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$L(2,1)$-LABELING OF CIRCULANT GRAPHS

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Abstract

An $L(2,1)$-labeling of a graph Γ is an assignment of non-negative integers to the vertices such that adjacent vertices receive labels that differ by at least 2, and those at a distance of two receive labels that differ by at least one. Let $\lambda_2^1(\Gamma)$ denote the least λ such that Γ admits an $L(2,1)$-labeling using labels from $\{0,1,\ldots,\lambda\}$. A Cayley graph of group G is called a circulant graph of order n, if $G=\mathbb{Z}_n$. In this paper initially we investigate the upper bound for the span of the $L(2,1)$-labeling for Cayley graphs on cyclic groups with "large" connection sets. Then we extend our observation and find the span of $L(2,1)$-labeling for any circulants of order n.

Keywords: graph coloring, $L(2,1)$-labeling, circulants.

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1. Introduction

The Frequency Assignment Problem (FAP) deals with assigning radio frequencies to the transmitters at different locations in a territory in such a manner that closely located transmitters receive frequencies that are sufficiently apart, so that these channels would not interfere with each other. This practical scenario can be realized as a graph theoretic problem by viewing each transmitter as a vertex of the graph. Consequently the associated vertex label corresponds to the radio frequency of the transmitter. Roberts [15] identified the difference between the terms "close" and "very close" in terms of edge-distance as follows. An edge is assumed to exist between two vertices if the corresponding transmitters are located "very close" physically; and two transmitters are said to be close if their corresponding vertices are at a distance of two. Motivated by this concept, Griggs and Yeh formulated $L(2,1)$-labeling of a graph [9].

Formally, an $L(2,1)$-labeling of a graph Γ is an assignment $f : V(\Gamma) \to \mathbb{Z}^+ \cup \{0\}$ such that

$$|f(x) - f(y)| \geq \begin{cases} 2, & \text{if } xy \in E(G), \\ 1, & \text{if } d(x,y) = 2, \end{cases}$$

where $d(x,y)$ is the distance between the vertices x and y. Let $\lambda_2^1(\Gamma)$ denote the least λ such that Γ admits an $L(2,1)$-labeling using labels from $\{0,1,\ldots,\lambda\}$. As bandwidth is a limited resource, the main target is in FAP is to come up with a frequency assignment using minimum number of frequencies, i.e. one needs to minimize the span of the labeling proposed. For convenience, without loss of generality, we consider the smallest label to be zero, so that the span is the highest label assigned. Many results have been published related to this problem and its variations [17, 11, 1, 5, 10, 12] over the past few decades. Extensive surveys on this topic could be found in [18, 7]. The determination of the exact value of $\lambda_2^1(\Gamma)$ for a given graph Γ is a very difficult task, it is an NP-hard problem to be precise [9, 7]. For this reason, researchers are trying to determine the bounds on $\lambda_2^1(\Gamma)$ instead, for different classes of graphs. An obvious lower bound is $\Delta + 1$ where Δ is the maximum degree of the graph. In 1992 Griggs and Yeh [9] conjectured that for any graph Γ, $\lambda_2^1(\Gamma) \leq \Delta^2$, where $\Delta \geq 2$. However, in the same paper [9] they showed that for any graph Γ, $\lambda_2^1(\Gamma) \leq \Delta^2 + 2\Delta$. Later this result was refined by Chang and Kuo [8] as $\lambda_2^1(\Gamma) \leq \Delta^2 + \Delta$. This result was proved asymptotically by Havet *et al.* in 2008 [11]. Even though the conjecture is proven to be true for a selected families of graphs, viz. paths, cycles, wheels [9], trees [8], [10], Cartesian product and the composition of graphs [17], generalized Petersen graphs [12], chordal graphs [16], etc., it is yet to be proved in general. There are very few graph classes for which $\lambda_2^1(\Gamma)$ can be calculated efficiently. These are paths, cycles, wheels, trees, generalized petersen graphs, etc. There exist large families of graphs for which it is still unknown whether the computation of $\lambda_2^1(G)$ is NP-complete or polynomially solvable (see [5, 14, 16, 3, 1]). Hence finding good upper bounds on $\lambda_2^1(\Gamma)$ have always been a very interesting problem in graph labeling.

$L(2,1)$ labeling of Cayley graphs were investigated by Zhao [19] on abelian groups and by Bahls [2] on more general groups. Recently, Li *et al.* [13] investigated the $L(2,1)$ labeling of cubic Cayley graphs on dihedral groups. We observed that compared to other families of graphs, $L(h,1)$-labeling of Cayley graphs has not been explored much. The circulant graphs, a particular class of Cayley graphs, always attracted mathematicians for their symmetry. To the best of our knowledge, for the first time we investigate the $L(2,1)$-labeling of circulants in this paper.

