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NEW FORMULAE FOR THE DECYCLING NUMBER OF GRAPHS

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Abstract

A set S of vertices of a graph G is called a decycling set if $G-S$ is acyclic. The minimum order of a decycling set is called the decycling number of G, and denoted by $\nabla(G)$. Our results include: (a) For any graph G,

$$\nabla(G) = n - \max_{T}\{\alpha(G - E(T))\},$$

where T is taken over all the spanning trees of G and $\alpha(G - E(T))$ is the independence number of the co-tree $G - E(T)$. This formula implies that computing the decycling number of a graph G is equivalent to finding a spanning tree in G such that its co-tree has the largest independence number. Applying the formula, the lower bounds for the decycling number of some (dense) graphs may be obtained. (b) For any decycling set S of a k-regular graph G,

$$|S| = \frac{1}{k-1}(\beta(G) + m(S)),$$

where $\beta(G) = |E(G)| - |V(G)| + 1$ and $m(S) = c + |E(S)| - 1$, c and $|E(S)|$ are, respectively, the number of components of $G - S$ and the number of

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edges in $G[S]$. Hence S is a ∇-set if and only if $m(S)$ is minimum, where ∇-set denotes a decycling set containing exactly $\nabla(G)$ vertices of G. This provides a new way to locate $\nabla(G)$ for k-regular graphs G. (c) 4-regular graphs G with the decycling number $\nabla(G) = \left\lceil \frac{\beta(G)}{3} \right\rceil$ are determined.

Keywords: decycling number, independence number, cycle rank, margin number.

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1. Introduction

Graphs throughout this paper are loopless and multiple edges are permitted. For general theoretic notations, we follow Diestel [4]. The minimum number of edges whose removal eliminates all cycles in a given graph has been known as the *cycle rank* of the graph, and this parameter has a simple expression $\beta(G) = |E(G)| - |V(G)| + w$ (see [9]), where w is the number of components of G. The corresponding problem of eliminating all cycles from a graph by means of deletion of vertices goes back at least to the work of Kirchhoff on spanning trees [10].

Let $G = (V, E)$ be a graph. We define a vertex set S of G to be a *decycling set* if $G - S$ is *cycle-free*. The cardinality of a minimum decycling set of G is called the *decycling number*, and denoted by $\nabla(G)$ (or ∇ for short). A decycling set containing exactly $\nabla(G)$ vertices of G is called a ∇-set. Vertices of a decycling set are labeled by " $\bullet$ " and the bold edges induce a spanning tree T of a graph G in the following figures. Let $m(S) = c + |E(S)| - 1$ be the *margin number* of a decycling set S, where c and $|E(S)|$ are, respectively, the number of components of $G - S$ and the number of edges in $G[S]$. Determining the decycling number is equivalent to finding the size of the largest induced forest of G proposed first by Erdös [5]. If $S \subseteq V(G)$ is a ∇-set, then $G - S$ is a largest induced forest of G. The problem of determining the decycling number of graphs have been proved to be NP-complete by Karp [11], even for general graphs such as bipartite graphs, planar graphs and perfect graphs, the decycling problem is very hard to solve. It is easy to see that $\nabla(G) = 0$ if and only if G is a forest, and $\nabla(G) = 1$ if and only if G has at least one cycle and a vertex on all of its cycles. One may see [2] as a brief survey.

In this paper, we consider the decycling problem from two new perspectives: the effects of (a) spanning trees and (b) the margin number, respectively, on the decycling number of graphs. Given a connected graph G and a surface P, we say that G can be *embedded* into P if there exists a polyhedron $\sum$ on P such that the 1-skeleton of $\sum$ has a subgraph homeomorphic to G. The components of $\sum - G$ are called the *faces* of the embedding. When each face is homeomorphic to an open disc, the embedding is called a *cellular*. The maximum genus, denoted by $\gamma_M(G)$, of a connected graph G is the largest genus of an orientable surface on which G admits a cellular embedding. Let T be a spanning tree of a connected graph G. The subgraph $G - E(T)$ of G is called a *co-tree* of G. Note that the number of edges in any co-tree of G is just the cycle rank $\beta(G)$. The *Betti deficiency* of G, denoted by $\xi(G)$, is defined the minimum number of odd components (i.e., the components containing odd number of edges) among co-trees of G. We call T a *Xuong-tree* if the number of odd components of $G - E(T)$ is $\xi(G)$. The following result of Xuong defines an edge-partition of a co-tree.

Lemma 1 [20]**.** *Let G be a connected graph and T_X be a Xuong-tree of G. Then there exists an edge-partition of $E(G) - E(T_X)$ as follows:*

$$E(G) - E(T_X) = \{e_1, e_2\} \cup \{e_3, e_4\} \cup \cdots \cup \{e_{2m-1}, e_{2m}\} \cup \{f_1, f_2, \ldots, f_s\},$$

where (1) $m = \gamma_M(G), s = \xi(G)$*;* (2) *for any* $i = 1, 2, \ldots, m$*,* $e_{2i-1} \cap e_{2i} \neq \emptyset$*, and* $\{f_1, f_2, \ldots, f_s\}$ *is a matching of* G*.*

Let T_X be a Xuong-tree and the edge-partition of $E(G)-E(T_X)$ be as defined in Lemma 1. Consider a set

$$S_X = \{u_i \,|\, u_i \in e_{2i-1} \cap e_{2i}, 1 \le i \le m\} \cup \{v_j \,|\, v_j \text{ is an end of } f_j, 1 \le j \le s\}.$$

Then $G - S_X$ contains no cycle (since removing S_X from G will eliminate all the possible fundamental cycles of G) and hence S_X is a decycling set of G, that is, $\nabla(G) \le |S_X|$.

