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INDEPENDENCE NUMBER, CONNECTIVITY AND ALL FRACTIONAL (a, b, k)-CRITICAL GRAPHS

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Abstract

Let G be a graph and a, b and k be nonnegative integers with $1 \leq a \leq b$. A graph G is defined as *all fractional* (a, b, k)*-critical* if after deleting any k vertices of G, the remaining graph has all fractional $[a, b]$-factors. In this paper, we prove that if $\kappa(G) \geq \max\left\{\frac{(b+1)^2+2k}{2}, \frac{(b+1)^2\alpha(G)+4ak}{4a}\right\}$, then G is all fractional (a, b, k)-critical. If $k = 0$, we improve the result given in [Filomat 29 (2015) 757–761]. Moreover, we show that this result is best possible in some sense.

Keywords: independence number, connectivity, fractional $[a, b]$-factor, fractional (a, b, k)-critical graph, all fractional (a, b, k)-critical graph.

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1. Introduction

All graphs considered here are finite, simple and undirected graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $x \in V(G)$, we use $d_G(x)$ and $N_G(x)$ to denote the degree and neighbourhood of x in G, respectively. For any $S \subseteq V(G)$, let $N_G(S)$ denote the union of $N_G(x)$ for each $x \in S$. We use $G[S]$ and $G - S$ to denote the subgraph of G induced by S and $V(G) - S$. A subset I of $V(G)$ is an *independent set* of G, if no two distinct vertices in I are adjacent. The cardinality of a maximum independent set in a graph G is called the *independence number* of G, denoted by $\alpha(G)$. A *vertex-cut* of a noncomplete

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graph G is a set of vertices of G such that $G-S$ is disconnected. A vertex-cut of minimum cardinality in G is called a *minimum vertex-cut* of G and this cardinality is called the *connectivity* of G and is denoted by $\kappa(G)$.

Let g, f be two integer-valued functions defined on $V(G)$ with $0 \leq g(x) \leq f(x)$ for all $x \in V(G)$. A *(g,f)-factor* of G is a spanning subgraph H of G satisfying $g(x) \leq d_H(x) \leq f(x)$ for all $x \in V(G)$. Let $a \leq b$ be two integers. A (g,f)-factor is called an *$[a,b]$-factor* if $g(x) \equiv a$ and $f(x) \equiv b$. Let $h : E(G) \to [0,1]$ be a function. If $g(x) \leq \sum_{x \in e} h(e) \leq f(x)$ holds for every $x \in V(G)$, then we call graph F with vertex set $V(G)$ and edge set E_h a *fractional (g,f)-factor* of G with indicator function h, where $E_h = \{e \in E(G) | h(e) > 0\}$. If $f(x) = g(x)$ for all $x \in V(G)$, then a fractional (g,f)-factor is called a *fractional f-factor.* If $g(x) \equiv a$ and $f(x) \equiv b$, then a fractional (g,f)-factor is called a *fractional $[a,b]$-factor.* Let p be an integer-valued function defined on $V(G)$ such that $g(x) \leq p(x) \leq f(x)$ for each $x \in V(G)$. We say that G has *all fractional (g,f)-factors* if G has a fractional p-factor for every p described above. If $g(x) \equiv a$ and $f(x) \equiv b$, then all fractional (g,f)-factors are said to be *all fractional $[a,b]$-factors.* A graph G is called an *all fractional (a,b,k)-critical* graph if after deleting any k vertices of G the remaining graph of G has all fractional $[a,b]$-factors.

Many authors have studied factors and fractional factors of graphs. For example, see [1, 3, 4, 5, 6, 7, 8, 9, 10, 13, 14]. Anstee [1] and Lu [6] gave necessary and sufficient conditions for a graph to have all fractional (g,f)-factors and all fractional $[a,b]$-factors, respectively. Liu *et al.* [5] proved the necessary and sufficient conditions for a graph to have a fractional (g,f)-factor. The following theorem, on the existence of fractional (g,f)-factors of graphs, is well known.