Definition. Let $\mathbb{Z}_n$ be a cyclic group and $S \subset \mathbb{Z}_n$ such that $0 \notin S$. Define a graph $\Gamma = \Gamma(\mathbb{Z}_n, S)$ by $V(\Gamma) = \mathbb{Z}_n$ and $E(\Gamma) = \{(u,v) : v - u \in S\}$. Such a graph is a *circulant graph of order* n with connection set S. Note that $S = S^{-1} = \{-s : s \in S\}$ for circulant graphs.

We focus our work primarily on "large" connection sets. Note that $|S| \leq n-1$, since $0 \notin S$ (no loops). Also when $|S| = n-1$, then Γ becomes a complete graph, and one can easily observe that $\lambda_2^1(K_n) = 2n-2$. It is clear that for the entire family of circulant graphs $\lambda_2^1(\Gamma)$ does not only follow the conjecture ($\lambda_2^1(\Gamma) \leq \Delta^2$), but also $\lambda_2^1(\Gamma) \leq 2n-2$ where $|S| \leq n-2$.

The rest of this paper is structured as follows. Section 2 consists of the main results in the form of theorems and supporting lemmas describing the upper bounds of the span for the circulants with the connection sets of cardinalities $n-2, n-3$, and $n-4$, respectively. In Section 3 we provide the algorithms for assignment of vertex labels to generalized cases for circulants with "large" connection sets followed by concluding remarks in Section 4.

2. Main Results

First we define the notations that will be used throughout this paper. For the vertices $i \in V(\Gamma)$ and $a \in V(\Gamma) \setminus S$, let $d = \gcd(n,a)$. Also $i \equiv \ell_i \pmod d$, and $\ell_i = q_i \frac{d}{2} + r_i$, $q_i, r_i \in \mathbb{Z}$. It can be observed that $q_i \in \{0,1\}$, and $r_i \in \{0,1,\ldots,\frac{d}{2}-1\}$. For any $a \in V(\Gamma) \setminus S$, let p_i and p' be the smallest non-negative integers for which $a \big| (np_i + (i-\ell_i))$ and $a \big| \left(np' + \frac{n}{2} - a - \frac{d}{2}\right)$, respectively. Note that p' is constant for fixed a. For convenience, we will use the notations $p_{ij} = p_i - p_j, \ell_{ij} = \ell_i - \ell_j, q_{ij} = q_i - q_j$ and $r_{ij} = r_i - r_j$ whenever required.

Theorem 1. *If* $|S| = n-2$*, then* $\lambda_2^1(\Gamma) \leq \frac{3n}{2} - 2$.

Proof. We begin this proof with the observation that n must be even, since $0 \notin S$, and $|S| = n-2$, $\frac{n}{2} \notin S$. Now we introduce the function $f : V(G) \to \mathbb{Z}_{\frac{3n}{2}-1}$.

$$f(i) = \begin{cases} 3i, & \text{if } i \in \{0,1,\ldots,\frac{n}{2}-1\}, \\ 3\left(i - \frac{n}{2}\right) + 1, & \text{if } i \in \left\{\frac{n}{2}, \frac{n}{2}+1, \ldots, n-1\right\}. \end{cases}$$

Our claim is $f \cong L(2,1)$, i.e., $|\, f(i) - f(j)\, | \geq 2$ if $j - i \in S$ and $|\, f(i) - f(j)\, | \geq 1$ if $\operatorname{dist}(i,j) = 2$, where $\operatorname{dist}(i,j)$ is defined as the length of the shortest path between the vertices $i,j \in V(\Gamma)$. Let us first consider that $i,j \in \{0,1,\ldots,\frac{n}{2}-1\}$, then $|f(i) - f(j)| = |3(i-j)| > 2$, for $i \neq j$ which clearly satisfies the requirement. Similarly the assumption $i,j \in \{\frac{n}{2}, \frac{n}{2}+1, \ldots, n-1\}$ also meets the requirement.

