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ANTI-RAMSEY NUMBER OF HANOI GRAPHS

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Abstract

Let $\mathrm{ar}(G, H)$ be the largest number of colors such that there exists an edge coloring of G with $\mathrm{ar}(G, H)$ colors such that each subgraph isomorphic to H has at least two edges in the same color. We call $\mathrm{ar}(G, H)$ the *anti-Ramsey number* for a pair of graphs (G, H). This notion was introduced by Erdős, Simonovits and Sós in 1973 and studied in numerous papers.

Hanoi graphs were introduced by Scorer, Grundy and Smith in 1944 as the model of the well known Tower of Hanoi puzzle.

In the paper we study the anti-Ramsey number of Hanoi graphs and consider them both as the graph G and H. Among others we present the exact value of the anti-Ramsey number in case when both graphs are constructed for the same number of pegs.

Keywords: anti-Ramsey number, rainbow number, Hanoi graph.

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1. Introduction

The graphs considered below will always be simple. Throughout the paper we use the standard graph theory notation (see, e.g., [8]). In particular, K_n is a complete graph on n vertices. The graph K_3 redis called a triangle. Let t be a positive integer and F be a graph. By the symbol tF we mean a graph consisting of t disjoint copies of the graph F. For two graphs F and G by $F \cup G$ we denote a sum of graphs, i.e., a graph with a vertex set $V(F) \cup V(G)$ and an edge set $E(F) \cup E(G)$. For a set S by $|S|$ we denote the cardinality of S and $S(n, k)$ is the Stirling number of the second kind. In the following subsections we give only a short overview on both anti-Ramsey numbers and Hanoi graphs. We provide also references for an interested reader.

1.1. Anti-Ramsey numbers

A subgraph of an edge-coloured graph is called *rainbow* if all of its edges have different colors. For graphs G and H the *anti-Ramsey number* $\mathrm{ar}(G,H)$ is the maximum number of colors which can be used in such an edge-colouring of G which avoids any rainbow copy of H. It means, equivalently, that in each edge-coloring of G with $\mathrm{ar}(G,H)+1$ colors a rainbow copy of H must appear. Anti-Ramsey numbers were introduced by Erdős *et al.* [9] and considered there in a classical case when $G = K_n$. Since then numerous results were established for a variety of graphs H, including among others, cycles [1, 19, 24], matchings [10, 13, 27] and trees [18, 20]. Later on different graphs were considered as a graph G, for instance bipartite graphs [4, 5, 23], hypercubes and a product of cycles [25] or complete split graphs [12]. Apart from a fixed graph, a set of triangulations played role of G [17, 21]. The paper of Fujita, Magnant and Ozeki [11] presents a survey of results in classical and nonclassical cases. From our point of view the results of Axenovich *et al.* [3] and Bode *et al.* [6] are the most similar in spirit to ours. In the mentioned papers host graphs and the graphs which we would like to avoid as a rainbow copy belong to the same class, namely hypercubes. Hanoi graphs, which we describe in the next paragraph, have a similar property. Larger Hanoi graphs can be recursively constructed from the smaller ones. Their fractal structure suits very well to the anti-Ramsey topic.

1.2. Hanoi graphs

Let n and p be integers. The famous Tower of Hanoi puzzle consists of $p \geq 1$ pegs and $n \geq 0$ discs of pairwise different diameters. At the beginning all discs are placed on the first peg ordered from the largest on the bottom to the smallest on the top. We call such a position a perfect state. The goal is to move all discs to the last peg and place them in the perfect state in such a way that in each move we can change the position of exactly one disc following the *divine rule*: one must not place a disc on a smaller one. Such a move is called a legal one. The question is about the minimal number of moves necessary to solve the puzzle. It is well-known that for the original puzzle with three pegs it is $2^n - 1$, where n is the number of discs. Later on the puzzle was generalized to four (Reve's puzzle), and then to more pegs. Recently Bousch has shown that the Frame-Stewart algorithm is optimal for $p = 4$ [7].

