NEW FORMULAE FOR THE DECYCLING NUMBER OF GRAPHS

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Abstract

A set \(S\) of vertices of a graph \(G\) is called a decycling set if \(G - S\) is acyclic. The minimum order of a decycling set is called the decycling number of \(G\), and denoted by \(\nabla(G)\). Our results include: (a) For any graph \(G\),

\[\nabla(G) = n - \max_T \{\alpha(G - E(T))\},\]

where \(T\) is taken over all the spanning trees of \(G\) and \(\alpha(G - E(T))\) is the independence number of the co-tree \(G - E(T)\). This formula implies that computing the decycling number of a graph \(G\) is equivalent to finding a spanning tree in \(G\) such that its co-tree has the largest independence number. Applying the formula, the lower bounds for the decycling number of some (dense) graphs may be obtained. (b) For any decycling set \(S\) of a \(k\)-regular graph \(G\),

\[|S| = \frac{1}{k - 1}(\beta(G) + m(S)),\]

where \(\beta(G) = |E(G)| - |V(G)| + 1\) and \(m(S) = c + |E(S)| - 1\), \(c\) and \(|E(S)|\) are, respectively, the number of components of \(G - S\) and the number of

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edges in $G[S]$. Hence $S$ is a $\nabla$-set if and only if $m(S)$ is minimum, where $\nabla$-set denotes a decycling set containing exactly $\nabla(G)$ vertices of $G$. This provides a new way to locate $\nabla(G)$ for $k$-regular graphs $G$. (c) 4-regular graphs $G$ with the decycling number $\nabla(G) = \lceil \frac{\beta(G)}{3} \rceil$ are determined.

**Keywords:** decycling number, independence number, cycle rank, margin number.

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1. **Introduction**

Graphs throughout this paper are loopless and multiple edges are permitted. For general theoretic notations, we follow Diestel [4]. The minimum number of edges whose removal eliminates all cycles in a given graph has been known as the cycle rank of the graph, and this parameter has a simple expression $\beta(G) = |E(G)| - |V(G)| + w$ (see [9]), where $w$ is the number of components of $G$. The corresponding problem of eliminating all cycles from a graph by means of deletion of vertices goes back at least to the work of Kirchhoff on spanning trees [10].

Let $G = (V, E)$ be a graph. We define a vertex set $S$ of $G$ to be a decycling set if $G - S$ is cycle-free. The cardinality of a minimum decycling set of $G$ is called the decycling number, and denoted by $\nabla(G)$ (or $\nabla$ for short). A decycling set containing exactly $\nabla(G)$ vertices of $G$ is called a $\nabla$-set. Vertices of a decycling set are labeled by “•” and the bold edges induce a spanning tree $T$ of a graph $G$ in the following figures. Let $m(S) = c + |E(S)| - 1$ be the margin number of a decycling set $S$, where $c$ and $|E(S)|$ are, respectively, the number of components of $G - S$ and the number of edges in $G[S]$. Determining the decycling number is equivalent to finding the size of the largest induced forest of $G$ proposed first by Erdős [5]. If $S \subseteq V(G)$ is a $\nabla$-set, then $G - S$ is a largest induced forest of $G$. The problem of determining the decycling number of graphs have been proved to be NP-complete by Karp [11], even for general graphs such as bipartite graphs, planar graphs and perfect graphs, the decycling problem is very hard to solve. It is easy to see that $\nabla(G) = 0$ if and only if $G$ is a forest, and $\nabla(G) = 1$ if and only if $G$ has at least one cycle and a vertex on all of its cycles. One may see [2] as a brief survey.

In this paper, we consider the decycling problem from two new perspectives: the effects of (a) spanning trees and (b) the margin number, respectively, on the decycling number of graphs. Given a connected graph $G$ and a surface $P$, we say that $G$ can be embedded into $P$ if there exists a polyhedron $\Sigma$ on $P$ such that the 1-skeleton of $\Sigma$ has a subgraph homeomorphic to $G$. The components of $\Sigma - G$ are called the faces of the embedding. When each face is homeomorphic to an
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open disc, the embedding is called a cellular. The maximum genus, denoted by $\gamma_M(G)$, of a connected graph $G$ is the largest genus of an orientable surface on which $G$ admits a cellular embedding. Let $T$ be a spanning tree of a connected graph $G$. The subgraph $G - E(T)$ of $G$ is called a co-tree of $G$. Note that the number of edges in any co-tree of $G$ is just the cycle rank $\beta(G)$. The Betti deficiency of $G$, denoted by $\xi(G)$, is defined as the minimum number of odd components (i.e., the components containing odd number of edges) among co-trees of $G$. We call $T$ a Xuong-tree if the number of odd components of $G - E(T)$ is $\xi(G)$. The following result of Xuong defines an edge-partition of a co-tree.

**Lemma 1** [20]. Let $G$ be a connected graph and $T_X$ be a Xuong-tree of $G$. Then there exists an edge-partition of $E(G) - E(T_X)$ as follows:

$$E(G) - E(T_X) = \{e_1, e_2\} \cup \{e_3, e_4\} \cup \cdots \cup \{e_{2m-1}, e_{2m}\} \cup \{f_1, f_2, \ldots, f_s\},$$

where (1) $m = \gamma_M(G)$, $s = \xi(G)$; (2) for any $i = 1, 2, \ldots, m$, $e_{2i-1} \cap e_{2i} \neq \emptyset$, and \{\{f_1, f_2, \ldots, f_s\} is a matching of $G$.

