DECOMPOSITION OF THE PRODUCT OF CYCLES
BASED ON DEGREE PARTITION

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Abstract

The Cartesian product of $n$ cycles is a $2n$-regular, $2n$-connected and bipan-
cyclic graph. Let $G$ be the Cartesian product of $n$ even cycles and let $2n = n_1 + n_2 + \cdots + n_k$ with $k \geq 2$ and $n_i \geq 2$ for each $i$. We prove that if $k = 2$, then $G$ can be decomposed into two spanning subgraphs $G_1$ and $G_2$ such that each $G_i$ is $n_i$-regular, $n_i$-connected, and bipancyclic or nearly bipancyclic. For $k > 2$, we establish that if all $n_i$ in the partition of $2n$ are even, then $G$ can be decomposed into $k$ spanning subgraphs $G_1, G_2, \ldots, G_k$ such that each $G_i$ is $n_i$-regular and $n_i$-connected. These results are analogous to the corresponding results for hypercubes.

Keywords: hypercube, Cartesian product, $n$-connected, regular, bipan-
cyclic, spanning subgraph.

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1. Introduction

The graphs considered in this paper are simple, undirected and finite. The Cartesian product of two graphs $G_1$ and $G_2$ is the graph $G_1 \square G_2$ with vertex set $V(G_1) \times V(G_2)$ in which $(u_1, u_2)$ is adjacent to $(v_1, v_2)$ if and only if $u_1$ is adjacent to $v_1$ in $G_1$ and $u_2 = v_2$, or $u_2$ is adjacent to $v_2$ in $G_2$ and $u_1 = v_1$. The $n$-dimensional hypercube $Q_n$ is the Cartesian product of $n$ copies of the complete graph $K_2$. Therefore $Q_n$ is the Cartesian product of $n/2$ copies of a cycle of length 4 when $n$ is even. The Cartesian product of cycles and hypercubes are popular interconnection network topologies (see [6, 11]). The hypercube $Q_n$ is
an $n$-regular and $n$-connected graph whereas the Cartesian product of $n$ cycles is a $2n$-regular and $2n$-connected graph (see [16]).

Unless mentioned otherwise, in the remainder of this paper product means the Cartesian product of graphs.

A cycle is even if its length is a positive even integer. A bipartite graph $G$ is bipancyclic if $G$ is either a cycle or has cycles of every even length from 4 to $|V(G)|$. A 3-regular graph is nearly bipancyclic if it has cycles of every even length from 4 to $|V(G)|$ except possibly for 4 and 8. The bipancyclicity property of a given network is an important factor in determining whether the network topology can simulate rings of various lengths.

Alspach et al. [1] proved that the product of cycles can be decomposed into Hamiltonian cycles. This result subsumes earlier results due to Kotzig [10] and Foregger [8] on Hamiltonian decomposition of the product of cycles. El-Zanati and Eynden [7] studied the decomposition of the product of cycles, each of length a power of 2, into non-spanning cycles. Borse et al. [4] proved that if $m \geq 2$ and $m$ divides $n$, then the product of $n$ even cycles can be decomposed into isomorphic, spanning, $m$-regular, $m$-connected subgraphs which are bipancyclic or nearly bipancyclic also. The analogous results for the class of hypercubes are obtained in [1, 4, 7].

Motivated by applications in parallel computing, Borse and Kandekar [3] considered the decomposition of the hypercube $Q_n$ into two regular spanning subgraphs according to the partition of $n$ into two parts and obtained the following result.

**Theorem 1.1.** Let $n, n_1, n_2 \geq 2$ be integers such that $n = n_1 + n_2$. Then the hypercube $Q_n$ can be decomposed into two spanning subgraphs $G_1$ and $G_2$ such that $G_i$ is $n_i$-regular and $n_i$-connected for $i = 1, 2$. Moreover, $G_i$ is bipancyclic if $n_i \neq 3$ and nearly bipancyclic if $n_i = 3$.

We extend this result to the class of the product of even cycles as follows.

**Theorem 1.2.** Let $n, n_1, n_2 \geq 2$ be integers such that $2n = n_1 + n_2$ and let $G$ be the product of $n$ even cycles. Then $G$ can be decomposed into two spanning subgraphs $G_1$ and $G_2$ such that $G_i$ is $n_i$-regular and $n_i$-connected for each $i = 1, 2$. Moreover, $G_i$ is bipancyclic if $n_i \neq 3$ and nearly bipancyclic if $n_i = 3$.

For the decomposition of $Q_n$ according to the general partition of $n$, Sonawane and Borse [15] proved the following result.

**Theorem 1.3** [15]. Let $k, n_1, n_2, \ldots, n_k \geq 2$ be integers such that at most one $n_i$ is odd and $n = n_1 + n_2 + \cdots + n_k$. Then $Q_n$ can be decomposed into $k$ spanning subgraphs $G_1, G_2, \ldots, G_k$ such that each $G_i$ is $n_i$-regular and $n_i$-connected.

We extend this result also to the class of the product of even cycles as follows.
Theorem 1.4. Let $n, k \geq 2$ and $n_1, n_2, \ldots, n_k \geq 1$ be integers such that $n = n_1 + n_2 + \cdots + n_k$ and $G$ be the product of $n$ even cycles. Then $G$ can be decomposed into $k$ spanning subgraphs $G_1, G_2, \ldots, G_k$ such that each $G_i$ is $2n_i$-regular and $2n_i$-connected.

We prove Theorem 1.2 in Section 2. The proof of Theorem 1.4 is given in Section 3.

2. Decomposition Into Two Subgraphs

In this section, we prove Theorem 1.2. Firstly, we prove this theorem for the special cases $n_1 = 2$ and $n_1 = 3$. The general case follows from these two cases.