In 1941 Scorer *et al.* [26] proposed a graph model of the Tower of Hanoi puzzle for the original puzzle with three pegs. It was generalized around the turn of millenium when the puzzle with more pegs became more and more popular (see, e.g., [15]). We describe the model in the general case. Firstly we label the pegs $0, 1, \ldots, p-1$ and the discs $1, 2, \ldots, n$. Each state which can be obtained by legal moves we call a regular state. To each regular state we assign a sequence

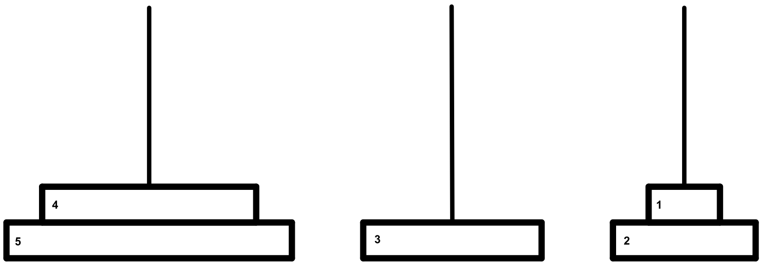


Figure 1. A regular state in Tower of Hanoi.

$(r_1, r_2, \ldots, r_n)$, where r_i is a label of the peg i-th disc is placed on. Each such a sequaence is a vertex of the Hanoi graph. For instance the sequence $(2, 2, 1, 0, 0)$ represents the state in Figure 1. Two vertices are adjacent if there is a legal move between the respective regular states. Considering the state in Figure 1 we have $N((2, 2, 1, 0, 0)) = \{(0, 2, 1, 0, 0), (1, 2, 1, 0, 0), (2, 2, 0, 0, 0)\}$. The vertex set is denoted by V_p^n, the edge set by E_p^n and the Hanoi graph built that way by H_p^n. As we mentioned Hanoi graphs were introduced to model the Tower of Hanoi puzzle and therefore, the length and uniqueness of the shortest path was widely considered, firstly between two perfect states and afterwards between two arbitrarily chosen regular states. We refer the readers interested in this aspect to the book of Hinz *et al.* [16] and the references therein. Since then different properties of Hanoi graphs have been studied which are not strictly connected with solving the puzzle, but rather with the structure of the graph. We quote here only some of them. For instance it has been shown by Arett and Doree [2] that the chromatic number of Hanoi graphs is equal to the number of pegs, by Hinz and Parisse [14] that they are first class graphs, that means that the chromatic index of Hanoi graphs is equal to its maximum degree. Due to Hinz and Parisse [15] we know that all Hanoi graphs are Hamiltonian, but only H_p^0, H_3^n, H_4^1 and H_4^2, $n, p \in \mathbb{N}_0$ are planar. Again we refer to [16] for some other properties, as well as the detailed history of the Tower of Hanoi puzzle and its graph model.

Theorem 1 [16, p. 190 and Proposition 5.23]**.** *Let* $n \geq 0$*,* $p \geq 3$*. Then*

$$| V_p^n |= p^n, \quad | E_p^n |= \frac{1}{2}\binom{p}{2}\left(p^n - (p-2)^n\right).$$

2. Anti-Ramsey Numbers for Hanoi Graphs

As we mentioned in the introduction the structure of Hanoi graphs fits very well to the anti-Ramsey number problem. Below we describe the recursive construction of them according to the number of discs. To avoid trivial situations in the following we always assume that $p \geq 3$ and $n \geq 1$. At the beginning, it is easy to observe that if we have only one disc then all moves are legal and therefore H_p^1 is isomorphic to K_p. Now consider the tower with $n-1$ discs and assume that we add the n-th disc (the largest). We can place it on any peg and then repeat any state from the smaller tower. It is equivalent that we can add any label from $\{0, 1, \ldots, p-1\}$ at the end of any sequence $(r_1, r_2, \ldots, r_{n-1})$ forming a sequence $(r_1, r_2, \ldots, r_n)$. Note that all moves which were legal in a smaller tower are also legal in that with n-th disc. Hence H_p^n consists of p copies of H_p^{n-1}. We add edges among these copies to represent the legal moves of the largest disc. See e.g. Figure 2 to see the construction of H_3^2 and H_4^2.

