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ORIENTED CHROMATIC NUMBER OF CARTESIAN PRODUCTS AND STRONG PRODUCTS OF PATHS

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Abstract

An oriented coloring of an oriented graph G is a homomorphism from G to H such that H is without selfloops and arcs in opposite directions. We shall say that H is a *coloring graph*. In this paper, we focus on oriented colorings of Cartesian products of two paths, called grids, and strong products of two paths, called strong-grids. We show that there exists a coloring graph with nine vertices that can be used to color every orientation of grids with five columns. We also show that there exists a strong-grid with two columns and its orientation which requires 11 colors for oriented coloring. Moreover, we show that every orientation of every strong-grid with three columns can be colored by 19 colors and that every orientation of every strong-grid with four columns can be colored by 43 colors. The above statements were proved with the help of computer programs.

Keywords: graph, oriented coloring, grid.

2010 Mathematics Subject Classification: 05C15.

1. Introduction

Let $G = (V(G), E(G))$ be a simple undirected graph. An *orientation* of G is a directed graph $\overrightarrow{G} = (V(\overrightarrow{G}), A(\overrightarrow{G}))$ obtained from G by ordering every edge $\{u, v\} \in E(G)$ either from u to v (resulting in an arc $(u, v) \in A(\overrightarrow{G})$), or conversely (yielding an arc $(v, u) \in A(\overrightarrow{G})$). In this paper, we shall deal with undirected graphs and their orientations. An orientation of a graph is called an oriented graph. An *oriented coloring* is a coloring c of the vertices of an oriented graph $\overrightarrow{G} = (V(\overrightarrow{G}), A(\overrightarrow{G}))$ such that

(i) no two neighbors have the same color,

(ii) for any two arcs (u, v) and $(y, z) \in A(\overrightarrow{G})$, if $c(u) = c(z)$ then $c(v) \neq c(y)$. In other words, if the arc (y, z) goes from color $c(y)$ to $c(z)$, then no other arc can go in the opposite direction, i.e., from $c(z)$ to $c(y)$.

With every oriented coloring c of $\overrightarrow{G}$ one can associate a digraph $\overrightarrow{H}_c$, called the *coloring graph of* $\overrightarrow{G}$, with set of vertices $V(\overrightarrow{H}_c) = \{c(x) : x \in V(G)\}$ and set of arcs $A(\overrightarrow{H}_c) = \{(c(x), c(y)) : (x, y) \in A(\overrightarrow{G})\}$. Due to conditions (i) and (ii), $\overrightarrow{H}_c$ is an oriented graph without loops and opposite arcs. An oriented coloring c can then be viewed as a homomorphism (that is an arc-preserving vertex mapping) from $\overrightarrow{G}$ to $\overrightarrow{H}_c$. In this case, $\overrightarrow{G}$ is said to be *colored* by $\overrightarrow{H}_c$. Similarly, every homomorphism from $\overrightarrow{G}$ to an oriented graph $\overrightarrow{H}$ can be viewed as a coloring of $\overrightarrow{G}$ using the vertices of $\overrightarrow{H}$ as colors. The *oriented chromatic number* $\overrightarrow{\chi}(\overrightarrow{G})$ of an oriented graph $\overrightarrow{G}$ is the smallest number of colors needed for its oriented coloring. The oriented chromatic number $\overrightarrow{\chi}(G)$ of an undirected graph G is the maximal chromatic number over all possible orientations of G. The oriented chromatic number of a family of graphs is the maximal chromatic number over all possible graphs of the family.

Oriented coloring has been studied in recent years [2, 5, 7, 8, 9, 11, 13, 14, 15, 17, 18], see [12] for a survey of the main results. Several authors established or bounded the oriented chromatic number for some families of graphs, such as oriented planar graphs [11], outerplanar graphs [14, 15], graphs with bounded degree three [7, 14, 17], k-trees [14], Halin graphs [4, 9], graphs with given excess [8] or grids [5, 18].

For a pair of undirected graphs G and H, the *Cartesian product* $G \Box H$ of G and H is the graph with vertex set $V(G) \times V(H)$ and where two vertices are adjacent if and only if they are equal in one coordinate and adjacent in the other. The *strong product* $G \boxtimes H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and where two vertices are adjacent if and only if they are adjacent in one coordinate and adjacent or equal in the other. We use P_k to denote the path on k vertices. In this paper we focus on the oriented chromatic number of Cartesian products of paths, called grids, and strong products of paths, called strong-grids.