Now it remains to consider the case when $i \in \{0,1,\ldots,\frac{n}{2}-1\}$, and $j \in \{\frac{n}{2}, \frac{n}{2}+1, \ldots, n-1\}$ or vice-versa. Without loss of generality we can assume the former one. In that case $|f(i) - f(j)| = \left|3\left(\frac{n}{2} + i - j\right) - 1\right|$. It can be easily verified that f is injective and $\frac{n}{2} \notin S$, $\left|3\left(\frac{n}{2}+i-j\right)-1\right| \neq 1$, for any choice of i, j. In particular $\mid f(i)-f(j) \mid \geq 2$ if $j-i \in S$ and $\mid f(i)-f(j) \mid \geq 1$ if $\operatorname{dist}(i,j)=2$. Moreover, it can be easily observed that $\max_{i \in \mathbb{Z}_n} f(i) = \frac{3n}{2}-2$, we find that $\lambda_2^1(\Gamma) \leq \frac{3n}{2}-2$. ∎

Next we will consider $|S| = n-3$, which is possible only if $\{0, a, n-a\} \notin S$, where a is any non-negative integer. For this specified connection set, we define the vertex labeling function f below, and prove that f provides an $L(2,1)$-labeling in Theorem 3, and Lemma 2.

$$f(i) = \frac{np_i + (i-\ell_i)}{a} + \ell_i\left(\frac{n}{d}+1\right).$$

Lemma 2. *For any $i \in Z_n$, there exists a non-negative integer $p \leq \frac{a}{d}$ such that $a \big| (np + (i-\ell_i))$.*

Proof. First note that $i - \ell_i = rd$, for some non-negative integer $r \leq \frac{n}{d}-1$. Hence $\frac{(np+(i-\ell_i))}{a} = \frac{(sp+r)}{t}$, where $s = \frac{n}{d}$, and $t = \frac{a}{d}$. Assume $\frac{(sp+r)}{t} = q$ which gives

$$-sp + tq = r. \tag{1}$$

Now for the choice $r = 1$, equation (1) reduces to $-sp+tq = 1$. But as $d = \gcd(n,a)$, we know $\gcd(s,t) = 1$. Thus by Euclidean Algorithm, there exists at least one integer solution (in terms of (p,q)) to the equation $-sp+tq = 1$. In addition we can view equation (1) as $-sp+tq = 1 \cdot r$, which also guarantees the existence of at least one integer solution to equation (1). Now it remains to show that there exist a non-negative integer p that satisfies equation (1).

As the existence is guaranteed for such integers p and q, we can also obtain infinitely many other integer solutions in the form of

$$\big(p + k\frac{t}{r}, q + k\frac{s}{r}\big),$$

where k is any integer. Starting with any integer solution (p,q), using the Euclidean Algorithm (or any other method), with the appropriate choice of k, we can obtain the smallest non-negative integer p such that $p < t = \frac{a}{d}$. This completes the proof. ∎

Theorem 3. *If $|S| = n-3$ (i.e., $\{0, a, n-a\} \notin S$ for any $a \in \mathbb{Z}^*_{\frac{n}{2}}$), then f defines an $L(2,1)$ labeling on Γ. Moreover, $\lambda_2^1(\Gamma) \leq n+d-2$.*

Proof. To prove that the function f described above, defines an $L(2,1)$ labeling on Γ with connection set S, we first need to show that f labels the vertices of Γ uniquely, and later we show that $|f(i)-f(j)| \geq 2$ when $(ij) \in E(G)$.

Claim 4. *f is injective.*

Proof. Let us start with the assumption that $f(i) = f(j)$ for some $i, j \in \mathbb{Z}_n$, where $i \neq j$. Without loss of generality we can assume that $j > i$. Now, $f(i) = f(j)$ implies

$$\frac{np_i + (i - \ell_i)}{a} + \ell_i\left(\frac{n}{d} + 1\right) = \frac{np_j + (j - \ell_j)}{a} + \ell_j\left(\frac{n}{d} + 1\right). \tag{2}$$

Equation (2), upon simplification, provides

$$(a(k+1) - 1)\ell_{ij} = np_{ij} + (j - i), \tag{3}$$

where $\frac{n}{d} = k$. Clearly, if $\ell_{ij} = 0$ then $j - i = np_{ij}$, which is only possible when $i = j$. Otherwise, (i.e., if $\ell_{ij} \neq 0$) in case of $p_{ij} = 0$, then from equation (3) we get $j - i \geq a(k+1) - 1 \geq \frac{an}{d} \geq n$, which is impossible. If $p_{ij} \neq 0$, then on simplifying equation (3) we get $j - i + np_{ij} = (a(k+1) - 1)\ell_{ij}$. As $a(k+1) > 1$ and $0 < (j - i) < n$, we can easily conclude that ℓ_{ij} and p_{ij} share the same sign (note that none of them is zero in this case). Without loss of generality we can assume that they are both positive. Upon simplifying equation (3) we get,

$$j - i = n\left(a\frac{\ell_{ij}}{d} - p_{ij}\right) + \ell_{ij}(a - 1). \tag{4}$$