For $n \geq 1$, let $[n] = \{1, 2, \ldots, n\}$. We define a particular type of 3-regular graph below.

Definition 2.1. Let $r, s \geq 4$ be even integers and let $W$ be the 3-regular graph with vertex set $V(W) = \{v_i^j : i \in [r]; j \in [s]\}$ and the edge set $E(W) = \{v_i^j v_{i+1}^j : i \in [r]; j \in [s]\} \cup \{v_i^j v_{i+1}^{j+1} : i = 1, 3, 5, \ldots, r - 1; j = 1, 3, 5, \ldots, s - 1\} \cup \{v_i^j v_{i+1}^{j+1} : i = 2, 4, 6, \ldots, r; j = 2, 4, 6, \ldots, s\}$, where $v_i^{s+1} = v_i^1$ and $v_{r+1}^j = v_1^j$ (see Figure 1). The graph $W$ is isomorphic to a honeycomb toroidal graph $H(s, r, 0)$ defined in [2].

![Figure 1. The graph W.](image)

We need the following lemmas.

Lemma 2.2 [4]. The graph $W$ defined above is 3-regular, 3-connected and nearly bipancyclic.
Lemma 2.3 [16]. Let $G_i$ be an $m_i$-regular and $m_i$-connected graph for $i = 1, 2$. Then the graph $G_1 \square G_2$ is $(m_1 + m_2)$-regular and $(m_1 + m_2)$-connected.

We prove the special case $n_1 = 2$ of Theorem 1.2 in the following proposition.

Proposition 2.4. Let $n \geq 2$ and let $G$ be the product of $n$ even cycles. Then $G$ has a Hamiltonian cycle $C$ such that $G - E(C)$ is a spanning, $(2n - 2)$-regular, $(2n - 2)$-connected and bipancyclic subgraph of $G$.

Proof. We prove the result by induction on $n$. The result holds for $n = 2$ as, by [1], the product of two cycles can be decomposed into two Hamiltonian cycles. Suppose $n \geq 3$. Let $G = C_1 \square C_2 \square \cdots \square C_n$, where $C_1, C_2, \ldots, C_n$ are even cycles. Let $H = C_1 \square C_2 \square \cdots \square C_{n-1}$, $|V(H)| = r$ and $|V(C_n)| = s$. Then $G = H \square C_n$ and further, $r$ and $s$ are even integers such that $r = |C_1||C_2| \cdots |C_{n-1}| \geq 4^{n-1} \geq 16$ and $s \geq 4$. Label the vertices of the cycle $C_n$ by $\{1, 2, \ldots, s\}$ so that $j$ is adjacent to $j + 1$ modulo $s$.

By induction, $H$ has a Hamiltonian cycle, say $Z$, such that $H - E(Z)$ is a spanning, $(2n - 4)$-regular, $(2n - 4)$-connected and bipancyclic subgraph of $H$. Label the vertices of the cycle $Z$ by the set $\{v_1, v_2, \ldots, v_r\}$ so that $v_p$ is adjacent to $v_{p+1\pmod{r}}$. For compactness, let $v_p^j$ denote the vertex $(v_p, j)$ of $H \square C_n$, let superscripts be computed modulo $s$ and let subscripts be computed modulo $r$. For $j \in [s]$, let $H^j$ be the copy of $H$ induced by the set $\{v_p^j : p \in [r]\}$ and let $Z^j$ be the copy of $Z$ in $H^j$. Let $F$ be the set of edges of $H \square C_n$ between the graphs $H^j$, that is, $F = \{v_p^jv_p^{j+1} : p \in [r], j \in [s]\}$. Then $G = H^1 \cup H^2 \cup \cdots \cup H^s \cup F$.

We now construct a Hamiltonian cycle $C$ by deleting one edge from $Z^j$ and adding one edge of $F$ between $H^j$ and $H^{j+1}$ for all $j$. Let $M = \{v_1^2v_2^2, v_1^3v_1^3, v_2^4v_2^4, v_1^5v_1^5, \ldots, v_2^{s-1}v_3, v_1^sv_1^s\}$ and let $C = \left( \bigcup_{j=1}^s (Z^j - v_1^jv_2^j) \right) \cup M$. Clearly, $C$ is a Hamiltonian cycle in $G$.

Let $K = G - E(C)$. Then $K$ is a spanning $(2n - 2)$-regular subgraph of $G$. Further, $K = \left( \bigcup_{j=1}^s \left( (H^j - Z^j) \cup \{v_1^jv_2^j\} \right) \right) \cup (F - M)$. We prove that $K$ is $(2n - 2)$-connected and bipancyclic.

Claim 1. $K$ is bipancyclic.

Proof. We prove the claim by constructing a spanning bipancyclic subgraph of $K$. Since $H^j - E(Z^j)$ is bipancyclic, it has a Hamiltonian cycle $X^j$ for $j \in [s]$. Therefore $V(X^j) = V(Z^j) = \{v_p^j, p \in [r]\}$. Let $J = (F - M) \cup (X^1 \cup X^2 \cup \cdots \cup X^s)$. Then $J$ is a spanning subgraph of $K$. Note that the edge $v_1^jv_2^j$ of $Z^j$ is a chord of $X^j$ in $H^j$ and so, a subpath of $X^j$ from $v_1^j$ to $v_2^j$ has odd length. Obtain a 3-regular spanning subgraph $W$ of $J$ by deleting alternate edges of $F$ between $X^j$ and $X^{j+1}$ starting from the edge $v_2^jv_2^{j+1}$ when $j$ is odd, and starting from
the edge \( v_1^j v_1^{j+1} \) when \( j \) is even. It is easy to see that \( W \) is isomorphic to the graph in Figure 1. By Lemma 2.2, \( W \) is nearly bipancyclic. Therefore \( W \) and hence \( J \) contains cycles of every even length from 10 to \( rs = |V(J)| \). The ladder graph in \( J \) formed by the paths \( X^1 - v_2^1 \) and \( X^2 - v_2^2 \) contains an \( l \)-cycle for any \( l \in \{4, 6, 8\} \). Thus \( J \) contains cycle of every even length from 4 to \( |V(J)| \) and so \( J \) is bipancyclic. As the graph \( J \) spans \( K \), the graph \( K \) is also bipancyclic.