In [5], Fertin, Raspaud and Roychowdhury have discussed bounds for the oriented chromatic number of $P_m \Box P_n$. They showed that

- $\overrightarrow{\chi}(P_m \Box P_n) \leq 11$, for every $m, n \geq 1$,
- there exists an orientation of $P_4 \Box P_5$ that requires 7 colors,
- $\overrightarrow{\chi}(P_2 \Box P_2) = 4$, $\overrightarrow{\chi}(P_2 \Box P_3) = 5$ and $\overrightarrow{\chi}(P_2 \Box P_n) = 6$, for $n \geq 4$,

- $\overrightarrow{\chi}(P_3 \Box P_3) = \overrightarrow{\chi}(P_3 \Box P_4) = \overrightarrow{\chi}(P_3 \Box P_5) = 6$, and $6 \leq \overrightarrow{\chi}(P_3 \Box P_n) \leq 7$, for every $n \geq 6$,
- $\overrightarrow{\chi}(P_4 \Box P_4) = 6$.

They also formulated the two following conjectures:

- every orientation of $P_m \Box P_n$ can be colored by seven colors,
- every orientation of $P_m \Box P_n$ can be colored by $\overrightarrow{T}_7$.

The coloring graph $\overrightarrow{T}_7$ is an oriented graph with set of vertices $V(\overrightarrow{T}_7) = \{0, 1, 2, \ldots, 6\}$ and set of arcs $A(\overrightarrow{T}_7) = \{(x, x + b \pmod 7) : x \in V(\overrightarrow{T}_7), b = 1, 2, \text{ or } 4\}$. Szepietowski and Targan [18] disproved the second conjecture by exhibiting an orientation of $P_5 \Box P_{33}$ that cannot be colored by $\overrightarrow{T}_7$. By the way, the oriented graph found in [18] can be colored by another coloring graph with 7 vertices. Furthermore, they showed that

- $\overrightarrow{\chi}(P_4 \Box P_n) = 7$, for every $n \geq 5$,
- $\overrightarrow{\chi}(P_3 \Box P_6) = 6$,
- $\overrightarrow{\chi}(P_3 \Box P_n) = 7$, for every $n \geq 7$.

Dybizbański and Nenca [3] disproved the first conjecture by exhibiting an orientation of $P_7 \Box P_{212}$ that requires 8 colors. However, the bounds for $P_5 \Box P_n$ were still $7 \leq \overrightarrow{\chi}(P_5 \Box P_n) \leq 11$.

Aravind, Narayanan and Subramanian [1] discussed the oriented chromatic number of strong products of paths. They showed that

- $8 \leq \overrightarrow{\chi}(P_2 \boxtimes P_n) \leq 11$, for every $n \geq 5$,
- $10 \leq \overrightarrow{\chi}(P_3 \boxtimes P_n) \leq 67$, for every $n \geq 5$.

Sopena [16] proved that

- $\overrightarrow{\chi}(P_k \boxtimes P_n) \leq 126$, for every $n, k \geq 3$.

The rest of the paper is organized as follows. In Section 2, we give definitions. In Section 3, we prove the following theorem.

Theorem 1.1. *For every $n \geq 5$, $\overrightarrow{\chi}(P_5 \Box P_n) \leq 9$.*

Moreover, we prove that the family of all orientations of all grids $P_5 \Box P_n$ can be colored with one coloring graph of order 9, which we call $\overrightarrow{H}_9$.

In Section 4, we show that there exists an orientation of $P_2 \boxtimes P_{398}$ which cannot be colored by any coloring graph with 10 vertices. This means that the lower bound for oriented chromatic number of strong-grids with two columns is the same as the upper bound obtained by Aravind, Narayanan and Subramanian in [1]. Moreover, we improve the bounds for the families of all orientations of the following strong-grids: strong-grids with three columns, denoted by $\mathcal{S}_3^{\rightarrow}$, and strong-grids with four columns denoted by $\mathcal{S}_4^{\rightarrow}$. We show that for every n, every orientation of $P_3 \boxtimes P_n$ can be colored by 19 colors and that there exists an n and an orientation of $P_3 \boxtimes P_n$ which requires 11 colors for oriented coloring. It follows that

Theorem 1.2. $11 \le \overrightarrow{\chi}(\mathcal{S}_3^{\rightarrow}) \le 19$.