Therefore, $\ell_{ij}(a - 1) \geq 0$. Now we consider the three different possibilities for $\frac{a\ell_{ij}}{d} - p_{ij}$. If $\frac{a\ell_{ij}}{d} - p_{ij} > 0$ then we get $j - i \geq n$, which is absurd. If $\frac{a\ell_{ij}}{d} - p_{ij} < 0$, then

$$\begin{aligned} j - i &= -\left(p_{ij} - \frac{a\ell_{ij}}{d}\right)n + \ell_{ij}(a-1) \leq -\left(p_{ij} - \frac{a\ell_{ij}}{d}\right)n + (d-1)(a-1) \\ &\leq \left(1 - \left(p_{ij} - \frac{a\ell_{ij}}{d}\right)\right)n - a < 0 \end{aligned}$$

which leads us to a contradiction as $1 - \left(p_{ij} - \frac{a\ell_{ij}}{d}\right) \leq 0$. The last possibility that we need to consider is $\ell_{ij}(a - 1) = 0$. In this case equation (4) simplifies to $j - i = \ell_{ij}(a - 1) = \frac{p_{ij}d}{a}(a - 1) = p_{ij}d - p_{ij}\frac{d}{a} = p_{ij}\left(d - \frac{1}{x}\right)$, where $x = \frac{a}{d}$. Obviously it leads to a contradiction since $p_{ij}(d - \frac{1}{x})$ is not an integer, as p_{ij} is a positive integer less than $x = \frac{a}{d}$ (Lemma 2). □

Claim 5. $|f(i) - f(j)| \geq 2$ *when* $(ij) \in E(G)$.

Proof. As we have already shown in the previous claim that f assigns distinct values to all the vertices of the graph Γ, it remains to be shown that $|f(i) -$

$f(j)| \neq 1$ for all $(ij) \in E(\Gamma)$. If possible we assume that $|f(i) - f(j)| = 1$ for some $i, j \in \mathbb{Z}_n$, where $j - i \in S$. This leads to

$$\left| \frac{np_i + (i - \ell_i)}{a} + \ell_i \left(\frac{n}{d} + 1 \right) - \frac{np_j + (i - \ell_j)}{a} - \ell_j \left(\frac{n}{d} + 1 \right) \right| = 1,$$

which simplifies to the following equation,

$$\left| np_{ij} - (j - i) - \ell_{ij} \left(a \left(\frac{n}{d} + 1 \right) - 1 \right) \right| = a. \tag{5}$$

If $p_{ij} = 0$, then equation (5) becomes

$$\left| (j - i) + \ell_{ij} \left(a \left(\frac{n}{d} + 1 \right) - 1 \right) \right| = a. \tag{6}$$

If $\ell_{ij} = 0$, then $j - i = a$, which is a contradiction since $a \notin S$. Otherwise $j - i = \pm a + \ell_{ij} \left(a \left(\frac{n}{d} + 1 \right) - 1 \right)$. Hence either $j - i \geq n$ or $j - i \leq -n$ (based on the sign of ℓ_{ij}) is a contradiction.

If $p_{ij} \neq 0$, then without loss of generality we assume that p_{ij} is positive. Once again if $\ell_{ij} = 0$, then equation (6) implies that $j - i = np_{ij} \pm a$, which is only possible if $j - i = n - a$, again this is absurd as $n - a \notin S$. We assume that $\ell_{ij} \neq 0$, which simplifies equation (6) to $j - i = np + \ell_{ij} \left(a \left(\frac{n}{d} + 1 \right) - 1 \right) \pm a$. If ℓ_{ij} is positive, then $j - i > n$, a contradiction. Hence considering $\ell_{ij} = -t$, where $t > 0$, we have

$$j - i = -n \left(a \frac{t}{d} - p \right) - ta + t \pm a. \tag{7}$$

Note that if $t \geq 2$, then from equation (7) $j - i \leq -n \left(2\frac{a}{d} - p \right) - 2a + 2 \pm a \leq -2n$, which is impossible. Hence the only option is to assume that $t = 1$, which simplifies equation (7) to $j - i = -n \left(\frac{a}{d} - p \right) - a + 1 \pm a$, which is possible only if $p = \frac{a}{d} - 1$. Thus we get $j - i = -n - a + 1 \pm a$, which means the value of $j - i$ is either $-n - 2a + 1$, or $-n + 1$. The former one is absurd, we can only consider the latter one, i.e., $i - j = n - 1$, which is equivalent to $j - i = 1$ in Γ. But in that case equation (7) implies $n = 0$, a contradiction. $\square$