\( \square \)

**Claim 2.** \( K \) is \((2n - 2)\)-connected.

**Proof.** Let \( D^j = (H^j - E(Z^j)) \cup \{v_1^j, v_2^j\} \) for \( j \in [s] \). Then \( K = \bigcup_{j=1}^s D^j \) \((F - M)\). Since \( H^j - E(Z^j) \) is \((2n - 4)\)-regular and \((2n - 4)\)-connected, \( D^j \) is \((2n - 4)\)-connected, and the degree of each of \( v_1^j \) and \( v_2^j \) in \( D^j \) is \( 2n - 3 \) and the degree of each of the remaining vertices in \( D^j \) is \( 2n - 4 \). For any \( j \), \( D^{j+1} \) and \( D^{j-1} \) are the two neighbouring subgraphs of \( D^j \) in the graph \( K \). Note that in \( K \) every vertex of \( D^j \) except \( v_1^j \) and \( v_2^j \) has neighbours in both \( D^{j+1} \) and \( D^{j-1} \), whereas each of \( v_1^j \) and \( v_2^j \) has a neighbour in exactly one of \( D^{j+1} \) and \( D^{j-1} \).

Let \( S \subseteq V(K) \) such that \(|S| \leq 2n - 3 \). To prove the claim, it suffices to prove that \( K - S \) is connected. As \( V(D^j) = V(H^j) \) and \( n \geq 3 \), \(|V(D^j)| \geq 4^{n-1} \geq 2n + 1 \geq |S| + 4 \). Hence there are at least three edges between \( D^j - S \) and \( D^{j+1} - S \) in \( K - S \) for any \( j \in [s] \). Therefore, if \( D^j - S \) is connected for all \( j \in [s] \), then \( K - S \) is connected. Suppose \( D^j - S \) is not connected for some \( j \). Without loss of generality, we may assume that \( D^1 - S \) is not connected. Then \( D^1 \) contains at least \( 2n - 4 \) vertices from \( S \). If \( S \subseteq V(D^1) \), then every vertex of \( D^1 - S \) has a neighbour in the connected graph \( K - V(D^1) \) and so \( K - S \) is connected. Suppose \( S \) is not a subset of \( V(D^1) \). Then \(|S| = 2n - 3 \) and \(|V(D^1) \cap S| = 2n - 4 \), and so only one vertex, say \( x \), from \( S \) is in \( V(K) - V(D^1) \). Let \( W = K - (V(D^1) \cup \{x\}) \). Then \( W \) is connected. Every vertex of \( D^1 - S \) except possibly \( v_1^1 \) and \( v_2^1 \) has a neighbour in \( W \). Since the degree of each of \( v_1^1 \) and \( v_2^1 \) is \( 2n - 3 \) in \( D^1 \), these vertices cannot be isolated in \( D^1 - S \). Hence the component of \( D^1 - S \) containing \( v_1^1 \) or \( v_2^1 \) has a neighbour in \( W \). This implies that \( K - S \) is connected.

Thus, \( C \) is a Hamiltonian cycle in \( G \) such that \( G - E(C) = K \) is a spanning, \((2n - 2)\)-regular, \((2n - 2)\)-connected and bipancyclic subgraph of \( G \).

\( \blacksquare \)

**Remark 2.5.** By Lemma 2.3, the product \( G \) of \( n \) cycles is \( 2n \)-regular and \( 2n \)-connected. If \( C \) is a cycle in \( G \), then the minimum degree of \( G - E(C) \) is \( 2n - 2 \) and hence \( G - E(C) \) cannot be \( k \)-connected for \( k = 2n - 1 \) or \( k = 2n \). However, the above proposition guarantees the existence of a cycle in \( G \) such that \( G - E(C) \) is \((2n - 2)\)-connected. Such a cycle is removable in \( G \). This result can be compared with an older theorem of Mader [13] which states that if \( H \) is a simple \( n \)-connected graph with minimum degree \( n + 2 \), then there is a cycle \( C \) in \( H \) such that \( H - E(C) \) is \( n \)-connected (also see [5, 9]).
We now prove the special case \( n = 3 \) of Theorem 1.2.

**Proposition 2.6.** Let \( n \geq 3 \) and let \( G \) be the product of \( n \) even cycles. Then \( G \) has a spanning, 3-regular, 3-connected and nearly bipancyclic subgraph \( W \) such that \( G - E(W) \) is a spanning, \((2n-3)\)-regular, \((2n-3)\)-connected subgraph of \( G \). Moreover, \( G - E(W) \) is bipancyclic if \( n \neq 3 \), and it is nearly bipancyclic otherwise.