Moreover, we prove that for every n, every orientation of $P_4 \boxtimes P_n$ can be colored by 43 colors and that there exists an n and an orientation of $P_4 \boxtimes P_n$ that requires 11 colors for oriented coloring. This means that

Theorem 1.3. $11 \le \overrightarrow{\chi}(\mathcal{S}_4^{\rightarrow}) \le 43$.

2. Definitions

Definition 2.1. Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two undirected graphs. Their *Cartesian product*, denoted by $G_1 \Box G_2$, is the graph where $V(G_1 \Box G_2) = V_1 \times V_2$ and $(u_1, u_2), (v_1, v_2) \in E(G_1 \Box G_2)$ if either $u_1 = v_1$ and $(u_2, v_2) \in E_2$, or $u_2 = v_2$ and $(u_1, v_1) \in E_1$. Their *strong product*, denoted by $G_1 \boxtimes G_2$, is the graph where $V(G_1 \boxtimes G_2) = V_1 \times V_2$ and $(u_1, u_2), (v_1, v_2) \in E(G_1 \boxtimes G_2)$ if either $u_1 = v_1$ and $(u_2, v_2) \in E_2$, or $u_2 = v_2$ and $(u_1, v_1) \in E_1$, or $(u_1, v_1) \in E_1$ and $(u_2, v_2) \in E_2$.

The Cartesian product of two paths $P_m \Box P_n$ is called the $m \times n$ *grid*. The strong product of two paths $P_m \boxtimes P_n$ is called the $m \times n$ *strong-grid*.

We shall say that $u \in V$ is a *source* (respectively a *sink*) if there is no arc incoming to u (respectively outgoing from u). A *tournament* is an orientation of an undirected complete graph.

Let p be a prime number such that $p \equiv 3 \pmod 4$, $d = \frac{p-1}{2}$, and $c_1, c_2, \ldots, c_d$ be the non-zero quadratic residues of p. The directed graph $\overrightarrow{T}_p$ with set of vertices $V(\overrightarrow{T}_p) = \{0, 1, \ldots, p-1\}$ and set of arcs $A(\overrightarrow{T}_p) = \{(x, x + c_i \pmod p) : x \in V(\overrightarrow{T}_p), 1 \le i \le d\}$ is called the *Paley tournament* of order p. It is easy to check that Paley tournaments are arc-transitive, i.e. for any two arcs (u, v), (x, y) in $A(\overrightarrow{T}_p)$, there exists an automorphism $f : \overrightarrow{T}_p \to \overrightarrow{T}_p$ satisfying $f(u) = x$, $f(v) = y$, and self-converse, i.e. T_p is isomorphic to its converse (a graph obtained by reversing each arc), see [6].

3. Cartesian Product of Paths

In this section we prove Theorem 1.1. First, we define the coloring graph $\overrightarrow{H}_9$ which is obtained from $\overrightarrow{T}_7$ by adding two vertices, one sink and one source.

More precisely, $V(\overrightarrow{H}_9) = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ and $(u, v) \in A(\overrightarrow{H}_9)$ if

- $u, v < 7$ and $v - u \equiv 1, 2,$ or $4 \pmod 7$, or
- $u = 7$, or
- $v = 8$.

Consider the grid $P_5 \square P_n$ with five columns and n rows. Let us denote by $v_i = (1, i)$, $w_i = (2, i)$, $x_i = (3, i)$, $y_i = (4, i)$, $z_i = (5, i)$ the five vertices in the i-th row, see Figure 1.

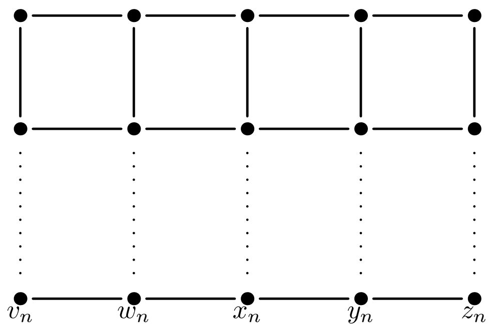

Figure 1. The grid $G(5, n)$.