Corollary 2. $\nabla(G) \le |S_X| \le \gamma_M(G) + \xi(G)$ *holds for every graph* G*.*

It is easy to see that the bound $|S_X|$ heavily depends on the choice of Xuong-tree T_X (since different T_X may lead to quiet different value of $|S_X|$). For instance, the wheel graph $W_{1,n} = K_1 \vee C_n$ with n spokes has $\nabla(W_{1,n}) = 2$. If one chooses a Xuong-tree $K_{1,n}$ as a spanning tree of $W_{1,n}$, then the corresponding $|S_X| = \lceil \frac{n}{2} \rceil$; meanwhile, a Hamilton path in $W_{1,n}$ will determine another S_X whose number of elements reaches the best value $\nabla(W_{1,n}) = 2$. Therefore, how to find a set $S_X \subseteq V(G)$ with the smallest size is a key to determine $\nabla(G)$.

This paper is organized as follows.

In Section 2, we show that $\nabla(G) = n - \max_T\{\alpha(G - E(T))\}$ holds for any graph G. This implies that determining the decycling number is equivalent to finding the largest independence number of a co-tree. So, finding the decycling number $\nabla(G)$, determining the largest independence number of a co-tree $G - E(T)$ and finding the size of a largest induced forest in a graph G are mutually equivalent. In this sense, finding the decycling number of a graph is very hard. Applying this formula, we may obtain lower bounds for the decycling number of some (dense) graphs. As an example, we prove that $\nabla(K_n - (E(T_1) \cup E(T_2) \cup \cdots \cup E(T_{k-1}))) \geq n - 2k$, where $T_1, T_2, \ldots, T_{k-1}$ are $k-1$ edge-disjoint spanning trees of a complete graph K_n. Many examples and applications are presented to show how to apply trees into identifying the decycling number of a graph.

In Section 3, we obtain an another formula to compute the decycling number of regular graphs. For any decycling set S of a k-regular graph G, we get that $|S| = \frac{1}{k-1}(\beta(G)+m(S))$. Obviously, S is a ∇-set if and only if $m(S)$ is minimum. Therefore, lower bounds for the decycling number of some (dense) graphs can be obtained. For a k-regular graph G, if $m(S) \geq 0$ for any ∇-set S of G, then $\nabla(G) \geq \frac{\beta(G)}{k-1}$. In many cases, these lower bounds may be tight (i.e., best possible) (see [3, 8, 12, 14–16, 18, 19]). Observe that for some (4-regular) graphs G of order n, there exists a decycling set S such that the margin number $m(S)$ is a linear function of n. For instance, a toroidal 4-regular graph G contains n disjoint $K_5 - e$'s (see Figure 4) whose decycling number of G is $2n+1$. It is easy to see that the margin nuber $m(S) = n+2$ for any ∇-set S of G. Moreover, we discuss some relationships between the decycling number and the large genus embeddings of graphs, and show the effects of spanning trees on such topics. In particular, we give a new and direct proof of a result due to Speckenmeyer [17] and thus solve an open problem of Speckenmeyer searching for an efficient algorithm to compute $Z(G)$, the cardinality of the maximum nonseparating independent set of G.

In Section 4, we investigate the extremal 4-regular graphs G with the decycling number $\nabla(G) = \left\lceil \frac{\beta(G)}{3} \right\rceil$. Our conclusion is that for any ∇-set S of graph G, there exists a spanning tree T in G such that elements of S are taken from the leaves of T with at most two exceptions (from the 2 or 3-degree vertices of T). Finally, we extend this result to general case.

2. A Formula Between the Decycling Number and the Independence Number

In this section, $\alpha(G)$ and $a(G)$ denote, respectively, the independence number and the number of vertices in a largest induced forest of a graph G.

Theorem 3. *Let G be a connected graph of order n. Then*

$$\nabla(G) = n - \max_T\{\alpha(G - E(T))\},$$

where T is taken over all spanning trees of G.

The above result reveals a relation among the decycling number, the independence number and the spanning trees in a graph and gives a new way to investigate these numbers.

Lemma 4. *Let G be a connected graph of order n. Then*

$$a(G) = \max_T\{\alpha(G - E(T))\},$$

where T is taken over all spanning trees of G.

Proof. Let F be a largest induced forest of G with $|F| = a(G)$. Then $V(G) - V(F)$ is a decycling set of G. Since G is connected, there exists a spanning tree T of G such that $F \subseteq T$, and $\alpha(G - E(T)) \geq |F|$. Hence,

$$a(G) \leq \max_T\{\alpha(G - E(T))\}.$$

Conversely, let T_1 be a spanning tree of G such that

$$\max_T\{\alpha(G - E(T))\} = \alpha(G - E(T_1),$$

and suppose that A is the largest independent set of $G - E(T_1)$, that is, $\mid A \mid = \alpha(G - E(T_1))$. When we recover the edges of T_1 into A, it induces a forest, and $G - A$ is a decycling set. Hence $|A| \leq a(G)$. That is,

$$\max_T\{\alpha(G - E(T))\} \leq a(G).$$

■

Proof of Theorem 3. By Lemma 4 and $a(G)+\nabla(G) = n$, the theorem follows. ■

From the proof of Theorem 3, one may see that if T is a spanning tree of G and A is the maximum independent set of the co-tree $G - E(T)$ (i.e., $|A| = \alpha(G-E(T))$), then $S = V(G)-A$ is a decycling set, and so $|V(G)-A|+|A| = n$, which means that $|A|$ is the largest among all spanning trees of G if and only if the corresponding decycling set $S = V(G) - A$ is minimum. Therefore, how to find a spanning tree T of G such that $\alpha(G-E(T))$ is the maximum is very crucial to computing the decycling number $\nabla(G)$ of G. In the following, we shall present several applications and examples to show the effects of the spanning trees on searching for the value $\nabla(G)$ of a graph G.