Theorem 1 [2]**.** *Let G be a graph, and let a, b and r be three nonnegative integers satisfying $1 \leq a \leq b-r$, and let g, f be two integer-valued functions defined on $V(G)$ with $a \leq g(x) \leq f(x) - r \leq b-r$ for every $x \in V(G)$. If*

$$\kappa(G) \geq \max\left\{\frac{(b+1)(b-r+1)}{2}, \frac{(b-r+1)^2\alpha(G)}{4(a+r)}\right\},$$

then G contains a fractional (g,f)-factor.

As far as we know, except a sufficient condition for graphs to be all fractional (a,b,k)-critical in terms of binding number $bind(G)$ in [11], there are few results for graphs to be all fractional (a,b,k)-critical. This is a motivation of this paper.

In this paper we use independent number and connectivity to obtain a new sufficient condition for a graph to be all fractional (a,b,k)-critical. The following theorem is the main result.

Theorem 2. *Let G be a graph and let a, b, k be nonnegative integers with $1 \leq a < b$. If $\kappa(G) \geq \max\left\{\frac{(b+1)^2+2k}{2}, \frac{(b+1)^2\alpha(G)+4ak}{4a}\right\}$, then G is all fractional (a,b,k)-critical.*

If $k = 0$ in Theorem 2, we can get the following corollary.

Corollary 3. *Let G be a graph and a, b nonnegative integers with $1 \leq a < b$. If $\kappa(G) \geq \max\left\{\frac{(b+1)^2}{2}, \frac{(b+1)^2\alpha(G)}{4a}\right\}$, then G has all fractional $[a, b]$-factors.*

2. The Proof of Theorem 2

Lemma 4 [12]**.** *Let a, b and k be nonnegative integers with $1 \leq a \leq b$, and let G be a graph of order n with $n \geq a+k+1$. Then G is all fractional (a, b, k)-critical if and only if for any $S \subseteq V(G)$ with $|S| \geq k$*

$$a|S| + \sum_{x \in T} d_{G-S}(x) - b|T| \geq ak,$$

where $T = \{x : x \in V(G) \backslash S, d_{G-S}(x) < b\}$.

***Proof of Theorem* 2.** Let G be a graph satisfying the hypothesis of Theorem 2. We prove the theorem by contradiction. Suppose that G is not all fractional (a, b, k)-critical. Then by Lemma 4, there exists a subset S of $V(G)$ with $|S| \geq k$ such that

$$a|S| + \sum_{x \in T} d_{G-S}(x) - b|T| < ak, \tag{1}$$

where $T = \{x : x \in V(G) \backslash S, d_{G-S}(x) < b\}$. Obviously, $T \neq \emptyset$. Otherwise,

$$a|S| + \sum_{x \in T} d_{G-S}(x) - b|T| = a|S| \geq ak,$$

contradicting to (1).

Now we consider the subgraph $G[T]$ of G induced by T. Set $T_1 = G[T]$. Choose $x_1 \in T_1$ with $d_{T_1}(x_1) = \delta(T_1)$ and $L_1 = N_{T_1}[x_1]$. Furthermore, for $i \geq 2$, choose $x_i \in T_i = T_1 - \bigcup_{1 \leq j < i} L_j$ with $d_{T_i}(x_i) = \delta(T_i)$ and $L_i = N_{T_i}[x_i]$. Set $|L_i| = d_i$. We continue these procedures until we reach the situation in which $T_i = \emptyset$ for some i, say for $i = r+1$. Following the above definition we know that $\{x_1, x_2, \ldots, x_r\}$ is an independent set of G. Obviously, $r \geq 1$ and $|T| = \sum_{1 \leq i \leq r} d_i$. Let $U = V(G) \setminus (S \cup T)$ and $\kappa(G - S) = t$.

Now, we prove the following claims.

Claim 1. $r > 1$ or $U \neq \emptyset$.

Otherwise, we get $r = 1$ and $U = \emptyset$.