Now it remains to show that $\lambda_2^1(\Gamma) \leq n + d - 2$. Note that f attains its maximum when both p_i and ℓ_i attain maximum, i.e., $p_i = \frac{a}{d} - 1$, whereas $\ell_i = d - 1$. So we have

$$\begin{aligned}
\max_{i\in\mathbb{Z}_n}\{f(i)\} &= \max_{i\in\mathbb{Z}_n}\left\{\frac{np_i+(i-\ell_i)}{a}+\ell_i\left(\frac{n}{d}+1\right)\right\}\\
&\le \max_{i\in\mathbb{Z}_n}\left\{\frac{n\left(\frac{a}{d}-1\right)+(i-(d-1))}{a}+(d-1)\left(\frac{n}{d}+1\right)\right\}\\
&= \max_{i\in\mathbb{Z}_n}\left\{\frac{n}{d}+\frac{-n+i-d+1}{a}+n+d-\frac{n}{d}-1\right\}\\
&= n+d-1-\min_{i\in\mathbb{Z}_n}\left\{\frac{n+d-1-i}{a}\right\}
\end{aligned}$$

Now in order to get the λ, we must find the smallest integer value of $\frac{n+d-1-i}{a}$ for all $i\in\mathbb{Z}_n$. As $i\le n-1$, we know that $\frac{n+d-1-i}{a}$ is always positive. Thus we can conclude that $\min_{i\in\mathbb{Z}_n}\left\{\frac{n+d-1-i}{a}\right\}\ge 1$. Finally we get $\max_{i\in\mathbb{Z}_n}\big\{f(i)\big\}\le n+d-2$. ■

Next we consider the case when $|S|=n-4$. First note that in this case n must be even, and the connection set S should be such that $\mathbb{Z}_n\setminus S=\{0,a,\frac{n}{2},n-a\}$ for some $a\in\mathbb{Z}_n^*$. Without loss of generality, we assume a is the smallest integer in the set $\mathbb{Z}_n^*\setminus S$. Let us first prove two lemmas before we propose the function that assign the labeling to Γ. The first one is easy to verify, so we skip the proof.

Lemma 6. *If a is coprime to n, then $\lambda_2^1(\Gamma)=n-1$.*

Lemma 7. *If $d\nmid\frac{n}{2}$, then d must be even.*

Proof. Let us consider the prime power decomposition of $n=2^{a_0}p_1^{a_1}\cdots p_n^{a_k}$, where $a_0\ge 1$. Let us also consider $d=2^{b_0}p_1^{b_1}\cdots p_n^{b_k}$, where $b_i\le a_i$ for all $i\in\{0,1,\dots,k\}$. But as $d\nmid\left(\frac{n}{2}\right)$, then it is clear that b has at least a prime factor that does not divide $\frac{n}{2}=2^{a_0-1}p_1^{a_1}\cdots p_n^{a_k}$, which is only possible when $b_0>a_0-1\ge 0$. Hence d is even integer. ■

For the sake of simplicity, for any $i\in V(\Gamma)$ we consider $C_i=\frac{np_i+(i-\ell_i)}{a}$, and $F_a=\frac{np'-a+\frac{(n-d)}{2}}{a}$. We can easily figure out the bounds for C_i, and F_a, such that $0\le C_i\le\frac{n}{d}-1$ for any $i\in V(\Gamma)$, and $0\le F_a\le\frac{n}{d}-2$, as $p_i,p'\le\frac{a}{d}-1$. Now the following function will assign $L(2,1)$-labeling to the graph G, which we will show in the following theorem.

$$f(i)=C_i+q_i(1+t_i)\frac{n}{d}+r_i\left(2\frac{n}{d}+1\right)-q_iF_a,$$

where $t_i=\begin{cases}0, & C_i\ge F_a,\\ 1, & C_i<F_a.\end{cases}$

We claim that this function assigns the $L(2,1)$-labeling to the circulants Γ with $|S| = n-4$, as well as the value of the $\lambda_2^1(\Gamma)$ is at most $n + \frac{d}{2} - 2$. We prove the second claim first and the former one in Theorem 10.

Theorem 8. *$\lambda(G) \leq n + \frac{d}{2} - 2$.*

Proof. First it is obvious that f attains its maximum when q_i, r_i attain their maximum values, i.e., $q_i = 1$, and $r_i = \frac{d}{2} - 1$, which immediately implies $\ell_i = d-1$. Also note that as $F_a \in \mathbb{Z}^+ \cup \{0\}$, $np' + \frac{n}{2} \geq a + \frac{d}{2}$. On the other hand, $p_i = \frac{a}{d} - 1$ gives us $\max\{C_i\} \geq F_a$, which implies that $t_i = 0$.