**Proof.** Let \( C_1, C_2, \ldots, C_n \) be even cycles and let \( G = C_1 \square C_2 \square \cdots \square C_n \). Write \( G \) as \( G = H \square C_n \), where \( H = C_1 \square C_2 \square \cdots \square C_{n-1} \). It follows from Proposition 2.4 that \( H \) has a decomposition into two subgraphs \( C \) and \( D \), where \( C \) is a Hamiltonian cycle, and \( D \) is spanning, \((2n-4)\)-regular, \((2n-4)\)-connected and bipancyclic. Obviously, \( H = D \cup C \). Let \( |V(C_n)| = s \) and \( |V(H)| = r \). Then, as in the proof of Proposition 2.4, we have \( G = H^1 \cup H^2 \cup \cdots \cup H^s \cup F \) with \( V(H^j) = \{ v^j_p : p \in [r] \} \) and \( F = \{ v^j_i v^{j+1}_i : i \in [r], j \in [s] \} \), where \( H^j \) is the copy of \( H \) corresponding to \( j \)th vertex of the cycle \( C_n \). Further, \( C^j \) is the copy of \( C \) in \( H^j \) with vertices \( v^j_1, v^j_2, \ldots, v^j_r, v^j_1 \) in order. Partition the edge set \( F \) into two parts \( F_1 \) and \( F_2 \), where \( F_1 = \{ v^j_i v^{j+1}_i : i = 1, 3, 5, \ldots, r - 1; j = 1, 3, 5, \ldots, s - 1 \} \cup \{ v^j_i v^{j+1}_i : i = 2, 4, 6, \ldots, r; j = 2, 4, 6, \ldots, s \} \) and \( F_2 = F - F_1 \).

We now construct a 3-regular subgraph \( W \) of \( G \) as required. Let \( W = C^1 \cup C^2 \cup \cdots \cup C^s \cup F_1 \). Then \( W \) is isomorphic to the graph in Figure 1. By Lemma 2.2, \( W \) is a 3-regular, 3-connected and nearly bipancyclic subgraph of \( G \). Let \( W' = G - E(W) \). Clearly, \( W' = D^1 \cup D^2 \cup \cdots \cup D^s \cup F_2 \), where \( D^j \) is the copy of \( D \) in \( H^j \). Further, \( W' \) is a spanning and \((2n-3)\)-regular subgraph of \( G \).

To complete the proof, it suffices to prove that \( W' \) is bipancyclic and \((2n-3)\)-connected.

Let \( Y^j \) be a Hamiltonian cycle in \( D^j \) for \( j \in [s] \). Let \( W'' = Y^1 \cup Y^2 \cup \cdots \cup Y^s \cup F_2 \). Then \( W'' \) is a spanning subgraph of \( W' \). Observe that \( W'' \) is isomorphic to \( W \) and so it is nearly bipancyclic. Hence \( W' \) is nearly bipancyclic. Suppose \( n \geq 4 \). Then the cycles of lengths 4 and 8 exist in the graph \( D^j \) and so in \( W' \). Hence \( W' \) contains cycles of every even length from 4 to \(|V(W')|\). Thus \( W' \) is bipancyclic in this case.

We now prove that \( W' \) is \((2n-3)\)-connected. Let \( S \subset W' \) with \(|S| \leq 2n-4 \). It is enough to prove that \( W' - S \) is connected. Suppose \( S \subset D^j \) for some \( j \). Then each component of \( D^j - S \) is joined to \( D^{j+1} \) or \( D^{j-1} \) and hence \( W' - S \) is connected. Suppose \( S \) intersects at least two of \( V(D^1), V(D^2), \ldots, V(D^s) \). Then \(|S \cap V(D^j)| < (2n-4)\) and hence \( D^j - S \) is connected as \( D^j \) is \((2n-4)\)-connected for each \( j \in [s] \). Through the edges of the matching \( F_2 \), half of the vertices of \( D^j \) have distinct neighbours in \( D^{j+1} \) and the remaining half have distinct neighbours in \( D^{j-1} \). Therefore the connected graph \( D^j - S \) is joined to each of \( D^{j+1} - S \) and \( D^{j-1} - S \) in \( W' \) by at least one edge. It implies that \( W' - S \) is connected. Thus \( W' \) is \((2n-3)\)-connected. This completes the proof. \( \blacksquare \)
The next lemma follows from the fact that the product of even cycles is bipancyclic (see [6]).

**Lemma 2.7.** If $G_1$ and $G_2$ are bipartite Hamiltonian graphs, then $G_1 \square G_2$ is bipancyclic.

We now prove Theorem 1.2.

**Proof of Theorem 1.2.** We may assume that $n_2 \geq n_1 \geq 2$. By Propositions 2.4 and 2.6, the result holds for $n_1 = 2$ and $n_1 = 3$. Therefore the result also holds for the cases $n = 2$ and $n = 3$. Suppose $n, n_1, n_2 \geq 4$. Assume that the result holds for all integers from 4 to $n - 1$. Let $G$ be the product of $n$ even cycles $C_1, C_2, \ldots, C_n$ and let $H = C_1 \square C_2 \square \cdots \square C_{n-2}$. Then $G = H \square (C_{n-1} \square C_n)$. Since $2n = n_1 + n_2$, we can express $2(n - 2)$ as $2(n - 2) = (n_1 - 2) + (n_2 - 2)$. Note that $n_1 - 2 \geq 2$ and $n_2 - 2 \geq 2$. Hence, by induction, $H$ has a decomposition into two spanning subgraphs $W_1$ and $W_2$ such that $W_i$ is $(n_i - 1)$-regular, $(n_i - 2)$-connected, and bipancyclic or nearly bipancyclic for $i = 1, 2$. Therefore, each $W_i$ contains a Hamiltonian cycle. By [1], the product of two cycles has a Hamiltonian decomposition. Hence $C_{n-1} \square C_n$ can be decomposed into two Hamiltonian cycles, say $Z_1$ and $Z_2$. This implies that $G = (W_1 \cup W_2) \square (Z_1 \cup Z_2) = (W_1 \square Z_1) \cup (W_2 \square Z_2) = G_1 \cup G_2$, where $G_1 = W_1 \square Z_1$ and $G_2 = W_2 \square Z_2$. Hence $G_1$ and $G_2$ are edge-disjoint spanning subgraphs of $G$ with $G = G_1 \cup G_2$. By Lemma 2.7, $G_i$ is bipancyclic and further, by Lemma 2.3, it is $n_i$-regular and $n_i$-connected for $i = 1, 2$. Thus $G_1$ and $G_2$ give a decomposition of $G$ as required.

