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ON THE HAMILTONIAN NUMBER OF A PLANE GRAPH

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Abstract

The Hamiltonian number of a connected graph is the minimum of the lengths of the closed spanning walks in the graph. In 1968, Grinberg published a necessary condition for the existence of a Hamiltonian cycle in a plane graph, formulated in terms of the degrees of its faces. We show how Grinberg's theorem can be adapted to provide a lower bound on the Hamiltonian number of a plane graph.

Keywords: Hamiltonian cycle, Hamiltonian walk, Hamiltonian number, Hamiltonian spectrum, Grinberg's theorem, planar graph.

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1. Introduction

A *walk* in a graph G is a sequence $v_0, e_1, v_1, \ldots, v_{k-1}, e_k, v_k$ of vertices v_i and edges e_i of G such that, for each $i \in \{1, \ldots, k\}$, the edge e_i has endpoints v_{i-1} and v_i; the *length* of a walk is the number of its edges. The walk is *closed* if $v_k = v_0$ and *spanning* if each vertex of G appears at least once in the sequence. In a closed walk and for $j \in \{1, \ldots, k-1\}$, the vertex v_j is a *repeat* if $v_j = v_i$ for some $i \in \{0, \ldots, j-1\}$. Thus the number of repeats in a closed spanning walk in G is the difference between the length of the walk and the order of G. For ease of notation, whenever G is simple, we will denote a walk in G by a sequence of adjacent vertices of G, since the intervening edges can be inferred.

A *Hamiltonian cycle* in a graph is a closed spanning walk that visits each vertex exactly once; a graph is called *Hamiltonian* provided that it contains a Hamiltonian cycle. While not every graph is Hamiltonian, every connected graph contains a closed spanning walk. A *Hamiltonian walk* is a closed spanning walk of minimum length. The *Hamiltonian number* of a connected graph G, denoted by $h(G)$, is the length of a Hamiltonian walk in G. Thus the Hamiltonian number of a graph can be thought of as a measure of how far the graph deviates from being Hamiltonian.

In 1968, Grinberg [11] published a necessary condition for the existence of a Hamiltonian cycle in a planar graph, formulated in terms of the degrees of its faces. The main goal of this paper is to show how Grinberg's theorem can be adapted to provide a lower bound on the Hamiltonian number of a plane graph. Before we state this theorem, we will place our work in context.

In general, determining the Hamiltonian number of a graph is difficult, but for a connected graph G of order n, the bounds $n \le h(G) \le 2(n-1)$ are easily obtained. A Hamiltonian walk in G must visit each vertex, which gives the lower bound. On the other hand, a pre-order, closed spanning walk in a spanning tree of G has length $2(n-1)$, yielding the upper bound. Over the years, much of the research on the Hamiltonian number has advanced along two fronts: developing tighter bounds for the Hamiltonian number in terms of natural graph parameters, or evaluating the Hamiltonian numbers of some special graphs or families of graphs.

Goodman and Hedetniemi [9] initiated the study of the Hamiltonian number of a graph. They proved, among other things, properties of Hamiltonian walks, upper and lower-bounds for the Hamiltonian number of a graph, and a formula for the Hamiltonian number of a complete n-partite graph. Their most accessible result is this: let G be a k-connected graph on n vertices with diameter d, then $h(G) \le 2(n-1) - \lfloor k/2 \rfloor (2d-2)$, which improves the elementary upper bound.

Soon after the publication of the seminal paper of Goodman and Hedetniemi, Bermond [3] published a theorem on the Hamiltonian number problem inspired by Ore's theorem. Ore's theorem gives a sufficient condition for a graph to be Hamiltonian in terms of the sums of the degrees of non-adjacent vertices; see, for example, Theorem 6.6 of [7]. Bermond showed the following: let G be a graph of order n and let $c \le n$; if $\deg(v) + \deg(w) \ge c$ for every pair of non-adjacent vertices v and w in $V(G)$, then $h(G) \le 2n - c$.