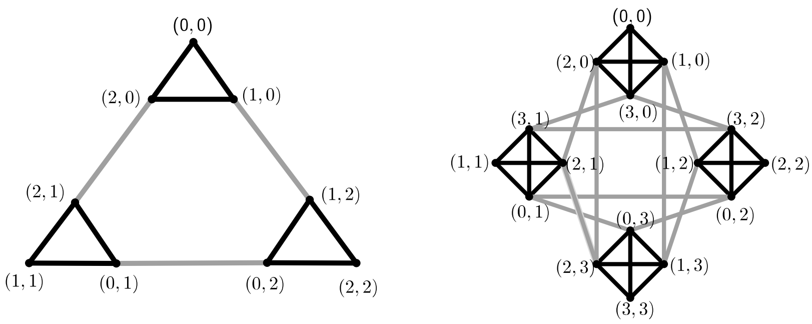


Figure 2. Construction of H_3^2 and H_4^2.

2.1. The exact value of $\mathrm{ar}\big(H_p^n, H_p^m\big)$

The recursive construction of Hanoi graphs allows us to fix p and consider $\mathrm{ar}(H_p^n, H_p^m)$. It is quite natural from the puzzle point of view when we have a fixed number of pegs and can add discs. It occurs that in this case we can easily determine the exact value of the anti-Ramsey number.

Theorem 2. *For a pair of Hanoi graphs H_p^n and H_p^m with $n \geq m > 0$, $p \geq 3$ we have*

$$\mathrm{ar}(H_p^n, H_p^m) = \frac{1}{2}\binom{p}{2}\left(p^n - (p-2)^n\right) - p^{n-m}.$$

Proof. Firstly, by the recursive construction, we observe that the graph H_p^n contains p^{n-m} copies of H_p^m. We construct the edge-coloring of H_p^n as follows. In each copy of H_p^m we color two arbitrarly chosen edges with the same color, but we use different colors in different copies. To each of remaining edges we use a new color. In such a coloring we do not obtain any rainbow H_p^m and we use

$\big(| E_p^m | -1\big) p^{n-m}$ colors on the edges of the copies H_p^m and $| E_p^n | -p^{n-m} | E_p^m |$ colors on the edges linking them. Hence, altogether we use

$$\big(| E_p^m | -1\big) p^{n-m} + | E_p^n | -p^{n-m} | E_p^m | = | E_p^n | -p^{n-m}$$

colors. So

$$\mathrm{ar}\big(H_p^n, H_p^m\big) \geq | E_p^n | -p^{n-m}.$$

Let us consider an arbitrary edge-coloring of H_p^n with $| E_p^n | -p^{n-m}+1$ colors. Note that we can use at most $| E_p^n | -p^{n-m} | E_p^m |$ colors to color edges linking copies H_p^m. So we use at least

$$| E_p^n | -p^{n-m} - | E_p^n | +p^{n-m} | E_p^m | +1 = p^{n-m} \big(| E_p^m | -1\big) + 1$$

colors which we use on the copies of H_p^m. Hence, by a strong version of the pigeonhole principle, we use

$$\left\lceil \frac{p^{n-m} \big(| E_p^m | -1\big) + 1}{p^{n-m}} \right\rceil = | E_p^m |$$

colors on at least one of these copies. This copy is rainbow, which by Theorem 1 completes the proof. ∎

2.2. Towards $\mathrm{ar}\big(H_p^n, H_q^m\big)$ for $p \geq q$ and $n \geq m$

Let us forget about the strict connection between Hanoi graphs and the Tower of Hanoi puzzle and look only on their stucture. Then we can change the number of pegs. Let us fix the number of discs n for a moment. It is not difficult to notice that H_q^n is a subgraph of H_p^n for $q \leq p$. It is enough to consider only vertices (and edges among them) which avoid some labels. It means that the pegs with these labels remain empty and can be ignored. In Figure 3 we see H_3^2 contained in H_4^2. Its vertices are marked with diamonds and the edges with dashed lines. Unfortunately not all copies of H_3^2 are obtained that way. We can for instance replace the vertex $(1,1)$ with the edges $\{(1,1),(0,1)\}$, $\{(1,1),(3,1)\}$ by the vertex $(2,1)$ with the edges $\{(2,1),(0,1)\}$, $\{(2,1),(3,1)\}$ to obtain another copy of H_3^2, but now all labels of pegs are present. Recalling a previous subsection we also note that H_p^n contains H_q^m for $p \geq q$ and $n \geq m$, so considering $\mathrm{ar}\big(H_p^n, H_q^m\big)$ is also a reasonable task but far more challenging. Below we make the first step in that direction.

