INDEPENDENCE NUMBER, CONNECTIVITY AND ALL FRACTIONAL \((a,b,k)\)-CRITICAL GRAPHS

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Abstract

Let \(G\) be a graph and \(a, b\) and \(k\) be nonnegative integers with \(1 \leq a \leq b\). A graph \(G\) is defined as all fractional \((a,b,k)\)-critical if after deleting any \(k\) vertices of \(G\), the remaining graph has all fractional \([a,b]\)-factors. In this paper, we prove that if \(\kappa(G) \geq \max\{\frac{(b+1)^2+2k}{2}, \frac{(b+1)^2n(G)+4ak}{4a}\}\), then \(G\) is all fractional \((a,b,k)\)-critical. If \(k = 0\), we improve the result given in [Filomat 29 (2015) 757–761]. Moreover, we show that this result is best possible in some sense.

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1. Introduction

All graphs considered here are finite, simple and undirected graphs. Let \(G\) be a graph with vertex set \(V(G)\) and edge set \(E(G)\). For a vertex \(x \in V(G)\), we use \(d_G(x)\) and \(N_G(x)\) to denote the degree and neighbourhood of \(x\) in \(G\), respectively. For any \(S \subseteq V(G)\), let \(N_G(S)\) denote the union of \(N_G(x)\) for each \(x \in S\). We use \(G[S]\) and \(G - S\) to denote the subgraph of \(G\) induced by \(S\) and \(V(G) - S\). A subset \(I\) of \(V(G)\) is an independent set of \(G\), if no two distinct vertices in \(I\) are adjacent. The cardinality of a maximum independent set in a graph \(G\) is called the independence number of \(G\), denoted by \(\alpha(G)\). A vertex-cut of a noncomplete

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Let $g, f$ be two integer-valued functions defined on $V(G)$ with $0 \leq g(x) \leq f(x)$ for all $x \in V(G)$. A $(g, f)$-factor of $G$ is a spanning subgraph $H$ of $G$ satisfying $g(x) \leq d_H(x) \leq f(x)$ for all $x \in V(G)$ . Let $a \leq b$ be two integers. A $(g, f)$-factor is called an $[a, b]$-factor if $g(x) \equiv a$ and $f(x) \equiv b$. Let $h : E(G) \to [0, 1]$ be a function. If $g(x) \leq \sum_{e \in E} h(e) \leq f(x)$ holds for every $x \in V(G)$, then we call graph $F$ with vertex set $V(F)$ and edge set $E_h$ a fractional $(g, f)$-factor of $G$ with indicator function $h$, where $E_h = \{e \in E(G) | h(e) > 0 \}$. If $f(x) = g(x)$ for all $x \in V(G)$, then a fractional $(g, f)$-factor is called a fractional $f$-factor. If $g(x) \equiv a$ and $f(x) \equiv b$, then a fractional $(g, f)$-factor is called a fractional $[a, b]$-factor. Let $p$ be an integer-valued function defined on $V(G)$ such that $g(x) \leq p(x) \leq f(x)$ for each $x \in V(G)$. We say that $G$ has all fractional $(g, f)$-factors if $G$ has a fractional $p$-factor for every $p$ described above. If $g(x) \equiv a$ and $f(x) \equiv b$, then all fractional $(g, f)$-factors are said to be all fractional $[a, b]$-factors. A graph $G$ is called an all fractional $(a, b, k)$-critical graph if after deleting any $k$ vertices of $G$ the remaining graph of $G$ has all fractional $[a, b]$-factors.

Many authors have studied factors and fractional factors of graphs. For example, see [1, 3, 4, 5, 6, 7, 8, 9, 10, 13, 14]. Anstee [1] and Lu [6] gave necessary and sufficient conditions for a graph to have all fractional $(g, f)$-factors and all fractional $[a, b]$-factors, respectively. Liu et al. [5] proved the necessary and sufficient conditions for a graph to have a fractional $(g, f)$-factor. The following theorem, on the existence of fractional $(g, f)$-factors of graphs, is well known.

**Theorem 1** [2]. Let $G$ be a graph, and let $a, b$ and $r$ be three nonnegative integers satisfying $1 \leq a \leq b - r$, and let $g, f$ be two integer-valued functions defined on $V(G)$ with $a \leq g(x) \leq f(x) - r \leq b - r$ for every $x \in V(G)$. If

$$\kappa(G) \geq \max \left\{ \frac{(b+1)(b-r+1)}{2}, \frac{(b-r+1)^2\alpha(G)}{4(a+r)} \right\},$$

then $G$ contains a fractional $(g, f)$-factor.

