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ARC FAULT TOLERANCE OF CARTESIAN PRODUCT OF REGULAR DIGRAPHS ON SUPER-RESTRICTED ARC-CONNECTIVITY

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Abstract

Let $D = (V(D), A(D))$ be a strongly connected digraph. An arc set $S \subseteq A(D)$ is a restricted arc-cut of D if $D - S$ has a non-trivial strong component D_1 such that $D - V(D_1)$ contains an arc. The restricted arc-connectivity $\lambda'(D)$ is the minimum cardinality over all restricted arc-cuts of D. In [C. Balbuena, P. García-Vázquez, A. Hansberg and L.P. Montejano, *On the super-restricted arc-connectivity of s-geodetic digraphs*, Networks **61** (2013) 20–28], Balbuena *et al.* introduced the concept of super-λ' digraphs. In this paper, we first introduce the concept of the arc fault tolerance of a digraph D on the super-λ' property. We define a super-λ' digraph D to be m-super-λ' if $D - S$ is still super-λ' for any $S \subseteq A(D)$ with $|S| \leq m$. The maximum value of such m, denoted by $S_{\lambda'}(D)$, is said to be the arc fault tolerance of D on the super-λ' property. $S_{\lambda'}(D)$ is an index to measure the reliability of networks. Next we provide a necessary and sufficient condition for the Cartesian product of regular digraphs to be super-λ'. Finally, we give the lower and upper bounds on $S_{\lambda'}(D)$ for the Cartesian product D of regular digraphs and give an example to show that the lower and upper bounds are best possible. In particular, the exact value of $S_{\lambda'}(D)$ is obtained in special cases.

Keywords: fault tolerance, restricted arc-connectivity, super-restricted arc-connectivity, Cartesian product, regular digraph.

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1. Introduction

In a multiprocessor system, processors communicate by exchanging messages through an interconnection network whose topology is often modeled by a graph or a digraph $D = (V(D), A(D))$, where the vertex set $V(D)$ corresponds to processors, and the edge set or the arc set $A(D)$ corresponds to communication links. The properties of the graph or digraph determine the system's working efficiency. One fundamental consideration in the design of networks is reliability. An edge (arc)-cut of a (strongly) connected (di)graph D is a set of edges (arcs) whose removal makes the remaining (di)graph no longer (strongly) connected. The edge (arc)-connectivity $\lambda(D)$ is the minimum cardinality over all edge (arc)-cuts of D. High edge (arc)-connectivity is desirable since such a (di)graph is more reliable. It is well known that $\lambda(D) \leq \delta(D)$, where $\delta(D)$ is the minimum degree of D. Hence a (di)graph D with $\lambda(D) = \delta(D)$ is said to be maximally edge (arc)-connected.

To design more reliable networks, besides the requirement of maximal edge (arc)-connectivity, it is also desirable that the number of minimum edge (arc)-cuts is as small as possible. For this purpose, Bauer *et al.* [1] defined the super-λ (di)graphs. A (strongly) connected (di)graph D is called a super edge (arc)-connected (di)graph, in short, a super-λ (di)graph, if every minimum edge (arc)-cut consists of edges (arcs) incident with one vertex. In order to estimate more precisely the reliability of networks, Esfahanian and Hakimi [6] introduced the concept of restricted edge-connectivity. A set of edges S in a connected graph G is a restricted edge-cut if $G-S$ is disconnected and contains no isolated vertex. The restricted edge-connectivity $\lambda'(G)$ is the minimum cardinality over all restricted edge-cuts of G. A connected graph G is called a super-restricted edge-connected graph, in short, a super-λ' graph, if every minimum restricted edge-cut consists of edges adjacent to one edge.

Recently, as a generalization of restricted edge-connectivity to digraphs, the concept of restricted arc-connectivity was introduced by Volkmann [11]. Let D be a strongly connected digraph. An arc set S of D is a restricted arc-cut of D if $D - S$ has a non-trivial strong component D_1, that means a strong component with order at least 2, such that $D - V(D_1)$ contains an arc. The restricted arc-connectivity $\lambda'(D)$ is the minimum cardinality over all restricted arc-cuts of D. A strongly connected digraph D is called λ'-connected if $\lambda'(D)$ exists. A restricted arc-cut S is called a λ'-cut if $|S| = \lambda'(D)$. For $u \in V(D)$, let $d^+(u) = d^+_D(u) = |\{v \in V(D) : uv \in A(D)\}|$ and $d^-(u) = d^-_D(u) = |\{v \in V(D) : vu \in A(D)\}|$. In [11], Volkmann proved that each strong digraph D of order $n \geq 4$ and girth $g = 2$ or $g = 3$ except some families of digraphs is λ'-connected and satisfies $\lambda(D) \leq \lambda'(D) \leq \xi(D)$, where $\xi(D)$ is defined as follows. If $C_g = u_1u_2 \cdots u_g u_1$ is a shortest cycle of D, then $\xi(C_g) = \min\left\{ \sum_{i=1}^{g} d^+(u_i) - g, \sum_{i=1}^{g} d^-(u_i) - g \right\}$ and $\xi(D) = \min\{\xi(C_g) : C_g \text{ is a shortest cycle of } D\}$.

For the investigation of $\lambda'(D)$, Wang and Lin [12] introduced the notion of arc-degree. For a pair X, Y of nonempty vertex sets of a digraph D, we define $(X, Y) = \{xy \in A(D) : x \in X, y \in Y\}$. If $Y = \overline{X} = V(D) \setminus X$, we write $\partial_D^+(X)$ or $\partial_D^-(Y)$ instead of (X, Y). When the digraph under consideration is obvious, we omit the subscript D and use $\partial^+(X)$ and $\partial^-(Y)$. Usually, we abbreviate $\partial^+(\{x\})$ and $\partial^-(\{x\})$ to $\partial^+(x)$ and $\partial^-(x)$, respectively. For any $xy \in A(D)$, the arc-degree of xy is defined as $\xi'(xy) = \min\{|\partial^+(\{x,y\})|, |\partial^-(\{x,y\})|, |\partial^+(x) \cup \partial^-(y)|, |\partial^-(x) \cup \partial^+(y)|\}$. The minimum arc-degree of D is $\xi'(D) = \min\{\xi'(xy) : xy \in A(D)\}$. The arc-degree of an arc $xy \in A(D)$ can be computed in terms of the degrees of vertices x and y. An arc $xy \in A(D)$ is a symmetric arc if $yx \in A(D)$. The set of symmetric arcs of D is denoted by $\mathrm{Sym}(D)$. If $xy \notin \mathrm{Sym}(D)$, then $\xi'(xy) = \min\{d^+(x) + d^+(y) - 1, d^-(x) + d^-(y) - 1, d^+(x) + d^-(y) - 1, d^-(x) + d^+(y)\}$. If $xy \in \mathrm{Sym}(D)$, then $\xi'(xy) = \min\{d^+(x) + d^+(y) - 2, d^-(x) + d^-(y) - 2, d^+(x)+d^-(y)-1, d^-(x)+d^+(y)-1\}$. By [12], $\xi'(D) \leq \xi(D)$ for many digraphs D, for example, for all the digraphs D with $\delta(D) \geq 3$.

More recently, Balbuena *et al.* [3] extended the notion of super-λ' graphs to digraphs as follows. A λ'-connected digraph D is called a super-restricted arc-connected digraph, in short, a super-λ' digraph, if and only if for every λ'-cut S there exists $xy \in A(D)$ such that $S \in \Omega_{xy} = \{\partial^+(\{x,y\}), \partial^-(\{x,y\}), \partial^+(x) \cup \partial^-(y), \partial^-(x) \cup \partial^+(y)\}$. In the same article, Balbuena *et al.* provided a sufficient condition for an s-geodetic digraph to be super-λ'. Super-restricted arc-connectivity is a more refined measure for the network reliability than restricted arc-connectivity.

A natural question is how many links of a super-λ' interconnection network an adversary needs to destroy such that the damaged network is not super-λ' any more. In fact, the similar question has been investigated for the super-λ (di)graphs in [4, 7, 8, 16]. In this paper, we study this problem for the super-λ' digraphs. For this purpose, we first introduce the following concepts.

Definition. Let m be a nonnegative integer. A super-λ' digraph D is m-super-λ' if $D - S$ is still super-λ' for any $S \subseteq A(D)$ with $|S| \leq m$.

Definition. The arc fault tolerance of a super-λ' digraph D on the super-λ' property, denoted by $S_{\lambda'}(D)$, is the integer m such that D is m-super-λ' but not $(m+1)$-super-λ'.

Example 1. Let C_4 be an undirected cycle of order 4, and let D be the digraph obtained from C_4 by replacing each edge of C_4 by two oppositely oriented arcs with the same ends. Then D is 0-super-λ'. For any $e \in A(D)$, $D - e$ is still super-λ'. So D is 1-super-λ'. For any $u \in V(D)$, let $\{e_1, e_2\} = \partial^+(u)$. Then D is not 2-super-λ' since $D - \{e_1, e_2\}$ is not strongly connected. Thus $S_{\lambda'}(D) = 1$.

Now, we answer the above question. An adversary needs to destroy at least $S_{\lambda'}(D)+1$ links for destroying the super-λ' property of an interconnection network D. $S_{\lambda'}(D)$ can be used to evaluate the reliability of an interconnection network D. Therefore, the determination of $S_{\lambda'}(D)$ is full of scientific significance as well as application value.

For designing large-scale interconnection networks, the Cartesian product is an important method to obtain large digraphs from smaller ones, with a number of parameters that can be easily calculated from the corresponding parameters of those small initial digraphs. The Cartesian product preserves many nice properties of the initial digraphs (see, for example, [14]). The Cartesian product of digraphs D_1 and D_2 is the digraph $D_1 \times D_2$ whose vertex set is $V(D_1) \times V(D_2)$ and whose arc set is the set of all pairs $(x_1, y_1)(x_2, y_2)$ such that either $x_1x_2 \in A(D_1)$ and $y_1 = y_2$, or $y_1y_2 \in A(D_2)$ and $x_1 = x_2$. In [5, 9, 10, 13, 14, 15, 17], the authors introduced some results about (arc) connectivity of the Cartesian product of digraphs.

For graph-theoretical terminology and notation not defined here we follow [2]. We only consider finite digraphs D without loops and multiple arcs. For a vertex u of D, $d^+_D(u)$ and $d^-_D(u)$ are called the out-degree and in-degree of u, respectively. If D has vertices $v_1, v_2, \ldots, v_n$, the sequence $(d^+_D(v_1), d^+_D(v_2), \ldots, d^+_D(v_n))$ is called an out-degree sequence of D. An in-degree sequence of D can be defined similarly. Let $\delta^+(D) = \min\{d^+_D(u) : u \in V(D)\}$ and $\delta^-(D) = \min\{d^-_D(u) : u \in V(D)\}$. Then $\delta(D) = \min\{\delta^+(D), \delta^-(D)\}$. For subsets X and X' of $V(D)$, denote by $D[X]$ the subdigraph of D induced by X and write $X \subset X'$ if X is properly contained in X'. Denote by $\overleftrightarrow{K}_n$ the complete digraph on n vertices. Any digraph with just one vertex is referred to as trivial. Let D and H be two digraphs. The union $D \cup H$ of D and H is the digraph with vertex set $V(D) \cup V(H)$ and arc set $A(D) \cup A(H)$. Let D_i be a digraph for $i = 1, 2, \ldots, n$. For simplicity, we write $\nu_i = |V(D_i)|$, $\lambda_i = \lambda(D_i)$, $\delta_i = \delta(D_i)$, $\delta^+_i = \delta^+(D_i)$ and $\delta^-_i = \delta^-(D_i)$. For $y \in V(D_2)$, we use D^y_1 to denote the subdigraph of $D_1 \times D_2$ induced by the vertex set $\{(x, y) : x \in V(D_1)\}$. Clearly, D^y_1 is isomorphic to D_1. D^x_2 can be defined similarly for $x \in V(D_1)$.

