*Discussiones Mathematicae*
*Graph Theory 39 (2019) 241–256*
*doi:10.7151/dmgt.2066*

# DECOMPOSITION OF THE PRODUCT OF CYCLES BASED ON DEGREE PARTITION

Y.M. Borse and S.R. Shaikh

*Department of Mathematics*
*Savitribai Phule Pune University*
*Pune-411007, India*

**e-mail:** ymborse11@gmail.com
shazia_31082@yahoo.co.in

**Abstract**

The Cartesian product of $n$ cycles is a $2n$-regular, $2n$-connected and bipancyclic graph. Let $G$ be the Cartesian product of $n$ even cycles and let $2n = n_1 + n_2 + \cdots + n_k$ with $k \geq 2$ and $n_i \geq 2$ for each $i$. We prove that if $k = 2$, then $G$ can be decomposed into two spanning subgraphs $G_1$ and $G_2$ such that each $G_i$ is $n_i$-regular, $n_i$-connected, and bipancyclic or nearly bipancyclic. For $k > 2$, we establish that if all $n_i$ in the partition of $2n$ are even, then $G$ can be decomposed into $k$ spanning subgraphs $G_1, G_2, \ldots, G_k$ such that each $G_i$ is $n_i$-regular and $n_i$-connected. These results are analogous to the corresponding results for hypercubes.

**Keywords:** hypercube, Cartesian product, $n$-connected, regular, bipancyclic, spanning subgraph.

**2010 Mathematics Subject Classification:** 05C40, 05C70, 68R10.

## 1. Introduction

The graphs considered in this paper are simple, undirected and finite. The *Cartesian product* of two graphs $G_1$ and $G_2$ is the graph $G_1 \square G_2$ with vertex set $V(G_1) \times V(G_2)$ in which $(u_1, u_2)$ is adjacent to $(v_1, v_2)$ if and only if $u_1$ is adjacent to $v_1$ in $G_1$ and $u_2 = v_2$, or $u_2$ is adjacent to $v_2$ in $G_2$ and $u_1 = v_1$. The $n$-dimensional hypercube $Q_n$ is the Cartesian product of $n$ copies the complete graph $K_2$. Therefore $Q_n$ is the Cartesian product of $n/2$ copies of a cycle of length 4 when $n$ is even. The Cartesian product of cycles and hypercubes are popular interconnection network topologies (see [6, 11]). The hypercube $Q_n$ is an $n$-regular and $n$-connected graph whereas the Cartesian product of $n$ cycles is a $2n$-regular and $2n$-connected graph (see [16]).

Unless mentioned otherwise, in the remainder of this paper product means the Cartesian product of graphs.

A cycle is *even* if its length is a positive even integer. A bipartite graph $G$ is *bipancyclic* if $G$ is either a cycle or has cycles of every even length from 4 to $|V(G)|$. A 3-regular graph is *nearly bipancyclic* if it has cycles of every even length from 4 to $|V(G)|$ except possibly for 4 and 8. The bipancyclicity property of a given network is an important factor in determining whether the network topology can simulate rings of various lengths.

Alspach *et al.* [1] proved that the product of cycles can be decomposed into Hamiltonian cycles. This result subsumes earlier results due to Kotzig [10] and Foregger [8] on Hamiltonian decomposition of the product of cycles. El-Zanati and Eynden [7] studied the decomposition of the product of cycles, each of length a power of 2, into non-spanning cycles. Borse *et al.* [4] proved that if $m \geq 2$ and $m$ divides $n$, then the product of $n$ even cycles can be decomposed into isomorphic, spanning, $m$-regular, $m$-connected subgraphs which are bipancyclic or nearly bipancyclic also. The analogous results for the class of hypercubes are obtained in [1, 4, 7].

Motivated by applications in parallel computing, Borse and Kandekar [3] considered the decomposition of the hypercube $Q_n$ into two regular spanning subgraphs according to the partition of $n$ into two parts and obtained the following result.

**Theorem 1.1.** *Let $n, n_1, n_2 \geq 2$ be integers such that $n = n_1 + n_2$. Then the hypercube $Q_n$ can be decomposed into two spanning subgraphs $G_1$ and $G_2$ such that $G_i$ is $n_i$-regular and $n_i$-connected for $i = 1, 2$. Moreover, $G_i$ is bipancyclic if $n_i \neq 3$ and nearly bipancyclic if $n_i = 3$.*

We extend this result to the class of the product of even cycles as follows.

**Theorem 1.2.** *Let $n, n_1, n_2 \geq 2$ be integers such that $2n = n_1 + n_2$ and let $G$ be the product of $n$ even cycles. Then $G$ can be decomposed into two spanning subgraphs $G_1$ and $G_2$ such that $G_i$ is $n_i$-regular and $n_i$-connected for each $i = 1, 2$. Moreover, $G_i$ is bipancyclic if $n_i \neq 3$ and nearly bipancyclic if $n_i = 3$.*

For the decomposition of $Q_n$ according to the general partition of $n$, Sonawane and Borse [15] proved the following result.

**Theorem 1.3** [15]**.** *Let $k, n_1, n_2, \ldots, n_k \geq 2$ be integers such that at most one $n_i$ is odd and $n = n_1 + n_2 + \cdots + n_k$. Then $Q_n$ can be decomposed into $k$ spanning subgraphs $G_1, G_2, \ldots, G_k$ such that each $G_i$ is $n_i$-regular and $n_i$-connected.*

We extend this result also to the class of the product of even cycles as follows.

**Theorem 1.4.** *Let $n, k \geq 2$ and $n_1, n_2, \ldots, n_k \geq 1$ be integers such that $n = n_1+n_2+\cdots+n_k$ and $G$ be the product of $n$ even cycles. Then $G$ can be decomposed into $k$ spanning subgraphs $G_1, G_2, \ldots, G_k$ such that each $G_i$ is $2n_i$-regular and $2n_i$-connected.*

We prove Theorem 1.2 in Section 2. The proof of Theorem 1.4 is given in Section 3.

## 2. Decomposition Into Two Subgraphs

In this section, we prove Theorem 1.2. Firstly, we prove this theorem for the special cases $n_1 = 2$ and $n_1 = 3$. The general case follows from these two cases.

For $n \geq 1$, let $[n] = \{1, 2, \ldots, n\}$. We define a particular type of 3-regular graph below.

