + 1 \geq \delta + 1$ for each $i \in \{1, \ldots, r\}$; so by (5), we have

$$|B_i| \geq l + 1 \geq k + 1. \tag{6}$$

As each block B_i contains at least l vertices that are non-cut for G, (6) implies that for each $i \in \{1, \ldots, r\}$ there is a class of $\mathcal{P}$ that intersects B_i in at least two vertices such that at least one of them is a non-cut-vertex, say v_1^i. Let $X = \{v_1^1, v_1^2, \ldots, v_1^r\}$ and consider a partition π of $V(G)$ of cardinality $k+1$ obtained from $\mathcal{P}$ by collecting vertices $v_1^1, v_1^2, \ldots, v_1^r$ of some classes of $\mathcal{P}$ to form a new class. Partition π with classes $\pi_1, \pi_2, \ldots, \pi_k, \pi_{k+1}$ is constructed as follows. For each $i \in \{1, \ldots, r\}$, let $\pi_i = U_i \setminus X_i$ where $X_i = X \cap U_i$ (X_i may be empty for some integers i), and $\pi_{k+1} = X$. It is a routine exercise to verify that $\pi = \{\pi_1, \pi_2, \ldots, \pi_k, \pi_{k+1}\}$ is a domatic partition of G, contradicting the assumption that $\mathcal{P}$ is a b-domatic partition of G. Thus

$$bd(G) \geq l + 1. \tag{7}$$

Now, we shall show that $bd(G) = k \leq l + 1$. When $l = \delta$, the last inequality is clearly true, and therefore by (7), we have $k = l + 1 = \delta + 1$. Now, assume that $l \leq \delta - 1$, and suppose without loss of generality that B_1 contains the smallest number of non-cut vertices in G. Then

$$l_1 = l \text{ and so } V(B_1) = \{v_1^1, v_2^1, \ldots, v_l^1, u_1^1, u_2^1, \ldots, u_{k_1}^1\}, \tag{8}$$

where $u_1^1, u_2^1, \ldots, u_{k_1}^1$ are the cut vertices of B_1. It is known that a vertex is a cut vertex if and only if it belongs to at least two blocks. Hence, without loss of generality, we may suppose that

$$\text{for } i \in \{1, \ldots, k_1\}, u_i^1 \in V(B_1) \cap V(B_{i+1}).$$

Let $s \geq 0$ be the number of blocks of G, that do not intersect B_1. Clearly $s < r$ and, if $s = 0$, each block B_j ($j \neq 1$) of G intersect B_1. If $s \geq 1$ we may suppose, without loss of generality, that

$$V(B_1) \cap V(B_j) = \emptyset \text{ for } j \in \{r - s + 1, \ldots, r\}.$$

Let $\mathcal{P} = \{U_1, U_2, \ldots, U_{l+1}\}$ be a partition of G of cardinality $l+1 \leq \delta$ defined according to the value of l as follows.

Case 1. $l \geq 2$.

- If $s \geq 1$, then $U_{l+1} = \{u_i^1 : 1 \leq i \leq k_1\} \cup \{v_1^j : r - s + 1 \leq j \leq r\}$; otherwise $U_{l+1} = \{u_i^1 : 1 \leq i \leq k_1\}$.
- $U_i = \{v_i^j : 1 \leq j \leq r\}$, $2 \leq i \leq l$.
- $U_1 = V(G) \setminus \left(\bigcup_{i=2}^{l+1} U_i\right)$.

Case 2. $l = 1$.

- If $s \geq 1$, then $U_2 = \{u_i^1 : 1 \leq i \leq k_1\} \cup \{v_1^j : r - s + 1 \leq j \leq r\}$; otherwise $U_2 = \{u_i^1 : 1 \leq i \leq k_1\}$.

• $U_1 = V(G) \setminus U_2$.