**Remark 2.8.** It is worth mentioning that Theorem 1.2 gives a partial solution to the following question due to Mader [14, p. 73].

Given any $n$-connected graph and $k \in \{1, 2, \ldots, n\}$, is there always a $k$-connected subgraph $H$ of $G$ so that $G - E(H)$ is $(n - k)$-connected?

3. **Decomposition Into $k$ Subgraphs**

In this section, we prove Theorem 1.4. Firstly, we give a construction of obtaining $I$-connected spanning subgraph of $C_1 \square C_2 \square \cdots \square C_n$ from the given $I$-connected spanning subgraph of $C_1 \square C_2 \square \cdots \square C_{n-1}$.

Suppose $n \geq 2$. Let $C_1, C_2, \ldots, C_n$ be even cycles, $H = C_1 \square C_2 \square \cdots \square C_{n-1}$ and $G = H \square C_n$. Let $|V(C_n)| = s$. Then $s \geq 4$. Let $H^j$ be the copy of $H$ in $G$ corresponding to $j$th vertex of the cycle $C_n$. Then, as in the proof of Proposition 2.4, $G = H^1 \cup H^2 \cup \cdots \cup H^s \cup F$, where $F = \bigcup_{j=1}^{s}(\{xy; x \in V(H^j), y \in V(H^{j+1})\})$. By Lemma 2.3, $H$ is $(2n - 2)$-connected. Let $l$ be an even integer such that $2 \leq l \leq 2n - 2$ and let $K$ be a spanning $l$-connected subgraph of $H$ in which $M = \{u_1u_2, u_3u_4, \ldots, u_{l-1}u_l\}$ is a matching consisting of $l/2$ edges $u_iu_{i+1}$. Let $K^j$
be the corresponding copy of $K$ in $H^j$ and let $M^j = \{u^j_1u^j_2, u^j_2u^j_4, \ldots, u^j_{l-1}u^j_l\}$ be the matching in $K^j$ corresponding to the matching $M$. Then $K^j$ is $l$-connected and $K^j - M^j$ is $l/2$-connected. In $G$, each vertex of $K^j$ is adjacent to the corresponding vertex of $K^j+1$ through an edge from $F$. Let $N = \{u^j_iu^{j+1}_i: j \in [s] \text{ and } j \text{ odd}; i \in [l] \text{ and } i \text{ odd}\} \cup \{u^j_iu^{j+1}_i: j \in [s] \text{ and } j \text{ even}; i \in [l] \text{ and } i \text{ even}\}$. Then $N \subseteq F$ and $N$ is a matching in $G$. Also, $V(N) = V(M^1) \cup V(M^2) \cup \cdots \cup V(M^s)$.

Let $W = \left( \bigcup_{j=1}^{s}(K^j - M^j) \right) \cup N$ (see Figure 2). Then $W$ is a spanning subgraph of $G$. We prove below that $W$ is $l$-connected also.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure2.png}
\caption{The spanning subgraph $W$ of $G$.}
\end{figure}

**Lemma 3.1.** The graph $W$ defined above is $l$-connected.

**Proof.** Let $S \subset V(W)$ with $|S| \leq l - 1$. It suffices to prove that $W - S$ is connected. As $V(W) = V(K^1) \cup V(K^2) \cup \cdots \cup V(K^s)$, $S \subset V(K^1) \cup V(K^2) \cup \cdots \cup V(K^s)$. Let $S^j = S \cap V(K^j)$ for $j \in [s]$. By $j + 1$ and $j - 1$, we mean $j + 1(\text{mod } s)$ and $j - 1(\text{mod } s)$, respectively.
Let $W^j = K^j - M^j$ for $j \in [s]$. Since $K^j$ is $l$-connected and matching $M^j$ contains $l/2$ edges, $W^j$ is $l/2$-connected and further, it contains all $l$ vertices of $M^j$ half of which have neighbours in $W^{j-1}$ while the remaining half have neighbours in $W^{j+1}$. Clearly, $W^j - S^j$ contains $l - |S^j| > |S| - |S^j| = (|S^1| + |S^2| + \cdots + |S^{j-1}| + |S^j| + |S^{j+1}| + \cdots + |S^s|) - |S^j| \geq |S^{j-1}| + |S^{j+1}|$ vertices of $M^j$. Therefore $W^j - S^j$ has at least one neighbour in $W^{j-1} - S^{j-1}$ or $W^{j+1} - S^{j+1}$. Further, if $W^j - S^j$ has no neighbour in $W^{j+1} - S^{j+1}$, then $|S^j| + |S^{j+1}| \geq l/2$.

We may assume that $|S^j| \geq |S^j|$ for all $j \in [s]$. Then $|S^j| < l/2$ for $j \neq 1$ as $|S| = \sum_{j=1}^s |S^j| < l$. Therefore, as $W^j$ is $l/2$-connected, $W^j - S^j$ is connected for $j \neq 1$.