Examining the structure of Hanoi graphs in more detailed way we can determine $\mathrm{ar}\big(H_p^n, H_3^1\big)$.

Lemma 3. *Let $n \geq 2$, $p \geq 3$ be integers. The edges of Hanoi graph H_p^n obtained by the moves of the largest disc induce a union of edge disjoint complete graphs, which can be expressed as $\bigcup_{s=1}^{p-2} \binom{p}{s} s! S(n-1,s) K_{p-s}$.*

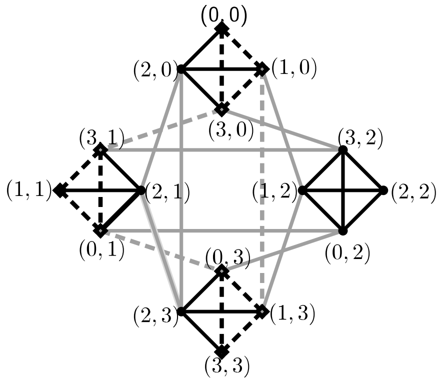


Figure 3. H_3^2 in H_4^2.

Proof. Consider the legal move of the largest disc from the vertex $(r_1, \ldots, r_{n-1}, i_1)$ to the vertex $(r_1, \ldots, r_{n-1}, i_2)$. It is possible only in case when none of $r_1, \ldots, r_{n-1}$ is equal either to i_1 or to i_2. It means that in the state $(r_1, \ldots, r_{n-1}, i_1)$ the peg i_2 is empty and reversely. Assume now that none of $r_1, \ldots, r_{n-1}$ is equal to any of $i_1, \ldots, i_s$, $s \geq 2$, and consider the vertices $(r_1, \ldots, r_{n-1}, i_j)$, where $j \in \{1, \ldots, s\}$ and $(r_1, \ldots, r_{n-1})$ is fixed. The largest disc can be moved to any of $(r_1, \ldots, r_{n-1}, i_k)$, where $k \neq j$ and $k \in \{1, \ldots, s\}$. It means that edges among these vertices form a complete graph K_s. It is easy to note that for different choices of indices $i_1, \ldots, i_s$ we obtain vertex disjoint copies of K_s. It is not difficult to observe that for a fixed choice of $i_1, \ldots, i_s$ we have $(p-s)!S(n-1, p-s)$ different sequences $(r_1, \ldots, r_{n-1})$. Hence we have $\binom{p}{s}(p-s)!S(n-1, p-s)$ copies of vertex disjoint K_s. It is obvious that $2 \leq s \leq p-1$. Therefore the graph induced by the moves of the largest disc can be described as $\bigcup_{s=2}^{p-1} \binom{p}{s}(p-s)!S(n-1, p-s)K_s$. Since $\binom{p}{s} = \binom{p}{p-s}$ we can also express it as $\bigcup_{s=1}^{p-2} \binom{p}{s} s!S(n-1, s)K_{p-s}$. ■

Remark 4. Using Lemma 3 we can express the number of edges of Hanoi graphs in a recursive way as follows.

$$\left|E\left(H_p^n\right)\right| = p\left|E\left(H_p^{n-1}\right)\right| + \sum_{s=1}^{p-2} \binom{p}{s} s!S(n-1, s)\binom{p-s}{2}.$$

It is shown in [22] that $\left|E\left(H_p^n\right)\right| = p\left|E\left(H_p^{n-1}\right)\right| + \binom{p}{2}(p-2)^{n-1}$.

Hence we obtain the following formula

$$\binom{p}{2}(p-2)^{n-1} = \sum_{s=1}^{p-2} \binom{p}{s} s!S(n-1, s)\binom{p-s}{2}.$$

For a similar formula expressing the number of edges of the whole Hanoi graph using Stirling numbers see also [22] and [16, Exercise 5.6].

We will also apply the theorem of Erdős, Simonovits and Sós about the anti-Ramsey number of a triangle in complete graphs.

Theorem 5 [9]**.** *Let* $m \geq 3$*. Then* $\operatorname{ar}(K_m, K_3) = m - 1$.