Example 1. Let T be a Hamilton path of a complete graph K_n. Then T is a spanning tree of K_n (see Figure 1(a)), and so $\alpha(K_n - E(T)) \geq 2$. By Theorem 3, $\nabla(K_n) \leq n - 2$. It deduces that $\nabla(K_n) = n - 2$ because of $\nabla(K_n) \geq n - 2$ (if we remove at most $n - 3$ vertices of K_n, then the resultant graph will contain a cycle).

Example 2. For a complete bipartite graph $K_{m,n}$, let $V(K_{m,n}) = V = X \cup Y$, where $X = \{x_1, x_2, \ldots, x_m\}$, $Y = \{y_1, y_2, \ldots, y_n\}$. Assume that $m \leq n$. We construct a spanning tree T of $K_{m,n}$ as follows: $E(T) = \{x_m y_j, x_i y_i \,|\, i = 1, 2, \ldots, m-1, j = 1, 2, \ldots, n\}$ (see Figure 1(b)). Then $\alpha(K_{m,n} - E(T)) \geq n+1$.

By Theorem 3, $\nabla(K_{m,n}) \leq m-1$. Since $\nabla(K_{m,n}) \geq m-1$ (otherwise, there is a cycle by removing at most $m-2$ vertices of $K_{m,n}$ in the resultant graph), we have $\nabla(K_{m,n}) = m-1$.

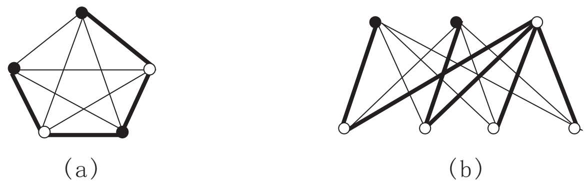


Figure 1. (a) $\nabla(K_5) = 5-2 = 3$; (b) $\nabla(K_{3,4}) = 3-1 = 2$.

Example 3. For a complete k-partite graph $K_{m_1,m_2,\ldots,m_k}$, let $V(K_{m_1,m_2,\ldots,m_k}) = V = X_1 \cup X_2 \cup \cdots \cup X_k$. Then $|X_i| = m_i, i = 1,2,\ldots,k$. Without loss of generality, suppose that $m_1 \leq m_2 \leq \cdots \leq m_k$. We construct a spanning tree T of $K_{m_1,m_2,\ldots,m_k}$ as follows: $E(T) = \{x_{1j}y, x_{11}z\}$, where $y \in V\backslash X_1$, z takes over the elements of $V\backslash X_1$, $x_{11}, x_{1j} \in X_1, j = 2,\ldots,m_1$. By Theorem 3, $\nabla(K_{m_1,m_2,\ldots,m_k} - E(T)) \leq \sum_{i=1}^{k} m_i - m_k - 1$. Since $\nabla(K_{m_1,m_2,\ldots,m_k} - E(T)) \geq \sum_{i=1}^{k} m_i - m_k - 1$ (otherwise, removing at most $\sum_{i=1}^{k} m_i - m_k - 2$ vertices of $K_{m_1,m_2,\ldots,m_k}$ will leave a cycle in the resultant graph), $\nabla(K_{m_1,m_2,\ldots,m_k}) = \sum_{i=1}^{k} m_i - m_k - 1$.

Example 4. Ren [16] proved that $\nabla(G) = \gamma_M(G) + \xi(G)$ for G being a cubic graph, where $\gamma_M(G)$ and $\xi(G)$ are the maximum genus and Betti deficiency of G, respectively. We consider a Xuong-tree T_X of a cubic graph G and an edge-partition of its co-tree $G - E(T_X)$ as defined in Lemma 1. Then the set S_X (as defined in Corollary 2) is a ∇-set since $G - S_X$ contains no cycle and $|S_X| = \gamma_M(G) + \xi(G)$. Now $V(G) - S_X$ is an independent set whose cardinality is $n - \max_T\{\alpha(G - E(T))\}$. Conversely, let T_1 be a spanning tree of G such that $\alpha(G - E(T_1)) = \max_T\{\alpha(G - E(T))\}$. Then there exists an independent set S of $G - E(T_1)$ with $|S| = \alpha(G - E(T_1))$ such that $G - E(T_1)$ contains an independent set A with $|A| = \alpha(G - E(T_1)) = \max_T\{\alpha(G - E(T))\}$. It is clear that $G[A]$ is a largest induced forest of G and $S = V(G) - A$ is a ∇-set. As shown in [16], T_1 is a Xuong-tree of G (since the number of odd components of $G - E(T_1)$ is $\xi(G)$).

Based on Theorem 3, many results and problems on the largest induced forests and the decycling set can be translated into one another. For instance, Albertson and Berman [1] posed the following conjecture.

Conjecture 5 [1]**.** *Every planar graph has an induced forest with at least half the vertices.*

The above conjecture can also be expressed into the following three forms.