$$\begin{aligned}
\max_{i\in\mathbb{Z}_n}\{f(i)\} &= \max_{i\in\mathbb{Z}_n}\left\{\frac{np_i + (i-\ell_i)}{a} + q_i(1+t_i)\frac{n}{d} + r_i\left(2\frac{n}{d}+1\right)\right.\\
&\quad \left. - \; q_i\frac{np' + \frac{n-d}{2} - a}{a}\right\}\\
&\leq \max_{i\in\mathbb{Z}_n}\left\{\frac{n\left(\frac{a}{d}-1\right) + (i-d+1)}{a} + \frac{n}{d} + \left(\frac{d}{2}-1\right)\left(2\frac{n}{d}+1\right)\right.\\
&\quad \left. - \; \frac{np' + \frac{n-d}{2} - a}{a}\right\}\\
&= n + \frac{d}{2} - \frac{1}{a}\min_{i\in\mathbb{Z}_n}\left\{n\left(p' + \frac{3}{2}\right) + \frac{d}{2} - i - 1\right\}\\
&= n + \frac{d}{2} - 1 - \frac{1}{a}\min_{i\in\mathbb{Z}_n}\left\{n + d - i - 1)\right\} \leq n + \frac{d}{2} - 2
\end{aligned}$$

as $\min_{i\in\mathbb{Z}_n}\left\{\frac{n+d-1-i}{a}\right\} \geq 1$, as we have shown in Theorem 3. ■

Now it remains to show that f assigns a $L(2,1)$-labeling to Γ.

Theorem 9. *If $|S| = n-4$, $d > 1$, and $d \nmid \frac{n}{2}$, where $d = \min\{\gcd(n,a)\}$, then f defines an $L(2,1)$ labeling on $\Gamma = Cir(n,S)$.*

Proof. Lemma 7 suggests that d cannot be odd, as $d \nmid \frac{n}{2}$. Also it is not difficult to verify that f is injective. It suffices to show that $|f(i) - f(j)| \geq 2$ when $(ij) \in E(G)$, i.e., $j - i \in S$. If possible let us assume that $|f(i) - f(j)| = 1$. Further, without loss of any generality, let us assume that $j > i$. Now $f(j) - f(i) = C_{ji} + \frac{n}{d}(q_j(t_j+1) - q_i(t_i+1)) + (r_j - r_i)\left(2\frac{n}{d}+1\right) - (q_j - q_i)F_a$, where we have set $C_{ji} = C_j - C_i$. Note that either one, or both of q_{ji} (i.e., $q_j - q_i$) and r_{ji} (i.e., $r_j - r_i$) is zero, or they share same signs. Without loss of generality we also assume that $f(j) - f(i) = -1$, which gives us

$$C_{ji} + \frac{n}{d}\big(q_j(t_j+1) - q_i(t_i+1)\big) + r_{ji}\left(2\frac{n}{d}+1\right) - q_{ji}F_a = -1. \tag{8}$$

Now if $t_i = t_j = t$, then it can be observed that for both the assumptions $r \geq 1$ and $r \leq 1$ we arrive at contradiction. The only remaining choice is $r = 0$. Further, $q_{ji} = 0$ implies $\ell_{ji} = 0$ and as a consequence we obtain $j - i = -np_{ji} \pm a$ which is absurd. Therefore, we must have $q_{ji} \in \{-1, 1\}$ and hence we reach $C_{ji} + q_{ji}\big((1+t)\frac{n}{d} - F_a\big) = -1$. If $t = 1$, then $C_{ji} = C_j - C_i < F_a$, which implies $C_{ji} + q_{ji}\left(2\frac{n}{d} - F_a\right) \geq 1$ if $q_{ji} = 1$, and $C_{ji} + q_{ji}\left(2\frac{n}{d} - F_a\right) \leq -2$ if $q_{ji} = -1$, as $F_a \leq \frac{n}{d} - 2$. On the other hand, if $t = 0$, and $q_{ji} = 1$, then $C_{ji} + \left(\frac{n}{d} - F_a\right) \geq C_j - C_i + \frac{n}{d} - C_j = \frac{n}{d} - C_i \geq 1$. If $q_{ji} = -1$ then equation (8) simplifies to $\frac{n(p_{ji}+p^{'}+\frac{1}{2})+(j-i)}{a} = \frac{n}{d}$, which is only possible when $j - i = \frac{n}{2}$, a contradiction.