As far as we know, except a sufficient condition for graphs to be all fractional $(a, b, k)$-critical in terms of binding number $\text{bind}(G)$ in [11], there are few results for graphs to be all fractional $(a, b, k)$-critical. This is a motivation of this paper.

In this paper we use independent number and connectivity to obtain a new sufficient condition for a graph to be all fractional $(a, b, k)$-critical. The following theorem is the main result.

**Theorem 2.** Let $G$ be a graph and let $a, b, k$ be nonnegative integers with $1 \leq a < b$. If $\kappa(G) \geq \max \left\{ \frac{(b+1)^2+2k}{2}, \frac{(b+1)^2\alpha(G)+4ak}{4a} \right\}$, then $G$ is all fractional $(a, b, k)$-critical.
If \( k = 0 \) in Theorem 2, we can get the following corollary.

**Corollary 3.** Let \( G \) be a graph and \( a, b \) nonnegative integers with \( 1 \leq a < b \). If \( \kappa(G) \geq \max \left\{ \frac{(b+1)^2}{2}, \frac{(b+1)^2\alpha(G)}{4a} \right\} \), then \( G \) has all fractional \([a, b]\)-factors.

## 2. The Proof of Theorem 2

**Lemma 4** [12]. Let \( a, b \) and \( k \) be nonnegative integers with \( 1 \leq a \leq b \), and let \( G \) be a graph of order \( n \) with \( n \geq a+k+1 \). Then \( G \) is all fractional \((a, b, k)\)-critical if and only if for any \( S \subseteq V(G) \) with \( |S| \geq k \)

\[
a|S| + \sum_{x \in T} d_{G-S}(x) - b|T| \geq ak,
\]

where \( T = \{ x : x \in V(G) \setminus S, d_{G-S}(x) < b \} \).

**Proof of Theorem 2.** Let \( G \) be a graph satisfying the hypothesis of Theorem 2. We prove the theorem by contradiction. Suppose that \( G \) is not all fractional \((a, b, k)\)-critical. Then by Lemma 4, there exists a subset \( S \) of \( V(G) \) with \( |S| \geq k \) such that

\[
(1) \quad a|S| + \sum_{x \in T} d_{G-S}(x) - b|T| < ak,
\]

where \( T = \{ x : x \in V(G) \setminus S, d_{G-S}(x) < b \} \). Obviously, \( T \neq \emptyset \). Otherwise,

\[
a|S| + \sum_{x \in T} d_{G-S}(x) - b|T| = a|S| \geq ak,
\]

contradicting to (1).

Now we consider the subgraph \( G[T] \) of \( G \) induced by \( T \). Set \( T_1 = G[T] \).

Choose \( x_1 \in T_1 \) with \( d_{T_1}(x_1) = \delta(T_1) \) and \( L_1 = N_{T_1}[x_1] \). Furthermore, for \( i \geq 2 \), choose \( x_i \in T_i = T_i - \bigcup_{1 \leq j < i} L_j \) with \( d_{T_i}(x_i) = \delta(T_i) \) and \( L_i = N_{T_i}[x_i] \). Set \( |L_i| = d_i \). We continue these procedures until we reach the situation in which \( T_i = \emptyset \) for some \( i \), say for \( i = r + 1 \). Following the above definition we know that \( \{x_1, x_2, \ldots, x_r\} \) is an independent set of \( G \). Obviously, \( r \geq 1 \) and \( |T| = \sum_{1 \leq i \leq r} d_i \).

Let \( U = V(G) \setminus (S \cup T) \) and \( \kappa(G - S) = t \).

Now, we prove the following claims.

**Claim 1.** \( r > 1 \) or \( U \neq \emptyset \).

Otherwise, we get \( r = 1 \) and \( U = \emptyset \).

First, we prove an inequality \( \frac{(a+b+1)^2}{4a} \leq \frac{(b+1)^2}{2} \), which is used later. In fact, this inequality is equivalent to \( 2(a+b+1)^2 - 4a(b+1)^2 \leq 0 \). Now, let
\[ f(a) = 2(a + b + 1)^2 - 4a(b + 1)^2, \]
and so
\[ f(a) = 12(a^2 + b^2 + 2a + 2b + 2ab + 1) - 4a(b^2 + 2b + 1) \]
\[ = 2a^2 + 2b^2 + 4a + 4b + 4ab + 2 - 4ab^2 - 8ab - 4a \]
\[ = 2a^2 + 2b^2 + 4ab + 2 - 4ab^2. \]

By differential, we get \( f'(a) = 4a - 4b - 4b^2 < 0 \). So \( f(a) \) is decreasing in \( 2 \leq a \leq b \) and we obtain
\[ f(a) \leq f(2) = 2(3 + b)^2 - 8(b + 1)^2 = 2(9 + b^2 + 6b - 8(b^2 + 1 + 2b) \]
\[ = 18 + 2b^2 + 12b - 8b^2 - 8 - 16b = 10 - 6b^2 - 4b \]
\[ = -2(3b^2 + 2b - 5) = -2(b - 1)(3b + 5) < 0, \]
which gives a proof of \( \frac{(a+b+1)^2}{4a} \leq \frac{(b+1)^2}{2} \).