Theorem 6. *Let G be a planar graph of order n. Then the following statements are mutually equivalent:*

(a) $\nabla(G) \leq \frac{n}{2}$;

(b) $|F| \geq \frac{n}{2}$ *holds for a largest induced forest F of G;*

(c) *There exists a spanning tree T in G such that $\alpha(G - E(T)) \geq \frac{n}{2}$.*

For a plane triangulation G, a plane with all faces are triangles. By Theorem 6(c), we may first find a spanning tree T of G to determine the independence number of its co-tree $G - E(T)$, and further to solve the decycling number of G. Since the number of edges of $G - E(T)$ is $2n - 5$, a natural idea is that the problem of the computation of the decycling number of a plane triangulation G can be put into the following problem.

Problem 7. *Determine the independence number for a planar graph G of order n with at most $2n - 5$ edges.*

In the literature, there are many results on the decycling number for sparse graphs such as 3 (or 4)-regular graphs, see [3, 8, 12, 14–16, 18, 19], but little is known for those with many edges (i.e., dense graphs). Theorem 3 offers a way to estimate the lower bounds for the decycling number of dense graphs. The following result is an application.

Theorem 8. *Let $T_1, T_2, \ldots, T_{k-1}$ be $k-1$ edge-disjoint spanning trees of a complete graph K_n. Then*

$$\nabla(K_n - (E(T_1) \cup E(T_2) \cup \cdots \cup E(T_{k-1}))) \geq n - 2k.$$

And the equality holds if and only if $K_n - (E(T_1) \cup E(T_2) \cup \cdots \cup E(T_{k-1}))$ contains a spanning tree T_k such that the graph $T_1 \cup T_2 \cup \cdots \cup T_{k-1} \cup T_k$ contains K_{2k}.

Proof. Let $T_1, T_2, \ldots, T_{k-1}$ be $k-1$ edge-disjoint spanning trees of a complete graph K_n. Assume (reductio ad absurdum) that T_k is a spanning tree of $K_n - (E(T_1) \cup E(T_2) \cup \cdots \cup E(T_{k-1}))$ such that $\alpha(K_n - (E(T_1) \cup E(T_2) \cup \cdots \cup E(T_{k-1}) \cup E(T_k))) \geq 2k + 1$. Then $T_1 \cup T_2 \cup \cdots \cup T_{k-1} \cup T_k$ contains K_{2k+1}, and hence

$$|\{E(T_1) \cup E(T_2) \cup \cdots \cup E(T_{k-1}) \cup E(T_k)\} \cap E(K_{2k+1})| \geq k(2k+1).$$

Color the edges of K_{2k+1} with k different colors, then the number of edges with the same color is not more than $2k$. Otherwise, there will exist a subgraph (induced by these edges) containing a cycle, which contraries to the number of edges of K_{2k+1}. By Theorem 3,

$$\begin{aligned} &\nabla(K_n - (E(T_1) \cup E(T_2) \cup \cdots \cup E(T_{k-1}))) \\ &= n - \max_{T_k}\{\alpha(K_n - (E(T_1) \cup E(T_2) \cup \cdots \cup E(T_{k-1}) \cup E(T_k)))\}, \end{aligned}$$

that is,

$$\max_{T_k}\{\alpha(K_n-(E(T_1)\cup E(T_2)\cup\cdots\cup E(T_{k-1})\cup E(T_k)))\}\leq 2k.$$

Let $\nabla(K_n-(E(T_1)\cup E(T_2)\cup\cdots\cup E(T_{k-1})))=n-2k$. By Theorem 3, there exists a spanning tree T_k of $K_n-(E(T_1)\cup E(T_2)\cup\cdots\cup E(T_{k-1}))$ such that $\alpha(K_n-(E(T_1)\cup E(T_2)\cup\cdots\cup E(T_{k-1})\cup E(T_k))=2k$. The edges of each T_i $(1\leq i\leq k)$ in K_{2k} form a spanning tree of K_{2k}. Conversely, for $k-1$ edge-disjoint spanning trees $T_1,T_2,\ldots,T_{k-1}$ of K_n, if there is a spanning tree T_k of $K_n-(E(T_1)\cup E(T_2)\cup\cdots\cup E(T_{k-1}))$ such that the subgraphs of K_n determined by these trees $T_1,T_2,\ldots,T_{k-1},T_k$ containing a complete graph K_{2k}, then $\alpha(K_n-(E(T_1)\cup E(T_2)\cup\cdots\cup E(T_{k-1})\cup E(T_k)))\geq 2k$. ■

3. A Formula Between the Decycling Number and the Margin Number

Let $E(S,G-S)$ be the set of edges such that each edge has one vertex in S and another one in $G-S$. $d_G(x)$ and $\Delta(G)$ (or Δ for short) represent the degree of vertex x and the maximum degree of G, respectively. In this section, we present another formula for the decycling number $\nabla(G)$ of a k-regular graph G.

Theorem 9. *Let S be a decycling set of G. Then*

$$\sum_{x\in S}(d_G(x)-1)=\beta(G)+m(S).$$

Proof. Let $S=\{x_1,x_2,\ldots,x_{|S|}\}$ be a decycling set of G. Then

$$\begin{aligned} q-\sum_{i=1}^{|S|}d_G(x_i)&=q-|E(S,G-S)|-2|E(S)|\\ &=p-|S|-c-|E(S)|, \end{aligned}$$

where $p=|V(G)|$, $q=|E(G)|$, c and $|E(S)|$ are, respectively, the number of components of $G-S$ and the number of edges of $G[S]$.

As $\beta(G)=q-p+1$,

$$\sum_{i=1}^{|S|}(d_G(x_i)-1)=\beta(G)+c+|E(S)|-1.$$

And for $m(S)=c+|E(S)|-1$, we have

$$\sum_{i=1}^{|S|}(d_G(x_i)-1)=\beta(G)+m(S).$$

This completes the proof. ■

Remark 1. Observe that if all the vertices of S have degree k, then $|S| = \frac{1}{k-1}(\beta(G) + m(S))$. In particular, if G is a k-regular graph, then S is a ∇-set if and only if $m(S)$ is minimum among all the decycling set S of G.