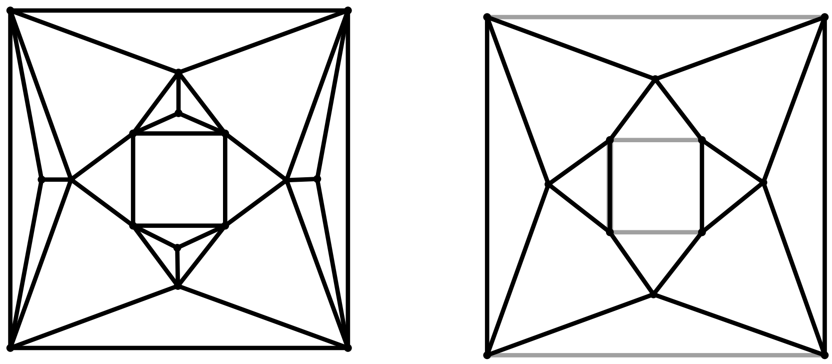
Now we are able to give a recursion formula for $\operatorname{ar}(H_p^n, H_3^1)$.

Theorem 6. *Let* $n \geq 2$ *and* $p \geq 3$ *be integers. Then*

$$\operatorname{ar}\left(H_p^n, H_3^1\right) = p \cdot \operatorname{ar}\left(H_p^{n-1}, H_3^1\right) + \sum_{s=1}^{p-2} \binom{p}{s} s! S(n-1, s)(p-s-1).$$

Proof. At the begining we remind the reader that H_3^1 is a triangle and the H_p^1 is a K_p. Hence, by Theorem 5, $\operatorname{ar}(H_p^1, H_3^1) = \operatorname{ar}(K_p, K_3) = p - 1$ so H_p^1 can be colored with $p-1$ colors without any rainbow triangle. We construct recursively the edge-coloring of H_p^n as follows. We color the edges for each of the p copies of H_p^{n-1} with $\operatorname{ar}(H_p^{n-1}, H_3^1)$ different colors without rainbow triangle (cf. recursive construction of Hanoi graphs). Now, by Lemma 3 and Theorem 5, to each of the $\binom{p}{s} s! S(n-1, s)$ copies of K_{p-s} we use $\operatorname{ar}(K_{p-s}, K_3) = p - s - 1$ new colors avoiding rainbow triangle (in case $s = p - 2$ the K_{p-s} is an edge which is not contained in any triangle, so each of these K_2s can contribute a new color to the coloring). Similarly to the proof of Theorem 2 we can conclude that we cannot use more then $p \cdot \operatorname{ar}(H_p^{n-1}, H_3^1) + \sum_{s=1}^{p-2} \binom{p}{s} s! S(n-1, s)(p-s-1)$ colors on the edges of H_p^n without producing a rainbow triangle. Assume we use one color more. When we partition $\sum_{s=1}^{p-2} \binom{p}{s} s! S(n-1, s)(p-s-1) + 1$ colors on the edges among copies of H_p^{n-1} then on at least one of complete graphs K_{p-s} we use more then $\operatorname{ar}(K_{p-s}, K_3)$ colors so a rainbow triangle appears. If we use $p \cdot \operatorname{ar}\left(H_p^{n-1}, H_3^1\right) + 1$ on the copies of H_p^{n-1} then on at least one of them we use $\operatorname{ar}\left(H_p^{n-1}, H_3^1\right) + 1$ colors so we obtain a rainbow triangle as well. A contradiction. ∎

It is not so easy to determine the exact value of the anti-Ramsey number for Hanoi graphs with different number of pegs even if the number of discs is equal, but greater than one. Below we consider the smallest such a case and show that $29 \leq \operatorname{ar}(H_4^2, H_3^2) < 34$. We need the following lemma first. Note that the graph H_3^2 contains three vertex disjoint triangles. Moreover each pair of these triangles is joined by exactly one edge and these three edges form a matching. We say that these three triangles are joined cyclically (cf. Figure 2). Additionally, as we mentioned, the graph H_4^2 is planar. It is easy to check that Figure 4(a) presents the graph H_4^2 as a plane graph.



(a) H_4^2 as a plane graph (b) A subgraph G of H_4^2 and its coloring

Figure 4. The graph H_4^2 and its subgraph G.