A regular network has the advantages of easy implementation and low cost when it is manufactured. Hence, in this paper, we focus on regular digraphs. A digraph D is k-regular if $d^+_D(v) = d^-_D(v) = k$ for all $v \in V(D)$; a regular digraph is one that is k-regular for some k. Let $D_1 \times D_2$ be the Cartesian product of regular digraphs D_1 and D_2. We provide a necessary and sufficient condition for $D_1 \times D_2$ to be super-λ' and give the lower and upper bounds on $S_{\lambda'}(D_1 \times D_2)$. An example shows that the lower and upper bounds are best possible. In particular, the exact value of $S_{\lambda'}(D_1 \times D_2)$ is obtained in special cases. These results are also generalized to the Cartesian product of n regular digraphs.

2. A Necessary and Sufficient Condition for the Cartesian Product of Regular Digraphs to be Super-λ'

We first give two lemmas.

Lemma 2 [12]. *Let D be a strongly connected digraph with $\delta^+(D) \geq 3$ or $\delta^-(D) \geq 3$. Then D is λ'-connected and $\lambda'(D) \leq \xi'(D)$.*

Lemma 3 [3]. *Let D be a λ'-connected digraph and let S be a λ'-cut of D. If D is not super-λ', then there exists a subset of vertices $X \subset V(D)$ such that $S = \partial^+(X)$ and both $D[X]$ and $D[\overline{X}]$ contain an arc.*

Theorem 4. *Let D_i be a strongly connected k_i-regular digraph with $k_i = \lambda_i \geq 2$ for $i = 1, 2$. Then $D_1 \times D_2$ is super-λ' if and only if $D_1 \times D_2 \ncong \overleftrightarrow{K}_n \times D$, where $n \geq 3$ and D is a strongly connected k-regular digraph with $k = \lambda(D) = 2$.*

Proof. *Necessity.* Let S be an arbitrary λ'-cut of $D_1 \times D_2$. Since $D_1 \times D_2$ is super-λ', there exists $xy \in A(D_1 \times D_2)$ such that $S \in \Omega_{xy} = \{\partial^+(\{x,y\}), \partial^-(\{x,y\}), \partial^+(x) \cup \partial^-(y), \partial^-(x) \cup \partial^+(y)\}$. Note that $\xi'(xy) = \min\{|\partial^+(\{x,y\})|, |\partial^-(\{x,y\})|, |\partial^+(x) \cup \partial^-(y)|, |\partial^-(x) \cup \partial^+(y)|\}$. Thus $\lambda'(D_1 \times D_2) = |S| \geq \xi'(xy) \geq \xi'(D_1 \times D_2)$. By $\delta(D_1 \times D_2) \geq 4$, Lemma 2 yields $\lambda'(D_1 \times D_2) \leq \xi'(D_1 \times D_2)$. Hence $\lambda'(D_1 \times D_2) = \xi'(D_1 \times D_2)$. Suppose that $D_1 \times D_2 \cong \overleftrightarrow{K}_n \times D$, where $n \geq 3$ and D is a strongly connected k-regular digraph with $k = \lambda(D) = 2$. Then $\overleftrightarrow{K}_n \times D$ is super-λ' and $\lambda'\big(\overleftrightarrow{K}_n \times D\big) = \xi'\big(\overleftrightarrow{K}_n \times D\big)$. Since $\overleftrightarrow{K}_n \times D$ is $(n+1)$-regular, for any $u \in V\big(\overleftrightarrow{K}_n \times D\big)$, $d^+(u) = d^-(u) = n+1$. Let $xy \in A\big(\overleftrightarrow{K}_n \times D\big)$. If $xy \notin \mathrm{Sym}\big(\overleftrightarrow{K}_n \times D\big)$, $\xi'(xy) = \min\{d^+(x)+d^+(y)-1, d^-(x)+d^-(y)-1, d^+(x)+d^-(y)-1, d^-(x)+d^+(y)\} = 2n+1$. If $xy \in \mathrm{Sym}\big(\overleftrightarrow{K}_n \times D\big)$, $\xi'(xy) = \min\{d^+(x)+d^+(y)-2, d^-(x)+d^-(y)-2, d^+(x)+d^-(y)-1, d^-(x)+d^+(y)-1\} = 2n$. Note that $\mathrm{Sym}\big(\overleftrightarrow{K}_n \times D\big) \neq \emptyset$. Thus $\xi'\big(\overleftrightarrow{K}_n \times D\big) = 2n$ and so $\lambda'\big(\overleftrightarrow{K}_n \times D\big) = 2n$. By $\lambda(D) = 2$, there exists a minimum arc-cut $\{y_1y_2, y_3y_4\}$ of D such that $D - \{y_1y_2, y_3y_4\}$ has t strong components $B_1, B_2, \ldots, B_t$. Denote $V\big(\overleftrightarrow{K}_n\big) = \{x_j : j = 1, 2, \ldots, n\}$ and denote by D^{x_j} the subdigraph of $\overleftrightarrow{K}_n \times D$ induced by the vertex set $\{(x_j, y) : y \in V(D)\}$. Clearly $D^{x_j} \cong D$. Thus $D^{x_j} - \big\{(x_j,y_1)(x_j,y_2), (x_j,y_3)(x_j,y_4)\big\}$ has t strong components. Let $S = \bigcup_{j=1}^n \big\{(x_j,y_1)(x_j,y_2), (x_j,y_3)(x_j,y_4)\big\}$. Then $|S| = 2n$ and $\overleftrightarrow{K}_n \times D - S$ has t strong components $\overleftrightarrow{K}_n \times B_1, \overleftrightarrow{K}_n \times B_2, \ldots, \overleftrightarrow{K}_n \times B_t$. By $n \geq 3$, S is clearly a restricted arc-cut of $\overleftrightarrow{K}_n \times D$. Note that $\lambda'\big(\overleftrightarrow{K}_n \times D\big) = 2n = |S|$. Thus S is a λ'-cut. By $n \geq 3$, for any $xy \in A(\overleftrightarrow{K}_n \times D)$, $S \notin \Omega_{xy}$, which implies that $\overleftrightarrow{K}_n \times D$ is not super-λ', a contradiction. We conclude that $D_1 \times D_2 \ncong \overleftrightarrow{K}_n \times D$.

Sufficiency. Let $D^* = D_1 \times D_2$. By Lemma 2, D^* is λ'-connected and $\lambda'(D^*) \leq \xi'(D^*)$. Suppose that D^* is not super-λ'. Then there exists a λ'-cut S

such that for any $xy \in A(D^*)$, $S \notin \Omega_{xy}$. By Lemma 3, there exists a subset of vertices $X \subset V(D^*)$ such that $S = \partial^+(X)$ and both $D^*[X]$ and $D^*[\overline{X}]$ contain an arc. Thus $|X| \geq 2$ and $|\overline{X}| \geq 2$. If $|X| = 2$, then there exists $uv \in A(D^*[X])$ such that $S = \partial^+(\{u, v\}) \in \Omega_{uv}$, a contradiction. Thus $|X| \geq 3$. Similarly, $|\overline{X}| \geq 3$. We give four claims.

Claim 1. $|X| \geq k_1 + k_2 - 1$.

Since D_i is k_i-regular for $i = 1, 2$, D^* is $(k_1 + k_2)$-regular. If $\mathrm{Sym}(D^*) \neq \emptyset$, then $|S| = \lambda'(D^*) \leq \xi'(D^*) = 2k_1 + 2k_2 - 2$. Note that $S = \partial^+(X)$. Thus

$$|X|(k_1 + k_2) - |X|(|X| - 1) \leq |S| \leq 2k_1 + 2k_2 - 2,$$

which implies that $(|X|-2)(k_1+k_2-|X|-1) \leq 0$. By $|X| \geq 3$, $|X| \geq k_1+k_2-1$.

If $\mathrm{Sym}(D^*) = \emptyset$, then $|S| = \lambda'(D^*) \leq \xi'(D^*) = 2k_1 + 2k_2 - 1$. Since $D^*[X]$ is not a complete digraph, $\sum_{v \in X} d^+_{D^*[X]}(v) \leq |X|(|X| - 1) - 1$. Hence

$$|X|(k_1 + k_2) - |X|(|X| - 1) + 1 \leq |S| \leq 2k_1 + 2k_2 - 1.$$

Similarly, $|X| \geq k_1 + k_2 - 1$. Claim 1 holds.

Claim 2. *For any $y \in V(D_2)$ and any $x \in V(D_1)$, $X \nsubseteq V(D_1^y)$ and $X \nsubseteq V(D_2^x)$.*

By contradiction. Suppose that $X \subseteq V(D_1^y)$ for some $y \in V(D_2)$. If $X \subset V(D_1^y)$, then, by Claim 1 and $k_i \geq 2$ for $i = 1, 2$, $\lambda'(D^*) = |S| \geq |X|k_2 + \lambda_1 \geq (k_1 + k_2 - 1)k_2 + k_1 \geq 2k_1 + 2k_2 + k_1 - 2 \geq 2k_1 + 2k_2 > \xi'(D^*)$, a contradiction. If $X = V(D_1^y)$, then $|S| = |X|k_2$. If $|X| \geq k_1 + k_2$ or $k_2 \geq 3$, then $|S| = |X|k_2 \geq 2k_1 + 2k_2$ or $|S| = |X|k_2 \geq 3(k_1 + k_2 - 1) > 2k_1 + 2k_2$. This means that $\lambda'(D^*) = |S| > \xi'(D^*)$, a contradiction. If $|X| = k_1 + k_2 - 1$ and $k_2 = 2$, then $|X| = |V(D_1^y)| = k_1 + 1$ and so D_1^y is a complete digraph. Thus $D_1 \times D_2 \cong \overleftrightarrow{K}_n \times D$, where $n = |X| \geq 3$ and D is a strongly connected k-regular digraph with $k = \lambda(D) = 2$, a contradiction. The case that $X \nsubseteq V(D_2^x)$ for any $x \in V(D_1)$ can be proved analogously. Claim 2 holds.

By Claim 2, X contains two vertices (x_1, y_1) and (x_2, y_2) such that $x_1 \neq x_2$ and $y_1 \neq y_2$. Let $\widetilde{D_1^{y_i}} = D_1^{y_i} - S$ and $\widetilde{D_2^{x_i}} = D_2^{x_i} - S$ for $i = 1, 2$.

Claim 3. *At least one graph of $\widetilde{D_1^{y_1}}$, $\widetilde{D_1^{y_2}}$, $\widetilde{D_2^{x_1}}$ and $\widetilde{D_2^{x_2}}$ is strongly connected.*

By contradiction. Suppose that $\widetilde{D_1^{y_i}}$ and $\widetilde{D_2^{x_i}}$ are not strongly connected for $i = 1, 2$. Then $\lambda'(D^*) = |S| \geq 2\lambda_1 + 2\lambda_2 = 2k_1 + 2k_2 > \xi'(D^*)$, a contradiction. Claim 3 holds.

By Claim 3, we may assume, without loss of generality, that $\widetilde{D_1^{y_1}}$ is strongly connected and so $V(D_1^{y_1}) \subseteq X$.

Claim 4. *For some $x \in V(D_1)$, $\widetilde{D_2^x}$ is strongly connected.*

By contradiction. Suppose that $\widetilde{D_2^x}$ is not strongly connected for any $x \in V(D_1)$. Then $|S| \geq \nu_1\lambda_2 = \nu_1 k_2$. If $\mathrm{Sym}(D^*) \neq \emptyset$, then $2k_1 + 2k_2 - 2 = \xi'(D^*) \geq |S| \geq \nu_1 k_2$. Thus $2k_1 - 2 \geq (\nu_1 - 2)k_2 \geq (k_1 - 1)k_2 \geq 2k_1 - 2$, which implies that all the inequalities become equalities. Hence $\nu_1 = k_1 + 1$ and $k_2 = 2$. This means that $D_1 \times D_2 \cong \overleftrightarrow{K}_n \times D$, where $n \geq 3$ and D is a strongly connected k-regular digraph with $k = \lambda(D) = 2$, a contradiction. If $\mathrm{Sym}(D^*) = \emptyset$, then $2k_1 + 2k_2 - 1 = \xi'(D^*) \geq |S| \geq \nu_1 k_2$. Thus $2k_1 - 1 \geq (\nu_1 - 2)k_2 \geq k_1 k_2 \geq 2k_1$, a contradiction. Claim 4 holds.