Remark that in either cases, each class of $\mathcal{P}$ intersect each block of G in at least one vertex. This means that $\mathcal{P}$ is a domatic partition of G. Observe also that for $i \in \{1, \ldots, k_1\}$, v_l^{i+1} is a private neighbor of u_i^1 with respect to U_{l+1}. In addition, each class U_i $(i = 1, \ldots, l)$ has a vertex v_i^1 that is adjacent to no vertex of its own class and to exactly one vertex from each of the classes U_j, $j \in \{1, \ldots, l\} \setminus \{i\}$. So, we conclude that v_1^1 is an isolated vertex in U_1 such that each of its neighbor is either isolated in its class or has a private neighbor with respect to U_{l+1}. Thus, in view of Theorem 7, $\mathcal{P}$ is a b-domatic partition of G, which means that $k \leq l + 1$. Hence, by (7), we get $k = l + 1$. ■

A *cactus graph* is a connected graph in which each block is either an edge or a cycle. A *friendship graph* F_n $(n \geq 2)$ is a cactus graph of order $2n + 1$ in which any two vertices have exactly one common neighbor.

In the following proposition, we prove that the b-domatic number of a cactus graph G in which every block has at least one vertex that is non-cut for G is equal to 2, except for K_3 and F_n $(n \geq 2)$.

Proposition 20. *Let G be a cactus graph such that each block has at least one vertex that is non-cut for G. Then $bd(G) = 2$ unless G is K_3 or F_n $(n \geq 2)$. In these cases, $bd(K_3) = bd(F_n) = 3$.*

Proof. Clearly $\delta(G) \leq 2$. Thus, by Proposition 3, $bd(G) = 2$ when $\delta(G) = 1$. So, assume that $\delta(G) = 2$. If G has a cycle of length at least 4, then G contains a non-cut vertex of degree 2 such that its neighbors form an independent set. Therefore, Theorem 9 yields $bd(G) = 2$. Now, assume that any cycle of G has length 3. Let l be as defined in Theorem 19 and let $r \geq 1$ be the number of blocks of G. If $r = 1$, then $G = K_3$ and so $bd(K_3) = 3$ by Proposition 6. Assume that $r \geq 2$. If $G = F_n$, then $l = 2$ and so $bd(F_n) = 3$ by Theorem 19. Otherwise $l = 1$ implying that $bd(G) = 2$ by Theorem 19 again. ■

4. Conclusion

In this paper, we have formulated and proved a sufficient condition for a given domatic partition of a graph to be b-domatic, however, we have shown that the converse is not true. Therefore, the necessary condition remains still an open problem.

We have also presented some infinite classes of graphs having b-domatic number equal to two and $\delta + 1$. In particular, we have determined the b-domatic number of block graph (cactus graph) G in which every block has at least one vertex that is non-cut for G. So it would be interesting to determine the b-domatic number for cacti and block graphs that contain at least one block whose vertices are all cut-vertices for G.

References

[1] S. Arumugam and K. Raja Chandrasekar, *Minimal dominating sets in maximum domatic partitions*, Australas. J. Combin. **52** (2012) 281–292.

[2] C. Berge, Graphs (North Holland, 1985).

[3] E.J. Cockayne and S.T. Hedetniemi, *Towards a theory of domination in graphs*, Networks **7** (1977) 247–261.
doi:10.1002/net.3230070305

[4] G.J. Chang, *The domatic number problem*, Discrete Math. **125** (1994) 115–122.
doi:10.1016/0012-365X(94)90151-1

[5] O. Favaron, *The b-domatic number of a graph*, Discuss. Math. Graph Theory **33** (2013) 747–757.
doi:10.7151/dmgt.1709

[6] F. Harary and S. Hedetniemi, *The achromatic number of a graph*, J. Combin. Theory **8** (1970) 154–161.
doi:10.1016/S0021-9800(70)80072-2

[7] R.W. Irving and D.F. Manlove, *The b-chromatic number of graphs*, Discrete Appl. Math. **91** (1999) 127–141.
doi:10.1016/S0166-218X(98)00146-2

[8] D.F. Manlove, Minimaximal and Maximinimal Optimization Problems: A Partial Order-Based Approach (Ph.D. Thesis, Tech. Rep. 27, Comp. Sci. Dept. Univ. Glasgow, Scotland, 1998).

[9] O. Ore, Theory of Graphs (Amer. Math. Soc, Providence, RI, 1962).

[10] S.H. Poon, W.C.K. Yen and C.T. Ung, *Domatic partition on several classes of graphs*, Combin. Optim Appl., Lecture Notes in Comput. Sci. **7402** (2012) 245–256.
doi:10.1007/978-3-642-31770-5_22

Received 1 February 2017
Revised 17 August 2017
Accepted 17 August 2017