INDEPENDENCE NUMBER, CONNECTIVITY AND ALL FRACTIONAL \((a, b, k)\)-CRITICAL GRAPHS

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Abstract

Let \(G\) be a graph and \(a, b\) and \(k\) be nonnegative integers with \(1 \leq a \leq b\). A graph \(G\) is defined as all fractional \((a, b, k)\)-critical if after deleting any \(k\) vertices of \(G\), the remaining graph has all fractional \([a, b]\)-factors. In this paper, we prove that if \(\kappa(G) \geq \max\left\{\frac{(b+1)^2+2k}{2}, \frac{(b+1)^2\alpha(G)+4ak}{4a}\right\}\), then \(G\) is all fractional \((a, b, k)\)-critical. If \(k = 0\), we improve the result given in [Filomat 29 (2015) 757–761]. Moreover, we show that this result is best possible in some sense.

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1. Introduction

All graphs considered here are finite, simple and undirected graphs. Let \(G\) be a graph with vertex set \(V(G)\) and edge set \(E(G)\). For a vertex \(x \in V(G)\), we use \(d_G(x)\) and \(N_G(x)\) to denote the degree and neighbourhood of \(x\) in \(G\), respectively. For any \(S \subseteq V(G)\), let \(N_G(S)\) denote the union of \(N_G(x)\) for each \(x \in S\). We use \(G[S]\) and \(G - S\) to denote the subgraph of \(G\) induced by \(S\) and \(V(G) - S\). A subset \(I\) of \(V(G)\) is an independent set of \(G\), if no two distinct vertices in \(I\) are adjacent. The cardinality of a maximum independent set in a graph \(G\) is called the independence number of \(G\), denoted by \(\alpha(G)\). A vertex-cut of a noncomplete

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graph $G$ is a set of vertices of $G$ such that $G - S$ is disconnected. A vertex-
cut of minimum cardinality in $G$ is called a minimum vertex-cut of $G$ and this
 cardinality is called the connectivity of $G$ and is denoted by $\kappa(G)$.

Let $g, f$ be two integer-valued functions defined on $V(G)$ with $0 \leq g(x) \leq f(x)$ for all $x \in V(G)$. A $(g, f)$-factor of $G$ is a spanning subgraph $H$ of $G$
satisfying $g(x) \leq d_H(x) \leq f(x)$ for all $x \in V(G)$. Let $a \leq b$ be two integers. A
$(g, f)$-factor is called an $[a, b]$-factor if $g(x) \equiv a$ and $f(x) \equiv b$. Let $h : E(G) \to [0, 1]$ be a function. If $g(x) \leq \sum_{e \in E_x} h(e) \leq f(x)$ holds for every $x \in V(G)$, then we
call graph $F$ with vertex set $V(G)$ and edge set $E_h$ a fractional $(g, f)$-factor of $G$
with indicator function $h$, where $E_h = \{ e \in E(G) | h(e) > 0 \}$. If $f(x) = g(x)$ for all
$x \in V(G)$, then a fractional $(g, f)$-factor is called a fractional $f$-factor. If $g(x) \equiv a$
and $f(x) \equiv b$, then a fractional $(g, f)$-factor is called a fractional $[a, b]$-factor. Let
$p$ be an integer-valued function defined on $V(G)$ such that $g(x) = p(x) \leq f(x)$
for each $x \in V(G)$. We say that $G$ has all fractional $(g, f)$-factors if $G$ has a fractional
$p$-factor for every $p$ described above. If $g(x) \equiv a$ and $f(x) \equiv b$, then all fractional $(g, f)$-factors are said to be all fractional $[a, b]$-factors. A graph $G$
is called an all fractional $(a, b, k)$-critical graph if after deleting any $k$ vertices of
$G$ the remaining graph of $G$ has all fractional $[a, b]$-factors.

Many authors have studied factors and fractional factors of graphs. For
example, see [1, 3, 4, 5, 6, 7, 8, 9, 10, 13, 14]. Anstee [1] and Lu [6] gave necessary
and sufficient conditions for a graph to have all fractional $(g, f)$-factors and all fractional
$[a, b]$-factors, respectively. Liu et al. [5] proved the necessary and sufficient conditions for a graph to have a fractional $(g, f)$-factor. The following
theorem, on the existence of fractional $(g, f)$-factors of graphs, is well known.

**Theorem 1** [2]. Let $G$ be a graph, and let $a, b$ and $r$ be three nonnegative integers
satisfying $1 \leq a \leq b - r$, and let $g, f$ be two integer-valued functions defined on
$V(G)$ with $a \leq g(x) \leq f(x) - r \leq b - r$ for every $x \in V(G)$. If

$$\kappa(G) \geq \max \left\{ \frac{(b + 1)(b - r + 1)}{2}, \frac{(b - r + 1)^2 \alpha(G)}{4(a + r)} \right\},$$

then $G$ contains a fractional $(g, f)$-factor.