**Definition 2.1.** Let $r, s \geq 4$ be even integers and let $W$ be the 3-regular graph with vertex set $V(W) = \{v_i^j : i \in [r];\ j \in [s]\}$ and the edge set $E(W) = \{v_i^j v_{i+1}^j : i \in [r]; j \in [s]\} \cup \{v_i^j v_i^{j+1} : i = 1, 3, 5, \ldots, r-1;\ j = 1, 3, 5, \ldots, s-1\} \cup \{v_i^j v_i^{j+1} : i = 2, 4, 6, \ldots, r;\ j = 2, 4, 6, \ldots, s\}$, where $v_i^{s+1} = v_i^1$ and $v_{r+1}^j = v_1^j$ (see Figure 1). The graph $W$ is isomorphic to a honeycomb toroidal graph $H(s, r, 0)$ defined in [2].

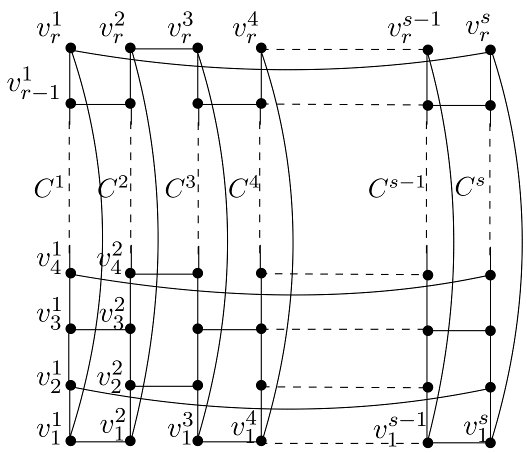


Figure 1. The graph $W$.

We need the following lemmas.

**Lemma 2.2** [4]**.** *The graph $W$ defined above is 3-regular, 3-connected and nearly bipancyclic.*

**Lemma 2.3** [16]. *Let $G_i$ be an $m_i$-regular and $m_i$-connected graph for $i = 1, 2$. Then the graph $G_1 \square G_2$ is $(m_1 + m_2)$-regular and $(m_1 + m_2)$-connected.*

We prove the special case $n_1 = 2$ of Theorem 1.2 in the following proposition.

**Proposition 2.4.** *Let $n \geq 2$ and let $G$ be the product of $n$ even cycles. Then $G$ has a Hamiltonian cycle $C$ such that $G - E(C)$ is a spanning, $(2n - 2)$-regular, $(2n - 2)$-connected and bipancyclic subgraph of $G$.*

***Proof.*** We prove the result by induction on $n$. The result holds for $n = 2$ as, by [1], the product of two cycles can be decomposed into two Hamiltonian cycles. Suppose $n \geq 3$. Let $G = C_1 \square C_2 \square \cdots \square C_n$, where $C_1, C_2, \ldots, C_n$ are even cycles. Let $H = C_1 \square C_2 \square \cdots \square C_{n-1}$, $|V(H)| = r$ and $|V(C_n)| = s$. Then $G = H \square C_n$ and further, $r$ and $s$ are even integers such that $r = |C_1||C_2| \cdots |C_{n-1}| \geq 4^{n-1} \geq 16$ and $s \geq 4$. Label the vertices of the cycle $C_n$ by $\{1, 2, \ldots, s\}$ so that $j$ is adjacent to $j + 1$ modulo $s$.

By induction, $H$ has a Hamiltonian cycle, say $Z$, such that $H - E(Z)$ is a spanning, $(2n - 4)$-regular, $(2n - 4)$-connected and bipancyclic subgraph of $H$. Label the vertices of the cycle $Z$ by the set $\{v_1, v_2, \ldots, v_r\}$ so that $v_p$ is adjacent to $v_{p+1 (\bmod r)}$. For compactness, let $v_p^j$ denote the vertex $(v_p, j)$ of $H \square C_n$, let superscripts be computed modulo $s$ with representative in $[s]$ and subscripts be modulo $r$ with representative in $[r]$. For $j \in [s]$, let $H^j$ be the copy of $H$ induced by the set $\{v_p^j : p \in [r]\}$ and let $Z^j$ be the copy of $Z$ in $H^j$. Let $F$ be the set of edges of $H \square C_n$ between the graphs $H^j$, that is, $F = \{v_p^j v_p^{j+1} : p \in [r],\ j \in [s]\}$. Then $G = H^1 \cup H^2 \cup \cdots \cup H^s \cup F$.

We now construct a Hamiltonian cycle $C$ by deleting one edge from $Z^j$ and adding one edge of $F$ between $H^j$ and $H^{j+1}$ for all $j$. Let $M = \{v_2^1 v_2^2, v_1^2 v_1^3, v_2^3 v_2^4, v_1^4 v_1^5, \ldots, v_2^{s-1} v_2^s, v_1^s v_1^1\}$ and let $C = \left(\bigcup_{j=1}^{s} (Z^j - v_1^j v_2^j)\right) \cup M$. Clearly, $C$ is a Hamiltonian cycle in $G$.

Let $K = G - E(C)$. Then $K$ is a spanning $(2n - 2)$-regular subgraph of $G$. Further, $K = \left(\bigcup_{j=1}^{s} \left((H^j - Z^j) \cup \{v_1^j v_2^j\}\right)\right) \cup (F - M)$. We prove that $K$ is $(2n - 2)$-connected and bipancyclic.

**Claim 1.** *$K$ is bipancyclic.*

***Proof.*** We prove the claim by constructing a spanning bipancyclic subgraph of $K$. Since $H^j - E(Z^j)$ is bipancyclic, it has a Hamiltonian cycle $X^j$ for $j \in [s]$. Therefore $V(X^j) = V(Z^j) = \{v_p^j : p \in [r]\}$. Let $J = (F - M) \cup (X^1 \cup X^2 \cup \cdots \cup X^s)$. Then $J$ is a spanning subgraph of $K$. Note that the edge $v_1^j v_2^j$ of $Z^j$ is a chord of $X^j$ in $H^j$ and so, a subpath of $X^j$ from $v_1^j$ to $v_2^j$ has odd length. Obtain a 3-regular spanning subgraph $W$ of $J$ by deleting alternate edges of $F$ between $X^j$ and $X^{j+1}$ starting from the edge $v_2^j v_2^{j+1}$ when $j$ is odd, and starting from the edge $v_1^j v_1^{j+1}$ when $j$ is even. It is easy to see that $W$ is isomorphic to the graph in Figure 1. By Lemma 2.2, $W$ is nearly bipancyclic. Therefore $W$ and hence $J$ contains cycles of every even length from 10 to $rs = |V(J)|$. The ladder graph in $J$ formed by the paths $X^1 - v_2^1$ and $X^2 - v_2^2$ contains an $l$-cycle for any $l \in \{4, 6, 8\}$. Thus $J$ contains cycle of every even length from 4 to $|V(J)|$ and so $J$ is bipancyclic. As the graph $J$ spans $K$, the graph $K$ is also bipancyclic. □