Chartrand, Thomas, Zhang, and Saenpholphat [8] introduced an alternative approach to the Hamiltonian number. Let G be a connected graph of order n. Given vertices u and v, let $d(u,v)$ denote the length of a shortest path from u to v. A cyclic ordering of the vertices of G is a permutation $s : v_1, v_2, \ldots, v_n, v_{n+1}$ of $V(G)$, where $v_{n+1} = v_1$. Given a cyclic ordering s, let $d(s) = \sum_{i=1}^{n} d(v_i, v_{i+1})$. The set $H(G) = \{d(s) : s \text{ is a cyclic ordering of } V(G)\}$ is called the *Hamiltonian spectrum* of G. Chartrand and his colleagues showed that $h(G) = \min H(G)$. This paper contains two other notable results: first, that a connected graph G of order n satisfies $h(G) \le 2(n-1)$ with equality if and only if G is a tree; second, that for each integer $n \ge 3$, every integer in the interval $[n, 2(n-1)]$ is the Hamiltonian number of some graph of order n. Král, Tong, and Zhu [12] and Liu [13] conducted additional research on the Hamiltonian spectra of graphs.

Various authors have studied the Hamiltonian number of special graphs and families of graphs. Punnim and Thaithae [19, 17] studied the Hamiltonian numbers of cubic graphs. A graph of order n with Hamiltonian number $n+1$ is called *almost Hamiltonian*. Punnim, Saenpholphat, and Thaithae [16] characterized the almost Hamiltonian cubic graphs and the almost Hamiltonian generalized Petersen graphs. Asano, Nishizeki, and Watanabe [2, 14] established a simple upper bound for the Hamiltonian number of a maximal planar graph of order $n \geq 3$ and created an algorithm for finding closed spanning walks in a graph with length close to its Hamiltonian number. Chang *et al.* [5] studied the Hamiltonian numbers of Möbius double-loop networks. The Hamiltonian number problem has a variety of cognates: Vacek [20, 21] analyzed open Hamiltonian walks; Araya and Wiener [1, 22] investigated hypohamiltonian graphs; Goodman, Hedetniemi, and Slater [10] studied the Hamiltonian completion problem; Chang and Tong [6] considered the Hamiltonian numbers of strongly connected digraphs; and, Okamoto, Zhang, and Saenpholphat [18, 15] studied the upper traceable numbers of graphs.

2. The Grinberg Number of a Plane Graph

The *boundary* of a face (region) of a plane graph is the subgraph induced by the edges adjacent to that face, and a *boundary walk* is a closed walk containing each of these edges. The *degree* of a face is the minimum length of a boundary walk. We will denote the degree of the face F by $\deg(F)$. Two faces of a plane graph are said to be *adjacent* if they share at least one common boundary edge. A vertex (or an edge) is said to be *incident* to a face if it lies on the boundary of the face.

Let G be a plane graph and let its faces (including its exterior face) be labeled $F_1, \ldots, F_N$. Let $\mathscr{G}(G)$ be the set of all nonnegative sums of the form

$$\sum_{i=1}^{N} \varepsilon_i \big(\deg(F_i) - 2 \big) \tag{1}$$

where $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N) \in \{-1, +1\}^N$, and let $\operatorname{gr}(G) = \min \mathscr{G}(G)$. We will call $\mathscr{G}(G)$ the *Grinberg set* of G and $\operatorname{gr}(G)$ the *Grinberg number* of G. To get a sense of this, consider the graphs pictured in Figure 1. The graph G_1 has 1 face of degree 18 and 5 faces of degree 6; thus, $\mathscr{G}(G_1) = \{4, 12, 20, 28, 36\}$, and $\operatorname{gr}(G_1) = 4$. The graph G_2 has 1 face of degree 7 and 3 faces of degree 3; thus, $\mathscr{G}(G_2) = \{2, 4, 6, 8\}$ and $\operatorname{gr}(G_2) = 2$. Finally, the graph G_3 has 3 faces of degree 6 and 2 faces of degree 4; thus, $\mathscr{G}(G_3) = \{0, 4, 8, 12, 16\}$ and $\operatorname{gr}(G_3) = 0$.

