DECOMPOSITION OF THE PRODUCT OF CYCLES BASED ON DEGREE PARTITION

Y.M. BORSE AND S.R. SHAIKH

Department of Mathematics
Savitribai Phule Pune University
Pune-411007, India

e-mail: ymborse11@gmail.com
shazia_31082@yahoo.co.in

Abstract

The Cartesian product of an even number of cycles is a 2n-regular, 2n-connected and bipancyclic graph. Let G be the Cartesian product of n even cycles and let $2n = n_1 + n_2 + \cdots + n_k$ with $k \geq 2$ and $n_i \geq 2$ for each $i$. We prove that if $k = 2$, then G can be decomposed into two spanning subgraphs $G_1$ and $G_2$ such that each $G_i$ is $n_i$-regular, $n_i$-connected, and bipancyclic or nearly bipancyclic. For $k > 2$, we establish that if all $n_i$ in the partition of $2n$ are even, then G can be decomposed into $k$ spanning subgraphs $G_1, G_2, \ldots, G_k$ such that each $G_i$ is $n_i$-regular and $n_i$-connected. These results are analogous to the corresponding results for hypercubes.

Keywords: hypercube, Cartesian product, $n$-connected, regular, bipancyclic, spanning subgraph.

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1. Introduction

The graphs considered in this paper are simple, undirected and finite. The Cartesian product of two graphs $G_1$ and $G_2$ is the graph $G_1 \square G_2$ with vertex set $V(G_1) \times V(G_2)$ in which $(u_1, u_2)$ is adjacent to $(v_1, v_2)$ if and only if $u_1$ is adjacent to $v_1$ in $G_1$ and $u_2 = v_2$, or $u_2$ is adjacent to $v_2$ in $G_2$ and $u_1 = v_1$. The $n$-dimensional hypercube $Q_n$ is the Cartesian product of $n$ copies of the complete graph $K_2$. Therefore $Q_n$ is the Cartesian product of $n/2$ copies of a cycle of length 4 when $n$ is even. The Cartesian product of cycles and hypercubes are popular interconnection network topologies (see [6, 11]). The hypercube $Q_n$ is
an $n$-regular and $n$-connected graph whereas the Cartesian product of $n$ cycles is a $2n$-regular and $2n$-connected graph (see [16]).

Unless mentioned otherwise, in the remainder of this paper product means the Cartesian product of graphs.

A cycle is even if its length is a positive even integer. A bipartite graph $G$ is bipancyclic if $G$ is either a cycle or has cycles of every even length from 4 to $|V(G)|$. A 3-regular graph is nearly bipancyclic if it has cycles of every even length from 4 to $|V(G)|$ except possibly for 4 and 8. The bipancyclicity property of a given network is an important factor in determining whether the network topology can simulate rings of various lengths.

Alspach et al. [1] proved that the product of cycles can be decomposed into Hamiltonian cycles. This result subsumes earlier results due to Kotzig [10] and Foregger [8] on Hamiltonian decomposition of the product of cycles. El-Zanati and Eynden [7] studied the decomposition of the product of cycles, each of length a power of 2, into non-spanning cycles. Borse et al. [4] proved that if $m \geq 2$ and $m$ divides $n$, then the product of $n$ even cycles can be decomposed into isomorphic, spanning, $m$-regular, $m$-connected subgraphs which are bipancyclic or nearly bipancyclic also. The analogous results for the class of hypercubes are obtained in [1, 4, 7].

Motivated by applications in parallel computing, Borse and Kandekar [3] considered the decomposition of the hypercube $Q_n$ into two regular spanning subgraphs according to the partition of $n$ into two parts and obtained the following result.

**Theorem 1.1.** Let $n, n_1, n_2 \geq 2$ be integers such that $n = n_1 + n_2$. Then the hypercube $Q_n$ can be decomposed into two spanning subgraphs $G_1$ and $G_2$ such that $G_i$ is $n_i$-regular and $n_i$-connected for $i = 1, 2$. Moreover, $G_i$ is bipancyclic if $n_i \neq 3$ and nearly bipancyclic if $n_i = 3$.

We extend this result to the class of the product of even cycles as follows.

**Theorem 1.2.** Let $n, n_1, n_2 \geq 2$ be integers such that $2n = n_1 + n_2$ and let $G$ be the product of $n$ even cycles. Then $G$ can be decomposed into two spanning subgraphs $G_1$ and $G_2$ such that $G_i$ is $n_i$-regular and $n_i$-connected for each $i = 1, 2$. Moreover, $G_i$ is bipancyclic if $n_i \neq 3$ and nearly bipancyclic if $n_i = 3$.

For the decomposition of $Q_n$ according to the general partition of $n$, Sonawane and Borse [15] proved the following result.

**Theorem 1.3** [15]. Let $k, n_1, n_2, \ldots, n_k \geq 2$ be integers such that at most one $n_i$ is odd and $n = n_1 + n_2 + \cdots + n_k$. Then $Q_n$ can be decomposed into $k$ spanning subgraphs $G_1, G_2, \ldots, G_k$ such that each $G_i$ is $n_i$-regular and $n_i$-connected.

We extend this result also to the class of the product of even cycles as follows.
**Theorem 1.4.** Let \( n, k \geq 2 \) and \( n_1, n_2, \ldots, n_k \geq 1 \) be integers such that \( n = n_1 + n_2 + \cdots + n_k \) and \( G \) be the product of \( n \) even cycles. Then \( G \) can be decomposed into \( k \) spanning subgraphs \( G_1, G_2, \ldots, G_k \) such that each \( G_i \) is \( 2n_i \)-regular and \( 2n_i \)-connected.