Suppose that $\overrightarrow{G}$ is an orientation of $P_5 \square P_n$. By $S(\overrightarrow{G})$ we denote the set of reachable colorings of the last row of $\overrightarrow{G}$, namely $S(\overrightarrow{G}) = \{(c_1, c_2, c_3, c_4, c_5)$: there exists a coloring $\gamma : V(\overrightarrow{G}) \to \overrightarrow{T}_7$, such that $\gamma(v_n) = c_1, \gamma(w_n) = c_2, \gamma(x_n) = c_3, \gamma(y_n) = c_4, \gamma(z_n) = c_5\}$.

Let $\Phi(\overrightarrow{G})$ be the vector $(\delta(v_n), \delta(x_n), \delta(z_n))$, where

$$\delta(u) = \begin{cases} 0, & \text{if } u \text{ is a sink or a source,} \\ 1, & \text{otherwise.} \end{cases}$$

Let $\mathcal{G}^{\to}$ denote the family of all orientations of all grids with five columns. Let us denote by $\mathcal{G}'^{\to}$ the set of orientation in $\mathcal{G}^{\to}$ without sinks or sources on first, third and last columns above the last row, i.e., $\mathcal{G}'^{\to} = \bigcup_{n=1}^{\infty} \{\overrightarrow{G} \in \mathcal{G}^{\to}$: $\overrightarrow{G}$ is an orientation of $P_5 \square P_n$, where for every $1 \le i < n$, none of v_i, x_i, z_i is a sink or a source$\}$. First, we show that every $\overrightarrow{G} \in \mathcal{G}'^{\to}$ can be colored by the Paley tournament $\overrightarrow{T}_7$.

Lemma 3.1. *For every $\overrightarrow{G} \in \mathcal{G}'^{\to}$, there exists a homomorphism $\gamma : \overrightarrow{G} \to \overrightarrow{T}_7$.*

Proof. We use an algorithm to check that the family $\{S(\overrightarrow{G}) : \overrightarrow{G} \in \mathcal{G}'^{\to}\}$ does not contain the empty set, which means that every $\overrightarrow{G} \in \mathcal{G}'^{\to}$ can be colored by $\overrightarrow{T}_7$. In order to do this, the algorithm looks through all possible pairs (S, Φ) such that there exists $\overrightarrow{G} \in \mathcal{G}'^{\rightarrow}$ for which $S(\overrightarrow{G}) = S$ and $\Phi(\overrightarrow{G}) = \Phi$. The algorithm uses a queue to stores such pairs. Since $\overrightarrow{T}_7$ is arc-transitive and self-converse, we can consider only those orientations of $\overrightarrow{G}$ where $(v_n, w_n) \in A(\overrightarrow{G})$ and only those colorings where $c(v_n) = 0$ and $c(w_n) = 1$ (see [18] for more details). Note that there are eight possible orientations of the last row with $(v_n, w_n) \in A(\overrightarrow{G})$ and $3 \times 3 \times 3 = 27$ possible colorings of the last row satysfying $c(v_n) = 0$ and $c(w_n) = 1$.

The algorithm starts with the grid $P_5 \Box P_1$. For every orientation $\overrightarrow{F}$ of $P_5 \Box P_1$, it computes the pair $(S(\overrightarrow{F}), \Phi(\overrightarrow{F}))$ and inserts it to the queue Q. Next, the algorithm takes one by one a pair (S, Φ) from the queue, which contains the information for some $\overrightarrow{G} \in \mathcal{G}'^{\rightarrow}$ and considers every graph $\overrightarrow{R} \in \mathcal{G}'^{\rightarrow}$ which can be built from $\overrightarrow{G}$ by adding one extra row and which does not contain a sink or a source in the first, the third and the fifth column above the last row. For each such graph $\overrightarrow{R}$, the algorithm computes the pair $(S(\overrightarrow{R}), \Phi(\overrightarrow{R}))$ and, provided it is a new one, inserts it to the queue Q. In order to compute the pair $(S(\overrightarrow{R}), \Phi(\overrightarrow{R}))$, the algorithm does not need to reconstruct the whole graph $\overrightarrow{G}$ in its memory and build the graph $\overrightarrow{R}$.