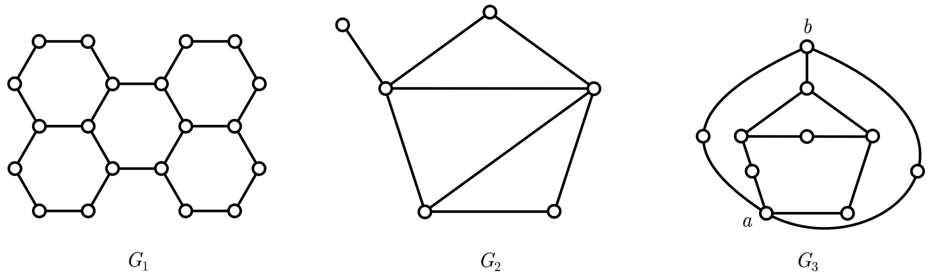

Figure 1. For these graphs, $\mathscr{G}(G_1) = \{4, 12, 20, 28, 36\}$, $\mathscr{G}(G_2) = \{2, 4, 6, 8\}$, and $\mathscr{G}(G_3) = \{0, 4, 8, 12, 16\}$; hence, $\text{gr}(G_1) = 4$, $\text{gr}(G_2) = 2$, and $\text{gr}(G_3) = 0$.

Given a graph G of order n, let $\text{rep}(G)$ denote the number of repeats in a Hamiltonian walk in G, that is, $\text{rep}(G) = h(G) - n$. For a plane graph G, Grinberg's theorem can be restated as follows: if $\text{rep}(G) = 0$, then $\text{gr}(G) = 0$; see [11]. Our main result can be seen as a natural extension of Grinberg's theorem.

Theorem 1. *Let G be a plane graph. Then, for some $\gamma \in \mathscr{G}(G)$ and nonnegative integer k,*

$$\text{rep}(G) = \frac{\gamma}{2} + 2k. \tag{2}$$

In particular,

$$\text{rep}(G) \geq \frac{\text{gr}(G)}{2}. \tag{3}$$

Proof. Our proof is an adaptation of the customary proof of Grinberg's theorem; see, for example, Theorem 18.2 of [4]. Hereafter let $V = \{v_1, \dots, v_n\}$ denote the vertex set of G and let σ be a closed spanning walk in G.

The walk σ in G induces a natural multigraph on V. Let G_σ have vertex set V. For each pair of vertices v_i and v_j in V, let there be as many edges in G_σ between v_i and v_j as there are occurrences of the edge v_iv_j in σ. We will call G_σ the *reduction* of G relative to σ. The graph G_σ is planar, Eulerian, and the number of edges in G_σ is equal to the length of σ.

Since G_σ is Eulerian, its dual graph is bipartite. Accordingly, we will label each face of G_σ with a $+$ or a $-$ sign as follows: the unbounded face is marked $+$; thereafter, if a face of G_σ is adjacent to (shares an edge with) a $+$ region, then it is marked $-$, and if a face of G_σ is adjacent to a $-$ region, then it is marked $+$. An example of a plane graph and its reduction relative to a closed spanning walk is presented in Figure 2.

There is a simple relationship between the number of faces of G_σ and the number of repeats in σ. For each $i \in \{1, \dots, n\}$, let m_i count the number of times

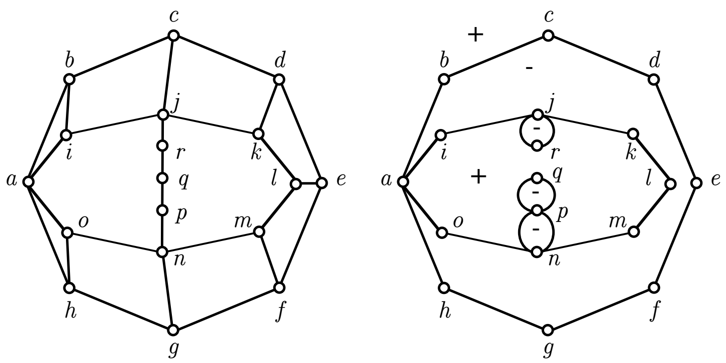

Figure 2. A plane graph (left) and its reduction (right) based on the closed spanning walk $a, b, c, d, e, f, g, h, a, i, j, r, j, k, l, m, n, p, q, p, n, o, a$. Notice that the walk has 4 repeats and its reduction has $4 + 2$ faces, in accord with equation (4).