As far as we know, except a sufficient condition for graphs to be all fractional
$(a, b, k)$-critical in terms of binding number $\text{bind}(G)$ in [11], there are few results
for graphs to be all fractional $(a, b, k)$-critical. This is a motivation of this paper.

In this paper we use independent number and connectivity to obtain a new
sufficient condition for a graph to be all fractional $(a, b, k)$-critical. The following
theorem is the main result.

**Theorem 2.** Let $G$ be a graph and let $a, b, k$ be nonnegative integers with $1 \leq a
< b$. If $\kappa(G) \geq \max \left\{ \frac{(b+1)^2+2k}{2}, \frac{(b+1)^2 \alpha(G)+4ak}{4a} \right\}$, then $G$ is all fractional $(a, b, k)$-
critical.
If $k = 0$ in Theorem 2, we can get the following corollary.

**Corollary 3.** Let $G$ be a graph and $a, b$ nonnegative integers with $1 \leq a < b$. If $\kappa(G) \geq \max \left\{ \frac{(b+1)^2}{2}, \frac{(b+1)^2\alpha(G)}{4a} \right\}$, then $G$ has all fractional $(a, b)$-factors.

### 2. The Proof of Theorem 2

**Lemma 4** [12]. Let $a, b$ and $k$ be nonnegative integers with $1 \leq a \leq b$, and let $G$ be a graph of order $n$ with $n \geq a + k + 1$. Then $G$ is all fractional $(a, b, k)$-critical if and only if for any $S \subseteq V(G)$ with $|S| \geq k$

$$a|S| + \sum_{x \in T} d_{G-S}(x) - b|T| \geq ak,$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) < b\}$.

**Proof of Theorem 2.** Let $G$ be a graph satisfying the hypothesis of Theorem 2. We prove the theorem by contradiction. Suppose that $G$ is not all fractional $(a, b, k)$-critical. Then by Lemma 4, there exists a subset $S$ of $V(G)$ with $|S| \geq k$ such that

$$a|S| + \sum_{x \in T} d_{G-S}(x) - b|T| < ak,$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) < b\}$. Obviously, $T \neq \emptyset$. Otherwise,

$$a|S| + \sum_{x \in T} d_{G-S}(x) - b|T| = a|S| \geq ak,$$

contradicting to (1).

Now we consider the subgraph $G[T]$ of $G$ induced by $T$. Set $T_1 = G[T]$. Choose $x_1 \in T_1$ with $d_{T_1}(x_1) = \delta(T_1)$ and $L_1 = N_{T_1}[x_1]$. Furthermore, for $i \geq 2$, choose $x_i \in T_i = T_{i-1} - \bigcup_{1 \leq j < i} L_j$ with $d_{T_i}(x_i) = \delta(T_i)$ and $L_i = N_{T_i}[x_i]$. Set $|L_i| = d_i$. We continue this procedures until we reach the situation in which $T_i = \emptyset$ for some $i$, say for $i = r + 1$. Following the above definition we know that $\{x_1, x_2, \ldots, x_r\}$ is an independent set of $G$. Obviously, $r \geq 1$ and $|T| = \sum_{1 \leq i \leq r} d_i$.

Let $U = V(G) \setminus (S \cup T)$ and $\kappa(G - S) = t$.

Now, we prove the following claims.

**Claim 1.** $r > 1$ or $U \neq \emptyset$.

Otherwise, we get $r = 1$ and $U = \emptyset$.

First, we prove an inequality $\frac{(a+b+1)^2}{4a} \leq \frac{(b+1)^2}{2}$, which is used later. In fact, this inequality is equivalent to $2(a + b + 1)^2 - 4a(b + 1)^2 \leq 0$. Now, let
\[ f(a) = 2(a + b + 1)^2 - 4a(b + 1)^2, \] and so
\[ f(a) = 12(a^2 + b^2 + 2a + 2b + 2ab + 1) - 4a(b^2 + 2b + 1) \]
\[ = 2a^2 + 2b^2 + 4a + 4b + 4ab + 2 - 4ab^2 - 8ab - 4a \]
\[ = 2a^2 + 2b^2 + 4b - 4ab + 2 - 4ab^2. \]

By differential, we get \( f'(a) = 4a - 4b - 4b^2 < 0 \). So \( f(a) \) is decreasing in \( 2 \leq a \leq b \)
and we obtain
\[ f(a) \leq f(2) = 2(3 + b)^2 - 8(b + 1)^2 = 2(9 + b^2 + 6b) - 8(b^2 + 1 + 2b) \]
\[ = 18 + 2b^2 + 12b - 8b^2 - 8 - 16b = 10 - 6b^2 - 4b \]
\[ = -2(3b^2 + 2b - 5) = -2(b - 1)(3b + 5) \leq 0, \]

which gives a proof of \( \frac{(a+b+1)^2}{4a} \leq \frac{(b+1)^2}{2} \).