Next we consider $t_i \neq t_j$. Without loss of generality we assume that $t_j = 1$ and $t_i = 0$ which simplifies equation (8) to

$$C_{ji} + \big(q_{ji} + q_j\big)\frac{n}{d} - q_{ji}F_a + r_{ji}\left(2\frac{n}{d} + 1\right) = -1.$$

Since q_{ji} and r_{ji} share same signs, we can easily observe that both $r_{ji} > 0$, and $r_{ji} < 0$ lead us to contradictions. Hence the only case we are going to consider is $r_{ji} = 0$. But in this case when $q_{ji} = 1$, we have $q_j = 1$ and $q_i = 0$, which gives us $F_a - C = 2\frac{n}{d}$, a contradiction. On the other hand if we consider $q_{ji} = -1$, i.e., $q_j = 0$ and $q_i = 1$, we arrive at $j - i = n\left(\frac{a}{d} - \left(p_{ji} + p^{'} + \frac{1}{2}\right)\right)$, which is again absurd.

Therefore, all the possibilities lead us to the conclusion that $|f(i) - f(j)| \neq 1$ and this confirms that f defines an $L(2,1)$-labeling on Γ. ■

3. Generalization

In this section we will generalize the result for any circulant, i.e., we provide a *way* to assign the vertex labeling that satisfies the $L(2,1)$ criteria (Algorithm (1) and (2)). Later in Theorem 10, we investigate the condition on the connection set S in order to have the exact value of $\lambda_2^1(\Gamma)$.

Algorithm (1) and Algorithm (2) together provide us the $L(2,1)$-labeling for circulants with *any* connection set S, such that $S^c = \mathbb{Z}_n \setminus S = \{a_1, a_2, \ldots, a_k, n - a_k, n - a_{k-1}, \ldots, n - a_2, n - a_1\}$. Immediately we can determine the upper bound for $\lambda_2^1(\Gamma)$ for those circulants. Here we denote $d_{a_1 a_2 \cdots a_k} = \gcd(a_1, a_2, \ldots, a_k)$. First Algorithm (1) takes the connection set S as input, immediately calculate the non-connection set $S^c = \{b_1, b_2, \ldots, b_{m'}\}$, and then finds the minimal non-connection set $S' = \{a_1, a_2, \ldots, a_m\} \subseteq S^c$; where $m \leq \frac{n-|S|}{2}$, and $\{a_1, a_2, \ldots, a_m\}$ is the smallest set such that $\gcd(a_1, a_2, \ldots, a_m) = \gcd(b_1, b_2, \ldots, b_{m'})$. Next Algorithm (2) assigns the $L(2,1)$-labeling to the circulant graph of order n. First of all, note that for any $a \in \mathbb{Z}_n$, the circulant graph $\Gamma = \Gamma(Z_n, \{a\})$ is $d = \gcd(n, a)$

many disconnected cycles of order $\frac{n}{2}d$. Also observe that the groups of integers modulo n; $\langle a_1 \rangle$ is a cyclic subgroup of $\mathbb{Z}_n$ with order $\frac{n}{d_{a_1}}$, and $\langle a_1, a_2 \rangle$ is a subgroup of Z_n with order $\frac{n}{d_{a_1 a_2}}$. It is easy to verify that $\langle a_1 \rangle \lhd \langle a_1, a_2 \rangle$, and hence $\frac{\langle a_1, a_2 \rangle}{\langle a_1 \rangle} = \frac{d_{a_1}}{d_{a_1 a_2}}$. Similarly for any $t \leq m$, $\frac{\langle a_1, a_2, \ldots, a_t \rangle}{\langle a_1, a_2, \ldots, a_{t-1} \rangle} = \frac{d_{a_1 a_2 \cdots a_t}}{d_{a_1 a_2 \cdots a_{t-1}}}$. According to Algorithm (2), we first label all vertices of $\left\{ i_1 a_1 | i_1 \in Z_{\frac{n}{d_{a_1}}} \right\} (\bmod n)$ with i_1. After labeling all the vertices of $\langle a_1 \rangle$ with labels $\{0, 1, \ldots, \frac{n}{d_1} - 1\}$, we label the vertex $a_2 - a_1$, as $\frac{n}{d}_1$, and follow the pattern of labeling vertices $a_2 + i_1 a_1$ as $\frac{n}{d}_1 + i_1 + 1$. It can be easily observed that this pattern of labeling can be repeated $\frac{d_1}{d_{a_1 a_2}}$ many times. Hence after labeling $\frac{n}{d_{a_1 a_2}}$ in this fashion, we then iterate this method for $\{a_3, a_4, \ldots, a_m\}$. Since according to Algorithm (2), consecutive labels are only being used in difference of $a_1, a_2, \ldots, a_{m-1}$ or a_m, any two adjacent vertices have the difference of labeling of at least 2. Hence Algorithm (2) provides an $L(2,1)$-labeling to the graph Γ.