**Claim 2.** *$K$ is $(2n-2)$-connected.*

***Proof.*** Let $D^j = (H^j - E(Z^j)) \cup \{v_1^j v_2^j\}$ for $j \in [s]$. Then $K = \left(\bigcup_{j=1}^{s} D^j\right) \cup (F - M)$. Since $H^j - E(Z^j)$ is $(2n-4)$-regular and $(2n-4)$-connected, $D^j$ is $(2n-4)$-connected, and the degree of each of $v_1^j$ and $v_2^j$ in $D^j$ is $2n-3$ and the degree of each of the remaining vertices in $D^j$ is $2n-4$. For any $j$, $D^{j+1}$ and $D^{j-1}$ are the two neighbouring subgraphs of $D^j$ in the graph $K$. Note that in $K$ every vertex of $D^j$ except $v_1^j$ and $v_2^j$ has neighbours in both $D^{j+1}$ and $D^{j-1}$, whereas each of $v_1^j$ and $v_2^j$ has a neighbour in exactly one of $D^{j+1}$ and $D^{j-1}$.

Let $S \subset V(K)$ such that $|S| \leq 2n-3$. To prove the claim, it suffices to prove that $K - S$ is connected. As $V(D^j) = V(H^j)$ and $n \geq 3$, $|V(D^j)| \geq 4^{n-1} \geq 2n+1 \geq |S|+4$. Hence there are at least three edges between $D^j - S$ and $D^{j+1} - S$ in $K - S$ for any $j \in [s]$. Therefore, if $D^j - S$ is connected for all $j \in [s]$, then $K - S$ is connected. Suppose $D^j - S$ is not connected for some $j$. Without loss of generality, we may assume that $D^1 - S$ is not connected. Then $D^1$ contains at least $2n-4$ vertices from $S$. If $S \subset V(D^1)$, then every vertex of $D^1 - S$ has a neighbour in the connected graph $K - V(D^1)$ and so $K - S$ is connected. Suppose $S$ is not a subset of $V(D^1)$. Then $|S| = 2n-3$ and $|V(D^1) \cap S| = 2n-4$, and so only one vertex, say $x$, from $S$ is in $V(K) - V(D^1)$. Let $W = K - (V(D^1) \cup \{x\})$. Then $W$ is connected. Every vertex of $D^1 - S$ except possibly $v_1^1$ and $v_2^1$ has a neighbour in $W$. Since the degree of each of $v_1^1$ and $v_2^1$ is $2n-3$ in $D^1$, these vertices cannot be isolated in $D^1 - S$. Hence the component of $D^1 - S$ containing $v_1^1$ or $v_2^1$ has a neighbour in $W$. This implies that $K - S$ is connected. □

Thus, $C$ is a Hamiltonian cycle in $G$ such that $G - E(C) = K$ is a spanning, $(2n-2)$-regular, $(2n-2)$-connected and bipancyclic subgraph of $G$. ■

**Remark 2.5.** By Lemma 2.3, the product $G$ of $n$ cycles is $2n$-regular and $2n$-connected. If $C$ is a cycle in $G$, then the minimum degree of $G - E(C)$ is $2n-2$ and hence $G - E(C)$ cannot be $k$-connected for $k = 2n-1$ or $k = 2n$. However, the above proposition guarantees the existence of a cycle in $G$ such that $G - E(C)$ is $(2n-2)$-connected. Such a cycle is *removable* in $G$. This result can be compared with an older theorem of Mader [13] which states that if $H$ is a simple $n$-connected graph with minimum degree $n+2$, then there is a cycle $C$ in $H$ such that $H - E(C)$ is $n$-connected (also see [5, 9]).

We now prove the special case $n_1 = 3$ of Theorem 1.2.

**Proposition 2.6.** *Let $n \geq 3$ and let $G$ be the product of $n$ even cycles. Then $G$ has a spanning, 3-regular, 3-connected and nearly bipancyclic subgraph $W$ such that $G - E(W)$ is a spanning, $(2n-3)$-regular, $(2n-3)$-connected subgraph of $G$. Moreover, $G - E(W)$ is bipancyclic if $n \neq 3$, and it is nearly bipancyclic otherwise.*

***Proof.*** Let $C_1, C_2, \ldots, C_n$ be even cycles and let $G = C_1 \square C_2 \square \cdots \square C_n$. Write $G$ as $G = H \square C_n$, where $H = C_1 \square C_2 \square \cdots \square C_{n-1}$. It follows from Proposition 2.4 that $H$ has a decomposition into two subgraphs $C$ and $D$, where $C$ is a Hamiltonian cycle, and $D$ is spanning, $(2n-4)$-regular, $(2n-4)$-connected and bipancyclic. Obviously, $H = D \cup C$. Let $|V(C_n)| = s$ and $|V(H)| = r$. Then, as in the proof of Proposition 2.4, we have $G = H^1 \cup H^2 \cup \cdots \cup H^s \cup F$ with $V(H^j) = \{v_p^j : p \in [r]\}$ and $F = \{v_i^j v_i^{j+1} : i \in [r],\ j \in [s]\}$, where $H^j$ is the copy of $H$ corresponding to $j$th vertex of the cycle $C_n$. Further, $C^j$ is the copy of $C$ in $H^j$ with vertices $v_1^j, v_2^j, \ldots, v_r^j, v_1^j$ in order. Partition the edge set $F$ into two parts $F_1$ and $F_2$, where $F_1 = \{v_i^j v_i^{j+1} : i = 1, 3, 5, \ldots, r-1;\ j = 1, 3, 5, \ldots, s-1\} \cup \{v_i^j v_i^{j+1} : i = 2, 4, 6, \ldots, r;\ j = 2, 4, 6, \ldots, s\}$ and $F_2 = F - F_1$.

We now construct a 3-regular subgraph $W$ of $G$ as required. Let $W = C^1 \cup C^2 \cup \cdots \cup C^s \cup F_1$. Then $W$ is isomorphic to the graph in Figure 1. By Lemma 2.2, $W$ is a 3-regular, 3-connected and nearly bipancyclic subgraph of $G$. Let $W' = G - E(W)$. Clearly, $W' = D^1 \cup D^2 \cup \cdots \cup D^s \cup F_2$, where $D^j$ is the copy of $D$ in $H^j$. Further, $W'$ is a spanning and $(2n-3)$-regular subgraph of $G$.