We prove Theorem 1.2 in Section 2. The proof of Theorem 1.4 is given in Section 3.

## 2. Decomposition Into Two Subgraphs

In this section, we prove Theorem 1.2. Firstly, we prove this theorem for the special cases \( n_1 = 2 \) and \( n_1 = 3 \). The general case follows from these two cases.

For \( n \geq 1 \), let \( [n] = \{1, 2, \ldots, n\} \). We define a particular type of 3-regular graph below.

**Definition 2.1.** Let \( r, s \geq 4 \) be even integers and let \( W \) be the 3-regular graph with vertex set \( V(W) = \{v_i^j : i \in [r]; j \in [s]\} \) and the edge set \( E(W) = \{v_i^j v_{i+1}^j : i \in [r]; j \in [s]\} \cup \{v_i^j v_i^{j+1} : i = 1, 3, 5, \ldots, r-1; j = 1, 3, 5, \ldots, s-1\} \cup \{v_i^j v_{i+1}^j : i = 2, 4, 6, \ldots, r; j = 2, 4, 6, \ldots, s\} \), where \( v_i^{s+1} = v_i^1 \) and \( v_{r+1}^j = v_1^j \) (see Figure 1). The graph \( W \) is isomorphic to a honeycomb toroidal graph \( H(s, r, 0) \) defined in [2].

![Figure 1. The graph W.](image)

We need the following lemmas.

**Lemma 2.2** [4]. The graph \( W \) defined above is 3-regular, 3-connected and nearly bipancyclic.
Lemma 2.3 [16]. Let \( G_i \) be an \( m_i \)-regular and \( m_i \)-connected graph for \( i = 1, 2 \). Then the graph \( G_1 \boxplus G_2 \) is \( (m_1 + m_2) \)-regular and \( (m_1 + m_2) \)-connected.

We prove the special case \( n_1 = 2 \) of Theorem 1.2 in the following proposition.

Proposition 2.4. Let \( n \geq 2 \) and let \( G \) be the product of \( n \) even cycles. Then \( G \) has a Hamiltonian cycle \( C \) such that \( G - E(C) \) is a spanning, \( (2n - 2) \)-regular, \( (2n - 2) \)-connected and bipancyclic subgraph of \( G \).

**Proof.** We prove the result by induction on \( n \). The result holds for \( n = 2 \) as, by [1], the product of two cycles can be decomposed into two Hamiltonian cycles. Suppose \( n \geq 3 \). Let \( G = C_1 \square C_2 \square \cdots \square C_n \), where \( C_1, C_2, \ldots, C_n \) are even cycles. Let \( H = C_1 \square C_2 \square \cdots \square C_{n-1} \), then \( |V(H)| = r \) and \( |V(C_n)| = s \). Then \( G = H \square C_n \) and further, \( r \) and \( s \) are even integers such that \( r = |C_1| \cdot |C_2| \cdot \cdots |C_{n-1}| \geq 4^{n-1} \geq 16 \) and \( s \geq 4 \). Label the vertices of the cycle \( C_n \) by \( \{1, 2, \ldots, s\} \) so that \( j \) is adjacent to \( j + 1 \) modulo \( s \).

By induction, \( H \) has a Hamiltonian cycle, say \( Z \), such that \( H - E(Z) \) is a spanning, \( (2n - 4) \)-regular, \( (2n - 4) \)-connected and bipancyclic subgraph of \( H \). Label the vertices of the cycle \( Z \) by the set \( \{v_1, v_2, \ldots, v_r\} \) so that \( v_p \) is adjacent to \( v_{p+1 \mod r} \). For compactness, let \( \upsilon^j_p \) denote the vertex \( (v_p, j) \) of \( H \square C_n \), let superscripts be computed modulo \( s \) and subscripts be modulo \( r \) with representative in \( [r] \). For \( j \in [s] \), let \( H^j \) be the copy of \( H \) induced by the set \( \{\upsilon^j_p : p \in [r]\} \) and let \( Z^j \) be the copy of \( Z \) in \( H^j \). Let \( F \) be the set of edges of \( H \square C_n \) between the graphs \( H^j \), that is, \( F = \{\upsilon^j_p \upsilon^j_{p+1} : p \in [r], j \in [s]\} \). Then \( G = H^1 \cup H^2 \cup \cdots \cup H^s \cup F \).

We now construct a Hamiltonian cycle \( C \) by deleting one edge from \( Z^j \) and adding one edge of \( F \) between \( H^j \) and \( H^{j+1} \) for all \( j \). Let \( M = \{v_1^j v_2^j, v_1^j v_3^j, v_2^j v_3^j, v_1^j v_4^j, \ldots, v_2^j v_3^j, v_4^j v_5^j\} \) and let \( C = \left( \bigcup_{j=1}^{s} (Z^j - v_1^j v_2^j) \right) \cup M \). Clearly, \( C \) is a Hamiltonian cycle in \( G \).

Let \( K = G - E(C) \). Then \( K \) is a spanning \( (2n - 2) \)-regular subgraph of \( G \). Further, \( K = \left( \bigcup_{j=1}^{s} (H^j - Z^j) \cup \{v_1^j v_2^j\} \right) \cup (F - M) \). We prove that \( K \) is \( (2n - 2) \)-connected and bipancyclic.