Let $T_X$ be a Xuong-tree and the edge-partition of $E(G) - E(T_X)$ be as defined in Lemma 1. Consider a set

$$S_X = \{u_i | u_i \in e_{2i-1} \cap e_{2i}, 1 \leq i \leq m\} \cup \{v_j | v_j \text{ is an end of } f_j, 1 \leq j \leq s\}.$$ 

Then $G - S_X$ contains no cycle (since removing $S_X$ from $G$ will eliminate all the possible fundamental cycles of $G$) and hence $S_X$ is a decycling set of $G$, that is, $\nabla(G) \leq |S_X|$.

**Corollary 2.** $\nabla(G) \leq |S_X| \leq \gamma_M(G) + \xi(G)$ holds for every graph $G$.

It is easy to see that the bound $|S_X|$ heavily depends on the choice of Xuong-tree $T_X$ (since different $T_X$ may lead to quite different value of $|S_X|$). For instance, the wheel graph $W_{1,n} = K_1 \cup C_n$ with $n$ spokes has $\nabla(W_{1,n}) = 2$. If one chooses a Xuong-tree $K_{1,n}$ as a spanning tree of $W_{1,n}$, then the corresponding $|S_X| = \lceil \frac{n}{2} \rceil$; meanwhile, a Hamilton path in $W_{1,n}$ will determine another $S_X$ whose number of elements reaches the best value $\nabla(W_{1,n}) = 2$. Therefore, how to find a set $S_X \subseteq V(G)$ with the smallest size is a key to determine $\nabla(G)$.

This paper is organized as follows.

In Section 2, we show that $\nabla(G) = n - \max_T \{\alpha(G - E(T))\}$ holds for any graph $G$. This implies that determining the decycling number is equivalent to finding the largest independence number of a co-tree. So, finding the decycling number $\nabla(G)$, determining the largest independence number of a co-tree $G - E(T)$ and finding the size of a largest induced forest in a graph $G$ are mutually equivalent. In this sense, finding the decycling number of a graph is very hard. Applying this formula, we may obtain lower bounds for the decycling number of
some (dense) graphs. As an example, we prove that \( \nabla(K_n - (E(T_1) \cup E(T_2) \cup \cdots \cup E(T_{k-1}))) \geq n - 2k \), where \( T_1, T_2, \ldots, T_{k-1} \) are \( k - 1 \) edge-disjoint spanning trees of a complete graph \( K_n \). Many examples and applications are presented to show how to apply trees into identifying the decycling number of a graph.

In Section 3, we obtain an another formula to compute the decycling number of regular graphs. For any decycling set \( S \) of a \( k \)-regular graph \( G \), we get that
\[
|S| = \frac{1}{k-1}(\beta(G) + m(S)).
\]

Obviously, \( S \) is a \( \nabla \)-set if and only if \( m(S) \) is minimum. Therefore, lower bounds for the decycling number of some (dense) graphs can be obtained. For a \( k \)-regular graph \( G \), if \( m(S) \geq 0 \) for any \( \nabla \)-set \( S \) of \( G \), then \( \nabla(G) \geq \frac{\beta(G)}{k-1} \). In many cases, these lower bounds may be tight (i.e., best possible) (see [3, 8, 12, 14–16, 18, 19]). Observe that for some (4-regular) graphs \( G \) of order \( n \), there exists a decycling set \( S \) such that the margin number \( m(S) \) is a linear function of \( n \). For instance, a toroidal 4-regular graph \( G \) contains \( n \) disjoint \( K_5 - e \)’s (see Figure 4) whose decycling number of \( G \) is \( 2n + 1 \). It is easy to see that the margin number \( m(S) = n + 2 \) for any \( \nabla \)-set \( S \) of \( G \). Moreover, we discuss some relationships between the decycling number and the large genus embeddings of graphs, and show the effects of spanning trees on such topics. In particular, we give a new and direct proof of a result due to Speckenmeyer [17] and thus solve an open problem of Speckenmeyer searching for an efficient algorithm to compute \( Z(G) \), the cardinality of the maximum nonseparating independent set of \( G \).

In Section 4, we investigate the extremal 4-regular graphs \( G \) with the decycling number \( \nabla(G) = \lceil \frac{\beta(G)}{3} \rceil \). Our conclusion is that for any \( \nabla \)-set \( S \) of graph \( G \), there exists a spanning tree \( T \) in \( G \) such that elements of \( S \) are taken from the leaves of \( T \) with at most two exceptions (from the 2 or 3-degree vertices of \( T \)). Finally, we extend this result to general case.

2. A Formula Between the Decycling Number and the Independence Number

In this section, \( \alpha(G) \) and \( a(G) \) denote, respectively, the independence number and the number of vertices in a largest induced forest of a graph \( G \).

**Theorem 3.** Let \( G \) be a connected graph of order \( n \). Then
\[
\nabla(G) = n - \max_T \{ \alpha(G - E(T)) \},
\]
where \( T \) is taken over all spanning trees of \( G \).

The above result reveals a relation among the decycling number, the independence number and the spanning trees in a graph and gives a new way to investigate these numbers.
Lemma 4. Let $G$ be a connected graph of order $n$. Then
\[ a(G) = \max_T \{ \alpha(G - E(T)) \}, \]
where $T$ is taken over all spanning trees of $G$.