First, we prove an inequality $\frac{(a+b+1)^2}{4a} \leq \frac{(b+1)^2}{2}$, which is used later. In fact, this inequality is equivalent to $2(a+b+1)^2 - 4a(b+1)^2 \leq 0$. Now, let

$f(a) = 2(a+b+1)^2 - 4a(b+1)^2$, and so

$$\begin{aligned} f(a) &= 12(a^2+b^2+2a+2b+2ab+1) - 4a(b^2+2b+1) \\ &= 2a^2+2b^2+4a+4b+4ab+2-4ab^2-8ab-4a \\ &= 2a^2+2b^2+4b-4ab+2-4ab^2. \end{aligned}$$

By differential, we get $f'(a) = 4a-4b-4b^2 < 0$. So $f(a)$ is decreasing in $2 \le a \le b$ and we obtain

$$\begin{aligned} f(a) \le f(2) &= 2(3+b)^2 - 8(b+1)^2 = 2(9+b^2+6b) - 8(b^2+1+2b) \\ &= 18+2b^2+12b-8b^2-8-16b = 10-6b^2-4b \\ &= -2(3b^2+2b-5) = -2(b-1)(3b+5) < 0, \end{aligned}$$

which gives a proof of $\frac{(a+b+1)^2}{4a} \le \frac{(b+1)^2}{2}$.

By (1), we have

$$ak > a|S| + \sum_{x\in T} d_{G-S}(x) - b|T| = a|S| + d_1(d_1-1) - bd_1,$$

so $|S| < \frac{-d_1^2+d_1+bd_1+ak}{a}$. Then,

$$\begin{aligned} |V(G)| = |S| + d_1 &< \frac{-d_1^2+d_1+bd_1+ak}{a} + d_1 = \frac{-d_1^2+d_1+bd_1+ad_1}{a} + k \\ &= \frac{-d_1^2+(a+b+1)d_1}{a} + k \le \frac{(a+b+1)^2}{4a} + k \le \frac{(b+1)^2}{2} + k, \end{aligned}$$

which contradicts the assumption that $|V(G)| > \kappa \ge \frac{(b+1)^2+2k}{2}$. This completes the proof of Claim 1.

Claim 2. $\sum_{x\in T} d_{G-S}(x) \ge \sum_{1\le i\le r}(d_i^2 - d_i) + \frac{rt}{2}$.

In fact, by the choice of x_i, we know that every vertex in L_i has degree at least $d_i - 1$ in T_i, which implies that $\sum_{1\le i\le r}(\sum_{x\in L_i} d_{T_i}(x)) \ge \sum_{1\le i\le r} d_i(d_i-1)$.

Because an edge joining $x \in L_i$ and $y \in L_j$ $(i<j)$ is counted only once, we obtain that

$$\sum_{x\in T} d_{G-S}(x) \ge \sum_{1\le i\le r}(d_i^2 - d_i) + \sum_{1\le i<j\le r} e_G(L_i, L_j) + e_G(T, U). \tag{2}$$

For each $L_i (1 \le i \le r)$, by $\kappa(G-S) = t$, we have

$$e_G(L_i, \bigcup_{j\ne i} L_j) + e_G(L_i, U) \ge t. \tag{3}$$

Summing up these inequalities for all i $(1 \le i \le r)$, we get

$$(4)\quad \sum_{1\le i\le r}\Big(e_G(L_i, \bigcup_{j\ne i} L_j) + e_G(L_i, U)\Big) = 2\sum_{1\le i<j\le r} e_G(L_i, L_j) + e_G(T, U) \ge rt.$$

According to (4), it is obvious that

$$(5)\quad \sum_{1\le i<j\le r} e_G(L_i, L_j) + e_G(T, U) \ge \frac{rt}{2}.$$

In terms of (2) and (5), we have

$$(6)\quad \sum_{x\in T} d_{G-S}(x) \ge \sum_{1\le i\le r}(d_i^2 - d_i) + \frac{rt}{2}.$$

This completes the proof of Claim 2.

Now we continue to prove the main theorem. Combining (1) and (6), obtain

$$\begin{aligned} ak &> a|S| + \sum_{x\in T} d_{G-S}(x) - b|T| \ge a|S| + \sum_{1\le i\le r}(d_i^2 - d_i) + \frac{rt}{2} - b\sum_{1\le i\le r} d_i \\ &= a|S| + \sum_{1\le i\le r}(d_i^2 - (b+1)d_i) + \frac{rt}{2} \ge a|S| - \frac{(b+1)^2 r}{4} + \frac{rt}{2}, \end{aligned}$$