To complete the proof, it suffices to prove that $W'$ is bipancyclic and $(2n-3)$-connected.

Let $Y^j$ be a Hamiltonian cycle in $D^j$ for $j \in [s]$. Let $W'' = Y^1 \cup Y^2 \cup \cdots \cup Y^s \cup F_2$. Then $W''$ is a spanning subgraph of $W'$. Observe that $W''$ is isomorphic to $W$ and so it is nearly bipancyclic. Hence $W'$ is nearly bipancyclic. Suppose $n \geq 4$. Then the cycles of lengths 4 and 8 exist in the graph $D^j$ and so in $W'$. Hence $W'$ contains cycles of every even length from 4 to $|V(W')|$. Thus $W'$ is bipancyclic in this case.

We now prove that $W'$ is $(2n-3)$-connected. Let $S \subset W'$ with $|S| \leq 2n-4$. It is enough to prove that $W' - S$ is connected. Suppose $S \subset D^j$ for some $j$. Then each component of $D^j - S$ is joined to $D^{j+1}$ or $D^{j-1}$ and hence $W' - S$ is connected. Suppose $S$ intersects at least two of $V(D^1), V(D^2), \ldots, V(D^s)$. Then $|S \cap V(D^j)| < (2n-4)$ and hence $D^j - S$ is connected as $D^j$ is $(2n-4)$-connected for each $j \in [s]$. Through the edges of the matching $F_2$, half of the vertices of $D^j$ have distinct neighbours in $D^{j+1}$ and the remaining half have distinct neighbours in $D^{j-1}$. Therefore the connected graph $D^j - S$ is joined to each of $D^{j+1} - S$ and $D^{j-1} - S$ in $W'$ by at least one edge. It implies that $W' - S$ is connected. Thus $W'$ is $(2n-3)$-connected. This completes the proof. ∎

The next lemma follows from the fact that the product of even cycles is bipancyclic (see [6]).

**Lemma 2.7.** *If $G_1$ and $G_2$ are bipartite Hamiltonian graphs, then $G_1 \square G_2$ is bipancyclic.*

We now prove Theorem 1.2.

***Proof of Theorem* 1.2.** We may assume that $n_2 \geq n_1 \geq 2$. By Propositions 2.4 and 2.6, the result holds for $n_1 = 2$ and $n_1 = 3$. Therefore the result also holds for the cases $n = 2$ and $n = 3$. Suppose $n, n_1, n_2 \geq 4$. Assume that the result holds for all integers from 4 to $n-1$. Let $G$ be the product of $n$ even cycles $C_1, C_2, \ldots, C_n$ and let $H = C_1 \square C_2 \square \cdots \square C_{n-2}$. Then $G = H \square (C_{n-1} \square C_n)$. Since $2n = n_1 + n_2$, we can express $2(n-2)$ as $2(n-2) = (n_1 - 2) + (n_2 - 2)$. Note that $n_1 - 2 \geq 2$ and $n_2 - 2 \geq 2$. Hence, by induction, $H$ has a decomposition into two spanning subgraphs $W_1$ and $W_2$ such that $W_i$ is $(n_i - 2)$-regular, $(n_i - 2)$-connected, and bipancyclic or nearly bipancyclic for $i = 1, 2$. Therefore, each $W_i$ contains a Hamiltonian cycle. By [1], the product of two cycles has a Hamiltonian decomposition. Hence $C_{n-1} \square C_n$ can be decomposed into two Hamiltonian cycles, say $Z_1$ and $Z_2$. This implies that $G = (W_1 \cup W_2) \square (Z_1 \cup Z_2) = (W_1 \square Z_1) \cup (W_2 \square Z_2) = G_1 \cup G_2$, where $G_1 = W_1 \square Z_1$ and $G_2 = W_2 \square Z_2$. Hence $G_1$ and $G_2$ are edge-disjoint spanning subgraphs of $G$ with $G = G_1 \cup G_2$. By Lemma 2.7, $G_i$ is bipancyclic and further, by Lemma 2.3, it is $n_i$-regular and $n_i$-connected for $i = 1, 2$. Thus $G_1$ and $G_2$ give a decomposition of $G$ as required. ■

**Remark 2.8.** It is worth mentioning that Theorem 1.2 gives a partial solution to the following question due to Mader [14, p. 73].

*Given any $n$-connected graph and $k \in \{1, 2, \ldots, n\}$, is there always a $k$-connected subgraph $H$ of $G$ so that $G - E(H)$ is $(n-k)$-connected ?*

## 3. Decomposition Into $k$ Subgraphs

In this section, we prove Theorem 1.4. Firstly, we give a construction of obtaining $I$-connected spanning subgraph of $C_1 \square C_2 \square \cdots \square C_n$ from the given $I$-connected spanning subgraph of $C_1 \square C_2 \square \cdots \square C_{n-1}$.

Suppose $n \geq 2$. Let $C_1, C_2, \ldots, C_n$ be even cycles, $H = C_1 \square C_2 \square \cdots \square C_{n-1}$ and $G = H \square C_n$. Let $|V(C_n)| = s$. Then $s \geq 4$. Let $H^j$ be the copy of $H$ in $G$ corresponding to $j$th vertex of the cycle $C_n$. Then, as in the proof of Proposition 2.4, $G = H^1 \cup H^2 \cup \cdots \cup H^s \cup F$, where $F = \bigcup_{j=1}^{s} (\{xy \colon x \in V(H^j), y \in V(H^{j+1})\})$. By Lemma 2.3, $H$ is $(2n-2)$-connected. Let $l$ be an even integer such that $2 \leq l \leq 2n-2$ and let $K$ be a spanning $l$-connected subgraph of $H$ in which $M = \{u_1u_2, u_3u_4, \ldots, u_{l-1}u_l\}$ is a matching consisting of $l/2$ edges $u_iu_{i+1}$. Let $K^j$

be the corresponding copy of $K$ in $H^j$ and let $M^j = \{u_1^j u_2^j, u_3^j u_4^j, \ldots, u_{l-1}^j u_l^j\}$ be the matching in $K^j$ corresponding to the matching $M$. Then $K^j$ is $l$-connected and $K^j - M^j$ is $l/2$-connected. In $G$, each vertex of $K^j$ is adjacent to the corresponding vertex of $K^{j+1}$ through an edge from $F$. Let $N = \{u_i^j u_i^{j+1} \colon j \in [s]$ and $j$ odd; $i \in [l]$ and $i$ odd$\} \cup \{u_i^j u_i^{j+1} \colon j \in [s]$ and $j$ even; $i \in [l]$ and $i$ even$\}$. Then $N \subseteq F$ and $N$ is a matching in $G$. Also, $V(N) = V(M^1) \cup V(M^2) \cup \cdots \cup V(M^s)$.