Although there exists some uncertain parameter like $m(S)$, this result provides a way to locate the value of $\nabla(G)$: once we find a decycling set S such that $m(S)$ reaches the minimum, then S is a ∇-set of G. We will show its applications in the discussion to come.

A simple corollary of Theorem 9 is:

Corollary 10. *Let G be a graph with maximum degree Δ which has the ∇-set such that each vertex of this set has degree Δ. Then*

$$\nabla(G) \geq \frac{\beta(G)}{\Delta - 1}. \tag{1}$$

Remark 2. (i) $\nabla(G) = \frac{\beta(G)}{\Delta-1}$ if and only if $m(S) = 0$ which means that for a ∇-set S of G, $G - S$ is a tree T_0 and $G[S]$ is an empty subgraph. In this case, $\nabla(G)$ has a strong combinatorial characterization: for any vertex $x \in S$ incident to a vertex $y \in V(T_0)$, insert the edge xy into T_0. This procedure determines a spanning tree T (it is in fact a Xuong-tree) of G (such that $\xi(G) = 0$) if we add $|S|$ edges into T_0. Therefore, deleting a vertex x of S will destroy $d_G(x) - 1$ fundamental cycles of G and deleting $\nabla(G)$ vertices of S will destroy all fundamental cycles of G;

(ii) the inequality (1) may be tight, see [3, 8, 12, 14–16, 18, 19].

The following examples show the formula of Theorem 9 applying on some types of regular graphs.

Example 5. Let S be a decycling set of a hypercube Q_n (a graph contains 2^n n-tuples of 0's and 1's as vertices with two vertices adjacent if they differ in exactly one position). Then

$$2^{n-1} - \frac{2^{n-1} - 1}{n - 1} \leq \nabla(Q_n) \leq 2^{n-1} - \frac{2^{n-1} - m(S) - 1}{n - 1}. \tag{2}$$

The inequalities in (2) are equalities for $n = 3, 4$ [3] (see Figure 2).

Proof. By the definition of Q_n, $\beta(Q_n) = (n - 2)2^{n-1} + 1$. By Corollary 10, $\nabla(Q_n) \geq 2^{n-1} - \frac{2^{n-1}-1}{n-1}$. Let S be a decycling set of Q_n. Then $|S| = 2^{n-1} - \frac{2^{n-1}-m(S)-1}{n-1}$, thus $\nabla(Q_n) \leq |S| = 2^{n-1} - \frac{2^{n-1}-m(S)-1}{n-1}$. ■

Remark 3. (i) Two spanning trees T_1 and T_2 which induced by the bold edges in Figure 2(a) and (b) of Q_n satisfying that $Q_n - E(T_i)$ $(i = 1, 2)$ has the largest independence number, respectively.

(ii) Focardi [6] proved that $2^{n-1} - \frac{2^{n-1}-1}{n-1} \leq \nabla(Q_n) \leq 2^{n-1} - \frac{2^{n-1}}{2(n-1)}$. The lower bound was also proved by Beineke and Vandell [3]. In fact, if the upper bound of Focardi's result is best possible, then $m(S) = 2^{n-2} - 1$ for any ∇-set S. Therefore, determining the decycling number of Q_n for larger n is very difficult.

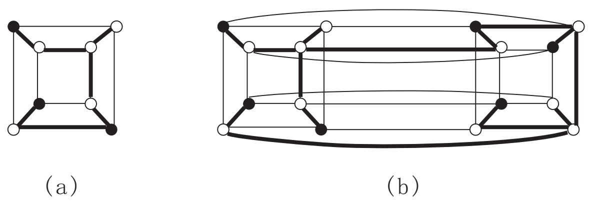


Figure 2. (a) Q_3 with $m(S) = 1$; (b) Q_4 with $m(S) = 1$.

Example 6. For any two cycles C_m and C_n, their *Cartesian product* is the graph $C_m \times C_n$ with vertex set $V(C_m \times C_n) = \{w_{ij} \mid i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\}$ and edge set $E(C_m \times C_n) = \{w_{ij}w_{rs} \mid i = r, v_jv_s \in E(C_n) \text{ or } j = s, u_iu_r \in E(C_m)\}$. Clearly, $C_m \times C_n$ is a 4-regular graph, see Figure 3. For any decycling set S of $C_m \times C_n$, we have

$$(3) \qquad \frac{mn+1}{3} \leq \nabla(C_m \times C_n) \leq \frac{mn + m(S) + 1}{3}, m, n \geq 3.$$

In particular, the bounds of $\nabla(C_m \times C_n)$ in (3) are sharp for $m = 3$ (or $n = 3$) (see Figure 3).