which implies that

$$(7)\quad ak > a|S| - \frac{(b+1)^2 r}{4} + \frac{rt}{2}.$$

Since $|S| \ge k$, from (7) we get that $-\frac{(b+1)^2 r}{4} + \frac{rt}{2} < 0$, which implies that

$$(8)\quad -\frac{(b+1)^2}{4} + \frac{t}{2} < 0.$$

By (7), (8), $\alpha(G) \ge \alpha(G[T]) \ge r$ and the assumption

$$\kappa(G) \ge \max\left\{\tfrac{(b+1)^2+2k}{2}, \tfrac{(b+1)^2\alpha(G)+4ak}{4a}\right\},$$

we get

$$\begin{aligned} ak &> a|S| - \frac{(b+1)^2 r}{4} + \frac{rt}{2} \ge a\left(\kappa(G) - t\right) - \frac{(b+1)^2}{4}\alpha(G) + \frac{t}{2}\alpha(G) \\ &\ge a\left(\kappa(G) - t\right) - \frac{(b+1)^2}{4}\frac{4a\kappa(G) - 4ak}{(b+1)^2} + \frac{t}{2}\frac{4a\kappa(G) - 4ak}{(b+1)^2} \\ &= at\Big(\frac{2\kappa(G) - 2k}{(b+1)^2} - 1\Big) + ak \ge at\Big(\frac{(b+1)^2 + 2k - 2k}{(b+1)^2} - 1\Big) + ak = ak, \end{aligned}$$

which is a contradiction. Therefore, G is all fractional (a, b, k)-critical. ∎

3. Remarks

Remark 1. Let us know that the condition $\kappa(G) \geq \frac{(b+1)^2+2k}{2}$ cannot be replaced by $\frac{(b+1)^2+2k}{2} - 1$. In fact, let $1 \leq a < b$ and $k \geq 0$ be three integers, and let $G = K_{\frac{(b+1)^2+2k}{2}-1} \vee \frac{a\left((b+1)^2-2\right)+2}{2b} K_1$. Let $S = K_{\frac{(b+1)^2+2k}{2}-1}$ and $T = \frac{a\left((b+1)^2-2\right)+2}{2b} K_1$. Obviously, $\kappa(G) = \frac{(b+1)^2+2k}{2} - 1 > k$, $|S| = \frac{(b+1)^2+2k}{2} - 1$, $|T| = \frac{a\left((b+1)^2-2\right)+2}{2b}$. So,

$$\begin{aligned} a|S| + d_{G-S}(T) - b|T| &= a\left(\frac{(b+1)^2+2k}{2} - 1\right) - b\frac{a\left((b+1)^2-2\right)+2}{2b} \\ &= a\frac{(b+1)^2}{2} + ak - a - b\frac{a\left((b+1)^2-2\right)+2}{2b} \\ &= ak - 1 < ak, \end{aligned}$$

a contradiction to Lemma 4, which implies that G is not all fractional (a,b,k)-critical.

Remark 2. The condition $\kappa(G) \geq \frac{(b+1)^2\alpha(G)+4ak}{4a}$ is equivalent to $a\kappa(G) \geq \frac{(b+1)^2\alpha(G)}{4} + ak$. Now we show that the condition $a\kappa(G) \geq \frac{(b+1)^2\alpha(G)}{4} + ak$ is best possible in the following sense. We cannot replace $a\kappa(G) \geq \frac{(b+1)^2\alpha(G)}{4} + ak$ by $a\kappa(G) \geq \frac{(b+1)^2\alpha(G)}{4} + ak - 1$, which is showed by the following example.

Let $b > a \geq 1$, $r \geq 1$ and $k \geq 0$ be four integers such that b is odd and $(\frac{b+1}{2})^2 r + ak - 1 \equiv 0 (\mathrm{mod}\ a)$. Let $G = K_p \vee rK_q$, where $p = \frac{(\frac{b+1}{2})^2 r + ak - 1}{a}$ and $q = \frac{b+1}{2}$. It is obvious that $\alpha(G) = r$ and $\kappa(G) = p = \frac{(\frac{b+1}{2})^2 r + ak - 1}{a}$. Let $S = V(K_p) \subseteq V(G)$ and $T = V(rK_q) \subseteq V(G)$, then $|S| = p = \frac{(\frac{b+1}{2})^2 r + ak - 1}{a} \geq k$ and $|T| = r\frac{b+1}{2}$. So, we have