Let $W = \left(\bigcup_{j=1}^s (K^j - M^j)\right) \cup N$ (see Figure 2). Then $W$ is a spanning subgraph of $G$. We prove below that $W$ is $l$-connected also.

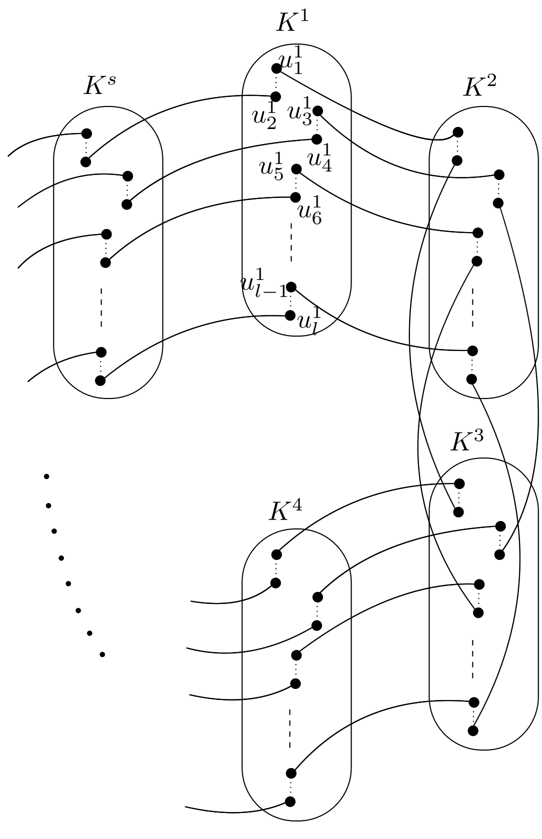


Figure 2. The spanning subgraph $W$ of $G$.

**Lemma 3.1.** *The graph $W$ defined above is $l$-connected.*

***Proof.*** Let $S \subset V(W)$ with $|S| \leq l-1$. It suffices to prove that $W - S$ is connected. As $V(W) = V(K^1) \cup V(K^2) \cup \cdots \cup V(K^s)$, $S \subset V(K^1) \cup V(K^2) \cup \cdots \cup V(K^s)$. Let $S^j = S \cap V(K^j)$ for $j \in [s]$. By $j+1$ and $j-1$, we mean $j+1 (\mathrm{mod}\ s)$ and $j-1 (\mathrm{mod}\ s)$, respectively.

Let $W^j = K^j - M^j$ for $j \in [s]$. Since $K^j$ is $l$-connected and matching $M^j$ contains $l/2$ edges, $W^j$ is $l/2$-connected and further, it contains all $l$ vertices of $M^j$ half of which have neighbours in $W^{j-1}$ while the remaining half have neighbours in $W^{j+1}$. Clearly, $W^j - S^j$ contains $l - |S^j| > |S| - |S^j| = (|S^1| + |S^2| + \cdots + |S^{j-1}| + |S^j| + |S^{j+1}| + \cdots + |S^s|) - |S^j| \geq |S^{j-1}| + |S^{j+1}|$ vertices of $M^j$. Therefore $W^j - S^j$ has at least one neighbour in $W^{j-1} - S^{j-1}$ or $W^{j+1} - S^{j+1}$. Further, if $W^j - S^j$ has no neighbour in $W^{j+1} - S^{j+1}$, then $|S^j| + |S^{j+1}| \geq l/2$.

We may assume that $|S^1| \geq |S^j|$ for all $j \in [s]$. Then $|S^j| < l/2$ for $j \neq 1$ as $|S| = \sum_{j=1}^{s} |S^j| < l$. Therefore, as $W^j$ is $l/2$-connected, $W^j - S^j$ is connected for $j \neq 1$.

Suppose $|S^1| \geq l/2$. Then $\sum_{j\neq 1} |S^j| < l/2$. Therefore $W^j - S^j$ has a neighbour in $W^{j+1} - S^{j+1}$ for $2 \leq j \leq s-1$. Let $T = W - (S \cup V(K^1))$. Then $T$ is a connected subgraph of $W$. We prove that every vertex of $W^1 - S^1$ has a neighbour in $T$. Let $D$ be a component of $W^1 - S^1$. If $W^1 - S^1$ is connected, then $D = W^1 - S^1$. Note that $W^1 - S^1 = (K^1 - M^1) - S^1 = (K^1 - S^1) - M^1$. Since $K^1$ is $l$-connected, $K^1 - S^1$ is $(l - |S^1|)$-connected. Therefore, if $W^1 - S^1$ is not connected, then $D$ contains at least $l - |S^1| > |S^2| + |S^s|$ vertices of the matching $M^1$. Each of these vertices has a neighbour in $W^2$ or $W^s$. In any case, $D$ has a neighbour in $W^2 - S^2$ or $W^s - S^s$ and so is in the connected graph $T$. This implies that $W - S$ is connected.

Suppose $|S^1| < l/2$. Then $|S^j| < l/2$ for all $j$. Hence $W^j - S^j$ is connected. Suppose $W^j - S^j$ has no neighbour in $W^{j+1} - S^{j+1}$ for some $j$. Then $W^{j-1} - S^{j-1}$ contains a neighbour of $W^j - S^j$. By the same argument, $W^{j+2} - S^{j+2}$ contains a neighbour of $W^{j+1} - S^{j+1}$. Suppose $W^i - S^i$ has no neighbour in $W^{i+1} - S^{i+1}$ for some $i \neq j$. Then $i \neq j-1, i \neq j+1$ and further, $|S^i| + |S^{i+1}| \geq l/2$. Therefore $|S| \geq |S^j| + |S^{j+1}| + |S^i| + |S^{i+1}| \geq l/2 + l/2 = l$, a contradiction. Hence, for any $i \neq j$, $W^i - S^i$ has neighbours in $W^{j+1} - S^{i+1}$. This implies that $W - S$ is connected. ■

**Definition 3.2.** Let $G$ be the product of $n$ even cycles. Suppose the diameter of $G$ is $d$. Let $v_0$ be an end-vertex of a path in $G$ of length $d$. Fix $v_0$. Let $V_0 = \{v_0\}$ and let $V_i = \{v \in V(G) : d(v_0, v) = i\}$ for $i \in [d]$, where $d(v_0, v)$ denotes the distance between $v_0$ and $v$ in $G$. Clearly, the sets $V_0, V_1, \ldots, V_d$ are mutually disjoint, non-empty and they partition the set $V(G)$. Let $K$ be a spanning subgraph of $G$. For $i \in [d]$, let $E_i(K) = \{xy \in E(K) : x \in V_{i-1},\ y \in V_i\}$. Then the edge sets $E_1(K), E_2(K), \ldots, E_d(K)$ are non-empty and mutually disjoint (see Figure 3).