By Claim 4, $V(D_2^x) \subseteq X$ or $V(D_2^x) \subseteq \overline{X}$. Since $V(D_1^{y_1}) \subseteq X$ and $V(D_2^x) \cap V(D_1^{y_1}) = \{(x, y_1)\}$, we see that $(x, y_1) \in X$ and so $V(D_2^x) \subseteq X$. Now, $V(D_1^{y_1}) \cup V(D_2^x) \subseteq X$. A similar argument can be used to establish that there exist two vertices $y' \in V(D_2)$ and $x' \in V(D_1)$ such that $V\big(D_1^{y'}\big) \cup V\big(D_2^{x'}\big) \subseteq \overline{X}$. Thus $V\big(D_1^{y_1}\big) \cap V\big(D_2^{x'}\big) = \{(x', y_1)\} \subseteq X \cap \overline{X}$, a contradiction. We conclude that $D_1 \times D_2$ is super-λ'. ■

3. $S_{\lambda'}(D)$ for the Cartesian Product of Regular Digraphs

For convenience, denote $\omega_D(X) = |\partial_D^+(X)| = |\partial^+(X)|$.

Lemma 5. *Let D be a strongly connected digraph with $\delta^+(D) \geq 3$ or $\delta^-(D) \geq 3$. If $\omega_D(X) > \xi'(D)$ holds for any $X \subseteq V(D)$ with $|X| \geq 3$ and $|\overline{X}| \geq 3$, then D is super-λ'.*

Proof. By Lemma 2, D is λ'-connected and $\lambda'(D) \leq \xi'(D)$. Suppose, to the contrary, that D is not super-λ'. Then there exists a λ'-cut S of D such that for any $xy \in A(D)$, $S \notin \Omega_{xy}$. By Lemma 3, there exists a subset of vertices $X \subset V(D)$ such that $S = \partial^+(X)$ and both $D[X]$ and $D[\overline{X}]$ contain an arc. Thus $|X| \geq 2$ and $|\overline{X}| \geq 2$. If $|X| = 2$, then there exists $uv \in A(D[X])$ such that $S = \partial^+(\{u, v\}) \in \Omega_{uv}$, a contradiction. Thus $|X| \geq 3$. Similarly, $|\overline{X}| \geq 3$. Hence $\xi'(D) \geq \lambda'(D) = |S| = \omega_D(X) > \xi'(D)$, a contradiction. We conclude that D is super-λ'. ■

Lemma 6. *Let D be a k-regular digraph. Then $\xi'(D - S) \leq \xi'(D)$ for any $S \subseteq A(D)$ with $|S| \leq k - 1$.*

Proof. Since D is k-regular, $\xi'(D) \geq 2k - 2$. By $|S| \leq \delta(D) - 1$, there exists $xy \in A(D - S)$ such that xy is adjacent to at least one arc a in S. Without loss of generality, assume that $a = xu$ with $u \in V(D)$ and $u \neq y$. Thus $\xi'(D - S) \leq \xi'(xy) \leq \min\big\{d_{D-S}^+(x) + d_{D-S}^+(y) - 1, d_{D-S}^-(x) + d_{D-S}^-(y) - 1, d_{D-S}^+(x) + d_{D-S}^-(y) - 1, d_{D-S}^-(x) + d_{D-S}^+(y)\big\} \leq \min\big\{d_D^+(x) + d_D^+(y) - 2, d_D^-(x) + d_D^-(y) - 1, d_D^+(x) + d_D^-(y) - 2, d_D^-(x) + d_D^+(y)\big\} = 2k - 2 \leq \xi'(D)$. ■

Lemma 7. *Let D be a digraph with $\delta(D) \geq 4$. Then $\delta^+(D-S) \geq 3$ or $\delta^-(D-S) \geq 3$ for any $S \subseteq A(D)$ with $|S| \leq \delta(D)-1$.*

Proof. If all the arcs in S are incident with a vertex x, then, for any $y \in V(D) \setminus \{x\}$, $\min\left\{d^+_{D-S}(y), d^-_{D-S}(y)\right\} \geq \delta(D)-1 \geq 3$. In order to prove that $\delta^+(D-S) \geq 3$ or $\delta^-(D-S) \geq 3$, it suffices to show that $d^+_{D-S}(x) \geq 3$ or $d^-_{D-S}(x) \geq 3$. Let $S = S_0 \cup S_1$ with $S_0 \cap S_1 = \emptyset$, $S_0 \subseteq \partial^+(x)$ and $S_1 \subseteq \partial^-(x)$. Then $|S_0|+|S_1| = |S|$ and so $|S_0| \leq \left\lfloor \frac{|S|}{2} \right\rfloor$ or $|S_1| \leq \left\lfloor \frac{|S|}{2} \right\rfloor$. Without loss of generality, assume that $|S_0| \leq \left\lfloor \frac{|S|}{2} \right\rfloor$. Then $d^+_{D-S}(x) = d^+_D(x) - |S_0| \geq d^+_D(x) - \left\lfloor \frac{|S|}{2} \right\rfloor \geq \delta(D) - \left\lfloor \frac{\delta(D)-1}{2} \right\rfloor = \left\lceil \frac{\delta(D)+1}{2} \right\rceil \geq \left\lceil \frac{4+1}{2} \right\rceil = 3$.

If exactly $|S|-1$ arcs in S are incident with a vertex x, then, for any $y \in V(D) \setminus \{x\}$, $d^+_{D-S}(y) \geq 3$ or $d^-_{D-S}(y) \geq 3$. In order to prove that $\delta^+(D-S) \geq 3$ or $\delta^-(D-S) \geq 3$, it suffices to show that $d^+_{D-S}(x) \geq 3$ and $d^-_{D-S}(x) \geq 3$. Let $S' = S \cap (\partial^+(x) \cup \partial^-(x))$. Then $|S'| = |S|-1$. If $S' \subseteq \partial^+(x)$, then $d^-_{D-S}(x) = d^-_D(x) \geq 4$. If $S' \subseteq \partial^-(x)$, then $d^+_{D-S}(x) = d^+_D(x) \geq 4$. Next consider the case $S' \nsubseteq \partial^+(x)$ and $S' \nsubseteq \partial^-(x)$. Let $S' = S'_0 \cup S'_1$ with $S'_0 \cap S'_1 = \emptyset$, $S'_0 \subseteq \partial^+(x)$ and $S'_1 \subseteq \partial^-(x)$. Then $|S'_0| \geq 1$, $|S'_1| \geq 1$ and $|S'_0|+|S'_1| = |S'|$. Note that $|S'| = |S|-1 \leq \delta(D)-2$. Thus $|S'_0| \leq \delta(D)-3$ and $|S'_1| \leq \delta(D)-3$, which implies that $d^+_{D-S}(x) = d^+_D(x) - |S'_0| \geq \delta(D) - (\delta(D)-3) = 3$ and $d^-_{D-S}(x) \geq \delta(D) - (\delta(D)-3) = 3$.

Suppose that at most $|S|-2$ arcs in S are incident with a vertex. If $|S|-2 = 1$, then any two arcs in S are not adjacent. It follows that for any $y \in V(D)$, $\min\left\{d^+_{D-S}(y), d^-_{D-S}(y)\right\} \geq \delta(D)-1 \geq 3$ and so $\delta(D-S) \geq 3$. If $|S|-2 \geq 2$, then for any $y \in V(D)$, $\min\left\{d^+_{D-S}(y), d^-_{D-S}(y)\right\} \geq \delta(D) - (|S|-2) \geq \delta(D) - (\delta(D)-3) = 3$ and so $\delta(D-S) \geq 3$. ■

Lemma 8. *Let $D = D_1 \times D_2$, where D_i is a strongly connected k_i-regular digraph with $k_i = \lambda_i \geq 3$ for $i = 1, 2$. Then $\omega_{D-S}(X) > \xi'(D-S)$ holds for any $S \subseteq A(D)$ and any $X \subseteq V(D)$ such that $|S| \leq k_1+k_2-1$, $|X| \geq 3$ and $X \subset V(D_1^{y_0})$ with $y_0 \in V(D_2)$ or $X \subset V(D_2^{x_0})$ with $x_0 \in V(D_1)$.*

Proof. Suppose, to the contrary, that there exists a subset S of $A(D)$ and a subset X of $V(D)$ satisfying the conditions of the lemma such that

$$\omega_{D-S}(X) \leq \xi'(D-S). \tag{1}$$

Case 1. $\mathrm{Sym}(D) \neq \emptyset$. Note that D is (k_1+k_2)-regular. Thus $\xi'(D) = 2k_1 + 2k_2 - 2$. By $|S| \leq k_1+k_2-1$, Lemma 6 yields $\xi'(D-S) \leq \xi'(D) = 2k_1+2k_2-2$. Thus $2k_1+2k_2-2 \geq \xi'(D-S) \geq \omega_{D-S}(X) \geq \omega_D(X) - |S| \geq \omega_D(X) - (k_1+k_2-1)$ and so

$$\omega_D(X) \leq 3k_1+3k_2-3. \tag{2}$$

Let $|X| = a \geq 3$. Assume that $X \subset V(D_1^{y_0})$ with $y_0 \in V(D_2)$. Then $\omega_D(X) = \omega_{D_1^{y_0}}(X) + \left|\left(X, \overline{V(D_1^{y_0})}\right)\right| = \omega_{D_1^{y_0}}(X) + \sum_{z\in X}\left|\left(\{z\}, \overline{V(D_1^{y_0})}\right)\right| = \omega_{D_1^{y_0}}(X) + ak_2$. Combining this with (2), we have

$$3k_1 \geq \omega_{D_1^{y_0}}(X) + (a-3)k_2 + 3. \tag{3}$$

Note that $a \geq 3$ and $k_2 \geq 3$. Thus (3) yields

$$3k_1 \geq \omega_{D_1^{y_0}}(X) + 3a - 6. \tag{4}$$

For any vertex $z \in X$, since $\left|\left(\{z\}, V\left(D_1^{y_0}\right) \setminus X\right)\right| = \left|\left(\{z\}, V\left(D_1^{y_0}\right)\right)\right| - \left|(\{z\}, X)\right| \geq k_1 - (a-1)$, we have $\omega_{D_1^{y_0}}(X) = \sum_{z\in X}\left|\left(\{z\}, V\left(D_1^{y_0}\right) \setminus X\right)\right| \geq a(k_1 - a + 1)$. Combining this with (4), we have

$$(a-3)k_1 \leq a^2 - 4a + 6. \tag{5}$$

Since $D_1^{y_0} \cong D_1$ and X is a nonempty proper subset of $V(D_1^{y_0})$, $\omega_{D_1^{y_0}}(X) \geq \lambda_1 = k_1$. By (4), we have

$$2k_1 \geq 3a - 6. \tag{6}$$

Combining (5) with (6), we have $(a-3)(3a-6) \leq 2a^2 - 8a + 12$ and so $1 \leq a \leq 6$. Note that $a \geq 3$. Thus $3 \leq a \leq 6$. Consider the following four cases.

Case 1.1. $a = 6$. Note that D is $(k_1 + k_2)$-regular and $D[X]$ has at most 30 arcs. Thus $\omega_D(X) \geq 6k_1 + 6k_2 - 30$. By (6), $k_1 \geq 6$ and so $k_1 + k_2 \geq 9$. If $\omega_D(X) \geq 6k_1 + 6k_2 - 29$, then $\omega_D(X) > 3k_1 + 3k_2 - 3$, contradicting (2). Thus $\omega_D(X) = 6k_1 + 6k_2 - 30$ and so $D[X] \cong \overleftrightarrow{K}_6$.

If $|S \cap (X, \overline{X})| \leq k_1 + k_2 - 2$, then $\omega_{D-S}(X) = \omega_D(X) - |S \cap (X, \overline{X})| \geq 6k_1 + 6k_2 - 30 - (k_1 + k_2 - 2) = 5k_1 + 5k_2 - 28 > 2k_1 + 2k_2 - 2 = \xi'(D) \geq \xi'(D-S)$, contradicting (1).