Suppose $|S^j| \geq l/2$. Then $\sum_{j \neq 1} |S^j| < l/2$. Therefore $W^j - S^j$ has a neighbour in $W^{j+1} - S^{j+1}$ for $2 \leq j \leq s - 1$. Let $T = W - (S \cup V(K^1))$. Then $T$ is a connected subgraph of $W$. We prove that every vertex of $W^1 - S^1$ has a neighbour in $T$. Let $D$ be a component of $W^1 - S^1$. If $W^1 - S^1$ is connected, then $D = W^1 - S^1$. Note that $W^1 - S^1 = (K^1 - M^1) - S^1 = (K^1 - S^1) - M^1$. Since $K^1$ is $l$-connected, $K^1 - S^1$ is $(l - |S^1|)$-connected. Therefore, if $W^1 - S^1$ is not connected, then $D$ contains at least $l - |S^1| > |S^2| + |S^s|$ vertices of the matching $M^1$. Each of these vertices has a neighbour in $W^2 - S^2$ or $W^s - S^s$ and so is in the connected graph $T$. This implies that $W - S$ is connected.

Suppose $|S^j| < l/2$. Then $|S^j| < l/2$ for all $j$. Hence $W^j - S^j$ is connected. Suppose $W^j - S^j$ has no neighbour in $W^{j+1} - S^{j+1}$ for some $j$. Then $W^{j-1} - S^{j-1}$ contains a neighbour of $W^j - S^j$. By the same argument, $W^{j+2} - S^{j+2}$ contains a neighbour of $W^{j+1} - S^{j+1}$. Suppose $W^i - S^i$ has no neighbour in $W^{i+1} - S^{i+1}$ for some $i \neq j$. Then $i \neq j - 1$, $i \neq j + 1$ and further, $|S^i| + |S^j| \geq l/2$. Therefore $|S| \geq |S^i| + |S^{j+1}| + |S^j| + |S^{j+1}| \geq l/2 + l/2 = l$, a contradiction. Hence, for any $i \neq j$, $W^i - S^i$ has neighbours in $W^{j+1} - S^{j+1}$. This implies that $W - S$ is connected.

\textbf{Definition 3.2.} Let $G$ be the product of $n$ even cycles. Suppose the diameter of $G$ is $d$. Let $x_0$ be an end-vertex of a path in $G$ of length $d$. Fix $v_0$. Let $V_0 = \{v_0\}$ and let $V_i = \{v \in V(G) : d(v_0, v) = i\}$ for $i \in [d]$, where $d(v_0, v)$ denotes the distance between $v_0$ and $v$ in $G$. Clearly, the sets $V_0, V_1, \ldots, V_d$ are mutually disjoint, non-empty and they partition the set $V(G)$. Let $K$ be a spanning subgraph of $G$. For $i \in [d]$, let $E_i(K) = \{xy \in E(K) : x \in V_{i-1}, y \in V_i\}$. Then the edge sets $E_1(K), E_2(K), \ldots, E_d(K)$ are non-empty and mutually disjoint (see Figure 3).

\textbf{Lemma 3.3.} Let $G$, $K$ and $E_i(K)$ be as in Definition 3.2. Then $E_1(K), E_2(K), \ldots, E_d(K)$ partition the edge set $E(K)$ of the graph $K$.

\textbf{Proof.} By definition of $V_i$, there is no edge in $G$ with one end-vertex in $V_i$ and the other in $V_{i'}$ when $|i - i'| \neq 1$. Suppose two vertices $x$ and $y$ of some
Let $P_x$ and $P_y$ be shortest paths in $G$ from $v_0$ to $x$, and $v_0$ to $y$ respectively. Then each of $P_x$ and $P_y$ takes exactly one vertex from each of $V_0, V_1, \ldots, V_i$. Therefore $P_x \cup P_y \cup \{xy\}$ contains an odd cycle in $G$, a contradiction to the fact that $G$ is bipartite. Thus each $V_i$ is independent. This implies that $E(K) = E_1(K) \cup E_2(K) \cup \cdots \cup E_d(K)$.

We need the following result.

**Lemma 3.4** [16]. Let $G_i$ be a graph with diameter $d_i$ for $i = 1, 2, \ldots, k$. Then the diameter of the graph $G_1 \Box G_2 \Box \cdots \Box G_k$ is $d_1 + d_2 + \cdots + d_k$.

We are all set to prove Theorem 1.4. This theorem is restated below for convenience.

**Theorem 3.5.** Let $G$ be the product of $n$ even cycles and let $n = n_1 + n_2 + \cdots + n_k$ with $k \geq 2$ and $n_i \geq 1$ for $i \in [k]$. Then $G$ can be decomposed into $k$ spanning subgraphs $G_1, G_2, \ldots, G_k$ such that each $G_i$ is $2n_i$-regular and $2n_i$-connected.

**Proof.** We prove the result by induction on $n$. Obviously, $n \geq k$. If $n = k$, then $G$ is the product of $k$ cycles and hence, by [1], $G$ can be decomposed into $k$ Hamiltonian cycles. Thus the result holds for $n = k$. 

Figure 3. A decomposition of $G$. 

We need the following result.
Suppose $n \geq k + 1$. Then $n_i \geq 2$ for some $i \in [k]$. Without loss of generality, we may assume that $n_k \geq 2$. Assume that the result holds for $n - 1$. Consider $n - 1 = n_1 + n_2 + \cdots + n_{k-1} + (n_k - 1)$. Let $G = C_1 \square C_2 \square \cdots \square C_n$, where $C_1, C_2, \ldots, C_n$ are even cycles. Let $|C_n| = s$ and let $H = C_1 \square C_2 \square \cdots \square C_{n-1}$. Then, as in the proof of Proposition 2.4, $G = H \square C_n = H^1 \cup H^2 \cup \cdots \cup H^s \cup F$, where $H^j$ is a copy of $H$ and $F = \bigcup_{j=1}^s \{xy : x \in V(H^j), y \in V(H^{j+1})\}$ with $H^{s+1} = H^1$.