vertex v_i is a repeat in the walk σ, that is, $m_i = (\deg_{G_\sigma} v_i)/2 - 1$. Let Φ count the number of faces of G_σ. We claim that

$$\Phi = 2 + \sum_{i=1}^{n} m_i. \tag{4}$$

Since the degree of the vertex v_i in G_σ is $2m_i + 2$, the number of edges in G_σ is $\sum_{i=1}^{n}(m_i + 1)$. By the Euler characteristic formula, $n - \sum_{i=1}^{n}(m_i + 1) + \Phi = 2$, which yields equation (4).

Our argument now moves into a second phase, culminating in a simple formula relating the degrees of the positively and negatively signed faces of G_σ. Let ℓ_+ denotes the number of positively signed faces of G_σ and label these faces by $\{A_i^+ : 1 \le i \le \ell_+\}$. Let ℓ_- and $\{A_i^- : 1 \le i \le \ell_-\}$ be defined likewise for the negatively signed faces of G_σ. Since each edge of G_σ is adjacent to a positively and a negatively signed face, it follows that $\sum_{i=1}^{\ell_+} \deg(A_i^+) = \sum_{i=1}^{\ell_-} \deg(A_i^-)$. Let $\Delta = \ell_- - \ell_+$. Then

$$\sum_{i=1}^{\ell_+} \big(\deg(A_i^+) - 2\big) - \sum_{i=1}^{\ell_-} \big(\deg(A_i^-) - 2\big) = 2\Delta. \tag{5}$$

We will modify this formula to incorporate the faces of G. Let the faces of G be labeled $\{F_i : 1 \le i \le N\}$. Each face of G is contained by a unique face of G_σ. For each $i \in \{1, \ldots, N\}$, let ε_i be sign of the face of G_σ that contains F_i. We will show that

$$\sum_{i=1}^{N} \varepsilon_i \big(\deg(F_i) - 2\big) = 2\Delta. \tag{6}$$

To verify this claim, we will follow Grinberg's strategy: we will add to G_σ, one at a time, those edges of G that were not traversed by σ. Such an edge must split a face of G_σ into two sub-faces, each with the same sign as the parent face. For the sake of argument, let us say that a face labeled A_i^+ is divided by an edge of G into two sub-faces, labeled $A_{i_1}^+$ and $A_{i_2}^+$. Since the two sub-faces share exactly one edge, we have

$$\deg(A_i^+) - 2 = \left(\deg(A_{i_1}^+) - 2\right) + \left(\deg(A_{i_2}^+) - 2\right).$$

Hence we can substitute $\left(\deg(A_{i_1}^+) - 2\right) + \left(\deg(A_{i_2}^+) - 2\right)$ for $\deg(A_i^+) - 2$ in equation (5) and retain equality. We continue this process until all of these edges have been added. We have almost arrived at equation (6). The only difference corresponds to those faces of G_σ that were created because an edge was traversed more than once by σ. Such a face has only two edges and thus contributes 0 to the sum. In this way, we have transformed equation (5) into equation (6).