**Proof.** Let $F$ be a largest induced forest of $G$ with $|F| = a(G)$. Then $V(G) - V(F)$ is a decycling set of $G$. Since $G$ is connected, there exists a spanning tree $T$ of $G$ such that $F \subseteq T$, and $\alpha(G - E(T)) \geq |F|$. Hence,
\[ a(G) \leq \max_T \{ \alpha(G - E(T)) \}. \]

Conversely, let $T_1$ be a spanning tree of $G$ such that
\[ \max_T \{ \alpha(G - E(T)) \} = \alpha(G - E(T_1)), \]
and suppose that $A$ is the largest independent set of $G - E(T_1)$, that is, $|A| = \alpha(G - E(T_1))$. When we recover the edges of $T_1$ into $A$, it induces a forest, and $G - A$ is a decycling set. Hence $|A| \leq a(G)$. That is,
\[ \max_T \{ \alpha(G - E(T)) \} \leq a(G). \]

**Proof of Theorem 3.** By Lemma 4 and $a(G) + \nabla(G) = n$, the theorem follows.

From the proof of Theorem 3, one may see that if $T$ is a spanning tree of $G$ and $A$ is the maximum independent set of the co-tree $G - E(T)$ (i.e., $|A| = \alpha(G - E(T))$), then $S = V(G) - A$ is a decycling set, and so $|V(G) - A| + |A| = n$, which means that $|A|$ is the largest among all spanning trees of $G$ if and only if the corresponding decycling set $S = V(G) - A$ is minimum. Therefore, how to find a spanning tree $T$ of $G$ such that $\alpha(G - E(T))$ is the maximum is very crucial to computing the decycling number $\nabla(G)$ of $G$. In the following, we shall present several applications and examples to show the effects of the spanning trees on searching for the value $\nabla(G)$ of a graph $G$.

**Example 1.** Let $T$ be a Hamilton path of a complete graph $K_n$. Then $T$ is a spanning tree of $K_n$ (see Figure 1(a)), and so $\alpha(K_n - E(T)) \geq 2$. By Theorem 3, $\nabla(K_n) \leq n - 2$. It deduces that $\nabla(K_n) = n - 2$ because of $\nabla(K_n) \geq n - 2$ (if we remove at most $n - 3$ vertices of $K_n$, then the resultant graph will contain a cycle).

**Example 2.** For a complete bipartite graph $K_{m,n}$, let $V(K_{m,n}) = V = X \cup Y$, where $X = \{x_1, x_2, \ldots, x_m\}, Y = \{y_1, y_2, \ldots, y_n\}$. Assume that $m \leq n$. We construct a spanning tree $T$ of $K_{m,n}$ as follows: $E(T) = \{x_i y_j, x_i y_{j+1} \mid i = 1, 2, \ldots, m-1, j = 1, 2, \ldots, n\}$ (see Figure 1(b)). Then $\alpha(K_{m,n} - E(T)) \geq n + 1$. 

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By Theorem 3, \( \nabla(K_{m,n}) \leq m - 1 \). Since \( \nabla(K_{m,n}) \geq m - 1 \) (otherwise, there is a cycle by removing at most \( m - 2 \) vertices of \( K_{m,n} \) in the resultant graph), we have \( \nabla(K_{m,n}) = m - 1 \).

![Figure 1](image_url)

**Example 3.** For a complete \( k \)-partite graph \( K_{m_1,m_2,...,m_k} \), let \( V(K_{m_1,m_2,...,m_k}) = V = X_1 \cup X_2 \cup \cdots \cup X_k \). Then \( |X_i| = m_i, i = 1,2,\ldots,k \). Without loss of generality, suppose that \( m_1 \leq m_2 \leq \cdots \leq m_k \). We construct a spanning tree \( T \) of \( K_{m_1,m_2,...,m_k} \) as follows: \( E(T) = \{x_1y_i, x_{11}z\} \), where \( y \in V \setminus X_1 \), \( z \) takes over the elements of \( V \setminus X_1 \), \( x_{11} \in X_1, j = 2,\ldots,m_1 \). By Theorem 3, \( \nabla(K_{m_1,m_2,...,m_k} - E(T)) \leq \sum_{i=1}^{k} m_i - m_k - 1 \). Since \( \nabla(K_{m_1,m_2,...,m_k} - E(T)) \geq \sum_{i=1}^{k} m_i - m_k - 1 \) (otherwise, removing at most \( \sum_{i=1}^{k} m_i - m_k - 2 \) vertices of \( K_{m_1,m_2,...,m_k} \) will leave a cycle in the resultant graph), \( \nabla(K_{m_1,m_2,...,m_k}) = \sum_{i=1}^{k} m_i - m_k - 1 \).

**Example 4.** Ren [16] proved that \( \nabla(G) = \gamma_M(G) + \xi(G) \) for \( G \) being a cubic graph, where \( \gamma_M(G) \) and \( \xi(G) \) are the maximum genus and Betti deficiency of \( G \), respectively. We consider a Xuong-tree \( T_X \) of a cubic graph \( G \) and an edge-partition of its co-tree \( G - E(T_X) \) as defined in Lemma 1. Then the set \( S_X \) (as defined in Corollary 2) is a \( \nabla \)-set since \( G - S_X \) contains no cycle and \( |S_X| = \gamma_M(G) + \xi(G) \). Now \( V(G) - S_X \) is an independent set whose cardinality is \( n - \max T \{ \alpha(G - E(T)) \} \). Conversely, let \( T_1 \) be a spanning tree of \( G \) such that \( \alpha(G - E(T_1)) = \max T \{ \alpha(G - E(T)) \} \). Then there exists an independent set \( S \) of \( G - E(T_1) \) with \( |S| = \alpha(G - E(T_1)) \) such that \( G - E(T_1) \) contains an independent set \( A \) with \( |A| = \alpha(G - E(T_1)) = \max T \{ \alpha(G - E(T)) \} \). It is clear that \( G[A] \) is a largest induced forest of \( G \) and \( S = V(G) - A \) is a \( \nabla \)-set. As shown in [16], \( T_1 \) is a Xuong-tree of \( G \) (since the number of odd components of \( G - E(T_1) \) is \( \xi(G) \)).

Based on Theorem 3, many results and problems on the largest induced forests and the decycling set can be translated into one another. For instance, Albertson and Berman [1] posed the following conjecture.