Proof. It is easy to see that $\beta(C_m \times C_n) = mn + 1$. By Corollary 10, $\nabla(C_m \times C_n) \geq \frac{1}{3}(mn + 1)$. Let S be a decycling set of $C_m \times C_n$. Then $\nabla(C_m \times C_n) \leq |S| = \frac{1}{3}(mn + m(S) + 1)$. When $m = 3$, $\frac{3n+1}{3} \leq \nabla(C_3 \times C_n) \leq \frac{3n+m(S)+1}{3}$, that is, $n + 1 \leq \nabla(C_3 \times C_n) \leq n + \frac{m(S)+1}{3}$, we can find a decycling set S such that $m(S) = 2$ (see Figure 3), then $\nabla(C_3 \times C_n) = n + 1$. ■

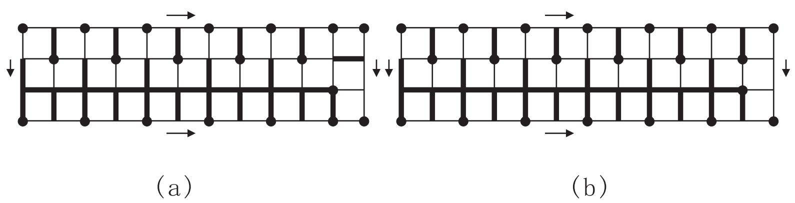


Figure 3. A drawing of $C_m \times C_n$ on the torus.

Remark 4. (i) The spanning trees T of $C_3 \times C_n$ in Figure 3(a) and (b) satisfy that $C_3 \times C_n - E(T)$ has the largest independence number, respectively.

(ii) Our result shows that $\nabla(C_m \times C_n) = \frac{mn+m(S)+1}{3}$ for S being a ∇-set of $C_m \times C_n$, which equals to Pike's result $\nabla(C_m \times C_n) = \lceil \frac{mn+2}{3} \rceil$ $(m, n \neq 4)$ when $m(S) \leq 1$ [13]. Therefore, this provides a way to locate the exact value of $\nabla(G)$ (to find a decycling set S with the minimum $m(S)$).

The formula of Theorem 9 also has some applications in topological graph theory.

A vertex set S is called a *nonseparating independent set* of G if S is an independent set of G and $G - S$ is connected. The cardinality of a maximum nonseparating independent set of G is denoted by $Z(G)$. The following result shows a close relation between nonseparating independence number $Z(G)$ and the maximum genus $\gamma_M(G)$ of a cubic graph G and makes an extention of a result due to Speckenmeyer [17], we give a new and direct proof of it via trees.

Theorem 11. *Let G be a cubic graph. Then*

(a) *$Z(G) = \gamma_M(G)$;*

(b) *for every maximum nonseparating independent set S of G, $G - S$ contains no two cycles sharing a vertex in common. Moreover, there exists a Xuong-tree T_X such that the elements in S are leaves of T_X.*

Proof. Let T_X be a Xuong-tree of G with an edge-partition of $G - E(T)$ as defined in Lemma 1. Then for $1 \leq i \leq \gamma_M(G)$, $e_{2i-1} \cap e_{2i} = \{u_i\}$ forms a set of independent vertices of G (which are leaves of T_X). Hence $\{u_1, u_2, \ldots, u_{\gamma_M(G)}\}$ is a nonseparating independent set of G. Therefore, $Z(G) \geq \gamma_M(G)$. To see the converse inequality, we consider a nonseparating independent set S. Then $G - S$ is connected. We may suppose further that $G - S$ is a tree T_0 and T is a spanning tree built in Remark 2(i). Then elements of S are leaves of T. After repeating the argument in Remark 2(i), we may see that $|S| \leq \gamma_M(G)$ which means that $Z(G) \leq \gamma_M(G)$. This proves (a).

Suppose that $G - S$ contains two cycles with one vertex in common. Then by Lemma 1, $\gamma_M(G - S) \geq 1$, which together with the construction of a largest genus embedding stated in the proof of (a), $\gamma_M(G) \geq Z(G) + 1 (= \gamma_M(G) + 1)$, a contradiction. Therefore, cycles of $G - S$ are independent. In fact, the number of cycles in $G - S$ is $\xi(G)$. In addition, any spanning tree T_0 of $G - S$ is a subgraph of a Xuong-tree T_X of G (as stated in the proof of (a)). This proves (b). ∎

By Theorem 11, we obtain a result which has been proved by Speckenmeyer in [17] as follows.

Corollary 12. *Let G be a cubic graph of order n. Then $\nabla(G) + Z(G) = \frac{n+2}{2}$.*

Furthermore, the formula (a) in Theorem 11, together with a result of Furst [7], provide an efficient way to compute the value $Z(G)$ for cubic graphs and solves an open problem raised in [17] searching for a polynomial time algorithm to decide $Z(G)$ and $\nabla(G)$.

Theorem 13. *Let S be a ∇-set of a k-regular graph G.*

(a) *If $m(S) = 0$, then for every ∇-set S, there exists a spanning tree T in G such that all vertices of S are leaves of T.*

(b) *If $m(S) = 0$, and $k \equiv 1 \pmod 2$, then $\nabla(G) = \frac{2}{k-1}\gamma_M(G)$, and each spanning tree T in* (a) *is a Xuong-tree of G.*

Proof. Let $S = \{x_1, x_2, \ldots, x_\nabla\}$ be a ∇-set of a k-regular graph G with $m(S) = 0$. Then $G[S]$ has no cycle and $G - S$ is a tree T_0. Suppose that y_i is a neighbor of x_i in T_0 $(i = 1, 2, \ldots, \nabla)$. Then $T = T_0 + \{e_i = x_iy_i \mid 1 \le i \le \nabla\}$ is a spanning tree of G and S is a subset of leaves of T. If $k \equiv 1 \pmod 2$, then we arrange the left $k-1$ edges (other than x_iy_i) into $\frac{k-1}{2}$ pairs for each i $(1 \le i \le \nabla)$ and thus it gives rise to an edge-partition of $G - E(T)$ with $\xi(G) = 0$. Notice that in Xuong's construction of the maximum genus embedding [20], each pair of adjacent edges in $G - E(T)$ will contribute a genus, the graph G may be embedded into an orientable surface with $\frac{k-1}{2}\nabla = \gamma_M(G)$ handles. This ends the proof of (b). As for (a), it follows from the discussion used in the proof of (a) of Theorem 11. ∎