By (1), we have
\[ ak > a|S| + \sum_{x \in T} d_{G-S}(x) - b|T| = a|S| + d_1(d_1 - 1) - bd_1, \]
so \( |S| < \frac{-d_1^2 + d_1 + bd_1 + ak}{a} \). Then,
\[ |V(G)| = |S| + d_1 < \frac{-d_1^2 + d_1 + bd_1 + ak}{a} + d_1 = \frac{-d_1^2 + d_1 + bd_1 + ad_1}{a} + k \]
\[ = \frac{-d_1^2 + (a + b + 1)d_1}{a} + k \leq \frac{(a + b + 1)^2}{4a} + k \leq \frac{(b + 1)^2}{2} + k, \]

which contradicts the assumption that \( |V(G)| > \kappa \geq \frac{(b+1)^2+2k}{2} \). This completes the proof of Claim 1.

**Claim 2.** \( \sum_{x \in T} d_{G-S}(x) \geq \sum_{1 \leq i \leq r} (d_i^2 - d_i) + \frac{r^2}{2}. \)

In fact, by the choice of \( x_i \), we know that every vertex in \( L_i \) has degree at least \( d_i - 1 \) in \( T_i \), which implies that \( \sum_{1 \leq i \leq r} (\sum_{x \in L_i} d_{T_i}(x)) \geq \sum_{1 \leq i \leq r} d_i(d_i - 1). \)

Because an edge joining \( x \in L_i \) and \( y \in L_j \) (\( i < j \)) is counted only once, we obtain that
\[ \sum_{x \in T} d_{G-S}(x) \geq \sum_{1 \leq i \leq r} (d_i^2 - d_i) + \sum_{1 \leq i < j \leq r} e_G(L_i, L_j) + e_G(T, U). \]

For each \( L_i(1 \leq i \leq r) \), by \( \kappa(G-S) = t \), we have
\[ e_G(L_i, \bigcup_{j \neq i} L_j) + e_G(L_i, U) \geq t. \]
Summing up these inequalities for all \( i \) \((1 \leq i \leq r)\), we get

\[
\sum_{1 \leq i \leq r} \left( e_G(L_i, \bigcup_{j \neq i} L_j) + e_G(L_i, T) \right) = 2 \sum_{1 \leq i < j \leq r} e_G(L_i, L_j) + e_G(T, U) \geq rt.
\]

According to (4), it is obvious that

\[
\sum_{1 \leq i < j \leq r} e_G(L_i, L_j) + e_G(T, U) \geq \frac{rt}{2}.
\]

In terms of (2) and (5), we have

\[
\sum_{x \in T} d_G(x) - S(x) \geq \sum_{1 \leq i \leq r} (d_i^2 - d_i) + \frac{rt}{2}.
\]

This completes the proof of Claim 2.

Now we continue to prove the main theorem. Combining (1) and (6), obtain

\[
ak > a|S| + \sum_{x \in T} d_{G-S}(x) - b|T| \geq a|S| + \sum_{1 \leq i \leq r} (d_i^2 - d_i) + \frac{rt}{2} - b \sum_{1 \leq i \leq r} d_i
\]

\[
= a|S| + \sum_{1 \leq i \leq r} (d_i^2 - (b + 1)d_i) + \frac{rt}{2} \geq a|S| - \frac{(b + 1)^2r}{4} + \frac{rt}{2},
\]

which implies that

\[
ak > a|S| - \frac{(b + 1)^2r}{4} + \frac{rt}{2}.
\]

Since \(|S| \geq k\), from (7) we get that \(-\frac{(b + 1)^2r}{4} + \frac{rt}{2} < 0\), which implies that

\[
\frac{(b + 1)^2}{4} + \frac{t}{2} < 0.
\]