**Conjecture 5** [1]. Every planar graph has an induced forest with at least half the vertices.

The above conjecture can also be expressed into the following three forms.
Theorem 6. Let $G$ be a planar graph of order $n$. Then the following statements are mutually equivalent:
(a) $\nabla(G) \leq \frac{n}{2}$;
(b) $|F| \geq \frac{n}{2}$ holds for a largest induced forest $F$ of $G$;
(c) There exists a spanning tree $T$ in $G$ such that $\alpha(G - E(T)) \geq \frac{n}{2}$.

For a plane triangulation $G$, a plane with all faces are triangles. By Theorem 6(c), we may first find a spanning tree $T$ of $G$ to determine the independence number of its co-tree $G - E(T)$, and further to solve the decycling number of $G$. Since the number of edges of $G - E(T)$ is $2n - 5$, a natural idea is that the problem of the computation of the decycling number of a plane triangulation $G$ can be put into the following problem.

Problem 7. Determine the independence number for a planar graph $G$ of order $n$ with at most $2n - 5$ edges.

In the literature, there are many results on the decycling number for sparse graphs such as 3 (or 4)-regular graphs, see [3, 8, 12, 14–16, 18, 19], but little is known for those with many edges (i.e., dense graphs). Theorem 3 offers a way to estimate the lower bounds for the decycling number of dense graphs. The following result is an application.

Theorem 8. Let $T_1, T_2, \ldots, T_{k-1}$ be $k - 1$ edge-disjoint spanning trees of a complete graph $K_n$. Then

$$\nabla(K_n - (E(T_1) \cup E(T_2) \cup \cdots \cup E(T_{k-1}))) \geq n - 2k.$$ 

And the equality holds if and only if $K_n - (E(T_1) \cup E(T_2) \cup \cdots \cup E(T_{k-1}))$ contains a spanning tree $T_k$ such that the graph $T_1 \cup T_2 \cup \cdots \cup T_{k-1} \cup T_k$ contains $K_{2k+1}$.

Proof. Let $T_1, T_2, \ldots, T_{k-1}$ be $k - 1$ edge-disjoint spanning trees of a complete graph $K_n$. Assume (reductio ad absurdum) that $T_k$ is a spanning tree of $K_n - (E(T_1) \cup E(T_2) \cup \cdots \cup E(T_{k-1}))$ such that $\alpha(K_n - (E(T_1) \cup E(T_2) \cup \cdots \cup E(T_{k-1}) \cup E(T_k))) \geq 2k + 1$. Then $T_1 \cup T_2 \cup \cdots \cup T_{k-1} \cup T_k$ contains $K_{2k+1}$, and hence

$$|\{E(T_1) \cup E(T_2) \cup \cdots \cup E(T_{k-1}) \cup E(T_k)\} \cap E(K_{2k+1})| \geq k(2k + 1).$$

Color the edges of $K_{2k+1}$ with $k$ different colors, then the number of edges with the same color is not more than $2k$. Otherwise, there will exist a subgraph (induced by these edges) containing a cycle, which contraries to the number of edges of $K_{2k+1}$. By Theorem 3,

$$\nabla(K_n - (E(T_1) \cup E(T_2) \cup \cdots \cup E(T_{k-1})))$$

$$= n - \max_{T_k} \{\alpha(K_n - (E(T_1) \cup E(T_2) \cup \cdots \cup E(T_{k-1}) \cup E(T_k)))\},$$

$$\geq n - \max_{T_k} \left\{\frac{n}{2} - \frac{n}{2} \right\} = n - \frac{n}{2} = \frac{n}{2}.$$
that is,
\[
\max_{T_k} \{\alpha(K_n - (E(T_1) \cup E(T_2) \cup \cdots \cup E(T_{k-1}) \cup E(T_k)))\} \leq 2k.
\]

Let \( \nabla(K_n - (E(T_1) \cup E(T_2) \cup \cdots \cup E(T_{k-1}))) = n - 2k \). By Theorem 3, there exists a spanning tree \( T_k \) of \( K_n - (E(T_1) \cup E(T_2) \cup \cdots \cup E(T_{k-1})) \) such that \( \alpha(K_n - (E(T_1) \cup E(T_2) \cup \cdots \cup E(T_{k-1}) \cup E(T_k))) = 2k \). The edges of each \( T_i \) \((1 \leq i \leq k)\) in \( K_{2k} \) form a spanning tree of \( K_{2k} \). Conversely, for \( k - 1 \) edge-disjoint spanning trees \( T_1, T_2, \ldots, T_{k-1} \) of \( K_n \), if there is a spanning tree \( T_k \) of \( K_n - (E(T_1) \cup E(T_2) \cup \cdots \cup E(T_{k-1})) \) such that the subgraphs of \( K_n \) determined by these trees \( T_1, T_2, \ldots, T_{k-1}, T_k \) containing a complete graph \( K_{2k} \), then \( \alpha(K_n - (E(T_1) \cup E(T_2) \cup \cdots \cup E(T_{k-1}) \cup E(T_k))) \geq 2k \).

\[\blacksquare\]

3. A Formula Between the Decycling Number and the Margin Number

Let \( E(S, G - S) \) be the set of edges such that each edge has one vertex in \( S \) and another one in \( G - S \). \( d_G(x) \) and \( \Delta(G) \) (or \( \Delta \) for short) represent the degree of vertex \( x \) and the maximum degree of \( G \), respectively. In this section, we present another formula for the decycling number \( \nabla(G) \) of a \( k \)-regular graph \( G \).

**Theorem 9.** Let \( S \) be a decycling set of \( G \). Then
\[
\sum_{x \in S} (d_G(x) - 1) = \beta(G) + m(S).
\]

**Proof.** Let \( S = \{x_1, x_2, \ldots, x_{|S|}\} \) be a decycling set of \( G \). Then
\[
q - \sum_{i=1}^{|S|} d_G(x_i) = q - |E(S, G - S)| - 2|E(S)| = p - |S| - c - |E(S)|,
\]
where \( p = |V(G)|, q = |E(G)|, c \) and \( |E(S)| \) are, respectively, the number of components of \( G - S \) and the number of edges of \( G[S] \).