Our proof is nearly complete. For simplicity, let $\nu = \ell^- - 1$, $\pi = \ell_+ - 1$, and $\rho = \sum_{i=1}^n m_i$, the number of repeats in σ. Recalling equation (4), it follows that $\pi + \nu = \rho$. According to equation (6) and the definition of the Grinberg set, there exists $\gamma \in \mathscr{G}(G)$ such that $|\pi - \nu| = \gamma/2$. Solving this system of equations, we find $\rho = \gamma/2 + 2\min\{\nu, \pi\}$. This result applies to a Hamiltonian walk, which yields equation (2). The inequality (3) follows from this, since $\mathrm{gr}(G)/2$ is the smallest element of $\mathscr{G}(G)$. ∎

3. Some Observations and Applications

Different planar embeddings of a graph may produce different Grinberg sets and Grinberg numbers. For example, for each positive integer m, let $B_m = P_{2m+1} \circ K_2$, the corona product of a path graph P_{2m+1} and a complete graph K_2. Two planar embeddings of B_1 are pictured in Figure 3, labeled $B_{1,O}$ (outerplanar) and $B_{1,C}$ (concentric). Observe that $\mathscr{G}(B_{1,O}) = \{8, 10, 12, 14\}$ and $\mathrm{gr}(B_{1,C}) = 8$, while $\mathscr{G}(B_{1,C}) = \{0, 2, 10, 12, 14\}$ and $\mathrm{gr}(B_{1,C}) = 0$.

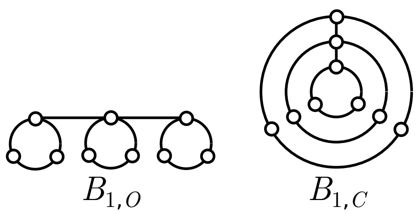

Figure 3. An outerplanar and a concentric embedding of the graph B_1.

This same family of graphs demonstrates that the difference between the left and right-hand sides of inequality (3) can be arbitrarily large. Observe that $\mathrm{gr}(B_{m,C}) = 0$. However, since $\mathrm{gr}(B_{m,O}) = 8m$, $\mathrm{rep}(B_{m,C}) = \mathrm{rep}(B_{m,O}) \geq 4m$.

As a consequence of inequality (3), if a plane graph G possesses a closed spanning walk with $\mathrm{gr}(G)/2$ repeats, then $\mathrm{rep}(G) = \mathrm{gr}(G)/2$. This is the case for the graphs G_1 and G_2 pictured in Figure 1, which satisfy $\mathrm{gr}(G_1) = 4$ and $\mathrm{gr}(G_2) = 2$. It is easy to find a closed spanning walk in G_1 with 2 repeats; thus, $\mathrm{rep}(G_1) = 2$. Likewise, it is easy to find a closed spanning walk in G_2 with 1 repeat; thus, $\mathrm{rep}(G_2) = 1$.

Whenever inequality (3) is strict, some additional work is required to determine $\mathrm{rep}(G)$, and equation (2) can be helpful in this undertaking. The graph G_3 pictured in Figure 1 has Grinberg set $\{0, 4, 8, 12, 16\}$. Thus, according to equation (2), $\mathrm{rep}(G_3)$ must be a nonnegative even integer. Since $G_3 - \{a, b\}$ has three components, G_3 is not Hamiltonian; thus, $\mathrm{rep}(G_3) \geq 2$. It is easy, however, to find a closed spanning walk with 2 repeats; thus, $\mathrm{rep}(G_3) = 2$.

Our next theorem can be used in conjunction with equation (2) to determine the number of repeats in a Hamiltonian walk. Let σ be a closed spanning walk in a plane graph G. Relative to σ, an edge of G is *even* provided that σ crosses it an even number of times; an edge of G is *odd* if it is not even. A vertex of G is *enveloped* if it has degree two and its incident edges are even.

Theorem 2. *Let σ be a closed spanning walk in a simple plane graph G. Then the number of repeats in σ must be at least the number of enveloped vertices.*

Proof. Let us assume that G has at least one enveloped vertex relative to σ and let $\{x_1, \ldots, x_k\}$ be the list of its enveloped vertices. Let a and b denote the vertices of G that are adjacent to x_1. Let G_1 be the plane graph obtained by deleting x_1 and inserting the edge ab. This process is known as *smoothing out* or *smoothing away* the vertex x_1. The closed spanning walk σ in G naturally generates a closed spanning walk σ_1 in G_1. First, in the listing of the vertices of G by σ, replace (one at a time) all occurrences of the string a, x_1, a with a and all occurrences of the string b, x_1, b with b. Next, replace all occurrences of the string a, x_1, b with a, b and all occurrences of the string b, x_1, a with b, a.