After running this algorithm, we have found that the algorithm stops with empty queue Q and no pair of the form $(\emptyset, \Phi)$ is reachable. This means that every $\overrightarrow{G} \in \mathcal{G}'^{\rightarrow}$ can be colored with $\overrightarrow{T}_7$. ∎

***Proof of Theorem* 1.1.** Let $\overrightarrow{G}$ be an orientation of $P_5 \Box P_n$, $n \geq 5$. We show that there exists a homomorphism γ such that $\gamma : \overrightarrow{G} \to \overrightarrow{H}_9$. First, we construct a new orientation $\overrightarrow{G}'$ of $P_5 \Box P_n$ by reversing some arcs in $\overrightarrow{G}$. More precisely, for every row i we have

- if v_i is a sink or a source, then we reverse the arc between v_i and w_i,
- if x_i is a sink or a source, then we reverse the arc between w_i and x_i,
- if z_i is a sink or a source, then we reverse the arc between y_i and z_i.

Note that every vertical arc remains unchanged. Moreover, for each vertex u in the first, the third or the fifth column, we reverse at most one arc incident with u. The other end of each reversed arc is in the second or fourth column thus the changes of orientation cannot create a new sink or a source in the first, the third or the fifth column. The graph $\overrightarrow{G}'$ does not have any sources or sinks in the first, third and fifth column. Hence, by Lemma 3.1, there is a coloring $\gamma' : \overrightarrow{G}' \to \overrightarrow{T}_7$. We construct the coloring $\gamma : \overrightarrow{G} \to \overrightarrow{H}_9$ in the following way.

- If v is in the second or the fourth column, then $\gamma(v) := \gamma'(v)$,
- if v is in the first, the third or the fifth column, and is not a sink or a source in $\overrightarrow{G}$, then $\gamma(v) := \gamma'(v)$,

- if v is in the first, the third or the fifth column, and is a sink or a source in $\overrightarrow{G}$, then
  - * $\gamma(v) := 7$ if v is a source in $\overrightarrow{G}$,
  - * $\gamma(v) := 8$ if v is a sink in $\overrightarrow{G}$.

In order to show that γ is an oriented coloring of $\overrightarrow{G}$, consider an arc $uv \in \overrightarrow{G}$. There are four possible cases.

- If the arc between u and v has been reversed, then one of its ends, say u, is a source (or a sink, respectively) in the first, the third, or the fifth column and receives color 7 (or 8, respectively). The other end v of the arc is in the second or the fourth column, thus it has a color from $\overrightarrow{T}_7$ and $(7, \gamma(v)) \in A(\overrightarrow{H}_9)$ (or $(\gamma(v), 8) \in A(\overrightarrow{H}_9)$, respectively).
- If the arc between u and v has not been reversed and the colors of u and v have not been changed, then these colors fit in $\overrightarrow{T}_7$, and they also fit in $\overrightarrow{H}_9$.
- If the arc between u and v has not been reversed but the color of one of its ends, say u, was changed, then it means that the vertex u is a sink or a source and has a color that matches the color of the vertex v ($\gamma(v) \in \overrightarrow{T}_7$).
- If the arc between u and v has not been reversed but the colors of both its ends were changed, then u and v belong to the same column (the first, the third or the fifth), and one of them is a sink and the other is a source. Their colors are 7 and 8, and fit in $\overrightarrow{H}_9$. ■

4. Strong Product of Paths

In this section, we focus on the strong products of paths $S_{k,n} = P_k \boxtimes P_n$, called strong-grids. Consider the strong-grid $S_{2,n}$ with 2 columns and n rows. Let us denote by $x_i = (1, i)$ and $y_i = (2, i)$ the two vertices in the i-th row.