If $|S \cap (X, \overline{X})| = k_1 + k_2 - 1$, then $\omega_{D-S}(X) = 5k_1 + 5k_2 - 29$. By $|S| \leq k_1 + k_2 - 1$, $S \cap A(D[X]) = \emptyset$. Note that $|S \cap (X, \overline{X})| = k_1 + k_2 - 1 \geq 8$. Thus there exists $xy \in \mathrm{Sym}(D[X])$ such that xy is adjacent to at least one arc in $S \cap (X, \overline{X})$. Thus $\xi'(D-S) \leq \xi'(xy) = \min\left\{d^+_{D-S}(x) + d^+_{D-S}(y) - 2, d^-_{D-S}(x) + d^-_{D-S}(y) - 2, d^+_{D-S}(x) + d^-_{D-S}(y) - 1, d^-_{D-S}(x) + d^+_{D-S}(y) - 1\right\} \leq d^+_{D-S}(x) + d^+_{D-S}(y) - 2 \leq d^+_D(x) + d^+_D(y) - 1 - 2 = 2k_1 + 2k_2 - 3 < 5k_1 + 5k_2 - 29 = \omega_{D-S}(X)$, contradicting (1).

Case 1.2. $a = 5$. Note that D is $(k_1 + k_2)$-regular and $D[X]$ has at most 20 arcs. Thus $\omega_D(X) \geq 5k_1 + 5k_2 - 20$. By (6), $k_1 \geq 5$ and so $k_1 + k_2 \geq 8$. If $\omega_D(X) \geq 5k_1 + 5k_2 - 18$, then $\omega_D(X) > 3k_1 + 3k_2 - 3$, contradicting (2). Thus $5k_1 + 5k_2 - 20 \leq \omega_D(X) \leq 5k_1 + 5k_2 - 19$. Since D is $(k_1 + k_2)$-regular, $\omega_D(X) = 5k_1 + 5k_2 - |A(D[X])|$ and so $19 \leq |A(D[X])| \leq 20$.

If $|S \cap (X, \overline{X})| \leq k_1 + k_2 - 3$, then $\omega_{D-S}(X) = \omega_D(X) - |S \cap (X, \overline{X})| \geq 4k_1 + 4k_2 - 17 > 2k_1 + 2k_2 - 2 = \xi'(D) \geq \xi'(D - S)$, contradicting (1).

If $k_1 + k_2 - 2 \leq |S \cap (X, \overline{X})| \leq k_1 + k_2 - 1$, then $\omega_{D-S}(X) \geq 4k_1 + 4k_2 - 19$. By $|S| \leq k_1 + k_2 - 1$, $|S \cap A(D[X])| \leq 1$. Note that $|S \cap (X, \overline{X})| \geq k_1 + k_2 - 2 \geq 6$ and $|X| = 5$. Thus there exists $x \in X$ such that $|S \cap (\{x\}, \overline{X})| \geq 2$. By $|X| = 5$, $19 \leq |A(D[X])| \leq 20$ and $|S \cap A(D[X])| \leq 1$, there exists $xz \in \mathrm{Sym}(D[X] - S)$. Thus $\xi'(D - S) \leq \xi'(xz) \leq d^+_{D-S}(x) + d^+_{D-S}(z) - 2 \leq d^+_D(x) + d^+_D(z) - 2 - 2 = 2k_1 + 2k_2 - 4 < 4k_1 + 4k_2 - 19 \leq \omega_{D-S}(X)$, contradicting (1).

Case 1.3. $a = 4$. Note that D is $(k_1 + k_2)$-regular and $D[X]$ has at most 12 arcs. Thus $\omega_D(X) \geq 4k_1 + 4k_2 - 12$. If $\omega_D(X) \geq 4k_1 + 4k_2 - 8$, then, by $k_1 + k_2 \geq 6$, $\omega_D(X) > 3k_1 + 3k_2 - 3$, contradicting (2). Thus $4k_1 + 4k_2 - 12 \leq \omega_D(X) \leq 4k_1 + 4k_2 - 9$.

Case 1.3.1. $4k_1 + 4k_2 - 12 \leq \omega_D(X) \leq 4k_1 + 4k_2 - 10$. Now $10 \leq |A(D[X])| \leq 12$ and so $10 \leq \Sigma_{v \in X} d^+_{D[X]}(v) \leq 12$. Thus the out-degree sequence of $D[X]$ may be $(3, 3, 3, 3), (2, 3, 3, 3), (2, 2, 3, 3)$, or $(1, 3, 3, 3)$. Since $X \subset V(D_1^{y_0})$, $|(X, V(D_1^{y_0}) \setminus X)| \geq \lambda_1 = k_1 \geq 3$. Combining this with the out-degree sequence of $D[X]$, there exists $u \in X$ such that $d^+_{D_1^{y_0}}(u) \geq 4$. Since $D_1^{y_0} \cong D_1$ and D_1 is k_1-regular, $k_1 \geq 4$ and so $k_1 + k_2 \geq 7$.

If $|S \cap (X, \overline{X})| \leq k_1 + k_2 - 4$, then $\omega_{D-S}(X) = \omega_D(X) - |S \cap (X, \overline{X})| \geq 3k_1 + 3k_2 - 8 > 2k_1 + 2k_2 - 2 = \xi'(D) \geq \xi'(D - S)$, contradicting (1).

If $|S \cap (X, \overline{X})| = k_1 + k_2 - 3$, then $\omega_{D-S}(X) \geq 3k_1 + 3k_2 - 9$ and $|S \cap A(D[X])| \leq 2$. Note that $|S \cap (X, \overline{X})| = k_1 + k_2 - 3 \geq 4$. Assume that there exists one of X, say x, such that $|S \cap (\{x\}, \overline{X})| \geq 2$. Combining $|S| \leq k_1 + k_2 - 1$ with $|S \cap (X, \overline{X})| = k_1 + k_2 - 3$, we have $|S \cap (\overline{X}, X)| \leq 2$. Since D is $(k_1 + k_2)$-regular and $k_1 + k_2 \geq 7$, there exists an arc $ux \in (\overline{X}, X)$ with $ux \notin S$. Thus $\xi'(D - S) \leq \xi'(ux) \leq d^+_{D-S}(u) + d^+_{D-S}(x) - 1 \leq d^+_D(u) + d^+_D(x) - 2 - 1 = 2k_1 + 2k_2 - 3 < 3k_1 + 3k_2 - 9 \leq \omega_{D-S}(X)$, contradicting (1). Assume that for any $x \in X$, $|S \cap (\{x\}, \overline{X})| \leq 1$. By $|S \cap (X, \overline{X})| \geq 4$, $|S \cap (\{x\}, \overline{X})| = 1$ for any $x \in X$. Note that $10 \leq |A(D[X])| \leq 12$ and $|S \cap A(D[X])| \leq 2$. Thus there exists $yz \in A(D[X] - S)$ and so $\xi'(D - S) \leq \xi'(yz) \leq d^+_{D-S}(y) + d^+_{D-S}(z) - 1 \leq 2k_1 + 2k_2 - 2 - 1 < 3k_1 + 3k_2 - 9 \leq \omega_{D-S}(X)$, contradicting (1).

If $|S \cap (X, \overline{X})| = k_1 + k_2 - 2$, then $\omega_{D-S}(X) \geq 3k_1 + 3k_2 - 10$ and $|S \cap A(D[X])| \leq 1$. By $|S \cap (X, \overline{X})| = k_1 + k_2 - 2 \geq 5$ and $|X| = 4$, there exists $x \in X$ such that $|S \cap (\{x\}, \overline{X})| \geq 2$. If there exists $xy \in \mathrm{Sym}(D[X] - S)$, then $\xi'(D - S) \leq \xi'(xy) \leq d^+_{D-S}(x) + d^+_{D-S}(y) - 2 \leq 2k_1 + 2k_2 - 2 - 2 < 3k_1 + 3k_2 - 10 \leq \omega_{D-S}(X)$, contradicting (1). Otherwise, by $10 \leq |A(D[X])| \leq 12$ and $|S \cap A(D[X])| \leq 1$, there exists $z \in X$ such that exactly one of $\{xz, zx\} \subseteq A(D[X])$, say xz, is in S. Thus $\xi'(D - S) \leq \xi'(zx) \leq d^+_{D-S}(z) + d^+_{D-S}(x) - 1 \leq 2k_1 + 2k_2 - 3 - 1 < 3k_1 + 3k_2 - 10 \leq \omega_{D-S}(X)$, contradicting (1).

If $|S\cap(X,\overline{X})| = k_1+k_2-1$, then $\omega_{D-S}(X) \geq 3k_1+3k_2-11$ and $S\cap A(D[X]) = \emptyset$. We first give a claim.

Claim. *If $|S\cap(X,\overline{X})| \geq 6$ and $|X| = 4$, then there exist $x, y \in X$ such that $|S\cap(\{x,y\},\overline{X})| \geq 4$.*

By contradiction. Suppose that for any $u, v \in X$, $|S\cap(\{u,v\},\overline{X})| \leq 3$. If $|S\cap(\{u,v\},\overline{X})| \leq 2$ for any $u, v \in X$, then, by $|X| = 4$, $|S\cap(X,\overline{X})| \leq 4$, contradicting $|S\cap(X,\overline{X})| \geq 6$. If there exist $u, v \in X$ such that $|S\cap(\{u,v\},\overline{X})| = 3$, then there exists one of $\{u, v\}$, say u, such that $|S\cap(\{u\},\overline{X})| \geq 2$. By $|S\cap(X,\overline{X})| \geq 6$, $|S\cap(X\setminus\{u,v\},\overline{X})| \geq 3$. Note that $|X\setminus\{u,v\}| = 2$. Thus there exists one of $X\setminus\{u,v\}$, say z, such that $|S\cap(\{z\},\overline{X})| \geq 2$ and so $|S\cap(\{u,z\},\overline{X})| \geq 4$, a contradiction. The claim holds.

Note that $|S\cap(X,\overline{X})| = k_1+k_2-1 \geq 6$ and $|X| = 4$. By Claim, there exist $x, y \in X$ such that $|S\cap(\{x,y\},\overline{X})| \geq 4$. If $xy \in A(D[X])$ or $yx \in A(D[X])$, then $\xi'(D-S) \leq d^+_{D-S}(x)+d^+_{D-S}(y)-1 \leq 2k_1+2k_2-4-1 < 3k_1+3k_2-11 \leq \omega_{D-S}(X)$, contradicting (1). Otherwise, by $10 \leq |A(D[X])| \leq 12$, every arc in $D[X]$ is symmetric. If $|S\cap(\{x,y\},\overline{X})| \geq 5$, then there exists one of $\{x, y\}$, say x, such that $|S\cap(\{x\},\overline{X})| \geq 3$. For any $z \in X$ with $z \neq y$, $xz \in \mathrm{Sym}(D[X])$. Thus $\xi'(D-S) \leq \xi'(xz) \leq d^+_{D-S}(x)+d^+_{D-S}(z)-2 \leq 2k_1+2k_2-3-2 < 3k_1+3k_2-11 \leq \omega_{D-S}(X)$, contradicting (1). Otherwise, $|S\cap(\{x,y\},\overline{X})| = 4$. There exists one of $\{x, y\}$, say x, such that $|S\cap(\{x\},\overline{X})| \geq 2$. By $|S\cap(X,\overline{X})| \geq 6$, $|S\cap(X\setminus\{x,y\},\overline{X})| \geq 2$. There exists $w \in X\setminus\{x,y\}$ such that $|S\cap(\{w\},\overline{X})| \geq 1$. Since $xw \in \mathrm{Sym}(D[X])$, $\xi'(D-S) \leq \xi'(xw) \leq d^+_{D-S}(x)+d^+_{D-S}(w)-2 \leq 2k_1+2k_2-3-2 < 3k_1+3k_2-11 \leq \omega_{D-S}(X)$, contradicting (1).

Case 1.3.2. $\omega_D(X) = 4k_1+4k_2-9$. If $|S\cap(X,\overline{X})| \leq k_1+k_2-2$, then, by $k_1+k_2 \geq 6$, $\omega_{D-S}(X) \geq 3k_1+3k_2-7 > 2k_1+2k_2-2 = \xi'(D) \geq \xi'(D-S)$, contradicting (1).

If $|S\cap(X,\overline{X})| = k_1+k_2-1$, then $\omega_{D-S}(X) = 3k_1+3k_2-8$ and $S\cap A(D[X]) = \emptyset$. By $|S\cap(X,\overline{X})| = k_1+k_2-1 \geq 5$ and $|X| = 4$, there exist $x \in X$ such that $|S\cap(\{x\},\overline{X})| \geq 2$. Note that $|A(D[X])| = 9$ and $S\cap A(D[X]) = \emptyset$. Thus there exists $y \in X$ such that $xy \in A(D[X])$ or $yx \in A(D[X])$. Thus $\xi'(D-S) \leq d^+_{D-S}(x)+d^+_{D-S}(y)-1 \leq d^+_D(x)+d^+_D(y)-2-1 = 2k_1+2k_2-3 < 3k_1+3k_2-8 = \omega_{D-S}(X)$, contradicting (1).