We claim that σ has at least one more repeat of vertices than σ_1. Since σ is a spanning walk, the vertex x_1 must occur at least once in σ and it must be preceded by a or b and followed by a or b. If a, x_1, a occurs in σ, then this repeat of a does not occur in σ_1. If b, x_1, b occurs in σ, then this repeat of b does not occur in σ_1. If either a, x_1, b or b, x_1, a occurs in σ, then, since the edge ax_1 is crossed an even number of times, this repeat of x_1 does not occur in σ_1. This process can be continued, one enveloped vertex at a time, until all of the enveloped vertices are smoothed out, resulting in a closed spanning walk σ_k in the graph G_k. The walk σ_k has at least k fewer repeats than σ, proving our result. ■

Here is an application of Theorem 2. The graph H pictured in Figure 4

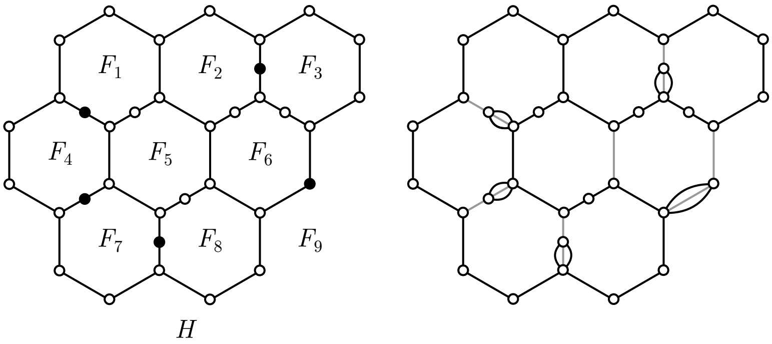

Figure 4. Although the graph H has Grinberg number 6, a Hamiltonian walk in H must have at least 5 repeats. A closed spanning walk with 5 repeats is pictured to the right.

has 8 interior faces of degree 8 and an exterior face of degree 20; thus, $\mathscr{G}(H) = \{6, 18, 30, 42, 54, 66\}$. According to equation (2), there exists a nonnegative integer k such that $\mathrm{rep}(H) = 3 + 2k$. If a closed spanning walk σ in H corresponds to a Grinberg number of 6, then σ partitions the faces of H so that face F_9 must have the same sign as either two or three of the octagonal faces and the remaining faces must have the opposite sign. In all cases, Theorem 2 demonstrates that more than three repeats must occur. For example, if F_9, F_6, and F_5 have the same sign and the remaining faces have the opposite sign, then there are 5 enveloped vertices, shown as the filled vertices in Figure 4. The same argument holds in each of the remaining cases. On the other hand, a closed spanning walk with 5 repeats can be readily found; see Figure 4. Thus, $\mathrm{rep}(H) = 5$.

The Grinberg number of plane graph can be easily calculated if one of its faces has large degree.

Theorem 3. *Let G be a plane graph of order n and let F be one of its faces. If $\deg(F) \geq n$, then $\mathrm{gr}(G) = 2(\deg(F) - n)$.*

Proof. Let e and f denote the number of edges and faces of G. Let $F_1, \ldots, F_f$ denote the faces of G with $F = F_1$. We will show that the sum in equation (1) is minimized for a particular choice of signs. Since the sum of the face degrees of G is two times the number of its edges,

$$\begin{aligned}\big(\deg(F_1) - 2\big) - \sum_{i=2}^{f} \big(\deg(F_i) - 2\big) &= 2\big(\deg(F) - 2\big) - \sum_{i=1}^{f} \big(\deg(F_i) - 2\big) \\ &= 2\big(\deg(F) - n\big) + 2\,(f - e + n - 2) \\ &= 2\big(\deg(F) - n\big).\end{aligned}$$