By (7), (8), \(\alpha(G) \geq \alpha(G[T]) \geq r\) and the assumption

\[
\kappa(G) \geq \max \left\{ \frac{(b + 1)^2 + 2k}{2}, \frac{(b + 1)^2\alpha(G) + 4ak}{4a} \right\},
\]

we get

\[
ak > a|S| - \frac{(b + 1)^2r}{4} + \frac{rt}{2} \geq a(\kappa(G) - t) - \frac{(b + 1)^2}{4}\alpha(G) + \frac{t}{2}\alpha(G)
\]

\[
\geq a(\kappa(G) - t) - \frac{(b + 1)^2}{4}4\alpha(G) - 4ak - \frac{t}{2}4\alpha(G) - 4ak
\]

\[
= at\left(\frac{2\kappa(G) - 2k}{(b + 1)^2} - 1\right) + ak \geq at\left(\frac{(b + 1)^2 + 2k - 2k}{(b + 1)^2} - 1\right) + ak = ak,
\]

which is a contradiction. Therefore, \(G\) is all fractional \((a, b, k)\)-critical. \(\square\)
3. Remarks

Remark 1. Let us know that the condition $\kappa(G) \geq \frac{(b+1)^2+2k}{2}$ cannot be replaced by $\frac{(b+1)^2+2k}{2} - 1$. In fact, let $1 \leq a < b$ and $k \geq 0$ be three integers, and let $G = K_{\frac{(b+1)^2+2k}{2}-1} \lor a \frac{(b+1)^2-2}{2b} K_1$. Let $S = K_{\frac{(b+1)^2+2k}{2}-1}$ and $T = a \frac{(b+1)^2-2}{2b} K_1$.

Obviously, $\kappa(G) = \frac{(b+1)^2+2k}{2} - 1 > k$, $|S| = \frac{(b+1)^2+2k}{2} - 1$, $|T| = a \frac{(b+1)^2-2}{2b} + 2$.

So,

$$a|S| + d_{G-S}(T) - b|T| = a \left( \frac{(b+1)^2+2k}{2} - 1 \right) - b a \left( \frac{(b+1)^2-2}{2b} + 2 \right) = a \frac{(b+1)^2}{2} + ak - a - b a \left( \frac{(b+1)^2-2}{2b} + 2 \right) = ak - 1 < ak,$$

a contradiction to Lemma 4, which implies that $G$ is not all fractional $(a, b, k)$-critical.

Remark 2. The condition $\kappa(G) \geq \frac{(b+1)^2 \alpha(G) + 4ak}{4}$ is equivalent to $\alpha \kappa(G) \geq \frac{(b+1)^2 \alpha(G) + 4ak}{4}$. Now we show that the condition $\alpha \kappa(G) \geq \frac{(b+1)^2 \alpha(G) + 4ak}{4}$ is best possible in the following sense. We cannot replace $\alpha \kappa(G) \geq \frac{(b+1)^2 \alpha(G) + 4ak}{4} + ak$ by $\alpha \kappa(G) \geq \frac{(b+1)^2 \alpha(G)}{4} + ak - 1$, which is showed by the following example.

Let $b > a \geq 1$, $r \geq 1$ and $k \geq 0$ be four integers such that $b$ is odd and $(\frac{b+1}{2})^2 r + ak - 1 \equiv 0 (\text{mod } a)$. Let $G = K_p \lor r K_q$, where $p = \frac{(b+1)^2 r + ak - 1}{a}$ and $q = \frac{b+1}{2}$. It is obvious that $\alpha(G) = r$ and $\kappa(G) = p = \frac{(b+1)^2 r + ak - 1}{a}$. Let $S = V(K_p) \subseteq V(G)$ and $T = V(r K_q) \subseteq V(G)$, then $|S| = p = \frac{(b+1)^2 r + ak - 1}{a} \geq k$ and $|T| = r \frac{b+1}{2}$. So, we have

$$a|S| + d_{G-S}(T) - b|T| = a \frac{\left( \frac{b+1}{2} \right)^2 r + ak - 1}{a} + r \left( \frac{b+1}{2} \right) \left( \frac{b+1}{2} - 1 \right) - br \left( \frac{b+1}{2} \right) = \left( \frac{b+1}{2} \right)^2 r + ak - 1 + r \left( \frac{b+1}{2} \right)^2 - r \left( \frac{b+1}{2} \right) - br \left( \frac{b+1}{2} \right) = \left( \frac{b+1}{2} \right)^2 r + ak - 1 + r \left( \frac{b+1}{2} \right)^2 - r \left( \frac{b+1}{2} \right) \left( 1 + b \right)$$
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\[
= \left(\frac{b+1}{2}\right)^2 r + ak - 1 + r \left(\frac{b+1}{2}\right)^2 - 2r \left(\frac{b+1}{2}\right)^2
= \left(\frac{b+1}{2}\right)^2 r + ak - 1 - r \left(\frac{b+1}{2}\right)^2 = ak - 1 < ak.
\]

In terms of Lemma 4, \(G\) is not all fractional \((a, b, k)\)-critical.

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