**Lemma 3.3.** *Let $G, K$ and $E_i(K)$ be as in Definition 3.2. Then $E_1(K), E_2(K), \ldots, E_d(K)$ partition the edge set $E(K)$ of the graph $K$.*

***Proof.*** By definition of $V_i$, there is no edge in $G$ with one end-vertex in $V_j$ and the other in $V_{j'}$ when $|j - j'| \neq 1$. Suppose two vertices $x$ and $y$ of some

$V_i$ are adjacent. Let $P_x$ and $P_y$ be shortest paths in $G$ from $v_0$ to $x$, and $v_0$ to $y$ respectively. Then each of $P_x$ and $P_y$ takes exactly one vertex from each of $V_0, V_1, \ldots, V_i$. Therefore $P_x \cup P_y \cup \{xy\}$ contains an odd cycle in $G$, a contradiction to the fact that $G$ is bipartite. Thus each $V_i$ is independent. This implies that $E(K) = E_1(K) \cup E_2(K) \cup \cdots \cup E_d(K)$. ■

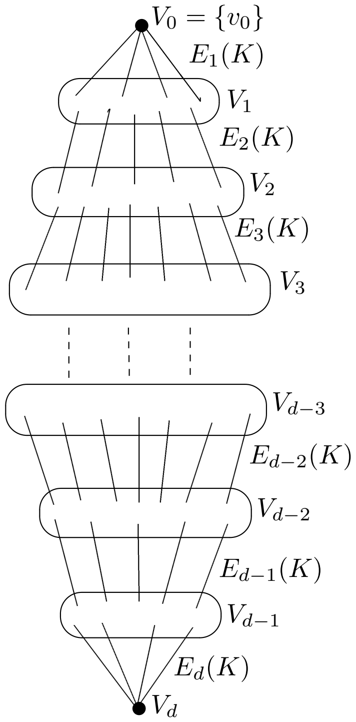


Figure 3. A decomposition of $G$.

We need the following result.

**Lemma 3.4** [16]**.** *Let $G_i$ be a graph with diameter $d_i$ for $i = 1, 2, \ldots, k$. Then the diameter of the graph $G_1 \square G_2 \square \cdots \square G_k$ is $d_1 + d_2 + \cdots + d_k$.*

We are all set to prove Theorem 1.4. This theorem is restated below for convenience.

**Theorem 3.5.** *Let $G$ be the product of $n$ even cycles and let $n = n_1 + n_2 + \cdots + n_k$ with $k \geq 2$ and $n_i \geq 1$ for $i \in [k]$. Then $G$ can be decomposed into $k$ spanning subgraphs $G_1, G_2, \ldots, G_k$ such that each $G_i$ is $2n_i$-regular and $2n_i$-connected.*

***Proof.*** We prove the result by induction on $n$. Obviously, $n \geq k$. If $n = k$, then $G$ is the product of $k$ cycles and hence, by [1], $G$ can be decomposed into $k$ Hamiltonian cycles. Thus the result holds for $n = k$.

Suppose $n \geq k+1$. Then $n_i \geq 2$ for some $i \in [k]$. Without loss of generality, we may assume that $n_k \geq 2$. Assume that the result holds for $n-1$. Consider $n-1 = n_1 + n_2 + \cdots + n_{k-1} + (n_k - 1)$. Let $G = C_1 \Box C_2 \Box \cdots \Box C_n$, where $C_1, C_2, \ldots, C_n$ are even cycles. Let $|C_n| = s$ and let $H = C_1 \Box C_2 \Box \cdots \Box C_{n-1}$. Then, as in the proof of Proposition 2.4, $G = H \Box C_n = H^1 \cup H^2 \cup \cdots \cup H^s \cup F$, where $H^j$ is a copy of $H$ and $F = \bigcup_{j=1}^{s}(\{xy\colon x \in V(H^j),\ y \in V(H^{j+1})\})$ with $H^{s+1} = H^1$.

By induction, $H$ can be decomposed into $k$ spanning subgraphs $H_1, H_2, \ldots,$ $H_k$ such that $H_i$ is $2n_i$-regular and $2n_i$-connected for $i \in [k-1]$, and $H_k$ is $2(n_k-1)$-regular and $2(n_k-1)$-connected.

Let $d$ be the diameter of $H$. Since each $C_i$ is an even cycle, the diameter of $C_i$ is $|C_i|/2 \geq 2$. Therefore, by Lemma 3.4, $d = \frac{|C_1|+|C_2|+\cdots+|C_{n-1}|}{2} \geq 2(n-1) = 2n-2 \geq 2n-2n_k$. Let $u_0$ be an end-vertex of a path in $H$ of length $d$. As in the Definition 3.2, we partition the vertex set $V(H)$ of $H$ into the sets $V_0(H), V_1(H), \ldots, V_d(H)$, where $V_0(H) = \{u_0\}$ and $V_i(H) = \{u \in V(H)\colon d(u, u_0) = i\}$ for $i \in [d]$. Since $H_i$ for $i \in [k]$ is a spanning subgraph of $H$, it follows from Lemma 3.3 that the edge set $E(H_i)$ of $H_i$ can be partitioned into the sets $E_1(H_i), E_2(H_i), \ldots, E_d(H_i)$, where $E_t(H_i) = \{xy \in E(H_i)\colon x \in V_{t-1},\ y \in V_t\}$ for $t \in [d]$. Note that if $e \in E_t(H_i)$ and $f \in E_{t'}(H_i)$ with $t' \geq t+2$, then $e$ and $f$ are vertex-disjoint (see Figure 3).

For $i \in [k-1]$, we obtain a matching $M_i$ of $H_i$ by choosing one edge from $n_i$ consecutive sets $E_{2t-1}(H_i)$ as follows.

Choose one edge from each of the sets $E_1(H_1), E_3(H_1), \ldots, E_{2n_1-1}(H_1)$ to get $M_1$. Thus, we let $M_1 = \{u_{t-1}u_t \in E_t(H_1)\colon t = 1, 3, 5, \ldots, 2n_1 - 1\}$. In general, we define $M_i = \{u_{t-1}u_t \in E_t(H_i)\colon t = 2p_i+1, 2p_i+3, \ldots, 2p_i + 2n_i - 1\}$, where $p_1 = 0$ and $p_i = n_1 + n_2 + n_3 + \cdots + n_{i-1}$ for $2 \leq i \leq k-1$.