By (1), we have
\[ ak > a|S| + \sum_{x \in T} d_{G-S}(x) - b|T| = a|S| + d_1(d_1 - 1) - bd_1, \]
so \( |S| < \frac{-d^2 + d_1 + bd_1 + ak}{a} \). Then,
\[ |V(G)| = |S| + d_1 < \frac{-d_1^2 + d_1 + bd_1 + ak}{a} + d_1 = \frac{-d_1^2 + d_1 + bd_1 + ad_1}{a} + k \]
\[ = \frac{-d_1^2 + (a + b + 1)d_1}{a} + k \leq \frac{(a + b + 1)^2}{4a} + k \leq \frac{(b + 1)^2}{2} + k, \]
which contradicts the assumption that \( |V(G)| > \kappa \geq \frac{(b+1)^2+2k}{2} \). This completes the proof of Claim 1.

Claim 2. \( \sum_{x \in T} d_{G-S}(x) \geq \sum_{1 \leq i \leq r} (d_i^2 - d_i) + \frac{r^2}{2} \).

In fact, by the choice of \( x_i \), we know that every vertex in \( L_i \) has degree at least \( d_i - 1 \) in \( T_i \), which implies that \( \sum_{1 \leq i \leq r} (\sum_{x \in L_i} d_{T_i}(x)) \geq \sum_{1 \leq i \leq r} d_i(d_i - 1) \).

Because an edge joining \( x \in L_i \) and \( y \in L_j \) (\( i < j \)) is counted only once, we obtain that
\[ (2) \quad \sum_{x \in T} d_{G-S}(x) \geq \sum_{1 \leq i \leq r} (d_i^2 - d_i) + \sum_{1 \leq i < j \leq r} e_G(L_i, L_j) + e_G(T, U). \]

For each \( L_i (1 \leq i \leq r) \), by \( \kappa(G - S) = t \), we have
\[ (3) \quad e_G(L_i, \bigcup_{j \neq i} L_j) + e_G(L_i, U) \geq t. \]
Summing up these inequalities for all $i$ ($1 \leq i \leq r$), we get

$$\sum_{1 \leq i \leq r} \left( e_G(L_i, \bigcup_{j \neq i} L_j) + e_G(L_i, U) \right) = 2 \sum_{1 \leq i < j \leq r} e_G(L_i, L_j) + e_G(T, U) \geq rt. \tag{4}$$

According to (4), it is obvious that

$$\sum_{1 \leq i < j \leq r} e_G(L_i, L_j) + e_G(T, U) \geq \frac{rt}{2}. \tag{5}$$

In terms of (2) and (5), we have

$$\sum_{x \in T} d_G(x) - S(x) \geq \sum_{1 \leq i \leq r} (d_i^2 - d_i) + \frac{rt}{2} \tag{6}$$

This completes the proof of Claim 2.

Now we continue to prove the main theorem. Combining (1) and (6), obtain

$$ak > a|S| + \sum_{x \in T} d_G(x) - b|T| \geq a|S| + \sum_{1 \leq i \leq r} (d_i^2 - d_i) + \frac{rt}{2} - b \sum_{1 \leq i \leq r} d_i$$

$$= a|S| + \sum_{1 \leq i \leq r} (d_i^2 - (b + 1)d_i) + \frac{rt}{2} \geq a|S| - \frac{(b + 1)^2r}{4} + \frac{rt}{2},$$

which implies that

$$ak > a|S| - \frac{(b + 1)^2r}{4} + \frac{rt}{2}. \tag{7}$$

Since $|S| \geq k$, from (7) we get that $-\frac{(b + 1)^2r}{4} + \frac{rt}{2} < 0$, which implies that

$$-\frac{(b + 1)^2}{4} + \frac{t}{2} < 0. \tag{8}$$

By (7), (8), $\alpha(G) \geq \alpha(G[T]) \geq r$ and the assumption $\kappa(G) \geq \max \left\{ \frac{(b + 1)^2 + 2k}{2}, \frac{(b + 1)^2 \alpha(G) + 4ak}{4a} \right\}$, we get

$$ak > a|S| - \frac{(b + 1)^2r}{4} + \frac{rt}{2} \geq a(\kappa(G) - t) - \frac{(b + 1)^2}{4} - \alpha(G) + t \frac{t}{2} \alpha(G)$$

$$\geq a(\kappa(G) - t) - \frac{(b + 1)^2}{4} \cdot \frac{4a\kappa(G) - 4ak}{(b + 1)^2} + \frac{t}{2} \frac{4a\kappa(G) - 4ak}{(b + 1)^2}$$

$$= a(t(2\kappa(G) - 2k) - 1) + ak \geq a(t\left(\frac{(b + 1)^2 + 2k - 2k}{(b + 1)^2} - 1\right) + ak = ak,$$

which is a contradiction. Therefore, $G$ is all fractional $(a, b, k)$-critical. \hfill \blacksquare
3. Remarks