As \( \beta(G) = q - p + 1, \)
\[
\sum_{i=1}^{|S|} (d_G(x_i) - 1) = \beta(G) + c + |E(S)| - 1.
\]

And for \( m(S) = c + |E(S)| - 1 \), we have
\[
\sum_{i=1}^{|S|} (d_G(x_i) - 1) = \beta(G) + m(S).
\]

This completes the proof. \[\blacksquare\]
Remark 1. Observe that if all the vertices of $S$ have degree $k$, then $|S| = \frac{1}{k-1}(\beta(G) + m(S))$. In particular, if $G$ is a $k$-regular graph, then $S$ is a $\nabla$-set if and only if $m(S)$ is minimum among all the decycling sets $S$ of $G$.

Although there exists some uncertain parameter like $m(S)$, this result provides a way to locate the value of $\nabla(G)$: once we find a decycling set $S$ such that $m(S)$ reaches the minimum, then $S$ is a $\nabla$-set of $G$. We will show its applications in the discussion to come.

A simple corollary of Theorem 9 is:

Corollary 10. Let $G$ be a graph with maximum degree $\Delta$ which has the $\nabla$-set such that each vertex of this set has degree $\Delta$. Then

$$\nabla(G) \geq \frac{\beta(G)}{\Delta - 1}.$$ (1)

Remark 2. (i) $\nabla(G) = \frac{\beta(G)}{\Delta - 1}$ if and only if $m(S) = 0$ which means that for a $\nabla$-set $S$ of $G$, $G - S$ is a tree $T_0$ and $G[S]$ is an empty subgraph. In this case, $\nabla(G)$ has a strong combinatorial characterization: for any vertex $x \in S$ incident to a vertex $y \in V(T_0)$, insert the edge $xy$ into $T_0$. This procedure determines a spanning tree $T$ (it is in fact a Xuong-tree) of $G$ (such that $\xi(G) = 0$) if we add $|S|$ edges into $T_0$. Therefore, deleting a vertex $x$ of $S$ will destroy $d_G(x) - 1$ fundamental cycles of $G$; deleting $\nabla(G)$ vertices of $S$ will destroy all fundamental cycles of $G$;

(ii) the inequality (1) may be tight, see [3, 8, 12, 14–16, 18, 19].

The following examples show the formula of Theorem 9 applying on some types of regular graphs.

Example 5. Let $S$ be a decycling set of a hypercube $Q_n$ (a graph contains $2^n$ $n$-tuples of 0’s and 1’s as vertices with two vertices adjacent if they differ in exactly one position). Then

$$2^{n-1} - \frac{2^{n-1} - 1}{n - 1} \leq \nabla(Q_n) \leq 2^{n-1} - \frac{2^{n-1} - m(S) - 1}{n - 1}.$$ (2)

The inequalities in (2) are equalities for $n = 3, 4$ [3] (see Figure 2).

Proof. By the definition of $Q_n$, $\beta(Q_n) = (n - 2)2^{n-1} + 1$. By Corollary 10, $\nabla(Q_n) \geq 2^{n-1} - \frac{2^{n-1} - 1}{n - 1}$. Let $S$ be a decycling set of $Q_n$. Then $|S| = 2^{n-1} - \frac{2^{n-1} - m(S) - 1}{n - 1}$, thus $\nabla(Q_n) \leq |S| = 2^{n-1} - \frac{2^{n-1} - m(S) - 1}{n - 1}$. 

Remark 3. (i) Two spanning trees $T_1$ and $T_2$ which induced by the bold edges in Figure 2(a) and (b) of $Q_n$ satisfying that $Q_n - E(T_i)$ ($i = 1, 2$) has the largest independence number, respectively.
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(ii) Focardi [6] proved that \(2^{n-1} - \frac{2^{n-1} - 1}{2^{n-1}} \leq \nabla(Q_n) \leq 2^{n-1} - \frac{2^{n-1} - 1}{2^{n-1}}\). The lower bound was also proved by Beineke and Vandell [3]. In fact, if the upper bound of Focardi’s result is best possible, then \(m(S) = 2^{n-2} - 1\) for any \(\nabla\)-set \(S\). Therefore, determining the decycling number of \(Q_n\) for larger \(n\) is very difficult.

Example 6. For any two cycles \(C_m\) and \(C_n\), their \textit{Cartesian product} is the graph 
\(C_m \times C_n\) with vertex set \(V(C_m \times C_n) = \{w_{ij} \mid i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\}\) and edge set 
\(E(C_m \times C_n) = \{w_{ij}w_{rs} \mid i = r, v_jv_s \in E(C_n) \text{ or } j = s, u_iu_r \in E(C_m)\}\).

Clearly, \(C_m \times C_n\) is a 4-regular graph, see Figure 3. For any decycling set \(S\) of 
\(C_m \times C_n\), we have

\[\frac{mn + 1}{3} \leq \nabla(C_m \times C_n) \leq \frac{mn + m(S) + 1}{3}, m, n \geq 3.\]

In particular, the bounds of \(\nabla(C_m \times C_n)\) in (3) are sharp for \(m = 3\) (or \(n = 3\)) (see Figure 3).