One case may appear if there is a vertex x of a decycling set S such that $d_G(x) < \Delta$. Then the formula of Theorem 9 will be invalid for this case. For instance, a grid of paths P_m and P_n is the graph $P_m \times P_n$ with vertex set $V(P_m \times P_n) = \{w_{ij} \mid i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\})$ and edge set $E(P_m \times P_n) = \{w_{ij}w_{rs} \mid i = r, v_jv_s \in E(P_n) \text{ or } j = s, u_iu_r \in E(P_m)\}$. We have to choose at least one vertex of degree 3 to eliminate the boundary cycle of $P_m \times P_n$. It is clear that any ∇-set S of the grid $P_m \times P_n$ does not need to contain a 2-degree vertex (since subdividing an edge of a graph does not change its decycling number). Therefore, we may only consider the ∇-set S whose vertices are of degree 4 or 3. Here, we slightly extend the formula of Theorem 9 as follows.

Theorem 14. *Let G be a non-regular graph with maximum degree Δ and S a decycling set of G. Suppose that d $(d < \Delta)$ is a fixed natural number with $S = S_\alpha \cup S_\beta$, $S_\alpha = \{x \mid d_G(x) = \Delta, x \in S\}$, $S_\beta = \{x \mid d_G(x) = d < \Delta, x \in S\}$. Then $|S| = \frac{1}{\Delta-1}(\beta(G) + (\Delta - d)\,|S_\beta| + m(S))$.*

Proof. Let S be a decycling set of a graph G. Similar to the proof of Theorem 9,

$$(\Delta - 1)(|S| - |S_\beta|) + (d-1)|S_\beta| = \beta(G) + c + |E(S)| - 1,$$

i.e.,

$$(\Delta - 1)|S| - (\Delta - d)|S_\beta| = \beta(G) + c + |E(S)| - 1,$$

and then

$$|S| = \frac{1}{\Delta - 1}(\beta(G) + (\Delta - d)|S_\beta| + m(S))$$

since $m(S) = c + |E(S)| - 1$. ∎

Corollary 15. *Let G be a non-regular graph with maximum degree Δ which has the ∇-set such that each vertex of this set has degree Δ. Then*

$$\nabla(G) \geq \frac{1}{\Delta-1}(\beta(G)+\Delta-d),$$

where d ($d<\Delta$) is the degree of some vertices of a ∇-set of G.

Example 7 (Cartesian product of two paths)**.** For any decycling set S of $P_m \times P_n$,

$$\frac{mn-m-n+2}{3} \leq \nabla(P_m \times P_n) \leq \frac{mn-m-n+|S_\beta|+m(S)+1}{3} \tag{4}$$

for $m,n \geq 3$.

The bounds of $\nabla(P_m \times P_n)$ in (4) are best possible for $n=4,6,7$ [3].

Proof. By the definition of $P_m \times P_n$, $\beta(P_m \times P_n) = mn-m-n+1$. By Corollary 15, it follows that $\nabla(P_m \times P_n) \geq \frac{1}{3}(mn-m-n+2)$. Let S be a decycling set of $P_m \times P_n$. Then $\nabla(P_m \times P_n) \leq |S| = \frac{1}{3}(mn-m-n+|S_\beta|+m(S)+1)$. ■

Remark 5. Some nonregular graphs may also have a ∇-set with a large margin number, such as the grid $P_5 \times P_n$. Beineke [3] proved that $\nabla(P_5 \times P_n) = \lfloor \frac{3n}{2} \rfloor - \lfloor \frac{n}{8} \rfloor - 1$. Together with Theorem 14, we get $m(S)+|S_\beta| = 3\left(\lfloor \frac{3n}{2} \rfloor - \lfloor \frac{n}{8} \rfloor\right) - 4n+1$, which tends to infinite as $n \to \infty$.

The above discussions imply that the margin number $m(S)$ of a decycling set S may be arbitrarily large for some regular graphs. For some (4-regular) graphs G of order n, there exists a decycling set S such that the margin number $m(S)$ is a linear function of n. For instance, a toroidal 4-regular graph G containing n disjoint $K_5 - e$'s (see Figure 4) whose decycling number of G is $2n+1$, and by formula (1), its margin number $m(S) = n+2$.

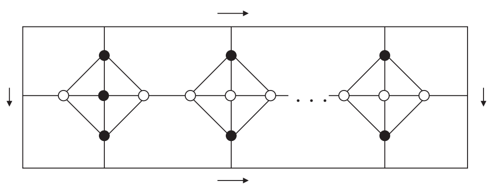


Figure 4. A toroidal 4-regular graph with $m(S) = n+2$.

On the other hand, suppose that S is a decycling set of a regular graph G, for any vertex $x \in S$, adding an edge to join x and $G-S$, this procedure determines a Xuong-tree T_X since $G-E(T_X)$ has no odd components (i.e., $\xi(G)=0$), which means that the elements of S are taken from the leaves of a Xuong-tree. This may be extended to the ∇-set S with the margin number $m(S)$ are of relative small, that is, when the margin number is small enough, the elements of S are taken from the leaves of a spanning tree T with few exceptions. We shall discuss this situation in Section 4.

4. 4-Regular Graphs

In this section we concentrate on studying the combinatorial structure of 4-regular graphs G with the decycling number $\nabla(G) = \left\lceil \frac{\beta(G)}{3} \right\rceil$.