Case 1.4. $a = 3$. Note that D is (k_1+k_2)-regular and $D[X]$ has at most 6 arcs. Thus $\omega_D(X) \geq 3k_1+3k_2-6$. By (2), $3k_1+3k_2-6 \leq \omega_D(X) \leq 3k_1+3k_2-3$.

Case 1.4.1. $\omega_D(X) = 3k_1+3k_2-6$. Now $D[X] \cong \overleftrightarrow{K}_3$. If $|S\cap(X,\overline{X})| \leq k_1+k_2-5$, then $\omega_{D-S}(X) = \omega_D(X)-|S\cap(X,\overline{X})| \geq 2k_1+2k_2-1 > 2k_1+2k_2-2 = \xi'(D) \geq \xi'(D-S)$, contradicting (1).

If $k_1+k_2-4 \leq |S\cap(X,\overline{X})| \leq k_1+k_2-3$, then $\omega_{D-S}(X) \geq 2k_1+2k_2-3$ and $|S\cap A(D[X])| \leq 3$. By $|S\cap(X,\overline{X})| \geq k_1+k_2-4 \geq 2$ and $|X| = 3$, there exist

$x, y \in X$ such that $|S \cap (\{x,y\}, \overline{X})| \geq 2$. Since $D[X] \cong \overleftrightarrow{K}_3$, $xy \in \mathrm{Sym}(D[X])$. If $xy \in \mathrm{Sym}(D[X]-S)$, then $\xi'(D-S) \leq \xi'(xy) \leq d^+_{D-S}(x) + d^+_{D-S}(y) - 2 \leq d^+_D(x) + d^+_D(y) - 2 - 2 = 2k_1 + 2k_2 - 4 < 2k_1 + 2k_2 - 3 \leq \omega_{D-S}(X)$, contradicting (1). If exactly one of $\{xy, yx\}$, say yx, is in S, then $\xi'(D-S) \leq \xi'(xy) \leq d^+_{D-S}(x) + d^+_{D-S}(y) - 1 \leq 2k_1 + 2k_2 - 3 - 1 < 2k_1 + 2k_2 - 3 \leq \omega_{D-S}(X)$, contradicting (1). Next consider $xy \in S$ and $yx \in S$. By $|S \cap (\{x,y\}, \overline{X})| \geq 2$, there exists one of $\{x,y\}$, say x, such that $|S \cap (\{x\}, \overline{X})| \geq 1$. Let $z = X \setminus \{x,y\}$. If $xz \in \mathrm{Sym}(D[X]-S)$, then $\xi'(D-S) \leq \xi'(xz) \leq d^+_{D-S}(x) + d^+_{D-S}(z) - 2 \leq 2k_1 + 2k_2 - 2 - 2 < 2k_1 + 2k_2 - 3 \leq \omega_{D-S}(X)$, contradicting (1). Otherwise, by $|S \cap A(D[X])| \leq 3$, exactly one of $\{xz, zx\}$, say zx, is in S. Thus $\xi'(D-S) \leq \xi'(xz) \leq d^+_{D-S}(x) + d^+_{D-S}(z) - 1 \leq 2k_1 + 2k_2 - 3 - 1 < 2k_1 + 2k_2 - 3 \leq \omega_{D-S}(X)$, contradicting (1).

If $|S \cap (X, \overline{X})| = k_1 + k_2 - 2$, then $\omega_{D-S}(X) = 2k_1 + 2k_2 - 4$ and $|S \cap A(D[X])| \leq 1$. Note that $|S \cap (X, \overline{X})| = k_1 + k_2 - 2 \geq 4$ and $|X| = 3$. Similar to the proof of Claim in Case 1.3.1, we can prove that there exist $x, y \in X$ such that $|S \cap (\{x,y\}, \overline{X})| \geq 3$. Since $D[X] \cong \overleftrightarrow{K}_3$, $xy \in \mathrm{Sym}(D[X])$. If $xy \in \mathrm{Sym}(D[X]-S)$, then $\xi'(D-S) \leq \xi'(xy) \leq d^+_{D-S}(x) + d^+_{D-S}(y) - 2 \leq 2k_1 + 2k_2 - 3 - 2 < 2k_1 + 2k_2 - 4 = \omega_{D-S}(X)$, contradicting (1). Otherwise, by $|S \cap A(D[X])| \leq 1$, exactly one of $\{xy, yx\}$, say yx, is in S. Thus $\xi'(D-S) \leq \xi'(xy) \leq d^+_{D-S}(x) + d^+_{D-S}(y) - 1 \leq 2k_1 + 2k_2 - 4 - 1 < 2k_1 + 2k_2 - 4 = \omega_{D-S}(X)$, contradicting (1).

If $|S \cap (X, \overline{X})| = k_1 + k_2 - 1$, then $\omega_{D-S}(X) = 2k_1 + 2k_2 - 5$ and $S \cap A(D[X]) = \emptyset$. Note that $|S \cap (X, \overline{X})| = k_1 + k_2 - 1 \geq 5$ and $|X| = 3$. Similar to the proof of Claim in Case 1.3.1, we can prove that there exist $x, y \in X$ such that $|S \cap (\{x,y\}, \overline{X})| \geq 4$. Since $D[X] \cong \overleftrightarrow{K}_3$ and $S \cap A(D[X]) = \emptyset$, $xy \in \mathrm{Sym}(D[X])$. Thus $\xi'(D-S) \leq \xi'(xy) \leq d^+_{D-S}(x) + d^+_{D-S}(y) - 2 \leq 2k_1 + 2k_2 - 4 - 2 < 2k_1 + 2k_2 - 5 = \omega_{D-S}(X)$, contradicting (1).

Case 1.4.2. $\omega_D(X) = 3k_1 + 3k_2 - 5$. Now $D[X] \cong D'$, where D' is a digraph obtained from $\overleftrightarrow{K}_3$ by deleting an arc.

If $|S \cap (X, \overline{X})| \leq k_1 + k_2 - 4$, then $\omega_{D-S}(X) \geq 2k_1 + 2k_2 - 1 > 2k_1 + 2k_2 - 2 = \xi'(D) \geq \xi'(D-S)$, contradicting (1).

If $k_1 + k_2 - 3 \leq |S \cap (X, \overline{X})| \leq k_1 + k_2 - 2$, then $\omega_{D-S}(X) \geq 2k_1 + 2k_2 - 3$ and $|S \cap A(D[X])| \leq 2$. By $|S \cap (X, \overline{X})| \geq k_1 + k_2 - 3 \geq 3$ and $|X| = 3$, there exist $x, y \in X$ such that $|S \cap (\{x,y\}, \overline{X})| \geq 2$. Since $D[X] \cong D'$, $xy \in A(D[X])$ or $yx \in A(D[X])$. Without loss of generality, suppose that $xy \in A(D[X])$. Consider the following two possibilities.

Assume that $xy \notin S$. Consider $yx \in A(D[X])$. If $yx \notin S$, then $xy \in \mathrm{Sym}(D[X]-S)$. Thus $\xi'(D-S) \leq \xi'(xy) \leq d^+_{D-S}(x) + d^+_{D-S}(y) - 2 \leq d^+_D(x) + d^+_D(y) - 2 - 2 = 2k_1 + 2k_2 - 4 < 2k_1 + 2k_2 - 3 \leq \omega_{D-S}(X)$, contradicting (1). If $yx \in S$, then $\xi'(D-S) \leq \xi'(xy) \leq d^+_{D-S}(x) + d^+_{D-S}(y) - 1 \leq 2k_1 + 2k_2 - 3 - 1 <$

$2k_1 + 2k_2 - 3 \leq \omega_{D-S}(X)$, contradicting (1). Consider $yx \notin A(D[X])$. Then, by $D[X] \cong D'$, every arc in $D[X] - xy$ is symmetric. If $|S \cap (\{x, y\}, \overline{X})| \geq 3$, then $\xi'(D-S) \leq \xi'(xy) \leq d^+_{D-S}(x) + d^+_{D-S}(y) - 1 \leq 2k_1 + 2k_2 - 3 - 1 < 2k_1 + 2k_2 - 3 \leq \omega_{D-S}(X)$, contradicting (1). Otherwise, $|S \cap (\{x, y\}, \overline{X})| = 2$. There exists one of $\{x, y\}$, say x, such that $|S \cap (\{x\}, \overline{X})| \geq 1$. Let $z = X \setminus \{x, y\}$. By $|S \cap (X, \overline{X})| \geq 3$, $|S \cap (\{z\}, \overline{X})| \geq 1$. If $xz \in \mathrm{Sym}(D[X] - S)$, then $\xi'(D-S) \leq \xi'(xz) \leq d^+_{D-S}(x) + d^+_{D-S}(z) - 2 \leq 2k_1 + 2k_2 - 2 - 2 < 2k_1 + 2k_2 - 3 \leq \omega_{D-S}(X)$, contradicting (1). If exactly one of $\{xz, zx\}$, say zx, is in S, then $\xi'(D-S) \leq \xi'(xz) \leq d^+_{D-S}(x) + d^+_{D-S}(z) - 1 \leq 2k_1 + 2k_2 - 3 - 1 < 2k_1 + 2k_2 - 3 \leq \omega_{D-S}(X)$, contradicting (1). If $xz \in S$ and $zx \in S$, then, by $|S \cap A(D[X])| \leq 2$, $yz \in \mathrm{Sym}(D[X] - S)$. Thus $\xi'(D-S) \leq \xi'(yz) \leq d^+_{D-S}(y) + d^+_{D-S}(z) - 2 \leq 2k_1 + 2k_2 - 2 - 2 < 2k_1 + 2k_2 - 3 \leq \omega_{D-S}(X)$, contradicting (1).

Assume that $xy \in S$. Consider $yx \in A(D[X])$. If $yx \notin S$, then $\xi'(D-S) \leq \xi'(yx) \leq d^+_{D-S}(y) + d^+_{D-S}(x) - 1 \leq 2k_1 + 2k_2 - 3 - 1 < 2k_1 + 2k_2 - 3 \leq \omega_{D-S}(X)$, contradicting (1). If $yx \in S$, then, by $|S \cap A(D[X])| \leq 2$, $S \cap A(D[X]) = \{xy, yx\}$. Note that $D[X] \cong D'$. Without loss of generality, suppose that $zy \notin A(D[X])$. Then $xz \in \mathrm{Sym}(D[X] - S)$. If $|S \cap (\{x\}, \overline{X})| \geq 1$, then $\xi'(D-S) \leq \xi'(xz) \leq d^+_{D-S}(x) + d^+_{D-S}(z) - 2 \leq 2k_1 + 2k_2 - 2 - 2 < 2k_1 + 2k_2 - 3 \leq \omega_{D-S}(X)$, contradicting (1). Otherwise, $|S \cap (\{y\}, \overline{X})| \geq 2$. Thus $\xi'(D-S) \leq \xi'(yz) \leq d^+_{D-S}(y) + d^+_{D-S}(z) - 1 \leq 2k_1 + 2k_2 - 3 - 1 < 2k_1 + 2k_2 - 3 \leq \omega_{D-S}(X)$, contradicting (1). Consider $yx \notin A(D[X])$. Then, by $D[X] \cong D'$, every arc in $D[X] - xy$ is symmetric. Suppose that $|S \cap (\{x\}, \overline{X})| \geq 2$. By $|S \cap A(D[X])| \leq 2$ and $xy \in S$, at least one of $\{xz, zx\}$, say xz, is not in S. Thus $\xi'(D-S) \leq \xi'(xz) \leq d^+_{D-S}(x) + d^+_{D-S}(z) - 1 \leq 2k_1 + 2k_2 - 3 - 1 < 2k_1 + 2k_2 - 3 \leq \omega_{D-S}(X)$, contradicting (1). Suppose that $|S \cap (\{x\}, \overline{X})| \leq 1$. Then, by $|S \cap (X, \overline{X})| \geq 3$, $|S \cap (\{y, z\}, \overline{X})| \geq 2$. If $yz \in \mathrm{Sym}(D[X] - S)$, then $\xi'(D-S) \leq \xi'(yz) \leq d^+_{D-S}(y) + d^+_{D-S}(z) - 2 \leq 2k_1 + 2k_2 - 2 - 2 < 2k_1 + 2k_2 - 3 \leq \omega_{D-S}(X)$, contradicting (1). Otherwise, by $|S \cap A(D[X])| \leq 2$ and $xy \in S$, exactly one of $\{yz, zy\}$, say zy, is in S. Thus $\xi'(D-S) \leq \xi'(yz) \leq d^+_{D-S}(y) + d^+_{D-S}(z) - 1 \leq 2k_1 + 2k_2 - 3 - 1 < 2k_1 + 2k_2 - 3 \leq \omega_{D-S}(X)$, contradicting (1).