Since $2\big(\deg(F) - n\big) \geq 0$, it follows that $\mathrm{gr}(G) = 2\big(\deg(F) - n\big)$, as was to be shown. ∎

The graph G_2 pictured in Figure 1 has 6 vertices and the degree of its exterior face is 7; thus, according to Theorem 3, $\mathrm{gr}(G_1) = 2$. A tree has a single face, and the degree of this face is twice the number of its edges. According to Theorem 3, the Grinberg number of a tree of order n, $n \geq 2$, is $2(n-2)$.

4. Some Remarks on Plane Almost Hamiltonian Graphs

As noted in the introduction, a graph of order n with Hamiltonian number $n+1$ is called *almost Hamiltonian*. For example, the graph G_2 pictured in Figure 1 is almost Hamiltonian. One consequence of Theorem 1 is that the Grinberg number of a plane almost Hamiltonian graph must be 0 or 2; our next theorem states that, in either case, 2 must be in its Grinberg set.

Theorem 4. *Let the graph G be planar and almost Hamiltonian. Then the Grinberg set of any planar embedding of G must contain* 2.

Proof. Let $\hat{G}$ be an embedding of G in the plane. A Hamiltonian walk in $\hat{G}$ contains one repeat; thus, according to equation (2), there exists a Grinberg number $\gamma \in \mathscr{G}(\hat{G})$ and a nonnegative integer k such that $1 = \gamma/2 + 2k$. It follows that $\gamma = 2$, as was to be shown. ■

It would be interesting to identify some distinguishing characteristics of plane almost Hamiltonian graphs that produce a Grinberg number of 2. Our next two theorems offer two such characteristics.

Theorem 5. *Let α and β be positive integers with* $\gcd(\alpha, \beta) \geq 2$. *Let G be a plane graph such that the degree of each face is in the set $\{\alpha k + \beta + 2 : k \geq 0, k \in \mathbb{Z}\}$. If G is almost Hamiltonian, then* $\mathrm{gr}(G) = 2$.

Proof. Let the faces of G be lableled $F_1, \ldots, F_N$. Let us assume, to the contrary, that $\mathrm{gr}(G) \neq 2$. Then, by Theorem 4, $\{0, 2\} \subset \mathscr{G}(G)$. It follows that there are choices of signs $(\varepsilon_1, \ldots, \varepsilon_N) \in \{-1, 1\}^N$ and $(\delta_1, \ldots, \delta_N) \in \{-1, 1\}^N$ such that

$$1 = \sum_{i=1}^{N} \left(\frac{\varepsilon_i - \delta_i}{2} \right) \big(\deg(F_i) - 2 \big).$$

Because of the condition on the degrees of the faces of G, this equation can be expressed as $\alpha p + \beta q = 1$ for some integers p and q, which contradicts the assumption that $\gcd(\alpha, \beta) \geq 2$. Thus $\mathrm{gr}(G) = 2$, as was to be shown. ■

For $\alpha = \beta = 2$, our theorem asserts that any plane almost Hamiltonian graph whose face degrees are in the set $\{2k + 4 : k \in \mathbb{Z}, k \geq 0\}$ must have Grinberg number 2. For each integer $m \geq 1$, the grid graph $P_{2m+1} \Box P_{2m+1}$ is such a graph.

Theorem 6. *Let G be a plane almost Hamiltonian graph for which all of its vertices belong to the outer face of the drawing. Then* $\text{gr}(G) = 2$.

Proof. Let F be the exterior face of G. Since G is not Hamiltonian, $\deg(F) > n$. By Theorem 3, $\text{gr}(G) = 2(\deg(F) - n) \geq 2$. However, since G is almost Hamiltonian, $\text{gr}(G) \leq 2$; thus, $\text{gr}(G) = 2$, as was to be shown. ■

The graph G_2 pictured in Figure 1 is an example of such a graph.

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