Theorem 16. *Let G be a 4-regular graph with $\nabla(G) = \left\lceil \frac{\beta(G)}{3} \right\rceil$. Then there exists a spanning tree T in G such that elements of any ∇-set of G are simply the leaves of T with at most two exceptions.*

Proof. Let S be a ∇-set of a 4-regular graph G. Assume that $\beta(G) = 3m + r$, $0 \leq r \leq 2$, m is a nonnegative integer. Then three claims arise.

Claim 1. *If $r = 0$, then S is a ∇-set if and only if $m(S) = 0$ and vertices of S are leaves of a spanning tree T of G.*

Proof. The first part follows from Theorem 9. Now suppose that $m(S) = 0$. Then $c = 1$ and $|E(S)| = 0$. We can construct a spanning tree T as we have reasoned in the proof of Theorem 9. It is clear that the elements of S are leaves of T. □

Claim 2. *If $r = 1$, then S is a ∇-set if and only if $m(S) = 2$ and vertices of S are leaves of a spanning tree T of G with at most two exceptions.*

Proof. The first part follows from Theorem 9. It is clear that $c+|E(S)| = 3$ since $m(S) = 2$. We construct a spanning tree T of G satisfying the above condition. There are three cases according to the values of c and $|E(S)|$.

Case 1. $c = 1$ and $|E(S)| = 2$. Since G is connected, there exists two edges, say $e_1 = ab$ and $e_2 = cd$ (possibly $b = c$) in $G[S]$. For each vertex $x \in S - \{a, b, c, d\}$, add edges $e = xy$, $e_3 = by$ and $e_4 = cy$ (prescribe $e_3 = e_4$ when $b = c$), into $G - S$, where $y \in G - S$. After this, we obtain a spanning tree T of G containing $G - S$ as its subgraph. Which satisfies the condition of Claim 2 (i.e., when $b \neq c$, S has two vertices b, c which are not leaves of T; if $b = c$, then the only exception of S is $b = c$).

Case 2. $c = 2$ and $|E(S)| = 1$. Without loss of generality, let Q_1, Q_2 be the two components of $G - S$ and $E(S) = \{xy\}, x, y \in S$. Since G is connected, there exists an edge $e = xy$ (possibly $x = y$) in $G[S]$. Two situations will appear to construct a spanning tree T of G: (a) If $x \neq y$, then (i) x and y join Q_1 and Q_2, respectively. Let $x_1 \in Q_1$ and $x_2 \in Q_2$ be such that $xx_1, yx_2 \in E(G)$. The edges xy, xx_1, yx_2, Q_1 and Q_2 form a tree T_1 containing Q_1, Q_2; (ii) x joins Q_1 and Q_2. Let $x_0 \in Q_1$ and $y_0 \in Q_2$ be such that $xx_0, xy_0 \in E(G)$. Then the edges xy, xx_0, xy_0, Q_1 and Q_2 form a tree T_1 containing Q_1, Q_2. For other vertices $z \in S - \{x, y\}$, we add an edge join z with $Q_1 \cup Q_2$. It is clear that such edges and T_1 form a spanning tree T of G. (b) If $x = y$, then there is an edge $e_0 = fg$ in $G[S]$ such that x joins Q_1 and Q_2, f joins $Q_1 \cup Q_2$, the remaining vertices of $S - \{x, f, g\}$ as did in the case of $x \neq y$, so we may construct a spanning tree T of G. The above spanning trees also satisfy the condition of Claim 2.

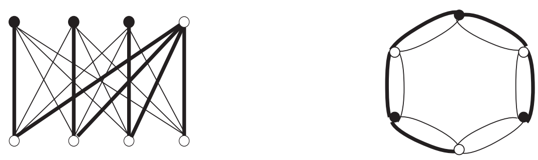
Case 3. $c = 3$ and $|E(S)| = 0$. Suppose that Q_1, Q_2 and Q_3 are three components of $G - S$. Then a spanning tree T of G will be constructed as follows. Since G is connected, there exist two vertices, say x and y (possibly $x = y$), in S such that x joins Q_1 and Q_2, y joins Q_2 and Q_3. This time we may also construct a spanning tree T of G which contains Q_1, Q_2 and Q_3 as we did in Case 2. And hence, when $x \neq y$, x and y are the only two vertices in S which are not leaves of T; if $x = y$, then x is the only vertex in S is not the leaf of T. □

Claim 3. *If $r = 2$, then S is a ∇-set if and only if $m(S) = 1$. Meanwhile, there exists a spanning tree T of G such that all (but at most one) vertices of S are leaves of T.*

Proof. The proof of Claim 3 is analogous to Claims 1 and 2, we omit its proof. □

Now the entire proof of the theorem is complete. ■

We give three examples of 4-regular graphs with $m(S) = 0, 1, 2$, respectively. See Figure 5(a), Figure 5(b) and Figure 2(b).



(a) (b)

Figure 5. (a) 4-regular graph with $m(S) = 0$; (b) 4-regular graph with $m(S) = 2$.

After a similar discussion in 4-regular graphs, we may extend Theorem 16 to general case.

Theorem 17. *Let G be a k-regular graph with $\nabla(G) = \left\lceil \frac{\beta(G)}{k-1} \right\rceil$ and $\beta(G) = m(k-1) + r$, $0 \leq r \leq k-2$, m is a nonnegative integer. Then S is a ∇-set of G*

if and only if

$$m(S) = \begin{cases} 0, & \text{for } r = 0, \\ k - r - 1, & \text{for otherwise.} \end{cases}$$

Moreover, there exists a spanning tree T in G such that elements of S are simply the leaves of T with at most $m(S)$ exceptions.

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