If $|S \cap (X, \overline{X})| = k_1 + k_2 - 1$, then $\omega_{D-S}(X) = 2k_1 + 2k_2 - 4$ and $S \cap A(D[X]) = \emptyset$. Note that $|S \cap (X, \overline{X})| = k_1 + k_2 - 1 \geq 5$ and $|X| = 3$. Similar to the proof of Claim in Case 1.3.1, we can prove that there exist $x, y \in X$ such that $|S \cap (\{x, y\}, \overline{X})| \geq 4$. Since $D[X] \cong D'$, $xy \in A(D[X])$ or $yx \in A(D[X])$. Thus $\xi'(D-S) \leq d^+_{D-S}(x) + d^+_{D-S}(y) - 1 \leq 2k_1 + 2k_2 - 4 - 1 < 2k_1 + 2k_2 - 4 = \omega_{D-S}(X)$, contradicting (1).

Case 1.4.3. $\omega_D(X) = 3k_1 + 3k_2 - 4$. If $|S \cap (X, \overline{X})| \leq k_1 + k_2 - 3$, then $\omega_{D-S}(X) \geq 2k_1 + 2k_2 - 1 > 2k_1 + 2k_2 - 2 = \xi'(D) \geq \xi'(D-S)$, contradicting (1).

If $k_1 + k_2 - 2 \leq |S \cap (X, \overline{X})| \leq k_1 + k_2 - 1$, then $\omega_{D-S}(X) \geq 2k_1 + 2k_2 - 3$ and $|S \cap A(D[X])| \leq 1$. Note that $|X| = 3$ and $|A(D[X])| = 4$. Thus there exists

$xy \in \mathrm{Sym}(D[X])$. Assume that $|S \cap (\{x,y\},\overline{X})| \geq 2$. If $xy \in \mathrm{Sym}(D[X]-S)$, then $\xi'(D-S) \leq \xi'(xy) \leq d^+_{D-S}(x) + d^+_{D-S}(y) - 2 \leq d^+_D(x) + d^+_D(y) - 2 - 2 = 2k_1 + 2k_2 - 4 < 2k_1 + 2k_2 - 3 \leq \omega_{D-S}(X)$, contradicting (1). Otherwise, by $|S \cap A(D[X])| \leq 1$, exactly one of $\{xy, yx\}$, say yx, is in S. Thus $\xi'(D-S) \leq \xi'(xy) \leq d^+_{D-S}(x) + d^+_{D-S}(y) - 1 \leq 2k_1 + 2k_2 - 3 - 1 < 2k_1 + 2k_2 - 3 \leq \omega_{D-S}(X)$, contradicting (1). Assume that $|S \cap (\{x,y\},\overline{X})| \leq 1$. Let $z = X \setminus \{x,y\}$. By $|S \cap (X,\overline{X})| \geq k_1 + k_2 - 2 \geq 4$, $|S \cap (\{z\},\overline{X})| \geq 3$. Note that $|A(D[X])| = 4$ and $|S \cap A(D[X])| \leq 1$. Thus at least one of $\{zx, xz, zy, yz\}$, say zx, is in $A(D[X]-S)$. This means that $\xi'(D-S) \leq \xi'(zx) \leq d^+_{D-S}(z) + d^+_{D-S}(x) - 1 \leq 2k_1 + 2k_2 - 3 - 1 < 2k_1 + 2k_2 - 3 \leq \omega_{D-S}(X)$, contradicting (1).

Case 1.4.4. $\omega_D(X) = 3k_1 + 3k_2 - 3$. If $|S \cap (X,\overline{X})| \leq k_1 + k_2 - 2$, then $\omega_{D-S}(X) \geq 2k_1 + 2k_2 - 1 > 2k_1 + 2k_2 - 2 = \xi'(D) \geq \xi'(D-S)$, contradicting (1).

If $|S \cap (X,\overline{X})| = k_1 + k_2 - 1$, then $\omega_{D-S}(X) = 2k_1 + 2k_2 - 2$ and $S \cap A(D[X]) = \emptyset$. By $|S \cap (X,\overline{X})| = k_1 + k_2 - 1 \geq 5$ and $|X| = 3$, there exist $x \in X$ such that $|S \cap (\{x\},\overline{X})| \geq 2$. Let $\{y,z\} = X \setminus \{x\}$. Note that $|A(D[X])| = 3$. Thus at least one of $\{xy, yx, xz, zx\}$, say xy, is in $A(D[X])$. This means that $\xi'(D-S) \leq \xi'(xy) \leq d^+_{D-S}(x) + d^+_{D-S}(y) - 1 \leq d^+_D(x) + d^+_D(y) - 2 - 1 = 2k_1 + 2k_2 - 3 < 2k_1 + 2k_2 - 2 = \omega_{D-S}(X)$, contradicting (1).

Case 2. $\mathrm{Sym}(D) = \emptyset$. Note that D is $(k_1 + k_2)$-regular. Thus $\xi'(D) = 2k_1 + 2k_2 - 1$. Similar to the first paragraph of the proof of Case 1, we can deduce that $3 \leq a \leq 7$. Moreover, similar to (2), we have

$$\omega_D(X) \leq 3k_1 + 3k_2 - 2. \tag{7}$$

Note that $\mathrm{Sym}(D) = \emptyset$. When $a = 4, 5, 6, 7$, $\omega_D(X) \geq 4k_1 + 4k_2 - 6, 5k_1 + 5k_2 - 10, 6k_1 + 6k_2 - 15, 7k_1 + 7k_2 - 21$, respectively. In all cases, by $k_1 + k_2 \geq 6$, $\omega_D(X) > 3k_1 + 3k_2 - 2$, contradicting (7).

If $a = 3$, then $\omega_D(X) \geq 3k_1 + 3k_2 - 3$. By (7), $3k_1 + 3k_2 - 3 \leq \omega_D(X) \leq 3k_1 + 3k_2 - 2$. If $|S \cap (X,\overline{X})| \leq k_1 + k_2 - 3$, then $\omega_{D-S}(X) = \omega_D(X) - |S \cap (X,\overline{X})| \geq 3k_1 + 3k_2 - 3 - (k_1 + k_2 - 3) = 2k_1 + 2k_2 > 2k_1 + 2k_2 - 1 = \xi'(D) \geq \xi'(D-S)$, contradicting (1). If $k_1 + k_2 - 2 \leq |S \cap (X,\overline{X})| \leq k_1 + k_2 - 1$, then $\omega_{D-S}(X) \geq 2k_1 + 2k_2 - 2$. By $|S \cap (X,\overline{X})| \geq k_1 + k_2 - 2 \geq 4$ and $|X| = 3$, there exists $x \in X$ such that $|S \cap (\{x\},\overline{X})| \geq 2$. Combining $|S| \leq k_1 + k_2 - 1$ with $|S \cap (X,\overline{X})| \geq k_1 + k_2 - 2$, we have $|S \cap (\overline{X}, X)| \leq 1$. Since D is $(k_1 + k_2)$-regular and $k_1 + k_2 \geq 6$, there exists an arc $ux \in (\overline{X}, X)$ with $ux \notin S$. Thus $\xi'(D-S) \leq \xi'(ux) \leq d^+_{D-S}(u) + d^+_{D-S}(x) - 1 \leq d^+_D(u) + d^+_D(x) - 2 - 1 = 2k_1 + 2k_2 - 3 < 2k_1 + 2k_2 - 2 \leq \omega_{D-S}(X)$, contradicting (1).

The case that $X \subset V(D_2^{x_0})$ with $x_0 \in V(D_1)$ can be proved similarly. Lemma 8 holds. ∎

Lemma 9. *Let $D = D_1 \times D_2$, where D_i is a strongly connected k_i-regular digraph with $k_i = \lambda_i \geq 3$ for $i = 1, 2$. Then $\omega_{D-S}(X) > \xi'(D-S)$ holds for any*

$S \subseteq A(D)$ and any $X \subseteq V(D)$ such that $|S| \le k_1 + k_2 - 1$ and $X = X_0 \cup X_1$, where $X_0 \cap X_1 = \emptyset$, $|X_0| \ge 1$, $|X_1| \ge 1$, $|X_0 \cup X_1| \ge 3$ and $X_0 \subset V(D_1^{y_0})$ (respectively, $X_0 \subset V(D_2^{x_0})$) and $X_1 \subset V(D_1^{y_1})$ (respectively, $X_1 \subset V(D_2^{x_1})$) with $\{y_0, y_1\} \subseteq V(D_2)$ (respectively, $\{x_0, x_1\} \subseteq V(D_1)$).

Proof. Suppose, to the contrary, that there exists a subset S of $A(D)$ and a subset X of $V(D)$ satisfying the conditions of the lemma such that

$$\omega_{D-S}(X) = \omega_{D-S}(X_0 \cup X_1) \le \xi'(D - S). \tag{8}$$

Case 1. $\mathrm{Sym}(D) \neq \emptyset$. Note that D is $(k_1 + k_2)$-regular. Thus $\xi'(D) = 2k_1 + 2k_2 - 2$. By $|S| \le k_1 + k_2 - 1$, Lemma 6 yields $\xi'(D - S) \le \xi'(D) = 2k_1 + 2k_2 - 2$. Thus $2k_1 + 2k_2 - 2 \ge \xi'(D - S) \ge \omega_{D-S}(X_0 \cup X_1) \ge \omega_D(X_0 \cup X_1) - |S| \ge \omega_D(X_0 \cup X_1) - (k_1 + k_2 - 1)$ and so

$$\omega_D(X_0 \cup X_1) \le 3k_1 + 3k_2 - 3. \tag{9}$$

Let $|X_0| = a$ and $|X_1| = b$. Then $a, b \ge 1$ and $a + b \ge 3$. Assume that $X_0 \subset V(D_1^{y_0})$ and $X_1 \subset V(D_1^{y_1})$ with $\{y_0, y_1\} \subseteq V(D_2)$. Then $\omega_D(X_0 \cup X_1) \ge \omega_{D_1^{y_0}}(X_0) + \left|\left(X_0, \overline{V(D_1^{y_0})}\right)\right| - a + \omega_{D_1^{y_1}}(X_1) + \left|\left(X_1, \overline{V(D_1^{y_1})}\right)\right| - b = \omega_{D_1^{y_0}}(X_0) + \omega_{D_1^{y_1}}(X_1) + ak_2 + bk_2 - a - b$. Combining this with (9), we have

$$3k_1 \ge \omega_{D_1^{y_0}}(X_0) + \omega_{D_1^{y_1}}(X_1) + (a + b - 3)k_2 - a - b + 3. \tag{10}$$

Note that $a + b \ge 3$ and $k_2 \ge 3$. Thus (10) yields

$$3k_1 \ge \omega_{D_1^{y_0}}(X_0) + \omega_{D_1^{y_1}}(X_1) + 2(a + b) - 6. \tag{11}$$

Note that $\omega_{D_1^{y_0}}(X_0) = \sum_{z \in X_0} |(\{z\}, V(D_1^{y_0}) \backslash X_0)| \ge a(k_1 - a + 1)$ and $\omega_{D_1^{y_1}}(X_1) \ge b(k_1 - b + 1)$. Combining this with (11), we have

$$(a + b - 3)k_1 \le a^2 + b^2 - 3(a + b) + 6. \tag{12}$$

Note that $\omega_{D_1^{y_0}}(X_0) \ge \lambda_1 = k_1$ and $\omega_{D_1^{y_1}}(X_1) \ge \lambda_1 = k_1$. By (11), $k_1 \ge 2(a + b) - 6$. Combining this with (12), we have

$$(a + b)^2 - 9(a + b) + 12 + 2ab \le 0, \tag{13}$$

that is, $(a + b - 2)(a + b - 7) + 2ab - 2 \le 0$. Since $a, b \ge 1$ and $a + b \ge 3$, $2ab - 2 > 0$. Thus $(a + b - 2)(a + b - 7) < 0$, which yields $2 < a + b < 7$ and so $3 \le a + b \le 6$.