Let $\overrightarrow{S}_{2,n}$ denote the set of orientations of the strong-grid $S_{2,n}$. Since every orientation of a strong-grid $S_{2,n}$ is isomorphic to another orientation with all horizontal edges going in the same direction, we consider only the later ones (see Figure 2), i.e., $\overrightarrow{S}'_{2,n} = \{\overrightarrow{S} \in \overrightarrow{S}_{2,n} : (x_i, y_i) \in A(\overrightarrow{S}), 1 \le i \le n\}$. Suppose that $\overrightarrow{S} \in \overrightarrow{S}'_{2,n}$. By $T(\overrightarrow{S}, \overrightarrow{H})$ we denote the set of reachable colorings on the last row of $\overrightarrow{S}$ by the coloring graph $\overrightarrow{H}$, i.e., $T(\overrightarrow{S}, \overrightarrow{H}) = \{(c_1, c_2) :$ there exists a coloring $\gamma : V(\overrightarrow{S}) \to \overrightarrow{H}$, such that $\gamma(x_n) = c_1$ and $\gamma(y_n) = c_2\}$.

We use an algorithm similar to the one used in Lemma 3.1 to prove the following theorem.

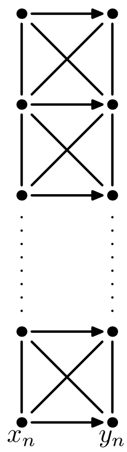

Figure 2. The strong-grid $S'_{2,n}$.

Theorem 4.1. *There exists an integer n such that $\overrightarrow{\chi}(P_2 \boxtimes P_n) = 11$.*

Proof. Aravind, Narayanan and Subramanian [1] proved that every strong-grid with two columns can be colored by the Paley tournament $\overrightarrow{T}_{11}$. We show that there exists an orientation of $P_2 \boxtimes P_{398}$ which cannot be colored by any coloring graph with ten vertices.

In order to construct a strong-grid that needs eleven colors for any oriented coloring, we use a `Extend` function. The `Extend` function for a given oriented strong-grid $\overrightarrow{S}_1$ and a coloring graph $\overrightarrow{H}$ returns a strong-grid $\overrightarrow{S}_2$, such that:

- $\overrightarrow{S}_2$ cannot be colored by $\overrightarrow{H}$.
- $\overrightarrow{S}_2$ is constructed by adding new rows to $\overrightarrow{S}_1$.
- If $\overrightarrow{S}_1$ cannot be colored by $\overrightarrow{H}$, then $\overrightarrow{S}_2$ is equal to $\overrightarrow{S}_1$.

The `Extend` function for a given $\overrightarrow{S}_1$ and $\overrightarrow{H}$ looks through all possible sets $T(\overrightarrow{S}, \overrightarrow{H})$, where $\overrightarrow{S}$ can be built from $\overrightarrow{S}_1$ by adding some extra rows. Similarly to the algorithm used in Lemma 3.1, the function uses a queue to stores such sets.

The function starts by computing the set $T(\overrightarrow{S}_1, \overrightarrow{H})$ and inserts it to the queue Q. Next, the algorithm takes one by one a set C from the queue Q and for every orientation of an extra row computes the set of colorings of the next row denoted by C' (similarly to Lemma 3.1). Then the set C' is inserted into the queue provided it is new one. Moreover, the function puts to an additional memory the triple consisting of the set C, the orientation of an extra row and the set C'. The algorithm stops when an empty set of colorings is reached. After reaching an empty set of colorings, the procedure reconstructs a grid $\overrightarrow{S}_2$, such that $T(\overrightarrow{S}_2, \overrightarrow{H}) = \emptyset$ and $\overrightarrow{S}_2$ is an extension of $\overrightarrow{S}_1$. To do this the algorithm uses the information kept in additional memory.

We use the `Extend` function with all non-isomorphic coloring graphs on ten vertices (there are 9733056 such tournaments). In the first run, we use a single arc. In the next run, we use the grid $\overrightarrow{S}$ returned in the previous run, and so on. It is easy to see that if $\overrightarrow{S}$ cannot be colored by $\overrightarrow{H}$, then any extension $\overrightarrow{S}'$ of $\overrightarrow{S}$ cannot be colored by $\overrightarrow{H}$. The result of the last run is a strong-grid $\overrightarrow{S}''$ that cannot be colored by any of the coloring graphs with ten vertices. The size of $\overrightarrow{S}''$ may vary depending on the order in which we consider non-isomorphic coloring graphs. Using `nauty` [10] to generate the list of all non-isomorphic coloring graphs of order 10, we found an orientation of $P_2 \boxtimes P_{398}$ that admits no oriented coloring with 10 colors. ∎