**Claim 1.** \( K \) is bipancyclic.

**Proof.** We prove the claim by constructing a spanning bipancyclic subgraph of \( K \). Since \( H^j - E(Z^j) \) is bipancyclic, it has a Hamiltonian cycle \( X^j \) for \( j \in [s] \). Therefore \( V(X^j) = V(Z^j) = \{v_1^j, p \in [r]\} \). Let \( J = (F - M) \cup X^1 \cup X^2 \cup \cdots \cup X^s \). Then \( J \) is a spanning subgraph of \( K \). Note that the edge \( v_1^j v_2^j \) of \( Z^j \) is a chord of \( X^j \) in \( H^j \) and so, a subpath of \( X^j \) from \( v_1^j \) to \( v_2^j \) has odd length. Obtain a 3-regular spanning subgraph \( W \) of \( J \) by deleting alternate edges of \( F \) between \( X^j \) and \( X^{j+1} \) starting from the edge \( v_2^j v_2^{j+1} \) when \( j \) is odd, and starting from
the edge $v_j^1 v_{j+1}^1$ when $j$ is even. It is easy to see that $W$ is isomorphic to the graph in Figure 1. By Lemma 2.2, $W$ is nearly bipancyclic. Therefore $W$ and hence $J$ contains cycles of every even length from 10 to $rs = |V(J)|$. The ladder graph in $J$ formed by the paths $X^1 - v_j^1$ and $X^2 - v_j^2$ contains an $(l)$-cycle for any $l \in \{4, 6, 8\}$. Thus $J$ contains cycle of every even length from 4 to $|V(J)|$ and so $J$ is bipancyclic. As the graph $J$ spans $K$, the graph $K$ is also bipancyclic. □

Claim 2. $K$ is $(2n - 2)$-connected.

**Proof.** Let $D^j = (H^j - E(Z^j)) \cup \{v_j^1 v_j^2\}$ for $j \in [s]$. Then $K = \left(\bigcup_{j=1}^{s} D^j\right) \cup (F - M)$. Since $H^j - E(Z^j)$ is $(2n - 4)$-regular and $(2n - 4)$-connected, $D^j$ is $(2n - 4)$-connected, and the degree of each of $v_j^1$ and $v_j^2$ in $D^j$ is $2n - 3$ and the degree of each of the remaining vertices in $D^j$ is $2n - 4$. For any $j$, $D^j + 1$ and $D^j - 1$ are the two neighbouring subgraphs of $D^j$ in the graph $K$. Note that in $K$ every vertex of $D^j$ except $v_j^1$ and $v_j^2$ has neighbours in both $D^j + 1$ and $D^j - 1$, whereas each of $v_j^1$ and $v_j^2$ has a neighbour in exactly one of $D^j + 1$ and $D^j - 1$.

Let $S \subset V(K)$ such that $|S| \leq 2n - 3$. To prove the claim, it suffices to prove that $K - S$ is connected. As $V(D^j) = V(H^j)$ and $n \geq 3, |V(D^j)| \geq 4n - 1 \geq 2n + 1 \geq |S| + 4$. Hence there are at least three edges between $D^j - S$ and $D^j + 1 - S$ in $K - S$ for any $j \in [s]$. Therefore, if $D^j - S$ is connected for all $j \in [s]$, then $K - S$ is connected. Suppose $D^j - S$ is not connected for some $j$. Without loss of generality, we may assume that $D^1 - S$ is not connected. Then $D^1$ contains at least $2n - 4$ vertices from $S$. If $S \subset V(D^1)$, then every vertex of $D^1 - S$ has a neighbour in the connected graph $K - V(D^1)$ and so $K - S$ is connected. Suppose $S$ is not a subset of $V(D^1)$. Then $|S| = 2n - 3$ and $|V(D^1) \cap S| = 2n - 4$, and so only one vertex, say $x$, from $S$ is in $V(K) - V(D^1)$. Let $W = K - (V(D^1) \cup \{x\})$. Then $W$ is connected. Every vertex of $D^1 - S$ except possibly $v_1^1$ and $v_2^1$ has a neighbour in $W$. Since the degree of each of $v_1^1$ and $v_2^1$ is $2n - 3$ in $D^1$, these vertices cannot be isolated in $D^1 - S$. Hence the component of $D^1 - S$ containing $v_1^1$ or $v_2^1$ has a neighbour in $W$. This implies that $K - S$ is connected. □

Thus, $C$ is a Hamiltonian cycle in $G$ such that $G - E(C) = K$ is a spanning, $(2n - 2)$-regular, $(2n - 2)$-connected and bipancyclic subgraph of $G$.

**Remark 2.5.** By Lemma 2.3, the product $G$ of $n$ cycles is $2n$-regular and $2n$-connected. If $C$ is a cycle in $G$, then the minimum degree of $G - E(C)$ is $2n - 2$ and hence $G - E(C)$ cannot be $k$-connected for $k = 2n - 1$ or $k = 2n$. However, the above proposition guarantees the existence of a cycle in $G$ such that $G - E(C)$ is $(2n - 2)$-connected. Such a cycle is removable in $G$. This result can be compared with an older theorem of Mader [13] which states that if $H$ is a simple $n$-connected graph with minimum degree $n + 2$, then there is a cycle $C$ in $H$ such that $H - E(C)$ is $n$-connected (also see [5, 9]).
We now prove the special case $n_1 = 3$ of Theorem 1.2.

**Proposition 2.6.** Let $n \geq 3$ and let $G$ be the product of $n$ even cycles. Then $G$ has a spanning, 3-regular, 3-connected and nearly bipancyclic subgraph $W$ such that $G - E(W)$ is a spanning, $(2n - 3)$-regular, $(2n - 3)$-connected subgraph of $G$. Moreover, $G - E(W)$ is bipancyclic if $n \neq 3$, and it is nearly bipancyclic otherwise.