Lemma 7. *Let G be a subgraph of H_4^2 presented in Figure* 4(b) *obtained from H_4^2 by deleting the vertices (i,i), $i = 1,2,3,4$. Then*

$$\operatorname{ar}\left(G, H_3^2\right) = 21.$$

Proof. In Figure 4(b) grey edges represent the same color, all remaining ones form a rainbow subgraph without grey edges. There are 21 colors used in this coloring. Note that at most two triangles without grey edges can be chosen to construct the H_3^2. Hence, exactly one triangle with a grey edge must be chosen for constructing the rainbow H_3^2. For each such a triangle T there is exactly one pair of triangles with "black" edges which form the H_3^2 with T. But both edges joining these triangles are grey. So there is no rainbow H_3^2 in the presented coloring.

Consider now an arbitrary edge-coloring of G with 22 colors. It is easy to note that at least six triangles from among eight must be rainbow. All possible positions of these triangles are presented in Figure 5 and noted by black lines. Two other triangles are represented by grey lines. It is easy to check that there is a rainbow H_3^2 in all three cases. The chosen triangles are marked by crosses and the edges joining them are dashed. It is obvious in the first two cases. In the last one firstly a grey dashed edge was chosen as the edge of color not appearing on these six triangles (there are at least four such edges) and then the rest of H_3^2 was constructed. ■

Note that we need almost as many colors as the number of edges of the graph G to obtain a rainbow copy of H_3^2. It means that although we have eight different copies of H_3^2 it is not so easy to have a rainbow one.

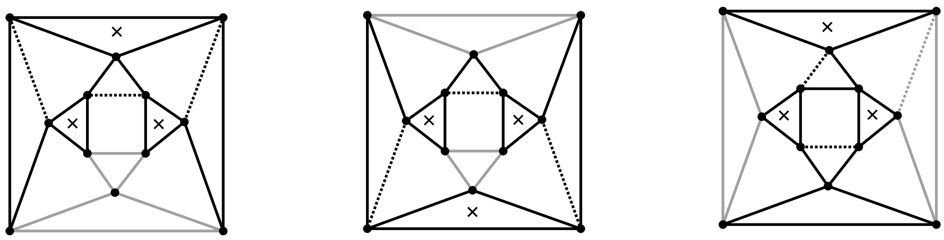


Figure 5. Positions of six rainbow triangles.

Theorem 8. $29 \leq \operatorname{ar}\left(H_4^2, H_3^2\right) < 34$.

Proof. Let us consider the coloring presented in Figure 6. We mean that all grey edges have the same color and the rest of the graph is rainbow without grey edges. So we use exactly 29 colors. To see that there is no rainbow H_3^2 in the presented coloring of H_4^2, consider the latter as consisting of four vertex-disjoint K_3s and four vertex-disjoint K_4s. Note that none of K_3s can be a triangle T of rainbow H_3^2, since (apart from exactly two triangles) any other triangle T' we choose has one of the following properties: T and T' have a vertex in common, T' has a grey edge, the only edge joining T and T' is grey, there is no edge joining T and T'. These two particular triangles are joined by a grey edge. Similar situation appears if we choose any triangle with the grey edge contained in K_4. Finally observe that no three triangles without grey edge are joined cyclically.

Now let us color the edges of H_4^2 with 34 colors. As we can use at most 12 colors on edges of 4-cliques with one end-vertex of degree 3, we must use at least 22 colors on the subgraph G and therefore by Lemma 7 a rainbow H_3^2 exists. ■

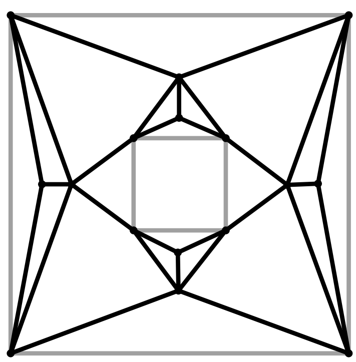


Figure 6. Coloring of H_4^2 without rainbow H_3^2.

We strongly believe that after a careful and complicated case consideration the upper bound might be decreased down to the lower bound. We also hope in possibility of elaborating at least a recursive formula for $\mathrm{ar}(H_p^n, H_q^m)$ for $p \geq q$ and $n \geq m$.

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