$$\begin{aligned} a|S| + d_{G-S}(T) - b|T| &= a\frac{\left(\frac{b+1}{2}\right)^2 r + ak - 1}{a} + r\left(\frac{b+1}{2}\right)\left(\frac{b+1}{2} - 1\right) \\ &\quad - br\left(\frac{b+1}{2}\right) \\ &= \left(\frac{b+1}{2}\right)^2 r + ak - 1 + r\left(\frac{b+1}{2}\right)^2 - r\left(\frac{b+1}{2}\right) \\ &\quad - br\left(\frac{b+1}{2}\right) \\ &= \left(\frac{b+1}{2}\right)^2 r + ak - 1 + r\left(\frac{b+1}{2}\right)^2 - r\left(\frac{b+1}{2}\right)(1+b) \end{aligned}$$

$$= \left(\frac{b+1}{2}\right)^2 r + ak - 1 + r\left(\frac{b+1}{2}\right)^2 - 2r\left(\frac{b+1}{2}\right)^2$$
$$= \left(\frac{b+1}{2}\right)^2 r + ak - 1 - r\left(\frac{b+1}{2}\right)^2 = ak - 1 < ak.$$

In terms of Lemma 4, G is not all fractional (a, b, k)-critical.

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REFERENCES

[1] R.P. Anstee, *Simplified existence theorems for (g, f)-factors*, Discrete Appl. Math. **27** (1990) 29–38.
doi:10.1016/0166-218X(90)90126-W

[2] Q. Bian and S. Zhou, *Independence number, connectivity and fractional (g, f)-factors in graphs*, Filomat **29** (2015) 757–761.
doi:10.2298/FIL1504757B

[3] J.R. Correa and M. Matamala, *Some remarks about factors of graphs*, J. Graph Theory **57** (2008) 265–274.
doi:10.1002/jgt.20284

[4] S. Liu and J. Cai, *Toughness and existence of fractional (g, f)-factors in graphs*, Ars Combin. **93** (2009) 305–311.

[5] G. Liu and L. Zhang, *Fractional (g, f)-factors of graphs*, Acta Math. Sci. Ser. B Engl. Ed. **21** (2001) 541–545.
doi:10.1016/S0252-9602(17)30443-5

[6] H. Lu, *Simplified existence theorems on all fractional $[a, b]$-factors*, Discrete Appl. Math. **161** (2013) 2075–2078.
doi:10.1016/j.dam.2013.02.006

[7] T. Niessen, *A characterization of graphs having all (g, f)-factors*, J. Combin. Theory Ser. B **72** (1998) 152–156.
doi:10.1006/jctb.1997.1797

[8] T. Nishimura, *Independence number, connectivity, and r-factors*, J. Graph Theory **13** (1989) 63–69.
doi:10.1002/jgt.3190130109

[9] S. Zhou, *Independence number, connectivity and (a, b, k)-critical graphs*, Discrete Math. **309** (2009) 4144–4148.
doi:10.1016/j.disc.2008.12.013

[10] S. Zhou, *Some results about component factors in graphs*, RAIRO Oper. Res., accepted.
doi:10.1051/ro/2017045

[11] S. Zhou, Q. Bian and Z. Sun, *Binding numbers for all fractional (a, b, k)-critical graphs*, Filomat **28** (2014) 709–713.
doi:10.2298/FIL1404709Z

[12] S. Zhou and Z. Sun, *On all fractional (a, b, k)-critical graphs*, Acta Math. Sin. Engl. Ser. **30** (2014) 696–702.
doi:10.1007/s10114-014-2629-2

[13] S. Zhou, Z. Sun and Z. Xu, *A result on r-orthogonal factorizations in digraphs*, European J. Combin. **65** (2017) 15–23.
doi:10.1016/j.ejc.2017.05.001

[14] S. Zhou, J. Wu and T. Zhang, *The existence of $P_{\geq 3}$-factor covered graphs*, Discuss. Math. Graph Theory **37** (2017) 1055–1065.
doi:10.7151/dmgt.1974

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