**Proof.** Let $C_1, C_2, \ldots, C_n$ be even cycles and let $G = C_1 \square C_2 \square \cdots \square C_n$. Write $G$ as $G = H \square C_n$, where $H = C_1 \square C_2 \square \cdots \square C_{n-1}$. It follows from Proposition 2.4 that $H$ has a decomposition into two subgraphs $C$ and $D$, where $C$ is a Hamiltonian cycle, and $D$ is spanning, $(2n - 4)$-regular, $(2n - 4)$-connected and bipancyclic. Obviously, $H = D \cup C$. Let $|V(C_n)| = s$ and $|V(H)| = r$. Then, as in the proof of Proposition 2.4, we have $G = H^1 \cup H^2 \cup \cdots \cup H^s \cup F$ with $V(H^j) = \{v_{ij}^p : p \in [r]\}$ and $F = \{v_{ij}^j v_{i+1}^j : i \in [r], \ j \in [s]\}$, where $H^j$ is the copy of $H$ corresponding to $j$th vertex of the cycle $C_n$. Further, $C^j$ is the copy of $C$ in $H^j$ with vertices $v_{1j}^j, v_{2j}^j, \ldots, v_{rj}^j$ in order. Partition the edge set $F$ into two parts $F_1$ and $F_2$, where $F_1 = \{v_{ij}^j v_{i+1}^j : i = 1, 3, 5, \ldots, r-1; j = 1, 3, 5, \ldots, s - 1\} \cup \{v_{ij}^j v_{i+1}^j : i = 2, 4, 6, \ldots, r; j = 2, 4, 6, \ldots, s\}$ and $F_2 = F - F_1$.

We now construct a 3-regular subgraph $W$ of $G$ as required. Let $W = C^1 \cup C^2 \cup \cdots \cup C^s \cup F_1$. Then $W$ is isomorphic to the graph in Figure 1. By Lemma 2.2, $W$ is a 3-regular, 3-connected and nearly bipancyclic subgraph of $G$. Let $W'' = G - E(W)$. Clearly, $W'' = D^1 \cup D^2 \cup \cdots \cup D^s \cup F_2$, where $D^j$ is the copy of $D$ in $H^j$. Further, $W'$ is a spanning and $(2n - 3)$-regular subgraph of $G$.

To complete the proof, it suffices to prove that $W'$ is bipancyclic and $(2n - 3)$-connected.

Let $Y^j$ be a Hamiltonian cycle in $D^j$ for $j \in [s]$. Let $W'' = Y^1 \cup Y^2 \cup \cdots \cup Y^s \cup F_2$. Then $W''$ is a spanning subgraph of $W'$. Observe that $W''$ is isomorphic to $W$ and so it is nearly bipancyclic. Hence $W'$ is nearly bipancyclic. Suppose $n \geq 4$. Then the cycles of lengths 4 and 8 exist in the graph $D^j$ and so in $W'$. Hence $W'$ contains cycles of every even length from 4 to $|V(W')|$. Thus $W'$ is bipancyclic in this case.

We now prove that $W'$ is $(2n - 3)$-connected. Let $S \subseteq W'$ with $|S| \leq 2n - 4$. It is enough to prove that $W' - S$ is connected. Suppose $S \subseteq D^j$ for some $j$. Then each component of $D^j - S$ is joined to $D^{j+1}$ or $D^{j-1}$ and hence $W' - S$ is connected. Suppose $S$ intersects at least two of $V(D^1), V(D^2), \ldots, V(D^s)$. Then $|S \cap V(D^j)| < (2n - 4)$ and hence $D^j - S$ is connected as $D^j$ is $(2n - 4)$-connected for each $j \in [s]$. Through the edges of the matching $F_2$, half of the vertices of $D^j$ have distinct neighbours in $D^{j+1}$ and the remaining half have distinct neighbours in $D^{j-1}$. Therefore the connected graph $D^j - S$ is joined to each of $D^{j+1} - S$ and $D^{j-1} - S$ in $W'$ by at least one edge. It implies that $W' - S$ is connected. Thus $W'$ is $(2n - 3)$-connected. This completes the proof. ■
The next lemma follows from the fact that the product of even cycles is bipancyclic (see [6]).

**Lemma 2.7.** If $G_1$ and $G_2$ are bipartite Hamiltonian graphs, then $G_1 \square G_2$ is bipancyclic.

We now prove Theorem 1.2.

**Proof of Theorem 1.2.** We may assume that $n_2 \geq n_1 \geq 2$. By Propositions 2.4 and 2.6, the result holds for $n_1 = 2$ and $n_1 = 3$. Therefore the result also holds for the cases $n = 2$ and $n = 3$. Suppose $n, n_1, n_2 \geq 4$. Assume that the result holds for all integers from 4 to $n - 1$. Let $G$ be the product of $n$ even cycles $C_1, C_2, \ldots, C_n$ and let $H = C_1 \square C_2 \square \cdots \square C_{n-2}$. Then $G = H \square (C_{n-1} \square C_n)$. Since $2n = n_1 + n_2$, we can express $2(n - 2)$ as $2(n - 2) = (n_1 - 2) + (n_2 - 2)$. Note that $n_1 - 2 \geq 2$ and $n_2 - 2 \geq 2$. Hence, by induction, $H$ has a decomposition into two spanning subgraphs $W_1$ and $W_2$ such that $W_1$ is $(n_1 - 2)$-regular, $(n_2 - 2)$-connected, and bipancyclic or nearly bipancyclic for $i = 1, 2$. Therefore, each $W_i$ contains a Hamiltonian cycle. By [1], the product of two cycles has a Hamiltonian decomposition. Hence $C_{n-1} \square C_n$ can be decomposed into two Hamiltonian cycles, say $Z_1$ and $Z_2$. This implies that $G = (W_1 \cup W_2) \square (Z_1 \cup Z_2) = (W_1 \square Z_1) \cup (W_2 \square Z_2) = G_1 \cup G_2$, where $G_1 = W_1 \square Z_1$ and $G_2 = W_2 \square Z_2$. Hence $G_1$ and $G_2$ are edge-disjoint spanning subgraphs of $G$ with $G = G_1 \cup G_2$. By Lemma 2.7, $G_i$ is bipancyclic and further, by Lemma 2.3, it is $n_i$-regular and $n_i$-connected for $i = 1, 2$. Thus $G_1$ and $G_2$ give a decomposition of $G$ as required.