Algorithm 1

```
1:  Input: The number n, and the connection set S.
2:  Output: a set {a_1, a_2, ..., a_m}
3:  procedure MINIMAL NON-CONNECTION SET
4:  Set Z^+_{n/2} \ S := {b_1, b_2, ..., b_{m'}}
5:  Compute d_{b_i} = gcd(n, b_i)
6:  Set a_1 := b_i such that d_{a_1} = min_{b_j ∈ Z^+_{n/2} \ S} gcd(n, b_j), d := d_{a_1}, k := 1
7:      if d = 1 then
8:          go to 13
9:      while d > d_{b_1 b_2 ··· b_{m'}} do
10:         Find b_t such that d_{a_1 a_2 ··· a_k b_t} = min_{s ∈ {1,2,...,m'}} gcd(a_1, a_2, ..., a_k, b_s)
11:         Set a_{k+1} ← b_t
12:         Set d ← gcd(a_1, a_2, ..., a_{k+1})
13:     print {a_1, a_2, ..., a_m}
```

Theorem 10. *Let $d = \min\{\gcd(n,a)\}$, and $S^c = \mathbb{Z}_n \setminus S$. Then if for all $a \in S^c$, there exist $s_1, s_2 \in S$ such that $a \equiv s_1 - s_2 \,(\bmod n)$, then*

$$\lambda_2^1(\Gamma) = \begin{cases} n + d - 2, & \textit{if } |S^c| \textit{is odd}, \\ n + \frac{d}{2} - 2, & \textit{if } |S^c| \textit{ is even}. \end{cases}$$

Proof. First we consider that $|S^c|$ is odd. From Theorem 3 it is obvious that $\lambda_2^1(\Gamma) \leq n + d - 2$. We just need to show that $\lambda_2^1(\Gamma) > n + d - 3$. If possible let us assume that $\lambda_2^1(\Gamma) \leq n + d - 3$.

Assume $d = 1$ which gives us $\lambda_2^1(\Gamma) \leq n-2$. But this value of $\lambda_2^1(\Gamma)$ clearly implies that at least two vertices $v_1, v_2 \in V(G)$ use the same label $m \in \{0, 1, \ldots, n-2\}$. Note that as they share same label, $v_2 - v_1 \notin S$, implies $v_2 - v_1 = a$, for some $a \in S^c$. But as $a \equiv s_1 - s_2 \pmod{n}$, for some $s_1, s_2 \in S$, distance between v_1 and v_2 is exactly 2, which is a contradiction. Next we assume that $d \geq 2$. It is easy to observe from Algorithm (2) that there will be exactly $d-1$ *jumps* in vertex labeling, hence there will be exactly $n-1$ labels available for n many vertices, which leads to the same contradiction. The case $|S^c|$ is similar to the other one. ■

Algorithm 2

Input: The number n, and $a_1, a_2, \ldots, a_m$.
Output: L(2,1)-labeling of the graph Γ
procedure Vertex Labeling
Set $i := 0$, $k := 0$, $d := d_{a_1 a_2 \cdots a_m}$
    while $d > 0$ **do**
        for $i_m := 0$ to $\frac{d_{a_1 a_2 \cdots a_{m-1}}}{d_{a_1 a_2 \cdots a_m}} - 1$ **do**
            for $i_{m-1} := 0$ to $\frac{d_{a_1 a_2 \cdots a_{m-2}}}{d_{a_1 a_2 \cdots a_{m-1}}} - 1$ **do**
                ....
                for $i_2 := 0$ to $\frac{d_{a_1}}{d_{a_1 a_2}} - 1$ **do**
                    $i \leftarrow (i + i_2 a_2 + i_3 a_3 + \cdots + i_m a_m) \pmod{n}$
                    for $i_1 := 0$ to $\frac{n}{d_{a_1}} - 1$ **do**
                        $i \leftarrow (i + i_1 a_1)|1|1 \pmod{n}$
                        $f(i) \leftarrow k$
                        $k \leftarrow k + 1$
                ....
        $d \leftarrow d - 1$
        $k \leftarrow k + 1$

4. Conclusion

In this paper we have worked on the $L(2,1)$-labeling of a new family of graphs, i.e., circulant graphs for large connection sets. We have provided the upper bound of the span for three specific cases, viz. $|S| = n-2, n-3$, and $n-4$. We have generalized the results for any large connection set in the form of algorithms. Moreover, we provided the condition for the exact span in these cases.

This was the first attempt to compute the span for $L(2,1)$-labeling of circulant graphs. Our present work could be extended in various directions in the future. For example one can find, $L(h,1)$, $L(h,k)$ or any other type of distance labeling for circulants, or even for generalized Cayley graphs. On the other hand, one can extend the similar technique to the other families of graphs, that are yet to be considered for $L(2,1)$-labeling.

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