If $a + b = 6$, then, by (13), we have $ab \le 3$, a contradiction. If $a + b = 5$, then, by (13), we have $ab \le 4$. Recall that $a, b \ge 1$. Thus $\{a, b\} = \{1, 4\}$. By the definition of Cartesian product, $D[X_0 \cup X_1]$ has at most 14 arcs. By $k_1 + k_2 \geq 6$, $\omega_D(X_0 \cup X_1) \geq 5k_1 + 5k_2 - 14 > 3k_1 + 3k_2 - 3$, contradicting (9).

If $a + b = 4$, then $\{a, b\}$ is equal to $\{1, 3\}$ or $\{2, 2\}$. By the definition of Cartesian product, $D[X_0 \cup X_1]$ has at most 8 arcs. By $k_1 + k_2 \geq 6$, $\omega_D(X_0 \cup X_1) \geq 4k_1 + 4k_2 - 8 > 3k_1 + 3k_2 - 3$, contradicting (9).

If $a + b = 3$, then $\{a, b\}$ is equal to $\{1, 2\}$. By the definition of Cartesian product, $D[X_0 \cup X_1]$ has at most 4 arcs and so $\omega_D(X_0 \cup X_1) \geq 3k_1 + 3k_2 - 4$. By (9), $3k_1 + 3k_2 - 4 \leq \omega_D(X_0 \cup X_1) = \omega_D(X) \leq 3k_1 + 3k_2 - 3$. Its proof is the same as the proof of Cases 1.4.3 and 1.4.4 of Lemma 8.

Case 2. $\mathrm{Sym}(D) = \emptyset$. Note that D is $(k_1 + k_2)$-regular. Thus $\xi'(D) = 2k_1 + 2k_2 - 1$. Similar to the first paragraph of the proof of Case 1, we can deduce that $3 \leq a + b \leq 7$. Moreover, similar to (9) and (13), we have

$$\omega_D(X_0 \cup X_1) \leq 3k_1 + 3k_2 - 2, \tag{14}$$

and

$$(a + b)^2 - 10(a + b) + 14 + 2ab \leq 0, \tag{15}$$

respectively.

If $a+b = 7$, then, by (15), we have $2ab \leq 7$, a contradiction. If $a+b = 6$, then, by (15), we have $ab \leq 5$. Recall that $a, b \geq 1$. Thus $\{a, b\} = \{1, 5\}$. Note that $\mathrm{Sym}(D) = \emptyset$. By the definition of Cartesian product, $D[X_0 \cup X_1]$ has at most 11 arcs. By $k_1 + k_2 \geq 6$, $\omega_D(X_0 \cup X_1) \geq 6k_1 + 6k_2 - 11 > 3k_1 + 3k_2 - 2$, contradicting (14). If $a + b = 5$, then, by (15), we have $2ab \leq 11$. Recall that $a, b \geq 1$. Thus $\{a, b\} = \{1, 4\}$. By the definition of Cartesian product, $D[X_0 \cup X_1]$ has at most 7 arcs. Thus $\omega_D(X_0 \cup X_1) \geq 5k_1 + 5k_2 - 7 > 3k_1 + 3k_2 - 2$, contradicting (14).

If $a + b = 4$, then $\{a, b\}$ is equal to $\{1, 3\}$ or $\{2, 2\}$. By the definition of Cartesian product, $D[X_0 \cup X_1]$ has at most 4 arcs. By $k_1 + k_2 \geq 6$, $\omega_D(X_0 \cup X_1) \geq 4k_1 + 4k_2 - 4 > 3k_1 + 3k_2 - 2$, contradicting (14).

If $a + b = 3$, then $\{a, b\}$ is equal to $\{1, 2\}$. By the definition of Cartesian product, $D[X_0 \cup X_1]$ has at most 2 arcs and so $\omega_D(X_0 \cup X_1) \geq 3k_1 + 3k_2 - 2$. By (14), $\omega_D(X_0 \cup X_1) = \omega_D(X) = 3k_1 + 3k_2 - 2$. If $|S \cap (X, \overline{X})| \leq k_1 + k_2 - 2$, then $\omega_{D-S}(X) \geq 2k_1 + 2k_2 > 2k_1 + 2k_2 - 1 = \xi'(D) \geq \xi'(D - S)$, contradicting (8). If $|S \cap (X, \overline{X})| = k_1 + k_2 - 1$, then $\omega_{D-S}(X) = 2k_1 + 2k_2 - 1$ and $S \cap A(D[X]) = \emptyset$. By $|S \cap (X, \overline{X})| = k_1 + k_2 - 1 \geq 5$, there exists $x \in X$ such that $|S \cap (\{x\}, \overline{X})| \geq 1$. Note that $|X| = 3$, $|A(D[X])| = 2$ and $\mathrm{Sym}(D) = \emptyset$. Thus there exists $y \in X$ such that $xy \in A(D[X])$ or $yx \in A(D[X])$. This means that $\xi'(D - S) \leq d^+_{D-S}(x) + d^+_{D-S}(y) - 1 \leq d^+_D(x) + d^+_D(y) - 1 - 1 = 2k_1 + 2k_2 - 2 < 2k_1 + 2k_2 - 1 = \omega_{D-S}(X)$, contradicting (8).

The case that $X_0 \subset V(D_2^{x_0})$ and $X_1 \subset V(D_2^{x_1})$ with $\{x_0, x_1\} \subseteq V(D_1)$ can be proved similarly. Lemma 9 holds. ■

By considering the out-degrees of vertices in X, we proved $\omega_{D-S}(X) > \xi'(D-S)$ in Lemmas 8 and 9. Note that $\partial^+_{D-S}(X) = \partial^-_{D-S}(\overline{X})$. Thus $\omega_{D-S}(X) = \partial^-_{D-S}(\overline{X})$. In the following, by considering the in-degrees of vertices in $\overline{X}$ and using the similar approaches to the two employed to prove Lemmas 8 and 9, we have:

Lemma 10. *Let $D = D_1 \times D_2$, where D_i is a strongly connected k_i-regular digraph with $k_i = \lambda_i \geq 3$ for $i = 1, 2$. Then $\omega_{D-S}(X) > \xi'(D-S)$ holds for any $S \subseteq A(D)$ and any $X \subseteq V(D)$ such that $|S| \leq k_1 + k_2 - 1$, $|\overline{X}| \geq 3$ and $\overline{X} \subset V(D_1^{y_0})$ with $y_0 \in V(D_2)$ or $\overline{X} \subset V(D_2^{x_0})$ with $x_0 \in V(D_1)$.*

Lemma 11. *Let $D = D_1 \times D_2$, where D_i is a strongly connected k_i-regular digraph with $k_i = \lambda_i \geq 3$ for $i = 1, 2$. Then $\omega_{D-S}(X) > \xi'(D-S)$ holds for any $S \subseteq A(D)$ and any $X \subseteq V(D)$ such that $|S| \leq k_1 + k_2 - 1$, $\overline{X} = X_0 \cup X_1$, where $X_0 \cap X_1 = \emptyset$, $|X_0| \geq 1$, $|X_1| \geq 1$, $|X_0 \cup X_1| \geq 3$ and $X_0 \subset V(D_1^{y_0})$ (respectively, $X_0 \subset V(D_2^{x_0})$) and $X_1 \subset V(D_1^{y_1})$ (respectively, $X_1 \subset V(D_2^{x_1})$) with $\{y_0, y_1\} \subseteq V(D_2)$ (respectively, $\{x_0, x_1\} \subseteq V(D_1)$).*

Lemma 12 [15]**.** *Let D_i be a nontrivial strongly connected digraph for $i = 1, 2$. Then $\lambda(D_1 \times D_2) = \min\left\{\delta_1^- + \delta_2^-, \delta_1^+ + \delta_2^+, \lambda_1\nu_2, \lambda_2\nu_1\right\}$.*

Theorem 13. *Let D_i be a strongly connected k_i-regular digraph with $k_i = \lambda_i \geq 3$ for $i = 1, 2$. Then* $\min\{k_1 + k_2 - 1, \nu_1 k_2 - 2k_1 - 2k_2, \nu_2 k_1 - 2k_1 - 2k_2\} \leq S_{\lambda'}(D_1 \times D_2) \leq k_1 + k_2 - 1$.

Proof. By $k_i \geq 3$ for $i = 1, 2$, Theorem 4 implies that $D_1 \times D_2$ is super-λ'. Note that $D_1 \times D_2$ is $(k_1 + k_2)$-regular. For any $x \in V(D_1 \times D_2)$, let S be the set of arcs with $S = \partial^+_{D_1 \times D_2}(x)$. Then $|S| = k_1 + k_2$. $D_1 \times D_2 - S$ is not super-λ' since $D_1 \times D_2 - S$ is not strongly connected. By the definition of $S_{\lambda'}(D_1 \times D_2)$, we have $S_{\lambda'}(D_1 \times D_2) \leq k_1 + k_2 - 1$.

Denote $m = \min\{k_1 + k_2 - 1, \nu_1 k_2 - 2k_1 - 2k_2, \nu_2 k_1 - 2k_1 - 2k_2\}$. Let $D = D_1 \times D_2$. To show that $S_{\lambda'}(D) \geq m$, it suffices to show that for any $S \subseteq A(D)$ with $|S| \leq m$, $D - S$ is still super-λ'. By Lemma 12, $\lambda(D) = \min\{\delta_1^- + \delta_2^-, \delta_1^+ + \delta_2^+, \lambda_1\nu_2, \lambda_2\nu_1\}$. Note that D_i is k_i-regular and $k_i = \lambda_i$ for $i = 1, 2$. Thus $\lambda_1\nu_2 = k_1\nu_2 \geq k_1(k_2 + 1) \geq k_1 + k_2$. Similarly, $\lambda_2\nu_1 \geq k_1 + k_2$. Hence $\lambda(D) = k_1 + k_2$. By $|S| \leq m \leq k_1 + k_2 - 1$, $D - S$ is strongly connected. Note that D is $(k_1 + k_2)$-regular and $|S| \leq k_1 + k_2 - 1$. Lemma 7 yields $\delta^+(D - S) \geq 3$ or $\delta^-(D - S) \geq 3$. Let X be any subset of $V(D - S)$ with $|X| \geq 3$ and $|\overline{X}| \geq 3$. Then, by Lemma 5, in order to show that $D - S$ is super-λ', it suffices to prove that $\omega_{D-S}(X) > \xi'(D - S)$ holds. By Lemma 6, $\xi'(D - S) \leq \xi'(D)$. Clearly, $\omega_{D-S}(X) \geq \omega_D(X) - |S| \geq \omega_D(X) - m$. If $\omega_D(X) > \xi'(D) + m$, then $\omega_{D-S}(X) > \xi'(D) \geq \xi'(D - S)$. In the following, we assume that

$$\omega_D(X) \leq \xi'(D) + m, \tag{16}$$

and prove that $\omega_{D-S}(X) > \xi'(D-S)$ holds in this case.

Denote $I_1 = \{x : x \in V(D_1)$ and $D_2^x - (X, \overline{X})$ is not strongly connected$\}$ and $I_2 = \{y : y \in V(D_2)$ and $D_1^y - (X, \overline{X})$ is not strongly connected$\}$. We give three claims.

Claim 1. *$|I_i| < \nu_i$ for $i = 1, 2$.*

By contradiction. Suppose, without loss of generality, that $|I_1| = \nu_1$. Then $D_2^x - (X, \overline{X})$ is not strongly connected for all $x \in V(D_1)$ and so $\omega_D(X) \geq \nu_1\lambda_2 = \nu_1 k_2$. Combining this with (16), we have

$$\nu_1 k_2 \leq \xi'(D) + m \leq 2k_1 + 2k_2 - 1 + \nu_1 k_2 - 2k_1 - 2k_2 = \nu_1 k_2 - 1,$$

a contradiction. Claim 1 holds.