**Remark 2.8.** It is worth mentioning that Theorem 1.2 gives a partial solution to the following question due to Mader [14, p. 73].

Given any $n$-connected graph and $k \in \{1, 2, \ldots, n\}$, is there always a $k$-connected subgraph $H$ of $G$ so that $G - E(H)$ is $(n - k)$-connected?

### 3. Decomposition Into $k$ Subgraphs

In this section, we prove Theorem 1.4. Firstly, we give a construction of obtaining $I$-connected spanning subgraph of $C_1 \square C_2 \square \cdots \square C_n$ from the given $I$-connected spanning subgraph of $C_1 \square C_2 \square \cdots \square C_{n-1}$.

Suppose $n \geq 2$. Let $C_1, C_2, \ldots, C_n$ be even cycles, $H = C_1 \square C_2 \square \cdots \square C_{n-1}$ and $G = H \square C_n$. Let $|V(C_n)| = s$. Then $s \geq 4$. Let $H^j$ be the copy of $H$ in $G$ corresponding to $j$th vertex of the cycle $C_n$. Then, as in the proof of Proposition 2.4, $G = H^1 \cup H^2 \cup \cdots \cup H^s \cup F$, where $F = \bigcup_{j=1}^{s} \{xy; x \in V(H^j), y \in V(H^{j+1})\}$. By Lemma 2.3, $H$ is $(2n - 2)$-connected. Let $l$ be an even integer such that $2 \leq l \leq 2n - 2$ and let $K$ be a spanning $l$-connected subgraph of $H$ in which $M = \{u_1 u_2, u_3 u_4, \ldots, u_{l-1} u_l\}$ is a matching consisting of $l/2$ edges $u_i u_{i+1}$. Let $K^j$
be the corresponding copy of \( K \) in \( H_j \) and let \( M^j = \{ u^1_j u^2_j, u^3_j u^4_j, \ldots, u^{l-1}_j u^j_l \} \) be the matching in \( K^j \) corresponding to the matching \( M \). Then \( K^j \) is \( l \)-connected and \( K^j - M^j \) is \( l/2 \)-connected. In \( G \), each vertex of \( K^j \) is adjacent to the corresponding vertex of \( K^j+1 \) through an edge from \( F \). Let \( N = \{ u^i_j u^{i+1}_j : j \in [s] \text{ and } j \text{ odd}; i \in [l] \text{ and } i \text{ odd} \} \cup \{ u^i_j u^{i+1}_j : j \in [s] \text{ and } j \text{ even}; i \in [l] \text{ and } i \text{ even} \} \). Then \( N \subseteq F \) and \( N \) is a matching in \( G \). Also, \( V(N) = V(M^1) \cup V(M^2) \cup \cdots \cup V(M^s) \).

Let \( W = \left( \bigcup_{j=1}^s (K^j - M^j) \right) \cup N \) (see Figure 2). Then \( W \) is a spanning subgraph of \( G \). We prove below that \( W \) is \( l \)-connected also.

**Figure 2.** The spanning subgraph \( W \) of \( G \).

**Lemma 3.1.** The graph \( W \) defined above is \( l \)-connected.

**Proof.** Let \( S \subset V(W) \) with \( |S| \leq l - 1 \). It suffices to prove that \( W - S \) is connected. As \( V(W) = V(K^1) \cup V(K^2) \cup \cdots \cup V(K^s) \), \( S \subset V(K^1) \cup V(K^2) \cup \cdots \cup V(K^s) \). Let \( S^j = S \cap V(K^j) \) for \( j \in [s] \). By \( j + 1 \) and \( j - 1 \), we mean \( j + 1(\mod s) \) and \( j - 1(\mod s) \), respectively.
Let $W^j = K^j - M^j$ for $j \in [s]$. Since $K^j$ is $l$-connected and matching $M^j$ contains $l/2$ edges, $W^j$ is $l/2$-connected and further, it contains all $l$ vertices of $M^j$ half of which have neighbours in $W^{j-1}$ while the remaining half have neighbours in $W^{j+1}$. Clearly, $W^j - S^j$ contains $l - |S^j| > |S| - |S^j| = (|S^1| + |S^2| + \cdots + |S^{j-1}| + |S^j| + |S^{j+1}| + \cdots + |S^s|) - |S^j| \geq |S^{j-1}| + |S^{j+1}|$ vertices of $M^j$. Therefore $W^j - S^j$ has at least one neighbour in $W^{j-1} - S^{j-1}$ or $W^{j+1} - S^{j+1}$. Further, if $W^j - S^j$ has no neighbour in $W^{j+1} - S^{j+1}$, then $|S^j| + |S^{j+1}| \geq l/2$. We may assume that $|S^1| \geq |S^j|$ for all $j \in [s]$. Then $|S^j| < l/2$ for $j \neq 1$ as $|S| = \sum_{j=1}^{s} |S^j| < l$. Therefore, as $W^j$ is $l/2$-connected, $W^j - S^j$ is connected for $j \neq 1$.