By induction, $H$ can be decomposed into $k$ spanning subgraphs $H_1, H_2, \ldots, H_k$ such that $H_i$ is $2n_i$-regular and $2n_i$-connected for $i \in [k-1]$, and $H_k$ is $2(n_k - 1)$-regular and $2(n_k - 1)$-connected.

Let $d$ be the diameter of $H$. Since each $C_i$ is an even cycle, the diameter of $C_i$ is $|C_i|/2 \geq 2$. Therefore, by Lemma 3.4, $d = \lceil |C_1| + |C_2| + \cdots + |C_{n-1}| \rceil \geq 2(n-1) = 2n-2 \geq 2n-2n_k$. Let $u_0$ be an end-vertex of a path in $H$ of length $d$. As in the Definition 3.2, we partition the vertex set $V(H)$ of $H$ into the sets $V_0(H), V_1(H), \ldots, V_d(H)$, where $V_0(H) = \{u_0\}$ and $V_i(H) = \{u \in V(H) : d(u, u_0) = i\}$ for $i \in [d]$. Since $H_i$ for $i \in [k]$ is a spanning subgraph of $H$, it follows from Lemma 3.3 that the edge set $E(H_i)$ of $H_i$ can be partitioned into the sets $E_1(H_i), E_2(H_i), \ldots, E_t(H_i)$, where $E_0(H_i) = \{xy \in E(H_i) : x \in V_{t-1}, y \in V_t\}$ for $t \in [d]$. Note that if $e \in E_0(H_i)$ and $f \in E_t(H_i)$ with $t' \geq t + 2$, then $e$ and $f$ are vertex-disjoint (see Figure 3).

For $i \in [k-1]$, we obtain a matching $M_i$ of $H_i$ by choosing one edge from each $n_i$ consecutive sets $E_{2t-1}(H_i)$ as follows.

Choose one edge from each of the sets $E_1(H_i), E_3(H_i), \ldots, E_{2n_i-1}(H_i)$ to get $M_1$. Thus, we let $M_1 = \{u_{t-1}u_t \in E_t(H_i) : t = 1, 3, 5, \ldots, 2n_i - 1\}$. In general, we define $M_i = \{u_{t-1}u_t \in E_t(H_i) : t = 2p_i + 1, 2p_i + 3, \ldots, 2p_i + 2n_i - 1\}$, where $p_1 = 0$ and $p_i = n_i + n_{i+1} + \cdots + n_{k-1}$ for $2 \leq i \leq k - 1$.

For $j \in [s]$, the graph $H^j$ is a copy of $H$. Let $H^j_i$ be the subgraph of $H^j$ corresponding to the subgraph $H_i$ of $H$ for $i \in [k]$. Therefore the graphs $H^1_i, H^2_i, \ldots, H^s_i$ decompose the graph $H^j$. Further, the edge set $E(H^j_i)$ has a partition into non-empty sets $E_1(H^j_i), E_2(H^j_i), \ldots, E_d(H^j_i)$. For $i \in [k-1]$, let $M^j_i$ be the matching in $H^j_i$ corresponding to the matching $M_i$ of $H$ and let $u^j_i$ be the vertex of $H^j$ corresponding to the vertex $u_t$ of $H$. Then $M^j_i = \{u^j_{t-1}u^j_t \in E_t(H^j_i) : t = 2p_i + 1, 2p_i + 3, \ldots, 2p_i + 2n_i - 1\}$. Let $M^j$ be the union of these $k-1$ matchings $M^j_i$. Therefore $M^j = \bigcup_{i=1}^{k-1} M^j_i = \{u^j_0u^j_1, u^j_2u^j_3, \ldots, u^j_{2n_1+\cdots+2n_{k-1}-2}u^j_{2n_1+\cdots+2n_{k-1}-1}\}$.

Clearly, $M^j$ is a matching in $H^j$.

We now construct the subgraphs $G_1, G_2, \ldots, G_k$ of $G$ which give a decomposition of $G$, as required.

**Construction of the graphs $G_i$ for $i \in [k]$.**

Let $i \in [k-1]$. We obtain $G_i$ from $H^1_i \cup \cdots \cup H^s_i$ by deleting the matching $M^j_i$ from $H^j_i$ for each $j$ and then adding a matching $D_i$ consisting of edges from the set
Given a graph $F$ having one end in $M_i^j$ and the other end in $M_i^{j+1}$ or $M_i^{j-1}$. More precisely, let

$$D_i = \{ u_i^j u_i^{j+1} : j = 1, 3, \ldots, s-1; \ t = 2p_i, 2p_i+2, \ldots, 2p_i+2n_i-2 \} \cup \{ u_i^j u_i^{j+1} : j = 2, 4, \ldots, s; \ t = 2p_i+1, 2p_i+3, \ldots, 2p_i+2n_i-1 \}.$$

Figure 4. The graph $G_i$.

For $i \in [k-1]$, let $G_i = \left( \bigcup_{j=1}^{s} (H_i^j - M_i^j) \right) \cup D_i$ (see Figure 4). Note that $D_i$ is a matching consisting of $n_i$ edges between $H^j$ and $H^{j+1}$ for each $j \in [s]$ and so the total number of edges in $D_i$ is $sn_i$.