Remark 1. Let us know that the condition \( \kappa(G) \geq \frac{(b+1)^2+2k}{2} \) cannot be replaced by \( \frac{(b+1)^2+2k}{2} - 1 \). In fact, let \( 1 \leq a < b \) and \( k \geq 0 \) be three integers, and let \( G = K_{\frac{(b+1)^2+2k}{2}} \cup a K_{\frac{(b+1)^2-2}{2b}} K_1 \). Let \( S = K_{\frac{(b+1)^2+2k}{2}} - 1 \) and \( T = a K_{\frac{(b+1)^2-2}{2b}} + K_1 \). Obviously, \( \kappa(G) = \frac{(b+1)^2+2k}{2} - 1 \), \( |S| = \frac{(b+1)^2+2k}{2} - 1 \), \( |T| = \frac{a K_{(b+1)^2-2} + 2}{2b} \). So,

\[
\begin{align*}
  a|S| + d_{G-S}(T) - b|T| &= a \left( \frac{(b+1)^2 + 2k}{2} - 1 \right) - b \frac{a K_{(b+1)^2-2} + 2}{2b} \\
  &= a \left( \frac{(b+1)^2}{2} + ak - a \right) - b \frac{a K_{(b+1)^2-2} + 2}{2b} \\
  &= ak - 1 < ak,
\end{align*}
\]

a contradiction to Lemma 4, which implies that \( G \) is not all fractional \((a, b, k)\)-critical.

Remark 2. The condition \( \kappa(G) \geq \frac{(b+1)^2 \alpha(G) + 4ak}{4a} \) is equivalent to \( a \kappa(G) \geq \frac{(b+1)^2 \alpha(G)}{4} + ak \). Now we show that the condition \( a \kappa(G) \geq \frac{(b+1)^2 \alpha(G)}{4} + ak \) is best possible in the following sense. We cannot replace \( a \kappa(G) \geq \frac{(b+1)^2 \alpha(G)}{4} + ak \) by \( a \kappa(G) \geq \frac{(b+1)^2 \alpha(G)}{4} + ak - 1 \), which is showed by the following example.

Let \( b > a \geq 1, r \geq 1 \) and \( k \geq 0 \) be four integers such that \( b \) is odd and \( \left( \frac{b+1}{2} \right)^2 r + ak - 1 \equiv 0(\text{mod } a) \). Let \( G = K_p \cup r K_q \), where \( p = \frac{(b+1)^2 r + ak - 1}{a} \) and \( q = \frac{b+1}{2} \). It is obvious that \( \alpha(G) = r \) and \( \kappa(G) = p = \frac{(b+1)^2 r + ak - 1}{a} \). Let \( S = V(K_p) \subseteq V(G) \) and \( T = V(r K_q) \subseteq V(G) \), then \( |S| = p = \frac{(b+1)^2 r + ak - 1}{a} \geq k \) and \( |T| = r \frac{b+1}{2} \). So, we have

\[
\begin{align*}
  a|S| + d_{G-S}(T) - b|T| &= a \frac{(b+1)^2 r + ak - 1}{a} + r \left( \frac{b+1}{2} \right) \left( \frac{b+1}{2} - 1 \right) \\
  &= \left( \frac{b+1}{2} \right)^2 r + ak - 1 + r \left( \frac{b+1}{2} \right)^2 - r \left( \frac{b+1}{2} \right) \\
  &= \left( \frac{b+1}{2} \right)^2 r + ak - 1 + r \left( \frac{b+1}{2} \right)^2 - r \left( \frac{b+1}{2} \right) (1+b)
\end{align*}
\]
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\[
= \left(\frac{b + 1}{2}\right)^2 r + ak - 1 + r \left(\frac{b + 1}{2}\right)^2 - 2r \left(\frac{b + 1}{2}\right)^2 \\
= \left(\frac{b + 1}{2}\right)^2 r + ak - 1 - r \left(\frac{b + 1}{2}\right)^2 = ak - 1 < ak.
\]

In terms of Lemma 4, \(G\) is not all fractional \((a, b, k)\)-critical.

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