Suppose $|S^1| \geq l/2$. Then $\sum_{j \neq 1} |S^j| < l/2$. Therefore $W^j - S^j$ has a neighbour in $W^{j+1} - S^{j+1}$ for $2 \leq j \leq s-1$. Let $T = W^- (S \cup V(K^1))$. Then $T$ is a connected subgraph of $W$. We prove that every vertex of $W^1 - S^1$ has a neighbour in $T$. Let $D$ be a component of $W^1 - S^1$. If $W^1 - S^1$ is connected, then $D = W^1 - S^1$. Note that $W^1 - S^1 = (K^1 - M^1) - S^1 = (K^1 - S^1) - M^1$. Since $K^1$ is $l$-connected, $K^1 - S^1$ is $(l - |S^1|)$-connected. Therefore, if $W^1 - S^1$ is not connected, then $D$ contains at least $l - |S^1| > |S^2| + |S^s|$ vertices of the matching $M^1$. Each of these vertices has a neighbour in $W^2$ or $W^s$. In any case, $D$ has a neighbour in $W^2 - S^2$ or $W^s - S^s$ and so is in the connected graph $T$. This implies that $W - S$ is connected.

Suppose $|S^1| < l/2$. Then $|S^j| < l/2$ for all $j$. Hence $W^j - S^j$ is connected. Suppose $W^j - S^j$ has no neighbour in $W^{j+1} - S^{j+1}$ for some $j$. Then $W^{j-1} - S^{j-1}$ contains a neighbour of $W^j - S^j$. By the same argument, $W^{j+2} - S^{j+2}$ contains a neighbour of $W^{j+1} - S^{j+1}$. Suppose $W^i - S^i$ has no neighbour in $W^{i+1} - S^{i+1}$ for some $i \neq j$. Then $i \neq j-1, i \neq j+1$ and further, $|S^1| + |S^i+1| \geq l/2$. Therefore $|S| \geq |S^1| + |S^i| + |S^i+1| = |S| + |S^i| + |S^i+1| \geq l/2 + l/2 = l$, a contradiction. Hence, for any $i \neq j$, $W^i - S^i$ has neighbours in $W^{j+1} - S^{j+1}$. This implies that $W - S$ is connected.

**Definition 3.2.** Let $G$ be the product of $n$ even cycles. Suppose the diameter of $G$ is $d$. Let $v_0$ be an end-vertex of a path in $G$ of length $d$. Fix $v_0$. Let $V_0 = \{v_0\}$ and let $V_i = \{v \in V(G) : d(v_0, v) = i\}$ for $i \in [d]$, where $d(v_0, v)$ denotes the distance between $v_0$ and $v$ in $G$. Clearly, the sets $V_0, V_1, \ldots, V_d$ are mutually disjoint, non-empty and they partition the set $V(G)$. Let $K$ be a spanning subgraph of $G$. For $i \in [d]$, let $E_i(K) = \{xy \in E(K) : x \in V_{i-1}, y \in V_i\}$. Then the edge sets $E_1(K), E_2(K), \ldots, E_d(K)$ are non-empty and mutually disjoint (see Figure 3).

**Lemma 3.3.** Let $G, K$ and $E_i(K)$ be as in Definition 3.2. Then $E_1(K), E_2(K), \ldots, E_d(K)$ partition the edge set $E(K)$ of the graph $K$.

**Proof.** By definition of $V_i$, there is no edge in $G$ with one end-vertex in $V_j$ and the other in $V_{j'}$ when $|j - j'| \neq 1$. Suppose two vertices $x$ and $y$ of some
$V_i$ are adjacent. Let $P_x$ and $P_y$ be shortest paths in $G$ from $v_0$ to $x$, and $v_0$ to $y$ respectively. Then each of $P_x$ and $P_y$ takes exactly one vertex from each of $V_0, V_1, \ldots, V_i$. Therefore $P_x \cup P_y \cup \{xy\}$ contains an odd cycle in $G$, a contradiction to the fact that $G$ is bipartite. Thus each $V_i$ is independent. This implies that $E(K) = E_1(K) \cup E_2(K) \cup \cdots \cup E_d(K)$.

We need the following result.

**Lemma 3.4** [16]. Let $G_i$ be a graph with diameter $d_i$ for $i = 1, 2, \ldots, k$. Then the diameter of the graph $G_1 \square G_2 \square \cdots \square G_k$ is $d_1 + d_2 + \cdots + d_k$.

We are all set to prove Theorem 1.4. This theorem is restated below for convenience.

**Theorem 3.5.** Let $G$ be the product of $n$ even cycles and let $n = n_1 + n_2 + \cdots + n_k$ with $k \geq 2$ and $n_i \geq 1$ for $i \in [k]$. Then $G$ can be decomposed into $k$ spanning subgraphs $G_1, G_2, \ldots, G_k$ such that each $G_i$ is $2n_i$-regular and $2n_i$-connected.