\textbf{Proof.} It is easy to see that \(\beta(C_m \times C_n) = mn + 1\). By Corollary 10, \(\nabla(C_m \times C_n) \geq \frac{1}{3}(mn + 1)\). Let \(S\) be a decycling set of \(C_m \times C_n\). Then \(\nabla(C_m \times C_n) \leq |S| = \frac{1}{3}(mn + m(S) + 1)\). When \(m = 3, \frac{3n+1}{3} \leq \nabla(C_3 \times C_n) \leq \frac{3n+3(S)+1}{3}\), that is, \(n + 1 \leq \nabla(C_3 \times C_n) \leq n + \frac{m(S)+1}{3}\), we can find a decycling set \(S\) such that \(m(S) = 2\) (see Figure 3), then \(\nabla(C_3 \times C_n) = n + 1\).

Remark 4. (i) The spanning trees \(T\) of \(C_3 \times C_n\) in Figure 3(a) and (b) satisfy that \(C_3 \times C_n - E(T)\) has the largest independence number, respectively.
(ii) Our result shows that \( \nabla(C_m \times C_n) = \frac{mn + m(S) + 1}{3} \) for \( S \) being a \( \nabla \)-set of \( C_m \times C_n \), which equals to Pike’s result \( \nabla(C_m \times C_n) = \left\lceil \frac{mn + 2}{3} \right\rceil \) when \( m(S) \leq 1 \) [13]. Therefore, this provides a way to locate the exact value of \( \nabla(G) \) (to find a decycling set \( S \) with the minimum \( m(S) \)).

The formula of Theorem 9 also has some applications in topological graph theory.

A vertex set \( S \) is called a nonseparating independent set of \( G \) if \( S \) is an independent set of \( G \) and \( G - S \) is connected. The cardinality of a maximum nonseparating independent set of \( G \) is denoted by \( Z(G) \). The following result shows a close relation between nonseparating independence number \( Z(G) \) and the maximum genus \( \gamma_M(G) \) of a cubic graph \( G \).

**Theorem 11.** Let \( G \) be a cubic graph. Then
\( Z(G) = \gamma_M(G) \);
\( G - S \) contains no two cycles sharing a vertex in common. Moreover, there exists a Xuong-tree \( T_X \) such that the elements in \( S \) are leaves of \( T_X \).

**Proof.** Let \( T_X \) be a Xuong-tree of \( G \) with an edge-partition of \( G - E(T) \) as defined in Lemma 1. Then for \( 1 \leq i \leq \gamma_M(G) \), \( e_{2i-1} \cap e_{2i} = \{u_i\} \) forms a set of independent vertices of \( G \) (which are leaves of \( T_X \)). Hence \( \{u_1, u_2, \ldots, u_{\gamma_M(G)}\} \) is a nonseparating independent set of \( G \). Therefore, \( Z(G) \geq \gamma_M(G) \). To see the converse inequality, we consider a nonseparating independent set \( S \). Then \( G - S \) is connected. We may suppose further that \( G - S \) is a tree \( T_0 \) and \( T \) is a spanning tree built in Remark 2(i). Then elements of \( S \) are leaves of \( T \). After repeating the argument in Remark 2(i), we may see that \( |S| \leq \gamma_M(G) \) which means that \( Z(G) \leq \gamma_M(G) \). This proves (a).

Suppose that \( G - S \) contains two cycles with one vertex in common. Then by Lemma 1, \( \gamma_M(G - S) \geq 1 \), which together with the construction of a largest genus embedding stated in the proof of (a), \( \gamma_M(G) \geq Z(G) + 1 (= \gamma_M(G) + 1) \), a contradiction. Therefore, cycles of \( G - S \) are independent. In fact, the number of cycles in \( G - S \) is \( \xi(G) \). In addition, any spanning tree \( T_0 \) of \( G - S \) is a subgraph of a Xuong-tree \( T_X \) of \( G \) (as stated in the proof of (a)). This proves (b). \( \blacksquare \)

By Theorem 11, we obtain a result which has been proved by Speckenmeyer in [17] as follows.

**Corollary 12.** Let \( G \) be a cubic graph of order \( n \). Then \( \nabla(G) + Z(G) = \frac{n + 2}{2} \).

Furthermore, the formula (a) in Theorem 11, together with a result of Furst [7], provide an efficient way to compute the value \( Z(G) \) for cubic graphs and solves an open problem raised in [17] searching for a polynomial time algorithm to decide \( Z(G) \) and \( \nabla(G) \).
Theorem 13. Let $S$ be a $\nabla$-set of a $k$-regular graph $G$.
(a) If $m(S) = 0$, then for every $\nabla$-set $S$, there exists a spanning tree $T$ in $G$ such that all vertices of $S$ are leaves of $T$.
(b) If $m(S) = 0$, and $k \equiv 1 \pmod{2}$, then $\nabla(G) = \frac{2}{k-1} \gamma_M(G)$, and each spanning tree $T$ in (a) is a Xuong-tree of $G$.

Proof. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a $\nabla$-set of a $k$-regular graph $G$ with $m(S) = 0$. Then $G[S]$ has no cycle and $G - S$ is a tree $T_0$. Suppose that $y_i$ is a neighbor of $x_i$ in $T_0$ ($i = 1, 2, \ldots, \nabla$). Then $T = T_0 + \{e_i = x_iy_i \mid 1 \leq i \leq \nabla\}$ is a spanning tree of $G$ and $S$ is a subset of leaves of $T$. If $k \equiv 1 \pmod{2}$, then we arrange the left $k - 1$ edges (other than $x_iy_i$) into $\frac{k-1}{2}$ pairs for each $i$ ($1 \leq i \leq \nabla$) and thus it gives rise to an edge-partition of $G - E(T)$ with $\xi(G) = 0$. Notice that in Xuong’s construction of the maximum genus embedding [20], each pair of adjacent edges in $G - E(T)$ will contribute a genus, the graph $G$ may be embedded into an orientable surface with $\frac{k-1}{2}\nabla = \gamma_M(G)$ handles. This ends the proof of (b). As for (a), it follows from the discussion used in the proof of (a) of Theorem 11.