For any $i \in [k-1]$, for $i \neq i'$, the graphs $H_i^j$ and $H_{i'}^j$ are edge-disjoint for each $j$. This implies that $G_1, G_2, \ldots, G_{k-1}$ are mutually edge-disjoint subgraphs of $G$. Since $H_i^j$ is a $2n_i$-regular and spanning subgraph of $H^j$, $G_i$ is also a $2n_i$-regular and spanning subgraph of $G$. Further, as $H_i^j$ is $2n_i$-connected, Lemma 3.1 implies that $G_i$ is also $2n_i$-connected.
Let $G_k = G - E(G_1 \cup G_2 \cup \cdots \cup G_{k-1})$. The graph $G_k$ is shown in Figure 5. It is easy to see that $G_k = \left( \bigcup_{j=1}^{s} (H_j^k \cup M^j) \right) \cup (F - D)$, where $D = \bigcup_{i=1}^{k-1} D_i$.

The edges of the matching $M^j$ are shown by the bold edges in Figure 5. Clearly, $D$ is a matching in $G$ consisting of $s(n_1 + n_2 + \cdots + n_{k-1}) = s(n - n_k)$ edges of $F$.

It follows that the graph $G_k$ is a spanning and $2n_k$-regular subgraph of $G$. Thus the graph $G$ decomposes into the spanning subgraphs $G_1, G_2, \ldots, G_k$.

It only remains to prove that the graph $G_k$ is $2n_k$-connected.

Claim. $G_k$ is $2n_k$-connected.

Proof. Let $S \subset V(G_k) = \bigcup_{j=1}^{s} V(H_j^i)$ such that $0 < |S| \leq 2n_k - 1$. It suffices to prove that $G_k - S$ is connected. Let $S_j = V(H_j^i) \cap S$ for $j \in [s]$. Since $V(H_j^i) = V(H^i), |V(H_j^i)| = |V(H^j)| = r = |C_1||C_2| \cdots |C_{n-1}| \geq 4^{n-1} = 2^{2n-2} =$
where proof.

This implies that $T = G_k - S$. This completes the proof.

\[2n - 2n_k + 2n_k - 2 = 2n - 2n_k - 2 > (2n - 2n_k)(2n_k - 1) \geq 2(n - n_k)|S| \geq (n - n_k) + |S| \geq (n - n_k) + |S|.

Hence, there are at least $r - (n - n_k) - |S| > 0$ edges between $H_k^j - S^j$ and $H_k^{j+1} - S^{j+1}$ in $G_k - S$.

Obviously at least one $S^j$ is non-empty. We may assume that $S^1 \neq \emptyset$ and $|S^1| \geq |S^j|$ for $j \in [s]$. Suppose two more $S^j$ are non-empty. Then $|S^j| < 2n_k - 2$ for $j \in [s]$ as $|S| \leq 2n_k - 1$. Hence each $H_k^j - S^j$ is connected as $H_k^j$ is $(2n_k - 2)$-connected. Further, $H_k^j - S^j$ is connected to $H_k^{j+1} - S^{j+1}$ by edges of $F - D$. This implies that $G_k - S$ is connected.

Suppose $S^j = \emptyset$ for all $j \neq 1$. Therefore $H_k^j - S^j = H_k^j$ is connected for all $j \neq 1$. Obviously, each vertex of $H_k^1 - S^1$ has a neighbour in $H_k^2$ or $H_k^3$. Hence $G_k - S$ is connected.

Suppose only one $S^j$ other than $S^1$ is nonempty. Suppose $S^2 \neq \emptyset$. Then $S = S^1 \cup S^2$. If $H_k^1 - S^1$ and $H_k^2 - S^2$ are connected, then they are connected to each other by an edge of $F - D$ and so $G_k - S$ is connected. Suppose $H_k^1 - S^1$ is not connected. Then $|S^1| = 2n_k - 2$ and $|S^2| = 1$ as $|S| \leq 2n_k - 1$ and $H_k^1$ is $(2n_k - 2)$-connected. This implies that $H_k^j - S^j$ for any $j \neq 1$ is connected. Let $T = G_k - (V(H_k^1) \cup S) = G_k - (V(H_k^1) \cup S^2)$. Then $T$ is connected. It suffices to prove that every component of $H_k^1 - S^1$ has a neighbour in $T$. Let $W$ be a component of $H_k^1 - S^1$ and let $v$ be a vertex of $W$. If $v$ has a neighbour in $H_k^s$, then we are through. Suppose $v$ has no neighbour in $H_k^s$. Then $v$ has a neighbour $v'$ in $H_k^2$. If $v' \notin S^2$, then also we are through. Suppose $v' \in S^2$. Then $S^2 = \{v'\}$. Also, $v$ is an end-vertex of an edge of the matching $M^1$. Therefore the degree of $v$ in $H_k^1$ is $2n_k - 1$. Hence $v$ has a neighbour $u$ in $H_k^1 - S^1$. Obviously, $u$ is in $W$. Further, $u$ has a neighbour in the subgraphs $H_k^s$ or $H_k^2 - S^2 = H_k^2 - \{v'\}$ of $T$. Thus $W$ has a neighbour in the connected graph $T$. Hence $G_k - S$ is connected.

Similarly, $G_k - S$ is connected when $S^s \neq \emptyset$.

Suppose $S^j \neq \emptyset$ for some $j \notin \{1, 2, s\}$. Then every component of $H_k^1 - S^1$ has a neighbour in $H_k^2$ or $H_k^3$. It follows that $G_k - S$ is connected. This proves the claim.

Thus, the graph $G$ decomposes into the spanning subgraphs $G_1, G_2, \ldots, G_k$, where $G_i$ is $2n_i$-regular and $2n_i$-connected for $i = 1, 2, \ldots, k$. This completes the proof.

\[\square\]

References

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