**Proof.** We prove the result by induction on $n$. Obviously, $n \geq k$. If $n = k$, then $G$ is the product of $k$ cycles and hence, by [1], $G$ can be decomposed into $k$ Hamiltonian cycles. Thus the result holds for $n = k$.
Suppose \( n \geq k + 1 \). Then \( n_i \geq 2 \) for some \( i \in [k] \). Without loss of generality, we may assume that \( n_k \geq 2 \). Assume that the result holds for \( n - 1 \). Consider \( n - 1 = n_1 + n_2 + \cdots + n_{k-1} + (n_k - 1) \). Let \( G = C_1 \sqcap C_2 \sqcap \cdots \sqcap C_n \), where \( C_1, C_2, \ldots, C_n \) are even cycles. Let \( |C_n| = s \) and let \( H = C_1 \sqcap C_2 \sqcap \cdots \sqcap C_{n-1} \). Then, as in the proof of Proposition 2.4, \( G = H \sqcap C_n = H^1 \cup H^2 \cup \cdots \cup H^s \cup F \), where \( H^j \) is a copy of \( H \) and \( F = \bigcup_{j=1}^{s} \{xy: x \in V(H^j), y \in V(H^{j+1})\} \) with \( H^{s+1} = H^1 \).

By induction, \( H \) can be decomposed into \( k \) spanning subgraphs \( H_1, H_2, \ldots, H_k \) such that \( H_i \) is \( 2n_i \)-regular and \( 2n_i \)-connected for \( i \in [k-1] \), and \( H_k \) is \( 2(n_k - 1) \)-regular and \( 2(n_k - 1) \)-connected.

Let \( d \) be the diameter of \( H \). Since each \( C_i \) is an even cycle, the diameter of \( C_i \) is \( |C_i|/2 \geq 2 \). Therefore, by Lemma 3.4, \( d = \frac{|C_1|+|C_2|+\cdots+|C_{n-1}|}{2} \geq 2(n-1) = 2n-2 \geq 2n-2n_k \). Let \( u_0 \) be an end-vertex of a path in \( H \) of length \( d \). As in the Definition 3.2, we partition the vertex set \( V(H) \) of \( H \) into the sets \( V_0(H), V_1(H), \ldots, V_d(H) \), where \( V_0(H) = \{u_0\} \) and \( V_i(H) = \{u \in V(H): d(u, u_0) = i\} \) for \( i \in [d] \). Since \( H_i \) for \( i \in [k] \) is a spanning subgraph of \( H \), it follows from Lemma 3.3 that the edge set \( E(H_i) \) of \( H_i \) can be partitioned into the sets \( E_1(H_i), E_2(H_i), \ldots, E_s(H_i) \), where \( E_0(H_i) = \{xy \in E(H_i): x \in V_{i-1}, y \in V_i\} \) for \( t \in [d] \). Note that if \( e \in E_t(H_i) \) and \( f \in E_{t'}(H_i) \) with \( t' \geq t + 2 \), then \( e \) and \( f \) are vertex-disjoint (see Figure 3).

For \( i \in [k-1] \), we obtain a matching \( M_i \) of \( H_i \) by choosing one edge from each of \( n_i \) consecutive sets \( E_{2n_i-1}(H_i) \) as follows.

Choose one edge from each of the sets \( E_1(H_i), E_3(H_i), \ldots, E_{2n_i-1}(H_i) \) to get \( M_i \). Thus, we let \( M_i = \{u_{t-1}u_t \in E_t(H_i): t = 1, 3, 5, \ldots, 2n_i-1\} \). In general, we define \( M_i = \{u_{t-1}u_t \in E_t(H_i): t = 2p_i+1, 2p_i+3, \ldots, 2p_i+2n_i-1\} \), where \( p_i = 0 \) and \( p_i = n_1 + n_2 + n_3 + \cdots + n_{i-1} \) for \( 2 \leq i \leq k-1 \).

For \( j \in [s] \), the graph \( H^j \) is a copy of \( H \). Let \( H^j \) be the subgraph of \( H^j \) corresponding to the subgraph \( H_i \) of \( H \) for \( i \in [k] \). Therefore the graphs \( H^1, H^2, \ldots, H_k \) decompose the graph \( H^1 \). Further, the edge set \( E(H^j) \) has a partition into non-empty sets \( E_1(H^j), E_2(H^j), \ldots, E_d(H^j) \). For \( i \in [k-1] \), let \( M^j_i \) be the matching in \( H^j_i \) corresponding to the matching \( M_i \) of \( H \) and let \( u^j_t \) be the vertex of \( H^j \) corresponding to the vertex \( u_t \) of \( H \). Then \( M^j_i = \{u^j_{t-1}u^j_t \in E_t(H^j_i): t = 2p_i+1, 2p_i+3, \ldots, 2p_i+2n_i-1\} \). Let \( M^j \) be the union of these \( k-1 \) matchings \( M^j_i \). Therefore \( M^j = \bigcup_{i=1}^{k-1} M^j_i = \left\{u^j_0u^j_1, u^j_2u^j_3, \ldots, u^j_{2n_1+\cdots+2n_{i-1}-2}u^j_{2n_1+\cdots+2n_{k-1}-1}\right\} \). Clearly, \( M^j \) is a matching in \( H^j \).

We now construct the subgraphs \( G_1, G_2, \ldots, G_k \) of \( G \) which give a decomposition of \( G \), as required.

Construction of the graphs \( G_i \) for \( i \in [k] \).

Let \( i \in [k-1] \). We obtain \( G_i \) from \( H^1_i \cup \cdots \cup H^s_i \) by deleting the matching \( M^j_i \) from \( H^j_i \) for each \( j \) and then adding a matching \( D_i \) consisting of edges from the set...
Having one end in $M^j_i$ and the other end in $M^{j+1}_i$ or $M^{j-1}_i$. More precisely, let $D_i = \{u^j_iu^{j+1}_i: j = 1, 3, \ldots, s-1; t = 2p_i, 2p_i+2, \ldots, 2p_i+2n_i-2\} \cup \{u^j_iu^{j+1}_i: j = 2, 4, \ldots, s; t = 2p_i+1, 2p_i+3, \ldots, 2p_i+2n_i-1\}$.