One case may appear if there is a vertex $x$ of a decycling set $S$ such that $d_G(x) < \Delta$. Then the formula of Theorem 9 will be invalid for this case. For instance, a grid of paths $P_m \times P_n$ is the graph $P_m \times P_n$ with vertex set $V(P_m \times P_n) = \{w_{ij} \mid i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\}$ and edge set $E(P_m \times P_n) = \{w_{ij}w_{rs} \mid i = r, v_jv_s \in E(P_n) \text{ or } j = s, u_iu_r \in E(P_m)\}$. We have to choose at least one vertex of degree 3 to eliminate the boundary cycle of $P_m \times P_n$. It is clear that any $\nabla$-set $S$ of the grid $P_m \times P_n$ does not need to contain a 2-degree vertex (since subdividing an edge of a graph does not change its decycling number). Therefore, we may only consider the $\nabla$-set $S$ whose vertices are of degree 4 or 3. Here, we slightly extend the formula of Theorem 9 as follows.

Theorem 14. Let $G$ be a non-regular graph with maximum degree $\Delta$ and $S$ a decycling set of $G$. Suppose that $d$ ($d < \Delta$) is a fixed natural number with $S = S_\alpha \cup S_\beta$, $S_\alpha = \{x \mid d_G(x) = \Delta, x \in S\}$, $S_\beta = \{x \mid d_G(x) = d < \Delta, x \in S\}$. Then $|S| = \frac{1}{\Delta - 1} (\beta(G) + (\Delta - d)|S_\beta| + m(S))$.

Proof. Let $S$ be a decycling set of a graph $G$. Similar to the proof of Theorem 9,

$$(\Delta - 1)|S| - |S_\beta| + (d - 1)|S_\beta| = \beta(G) + c + |E(S)| - 1,$$

i.e.,

$$(\Delta - 1)|S| - (\Delta - d)|S_\beta| = \beta(G) + c + |E(S)| - 1,$$

and then

$$|S| = \frac{1}{\Delta - 1} (\beta(G) + (\Delta - d)|S_\beta| + m(S))$$

since $m(S) = c + |E(S)| - 1$. 

\[\square\]
Corollary 15. Let $G$ be a non-regular graph with maximum degree $\Delta$ which has the $\nabla$-set such that each vertex of this set has degree $\Delta$. Then

$$\nabla(G) \geq \frac{1}{\Delta - 1}(\beta(G) + \Delta - d),$$

where $d$ ($d < \Delta$) is the degree of some vertices of a $\nabla$-set of $G$.

Example 7 (Cartesian product of two paths). For any decycling set $S$ of $P_m \times P_n$,

$$\frac{mn - m - n + 2}{3} \leq \nabla(P_m \times P_n) \leq \frac{mn - m - n + |S_\beta| + m(S) + 1}{3},$$

for $m, n \geq 3$.

The bounds of $\nabla(P_m \times P_n)$ in (4) are best possible for $n = 4, 6, 7 \ [3]$.

Proof. By the definition of $P_m \times P_n$, $\beta(P_m \times P_n) = mn - m - n + 1$. By Corollary 15, it follows that $\nabla(P_m \times P_n) \geq \frac{1}{3}(mn - m - n + 2)$. Let $S$ be a decycling set of $P_m \times P_n$. Then $\nabla(P_m \times P_n) \leq |S| = \frac{1}{3}(mn - m - n + |S_\beta| + m(S) + 1)$. □

Remark 5. Some nonregular graphs may also have a $\nabla$-set with a large margin number, such as the grid $P_5 \times P_n$. Beineke [3] proved that $\nabla(P_5 \times P_n) = \left\lfloor \frac{3n}{2} \right\rfloor - \left\lceil \frac{n}{5} \right\rceil - 1$. Together with Theorem 14, we get $m(S) + |S_\beta| = 3 \left( \left\lfloor \frac{3n}{2} \right\rfloor - \left\lceil \frac{n}{5} \right\rceil \right) - 4n + 1$, which tends to infinite as $n \to \infty$.

The above discussions imply that the margin number $m(S)$ of a decycling set $S$ may be arbitrarily large for some regular graphs. For some (4-regular) graphs $G$ of order $n$, there exists a decycling set $S$ such that the margin number $m(S)$ is a linear function of $n$. For instance, a toroidal 4-regular graph $G$ containing $n$ disjoint $K_5 - e$'s (see Figure 4) whose decycling number of $G$ is $2n + 1$, and by formula (1), its margin number $m(S) = n + 2$.

![Figure 4. A toroidal 4-regular graph with $m(S) = n + 2$.](image)

On the other hand, suppose that $S$ is a decycling set of a regular graph $G$, for any vertex $x \in S$, adding an edge to join $x$ and $G - S$, this procedure determines a Xuong-tree $T_X$ since $G - E(T_X)$ has no odd components (i.e., $\xi(G) = 0$), which means that the elements of $S$ are taken from the leaves of a Xuong-tree. This may be extended to the $\nabla$-set $S$ with the margin number $m(S)$ are of relative
small, that is, when the margin number is small enough, the elements of $S$ are taken from the leaves of a spanning tree $T$ with few exceptions. We shall discuss this situation in Section 4.

4. 4-Regular Graphs

In this section we concentrate on studying the combinatorial structure of 4-regular graphs $G$ with the decycling number $\nabla(G) = \lceil \frac{\beta(G)}{3} \rceil$.

Theorem 16. Let $G$ be a 4-regular graph with $\nabla(G) = \lceil \frac{\beta(G)}{3} \rceil$. Then there exists a spanning tree $T$ in $G$ such that elements of any $\nabla$-set of $G$ are simply the leaves of $T$ with at most two exceptions.