4.1. Proof of Theorem 1.2.

The lower bound for $\overrightarrow{\chi}(\mathcal{S}_3^{\rightarrow})$ follows from Theorem 4.1. We use an algorithm similar to the one used in Section 3 to show that for any $n \geq 1$ any orientation of $P_3 \boxtimes P_n$ can be colored by the Paley tournament $\overrightarrow{T}_{19}$. Consider now the grid $S_{3,n}$ with three columns and n rows. Let us denote by $x_i = (1, i)$, $y_i = (2, i)$, $z_i = (3, i)$ the three vertices in the i-th row, see Figure 3.

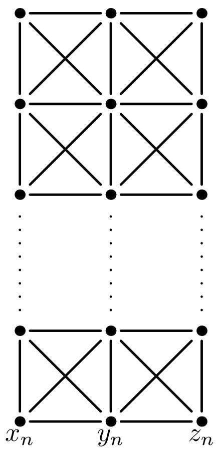

Figure 3. The strong-grid $S_{3,n}$.

Let $\mathcal{S}_3^{\rightarrow}$ denote the set of orientations of all strong-grids with three columns. Let $\overrightarrow{S}$ be an orientation of $P_3 \boxtimes P_n$, for some $n \geq 1$, and let $T(\overrightarrow{S})$ be the set of reachable distinct colorings on the last row of $\overrightarrow{S}$ by $\overrightarrow{T}_{19}$, i.e., $T(\overrightarrow{S}) = \{(c_1, c_2, c_3)$: there exists a coloring $\gamma : V(\overrightarrow{S}) \rightarrow \overrightarrow{T}_{19}$ such that $\gamma(x_n) = c_1$, $\gamma(y_n) = c_2, \gamma(z_n) = c_3\}$. Once again, since $\overrightarrow{T}_{19}$ is arc-transitive and self-converse, we can consider only those orientations of $\overrightarrow{S}$ where $(x_n, y_n) \in A(\overrightarrow{S})$, and only those colorings where $c(x_n) = 0$ and $c(y_n) = 1$. Thus, there are only two orientations of the last row with $(x_n, y_n) \in \overrightarrow{S}$ and only nine colorings of the last row satisfying $c(x_n) = 0$ and $c(y_n) = 1$.

The algorithm starts with the strong-grid $P_3 \boxtimes P_1$. For every orientation $\overrightarrow{S}$ of $P_3 \boxtimes P_1$ it computes the set $T(\overrightarrow{S})$ and inserts it into the queue Q. Next, the algorithm takes one by one a set T from the queue. The set T is the set of colorings of the last row of some $\overrightarrow{S} \in \mathcal{S}_3^{\rightarrow}$. The algorithm considers every graph $\overrightarrow{R} \in \mathcal{S}_3^{\rightarrow}$ which can be built from $\overrightarrow{S}$ by adding one extra row. For each such a graph $\overrightarrow{R}$ the algorithm computes the set $T(\overrightarrow{R})$ and, provided it is new one, inserts it into the queue Q. Once again the algorithm does not need to reconstruct the whole graph $\overrightarrow{R}$ to compute the set $T(\overrightarrow{R})$. After running the algorithm, we have found that the algorithm stops with the empty queue Q and the empty set of colorings of the last row is not reached. This means that for every n, every orientation of $P_3 \boxtimes P_n$ can be colored by the Paley tournament $\overrightarrow{T}_{19}$.

4.2. Proof of Theorem 1.3.

Consider now the strong-grid $S_{4,2} = P_4 \boxtimes P_2$. Let us denote by $v = (1,1)$, $w = (2,1)$, $x = (3,1)$, $y = (4,1)$ the vertices of the first row of $S_{4,2}$ and by $v' = (1,2)$, $w' = (2,2)$, $x' = (3,2)$, $y' = (4,2)$ the vertices of second row, see Figure 4.

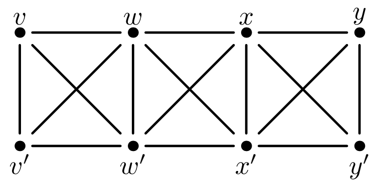

Figure 4. The strong-grid $S_{4,2}$.