For $i \in [k-1]$, let $G_i = \left( \bigcup_{j=1}^{s} (H^j_i - M^j_i) \right) \cup D_i$ (see Figure 4). Note that $D_i$ is a matching consisting of $n_i$ edges between $H^j$ and $H^{j+1}$ for each $j \in [s]$ and so the total number of edges in $D_i$ is $sn_i$.

For any $i \in [k-1]$ and $i' \in [k-1]$ with $i \neq i'$, the graphs $H^j_i$ and $H^j_{i'}$ are edge-disjoint for each $j$. This implies that $G_1, G_2, \ldots, G_{k-1}$ are mutually edge-disjoint subgraphs of $G$. Since $H^j_i$ is a $2n_i$-regular and spanning subgraph of $H^j$, $G_i$ is also a $2n_i$-regular and spanning subgraph of $G$. Further, as $H^j_i$ is $2n_i$-connected, Lemma 3.1 implies that $G_i$ is also $2n_i$-connected.
Let $G_k = G - E(G_1 \cup G_2 \cup \cdots \cup G_{k-1})$. The graph $G_k$ is shown in Figure 5.

It is easy to see that $G_k = \left( \bigcup_{j=1}^{s} (H_j^k \cup M^j) \right) \cup (F - D)$, where $D = \bigcup_{i=1}^{k-1} D_i$.

The edges of the matching $M^j$ are shown by the bold edges in Figure 5. Clearly, $D$ is a matching in $G$ consisting of $s(n_1 + n_2 + \cdots + n_{k-1}) = s(n - n_k)$ edges of $F$.

It follows that the graph $G_k$ is a spanning and $2n_k$-regular subgraph of $G$. Thus the graph $G$ decomposes into the spanning subgraphs $G_1, G_2, \ldots, G_k$.

It only remains to prove that the graph $G_k$ is $2n_k$-connected.

Claim. $G_k$ is $2n_k$-connected.

Proof. Let $S \subset V(G_k) = \bigcup_{j=1}^{s} V(H^j)$ such that $0 < |S| \leq 2n_k - 1$. It suffices to prove that $G_k - S$ is connected. Let $S^j = V(H_k^j) \cap S$ for $j \in [s]$. Since $V(H_k^1) = V(H^1), |V(H_k^1)| = |V(H^1)| = r = |C_1||C_2|\cdots|C_{n-1}| \geq 4^{n-1} = 2^{2n-2} = \ldots$
\[2^{2n-2nk+2} = \frac{2^{2n-2nk}}{2^{2nk-2}} > (2n - 2nk)(2nk - 1) \geq (n - nk)|S| \geq (n - nk) + \ldots\]

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\[\text{\textbf{Proof.}}\]

\[v \in H\]

This implies that \(v \notin S\). Therefore \(H_k^1 - S = H_k^2\) is connected for all \(j \neq 1\). Obviously, each vertex of \(H_k^1 - S\) has a neighbour in \(H_k^2\) or \(H_k^s\). Hence \(G_k - S\) is connected.

Suppose only one \(S^j\) other than \(S^1\) is nonempty. Suppose \(S^2 \neq \emptyset\). Then \(S = S^1 \cup S^2\). If \(H_k^1 - S^1\) and \(H_k^2 - S^2\) are connected, then they are connected to each other by an edge of \(F - D\) and so \(G_k - S\) is connected. Suppose \(H_k^1 - S^1\) is not connected. Then \(|S^1| = 2nk - 2\) and \(|S^2| = 1\) as \(|S| \leq 2nk - 1\) and \(H_k^1\) is \((2nk - 2)\)-connected. This implies that \(H_k^j - S^j\) for any \(j \neq 1\) is connected. Let \(T = G_k - (V(H_k^1) \cup S) = G_k - (V(H_k^1) \cup S^2)\). Then \(T\) is connected. It suffices to prove that every component of \(H_k^1 - S^1\) has a neighbour in \(T\). Let \(W\) be a component of \(H_k^1 - S^1\) and let \(v\) be a vertex of \(W\). If \(v\) has a neighbour in \(H_k^s\), then we are through. Suppose \(v\) has no neighbour in \(H_k^s\). Then \(v\) has a neighbour \(v'\) in \(H_k^2\). If \(v' \notin S^2\), then also we are through. Suppose \(v' \in S^2\). Then \(S^2 = \{v'\}\).

Also, \(v\) is an end-vertex of an edge of the matching \(M^1\). Therefore the degree of \(v\) in \(H_k^1\) is \(2nk - 1\). Hence \(v\) has a neighbour \(u\) in \(H_k^1 - S^1\). Obviously, \(u\) is in \(W\). Further, \(u\) has a neighbour in the subgraphs \(H_k^s\) or \(H_k^2 - S^2 = H_k^2 - \{v'\}\) of \(T\). Thus \(W\) has a neighbour in the connected graph \(T\). Hence \(G_k - S\) is connected.

Similarly, \(G_k - S\) is connected when \(S^s \neq \emptyset\).

Suppose \(S^j \neq \emptyset\) for some \(j \notin \{1, 2, s\}\). Then every component of \(H_k^1 - S^1\) has a neighbour in \(H_k^s\) or \(H_k^2\). It follows that \(G_k - S\) is connected. This proves the claim.

\[\square\]

Thus, the graph \(G\) decomposes into the spanning subgraphs \(G_1, G_2, \ldots, G_k\), where \(G_i\) is \(2n_i\)-regular and \(2n_i\)-connected for \(i = 1, 2, \ldots, k\). This completes the proof.

\[\blacksquare\]

\textbf{References}

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