Proof. Let $S$ be a $\nabla$-set of a 4-regular graph $G$. Assume that $\beta(G) = 3m + r$, $0 \leq r \leq 2$, $m$ is a nonnegative integer. Then three claims arise.

Claim 1. If $r = 0$, then $S$ is a $\nabla$-set if and only if $m(S) = 0$ and vertices of $S$ are leaves of a spanning tree $T$ of $G$.

Proof. The first part follows from Theorem 9. Now suppose that $m(S) = 0$. Then $c = 1$ and $|E(S)| = 0$. We can construct a spanning tree $T$ as we have reasoned in the proof of Theorem 9. It is clear that the elements of $S$ are leaves of $T$.

Claim 2. If $r = 1$, then $S$ is a $\nabla$-set if and only if $m(S) = 2$ and vertices of $S$ are leaves of a spanning tree $T$ of $G$ with at most two exceptions.

Proof. The first part follows from Theorem 9. It is clear that $c + |E(S)| = 3$ since $m(S) = 2$. We construct a spanning tree $T$ of $G$ satisfying the above condition. There are three cases according to the values of $c$ and $|E(S)|$.

Case 1. $c = 1$ and $|E(S)| = 2$. Since $G$ is connected, there exists two edges, say $e_1 = ab$ and $e_2 = cd$ (possibly $b = c$) in $G[S]$. For each vertex $x \in S - \{a, b, c, d\}$, add edges $e = xy$, $e_3 = by$ and $e_4 = cy$ (prescribe $e_3 = e_4$ when $b = c$), into $G - S$, where $y \in G - S$. After this, we obtain a spanning tree $T$ of $G$ containing $G - S$ as its subgraph. Which satisfies the condition of Claim 2 (i.e., when $b \neq c$, $S$ has two vertices $b, c$ which are not leaves of $T$; if $b = c$, then the only exception of $S$ is $b = c$).

Case 2. $c = 2$ and $|E(S)| = 1$. Without loss of generality, let $Q_1, Q_2$ be the two components of $G - S$ and $E(S) = \{xy\}, x, y \in S$. Since $G$ is connected, there exists an edge $e = xy$ (possibly $x = y$) in $G[S]$. Two situations will appear to construct a spanning tree $T$ of $G$: (a) If $x \neq y$, then (i) $x$ and $y$ join $Q_1$ and $Q_2,$
respectively. Let $x_1 \in Q_1$ and $x_2 \in Q_2$ be such that $xx_1, yx_2 \in E(G)$. The edges $xy, xx_1, yx_2, Q_1$ and $Q_2$ form a tree $T_1$ containing $Q_1, Q_2$; (ii) $x$ joins $Q_1$ and $Q_2$. Let $x_0 \in Q_1$ and $y_0 \in Q_2$ be such that $xx_0, xy_0 \in E(G)$. Then the edges $xy, xx_0, xy_0$, $Q_1$ and $Q_2$ form a tree $T_1$ containing $Q_1, Q_2$. For other vertices $z \in S - \{x, y\}$, we add an edge join $z$ with $Q_1 \cup Q_2$. It is clear that such edges and $T_1$ form a spanning tree $T$ of $G$. (b) If $x = y$, then there is an edge $e_0 = fg$ in $G[S]$ such that $x$ joins $Q_1$ and $Q_2$, $f$ joins $Q_1 \cup Q_2$, the remaining vertices of $S - \{x, f, g\}$ as did in the case of $x \neq y$, so we may construct a spanning tree $T$ of $G$. The above spanning trees also satisfy the condition of Claim 2.

Case 3. $c = 3$ and $|E(S)| = 0$. Suppose that $Q_1, Q_2$ and $Q_3$ are three components of $G - S$. Then a spanning tree $T$ of $G$ will be constructed as follows. Since $G$ is connected, there exist two vertices, say $x$ and $y$ (possibly $x = y$), in $S$ such that $x$ joins $Q_1$ and $Q_2$, $y$ joins $Q_2$ and $Q_3$. This time we may also construct a spanning tree $T$ of $G$ which contains $Q_1, Q_2$ and $Q_3$ as we did in Case 2. And hence, when $x \neq y$, $x$ and $y$ are the only two vertices in $S$ which are not leaves of $T$; if $x = y$, then $x$ is the only vertex in $S$ is not the leaf of $T$. □

**Claim 3.** If $r = 2$, then $S$ is a $\nabla$-set if and only if $m(S) = 1$. Meanwhile, there exists a spanning tree $T$ of $G$ such that all (but at most one) vertices of $S$ are leaves of $T$.

**Proof.** The proof of Claim 3 is analogous to Claims 1 and 2, we omit its proof. □

Now the entire proof of the theorem is complete.

We give three examples of 4-regular graphs with $m(S) = 0, 1, 2$, respectively. See Figure 5(a), Figure 5(b) and Figure 2(b).

![Figure 5](image)

Figure 5. (a) 4-regular graph with $m(S) = 0$; (b) 4-regular graph with $m(S) = 2$.

After a similar discussion in 4-regular graphs, we may extend Theorem 16 to general case.

**Theorem 17.** Let $G$ be a $k$-regular graph with $\nabla(G) = \lceil \frac{\beta(G)}{k-1} \rceil$ and $\beta(G) = m(k-1) + r$, $0 \leq r \leq k-2$, $m$ is a nonnegative integer. Then $S$ is a $\nabla$-set of $G$
if and only if

\[ m(S) = \begin{cases} 
0, & \text{for } r = 0, \\
k - r - 1, & \text{for otherwise.}
\end{cases} \]

Moreover, there exists a spanning tree \( T \) in \( G \) such that elements of \( S \) are simply the leaves of \( T \) with at most \( m(S) \) exceptions.

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