The lower bound for $\overrightarrow{\chi}(\mathcal{S}_4^{\rightarrow})$ follows from Theorem 4.1. To prove the upper bound we shall use the following property of the Paley tournament $\overrightarrow{T}_{43}$.

Lemma 4.2. *For any orientation $\overrightarrow{S}$ of $S_{4,2}$, if the first row of $\overrightarrow{S}$ can be colored by $\overrightarrow{T}_{43}$ with colors (c_v, c_w, c_x, c_y), where $c_v \neq c_x$ and $c_w \neq c_y$, then there is a coloring $\gamma : \overrightarrow{S} \to \overrightarrow{T}_{43}$, such that $\gamma(v) = c_v$, $\gamma(w) = c_w$, $\gamma(x) = c_x$, $\gamma(y) = c_y$, and moreover $\gamma(v') \neq \gamma(x')$ and $\gamma(w') \neq \gamma(y')$.*

Proof. We use a computer algorithm to check the above property. Once again, since $\overrightarrow{T}_{43}$ is arc-transitive and self-converse, we can consider only those orientations of $S_{4,2}$ where $(u, v) \in A(S_{4,2})$ and only those colorings where $c_u = 0$ and $c_v = 1$. ∎

Let $\overrightarrow{S}$ be any orientation of $P_4 \boxtimes P_n$. Using Lemma 4.2, we can color $\overrightarrow{S}$ row by row.

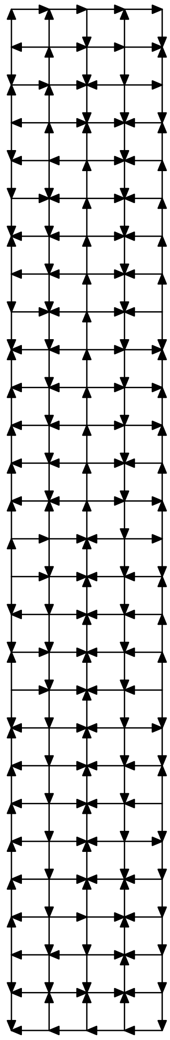

Figure 5. The oriented grid $\overrightarrow{G_5}$.

5. Conclusion

In this paper, we have proved that every orientation of $P_5 \square P_n$ can be colored by the coloring graph $\overrightarrow{H}_9$, see Section 3. However, $\overrightarrow{H}_9$ cannot be used to color every orientation of every grid $P_m \square P_n$. To prove that, we use an algorithm similar to the algorithm used in Lemma 3.1 and construct an orientation $\overrightarrow{G}_5$ of the grid

$P_5 \square P_{28}$ (see Figure 5) without sink and source vertices on the second, third and fourth column, which cannot be colored by the Paley tournament $\overrightarrow{T}_7$. One can now easily construct an orientation of $P_7 \square P_{28}$, by adding two extra columns in G_5: before the first one - column 0, and after the fifth one - column 6. The arcs incident with all vertices in additional columns are oriented in such a way that there is no sink and source vertices in the first and the fifth column. The resulting orientation cannot be colored by $\overrightarrow{H}_9$, as we are not able to use color 7 or 8 to color any vertex from column 1–5. We can only use colors from $\overrightarrow{T}_7$ but the oriented grid G_5 cannot be colored by $\overrightarrow{T}_7$.

On the website https://inf.ug.edu.pl/grids/ we posted the grid $\overrightarrow{G_5}$, and the strong-grid $P_2 \boxtimes P_{398}$ that requires 11 colors. On the same site we posted sample C++ programs that can be used to verify those grids and programs that we have used to prove property of $\overrightarrow{T}_{43}$ (Lemma 4.2), $\overrightarrow{T}_{19}$ (Theorem 1.2), and $\overrightarrow{T}_7$ (Lemma 3.1).

Every time when we use Paley tournaments to color certain kind of grids we check that no Paley tournament of smaller order has the expected property. For example $\overrightarrow{T}_{43}$ is the smallest Paley tournament with the property described in Lemma 4.2. The fact that this property is not true for $\overrightarrow{T}_{31}$ does not mean that any orientation of $P_4 \boxtimes P_n$ cannot be colored by $\overrightarrow{T}_{31}$.

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