For $j \in [s]$, the graph $H^j$ is a copy of $H$. Let $H_i^j$ be the subgraph of $H^j$ corresponding to the subgraph $H_i$ of $H$ for $i \in [k]$. Therefore the graphs $H_1^j, H_2^j, \ldots, H_k^j$ decompose the graph $H^j$. Further, the edge set $E(H_i^j)$ has a partition into non-empty sets $E_1(H_i^j), E_2(H_i^j), \ldots, E_d(H_i^j)$. For $i \in [k-1]$, let $M_i^j$ be the matching in $H_i^j$ corresponding to the matching $M_i$ of $H$ and let $u_t^j$ be the vertex of $H^j$ corresponding to the vertex $u_t$ of $H$. Then $M_i^j = \{u_{t-1}^j u_t^j \in E_t(H_i^j)\colon t = 2p_i+1, 2p_i+3, \ldots, 2p_i+2n_i-1\}$. Let $M^j$ be the union of these $k-1$ matchings $M_i^j$. Therefore $M^j = \bigcup_{i=1}^{k-1} M_i^j = \left\{u_0^j u_1^j, u_2^j u_3^j, \ldots, u_{2n_1+\cdots+2n_{k-1}-2}^j u_{2n_1+\cdots+2n_{k-1}-1}^j\right\}$. Clearly, $M^j$ is a matching in $H^j$.

We now construct the subgraphs $G_1, G_2, \ldots, G_k$ of $G$ which give a decomposition of $G$, as required.

**Construction of the graphs $G_i$ for $i \in [k]$.**

Let $i \in [k-1]$. We obtain $G_i$ from $H_i^1 \cup \cdots \cup H_i^s$ by deleting the matching $M_i^j$ from $H_i^j$ for each $j$ and then adding a matching $D_i$ consisting of edges from the set

$F$ having one end in $M_i^j$ and the other end in $M_i^{j+1}$ or $M_i^{j-1}$. More precisely, let $D_i = \{u_t^j u_t^{j+1} : j = 1, 3, \ldots, s-1;\ t = 2p_i, 2p_i+2, \ldots, 2p_i+2n_i-2\} \cup \{u_t^j u_t^{j+1} : j = 2, 4, \ldots, s;\ t = 2p_i+1, 2p_i+3, \ldots, 2p_i+2n_i-1\}$.

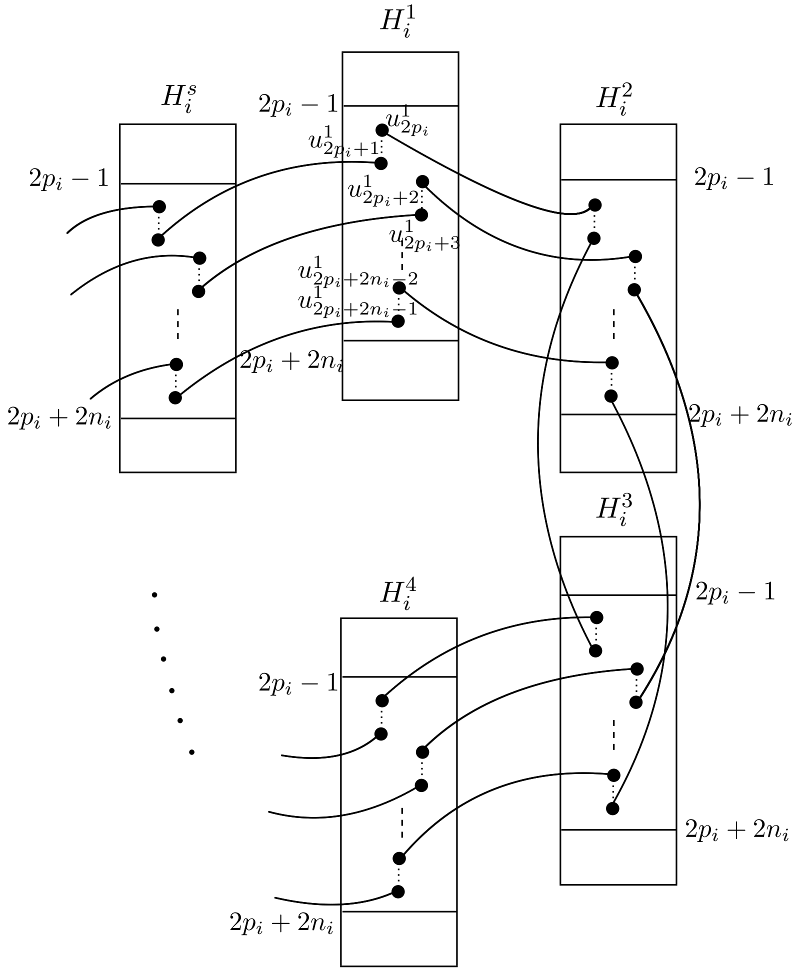


Figure 4. The graph $G_i$.

For $i \in [k-1]$, let $G_i = \left(\bigcup_{j=1}^{s}(H_i^j - M_i^j)\right) \cup D_i$ (see Figure 4). Note that $D_i$ is a matching consisting of $n_i$ edges between $H^j$and $H^{j+1}$ for each $j \in [s]$ and so the total number of edges in $D_i$ is $sn_i$.

For any $i \in [k-1]$ and $i' \in [k-1]$ with $i \neq i'$, the graphs $H_i^j$ and $H_{i'}^j$ are edge-disjoint for each $j$. This implies that $G_1, G_2, \ldots, G_{k-1}$ are mutually edge-disjoint subgraphs of $G$. Since $H_i^j$ is a $2n_i$-regular and spanning subgraph of $H^j$, $G_i$ is also a $2n_i$-regular and spanning subgraph of $G$. Further, as $H_i^j$ is $2n_i$-connected, Lemma 3.1 implies that $G_i$ is also $2n_i$-connected.

Let $G_k = G - E(G_1 \cup G_2 \cup \cdots \cup G_{k-1})$. The graph $G_k$ is shown in Figure 5. It is easy to see that $G_k = \left(\bigcup_{j=1}^{s}(H_k^j \cup M^j)\right) \cup (F - D)$, where $D = \bigcup_{i=1}^{k-1} D_i$. The edges of the matching $M^j$ are shown by the bold edges in Figure 5. Clearly, $D$ is a matching in $G$ consisting of $s(n_1 + n_2 + \cdots + n_{k-1}) = s(n - n_k)$ edges of $F$.

It follows that the graph $G_k$ is a spanning and $2n_k$-regular subgraph of $G$. Thus the graph $G$ decomposes into the spanning subgraphs $G_1, G_2, \ldots, G_k$.

It only remains to prove that the graph $G_k$ is $2n_k$-connected.

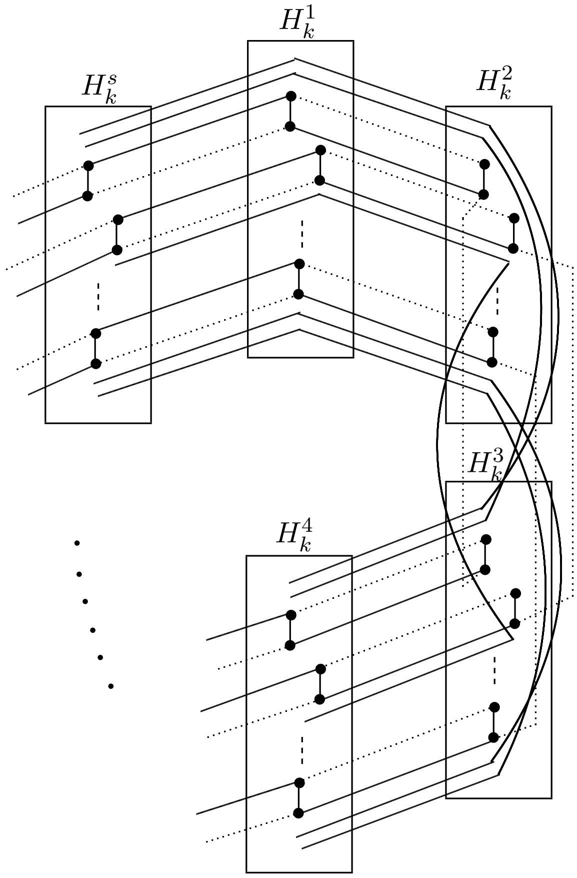


Figure 5. The graph $G_k$.

**Claim.** *$G_k$ is $2n_k$-connected.*

***Proof.*** Let $S \subset V(G_k) = \bigcup_{j=1}^{s} V(H^j)$ such that $0 < |S| \leq 2n_k - 1$. It suffices to prove that $G_k - S$ is connected. Let $S^j = V(H_k^j) \cap S$ for $j \in [s]$. Since $V(H_k^j) = V(H^j), |V(H_k^j)| = |V(H^j)| = r = |C_1||C_2| \cdots |C_{n-1}| \geq 4^{n-1} = 2^{2n-2} =$

$2^{2n-2n_k+2n_k-2} = 2^{2n-2n_k} \cdot 2^{2n_k-2} > (2n-2n_k)(2n_k-1) \geq 2(n-n_k)|S| \geq (n-n_k)+|S|$. Hence, there are at least $r-(n-n_k)-|S| > 0$ edges between $H_k^j - S^j$ and $H_k^{j+1} - S^{j+1}$ in $G_k - S$.

Obviously at least one $S^j$ is non-empty. We may assume that $S^1 \neq \emptyset$ and $|S^1| \geq |S^j|$ for $j \in [s]$. Suppose two more $S^j$ are non-empty. Then $|S^j| < 2n_k - 2$ for $j \in [s]$ as $|S| \leq 2n_k - 1$. Hence each $H_k^j - S^j$ is connected as $H_k^j$ is $(2n_k-2)$-connected. Further, $H_k^j - S^j$ is connected to $H_k^{j+1} - S^{j+1}$ by edges of $F - D$. This implies that $G_k - S$ is connected.

Suppose $S^j = \emptyset$ for all $j \neq 1$. Therefore $H_k^j - S^j = H_k^j$ is connected for all $j \neq 1$. Obviously, each vertex of $H_k^1 - S^1$ has a neighbour in $H_k^2$ or $H_k^s$. Hence $G_k - S$ is connected.

Suppose only one $S^j$ other than $S^1$ is nonempty. Suppose $S^2 \neq \emptyset$. Then $S = S^1 \cup S^2$. If $H_k^1 - S^1$ and $H_k^2 - S^2$ are connected, then they are connected to each other by an edge of $F - D$ and so $G_k - S$ is connected. Suppose $H_k^1 - S^1$ is not connected. Then $|S^1| = 2n_k - 2$ and $|S^2| = 1$ as $|S| \leq 2n_k - 1$ and $H_k^1$ is $(2n_k - 2)$-connected. This implies that $H_k^j - S^j$ for any $j \neq 1$ is connected. Let $T = G_k - (V(H_k^1) \cup S) = G_k - (V(H_k^1) \cup S^2)$. Then $T$ is connected. It suffices to prove that every component of $H_k^1 - S^1$ has a neighbour in $T$. Let $W$ be a component of $H_k^1 - S^1$ and let $v$ be a vertex of $W$. If $v$ has a neighbour in $H_k^s$, then we are through. Suppose $v$ has no neighbour in $H_k^s$. Then $v$ has a neighbour $v'$ in $H_k^2$. If $v' \notin S^2$, then also we are through. Suppose $v' \in S^2$. Then $S^2 = \{v'\}$. Also, $v$ is an end-vertex of an edge of the matching $M^1$. Therefore the degree of $v$ in $H_k^1$ is $2n_k - 1$. Hence $v$ has a neighbour $u$ in $H_k^1 - S^1$. Obviously, $u$ is in $W$. Further, $u$ has a neighbour in the subgraphs $H_k^s$ or $H_k^2 - S^2 = H_k^2 - \{v'\}$ of $T$. Thus $W$ has a neighbour in the connected graph $T$. Hence $G_k - S$ is connected.

Similarly, $G_k - S$ is connected when $S^s \neq \emptyset$.

Suppose $S^j \neq \emptyset$ for some $j \notin \{1, 2, s\}$. Then every component of $H_k^1 - S^1$ has a neighbour in $H_k^2$ or $H_k^s$. It follows that $G_k - S$ is connected. This proves the claim. □

Thus, the graph $G$ decomposes into the spanning subgraphs $G_1, G_2, \ldots, G_k$, where $G_i$ is $2n_i$-regular and $2n_i$-connected for $i = 1, 2, \ldots, k$. This completes the proof. ■

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Received 10 October 2016
Revised 21 July 2017
Accepted 21 July 2017