Claim 2. *$|I_i| \geq 1$ for $i = 1, 2$.*

By contradiction. Suppose, without loss of generality, that $|I_1| = 0$. Then $D_2^x - (X, \overline{X})$ is strongly connected for all $x \in V(D_1)$. By Claim 1, there exists $y \in V(D_2)$ such that $D_1^y - (X, \overline{X})$ is strongly connected. Thus we have that $D - (X, \overline{X})$ is strongly connected, a contradiction. Claim 2 holds.

Claim 3. *$|I_1| \leq 2$ or $|I_2| \leq 2$.*

By contradiction. Suppose that $|I_1| \geq 3$ and $|I_2| \geq 3$. Then

$$\begin{aligned} \omega_D(X) &\geq 3\lambda_1 + 3\lambda_2 = 3k_1 + 3k_2 \\ &= 2k_1 + 2k_2 - 1 + k_1 + k_2 - 1 + 2 \geq \xi'(D) + m + 2, \end{aligned}$$

contradicting (16). Claim 3 holds.

By Claims 2 and 3, we assume, without loss of generality, that $1 \leq |I_2| \leq 2$. We consider the following two cases.

Case 1. $|I_2| = 1$. Let $I_2 = \{y_0\}$. Then $D_1^{y_0} - (X, \overline{X})$ is not strongly connected and $D_1^y - (X, \overline{X})$ is strongly connected for all $y \in V(D_2) \setminus \{y_0\}$. By Claim 1, there exists $x \in V(D_1)$ such that $D_2^x - (X, \overline{X})$ is strongly connected. Thus $D_2^x \cup \big(\bigcup_{y \in V(D_2)\setminus\{y_0\}} D_1^y\big) - (X, \overline{X})$ is strongly connected and so is contained in $D[X]$ or $D[\overline{X}]$. If $D_2^x \cup \big(\bigcup_{y \in V(D_2)\setminus\{y_0\}} D_1^y\big) - (X, \overline{X})$ is contained in $D[\overline{X}]$, then $X \subseteq V(D_1^{y_0})$. Since $D_1^{y_0} - (X, \overline{X})$ is not strongly connected, $X \subset V(D_1^{y_0})$. By Lemma 8, we have $\omega_{D-S}(X) > \xi'(D-S)$. The theorem holds in this case. If $D_2^x \cup \big(\bigcup_{y \in V(D_2)\setminus\{y_0\}} D_1^y\big) - (X, \overline{X})$ is contained in $D[X]$, then $\overline{X} \subseteq V(D_1^{y_0})$. Since $D_1^{y_0} - (X, \overline{X})$ is not strongly connected, $\overline{X} \subset V(D_1^{y_0})$. By Lemma 10, we have $\omega_{D-S}(X) > \xi'(D-S)$. The theorem holds in this case.

Case 2. $|I_2| = 2$. Let $I_2 = \{y_0, y_1\}$. Then $D_1^{y_0} - (X, \overline{X})$ and $D_1^{y_1} - (X, \overline{X})$ are not strongly connected, but $D_1^y - (X, \overline{X})$ is strongly connected for all $y \in V(D_2) \setminus$

$\{y_0, y_1\}$. By Claim 1, there exists $x \in V(D_1)$ such that $D_2^x - (X, \overline{X})$ is strongly connected. Thus $D_2^x \cup \left(\bigcup_{y\in V(D_2)\setminus\{y_0,y_1\}} D_1^y\right) - (X, \overline{X})$ is strongly connected and so is contained in $D[X]$ or $D[\overline{X}]$. If $D_2^x \cup \left(\bigcup_{y\in V(D_2)\setminus\{y_0,y_1\}} D_1^y\right) - (X, \overline{X})$ is contained in $D[\overline{X}]$, then $X \subseteq V(D_1^{y_0} \cup D_1^{y_1})$. Let $X = X_0 \cup X_1$ with $X_0 \cap X_1 = \emptyset$, $X_0 \subseteq V(D_1^{y_0})$ and $X_1 \subseteq V(D_1^{y_1})$. Then $|X_0 \cup X_1| = |X| \geq 3$. Since $D_1^{y_0} - (X, \overline{X})$ and $D_1^{y_1} - (X, \overline{X})$ are not strongly connected, $|X_0| \geq 1$, $|X_1| \geq 1$, $X_0 \subset V(D_1^{y_0})$ and $X_1 \subset V(D_1^{y_1})$. By Lemma 9, we have $\omega_{D-S}(X) > \xi'(D-S)$. The theorem holds in this case. If $D_2^x \cup \left(\bigcup_{y\in V(D_2)\setminus\{y_0,y_1\}} D_1^y\right) - (X, \overline{X})$ is contained in $D[X]$, then $\overline{X} \subseteq V(D_1^{y_0} \cup D_1^{y_1})$. Let $\overline{X} = X_0 \cup X_1$ with $X_0 \cap X_1 = \emptyset$, $X_0 \subseteq V(D_1^{y_0})$ and $X_1 \subseteq V(D_1^{y_1})$. Then $|X_0 \cup X_1| = |\overline{X}| \geq 3$. Since $D_1^{y_0} - (X, \overline{X})$ and $D_1^{y_1} - (X, \overline{X})$ are not strongly connected, $|X_0| \geq 1$, $|X_1| \geq 1$, $X_0 \subset V(D_1^{y_0})$ and $X_1 \subset V(D_1^{y_1})$. By Lemma 11, we have $\omega_{D-S}(X) > \xi'(D-S)$. The theorem holds in this case. ■

Remark 14. The lower and upper bounds on $S_{\lambda'}(D_1 \times D_2)$ in Theorem 13 are best possible. The reasons are as follows. Let $D_1 \cong \overleftrightarrow{K}_5$ and $D_2 \cong \overleftrightarrow{K}_8$. Then D_i is a strongly connected k_i-regular graph with $k_i = \lambda_i \geq 3$ for $i = 1, 2$. Clearly $\min\{k_1 + k_2 - 1, \nu_1 k_2 - 2k_1 - 2k_2, \nu_2 k_1 - 2k_1 - 2k_2\} = 10 = k_1 + k_2 - 1 = \nu_2 k_1 - 2k_1 - 2k_2$. By Theorem 13, $S_{\lambda'}(D_1 \times D_2) = k_1 + k_2 - 1 = \min\{k_1 + k_2 - 1, \nu_1 k_2 - 2k_1 - 2k_2, \nu_2 k_1 - 2k_1 - 2k_2\} = \nu_2 k_1 - 2k_1 - 2k_2$, which implies that the lower and upper bounds on $S_{\lambda'}(D_1 \times D_2)$ in Theorem 13 are attainable.

Corollary 15. *Let D_i be a strongly connected k_i-regular digraph with $k_i = \lambda_i \geq 3$ for $i = 1, 2$. Then $S_{\lambda'}(D_1 \times D_2) = k_1 + k_2 - 1$ if one of the following conditions holds:*

(a) *$k_1 \geq 5$ and $k_2 \geq 5$,*

(b) *D_1 and D_2 are not complete digraphs and $k_1, k_2 \geq 4$.*

Proof. By Theorem 13, $\min\{k_1 + k_2 - 1, \nu_1 k_2 - 2k_1 - 2k_2, \nu_2 k_1 - 2k_1 - 2k_2\} \leq S_{\lambda'}(D_1 \times D_2) \leq k_1 + k_2 - 1$. It suffices to show that $\nu_1 k_2 - 2k_1 - 2k_2 \geq k_1 + k_2 - 1$ and $\nu_2 k_1 - 2k_1 - 2k_2 \geq k_1 + k_2 - 1$.

(a) Note that $\nu_1 k_2 - 2k_1 - 2k_2 - (k_1 - k_2 - 1) = \nu_1 k_2 - 3k_1 - 3k_2 + 1 \geq (k_1 + 1)k_2 - 3k_1 - 3k_2 + 1 = k_1 k_2 - 3k_1 - 2k_2 + 1 = (k_1 - 2)(k_2 - 3) - 5 > 0$ because $k_1 \geq 5$ and $k_2 \geq 5$. Thus $\nu_1 k_2 - 2k_1 - 2k_2 > k_1 + k_2 - 1$. Similarly, $\nu_2 k_1 - 2k_1 - 2k_2 > k_1 + k_2 - 1$.

(b) Since D_1 is not a complete digraph, $\nu_1 \geq k_1 + 2$. Thus $\nu_1 k_2 - 2k_1 - 2k_2 - (k_1 - k_2 - 1) = \nu_1 k_2 - 3k_1 - 3k_2 + 1 \geq (k_1 + 2)k_2 - 3k_1 - 3k_2 + 1 = k_1 k_2 - 3k_1 - k_2 + 1 = (k_1 - 1)(k_2 - 3) - 2 > 0$ because $k_1 \geq 4$ and $k_2 \geq 4$. Thus $\nu_1 k_2 - 2k_1 - 2k_2 > k_1 + k_2 - 1$. Similarly, $\nu_2 k_1 - 2k_1 - 2k_2 > k_1 + k_2 - 1$. ■

In fact, Theorem 13 can be generalized to the Cartesian product of n strongly connected regular digraphs. We need the following lemma.

Lemma 16. *Let D_i be a strongly connected k_i-regular digraph with $k_i = \lambda_i \geq 3$ for $i = 1, 2, \ldots, n$. Then $D_1 \times D_2 \times \cdots \times D_n$ is super-λ' and $\lambda(D_1 \times D_2 \times \cdots \times D_n) = k_1 + k_2 + \cdots + k_n$.*

Proof. We first prove that $\lambda(D_1 \times D_2 \times \cdots \times D_n) = k_1 + k_2 + \cdots + k_n$ by induction on n. If $n = 2$, then, by Lemma 12, $\lambda(D_1 \times D_2) = \min\{\delta_1^- + \delta_2^-, \delta_1^+ + \delta_2^+, \lambda_1\nu_2, \lambda_2\nu_1\}$. Note that D_i is k_i-regular and $k_i = \lambda_i$ for $i = 1, 2$. Thus $\lambda_1\nu_2 = k_1\nu_2 \geq k_1(k_2+1) \geq k_1+k_2$. Similarly, $\lambda_2\nu_1 \geq k_1+k_2$. Hence $\lambda(D_1 \times D_2) = k_1+k_2$. Suppose that $n \geq 3$ and $\lambda(D_1 \times D_2 \times \cdots \times D_{n-1}) = k_1 + k_2 + \cdots + k_{n-1}$. Note $D_1 \times D_2$ is a strongly connected $(k_1 + k_2)$-regular digraph with $k_1 + k_2 = \lambda(D_1 \times D_2)$. Thus, by the induction hypothesis, $\lambda(D_1 \times D_2 \times \cdots \times D_n) = \lambda((D_1 \times D_2) \times D_3 \times \cdots \times D_n) = (k_1 + k_2) + k_3 + \cdots + k_n = k_1 + k_2 + \cdots + k_n$. Next we prove that $D_1 \times D_2 \times \cdots \times D_n$ is super-λ'. Note that $D_1 \times D_2 \times \cdots \times D_{n-1}$ is a strongly connected $(k_1 + k_2 + \cdots + k_{n-1})$-regular digraph with $k_1 + k_2 + \cdots + k_{n-1} = \lambda(D_1 \times D_2 \times \cdots \times D_{n-1})$. By $k_i \geq 3$, Theorem 4 implies that $D_1 \times D_2 \times \cdots \times D_n = (D_1 \times D_2 \times \cdots \times D_{n-1}) \times D_n$ is super-λ'. ■

Theorem 17. *Let D_i be a strongly connected k_i-regular digraph with $k_i = \lambda_i \geq 3$ for $i = 1, 2, \ldots, n$. Then $\min_{1\leq i\leq n}\big\{\sum_{j=1}^n k_j - 1, \nu_i\big(\sum_{j=1}^n k_j - k_i\big) - 2\sum_{j=1}^n k_j\big\} \leq S_{\lambda'}(D_1 \times D_2 \times \cdots \times D_n) \leq \sum_{j=1}^n k_j - 1$.*

Proof. By Lemma 16, $D_1 \times D_2 \times \cdots \times D_n$ is super-λ'. Note that $D_1 \times D_2 \times \cdots \times D_{n-1}$ is a strongly connected $(k_1 + k_2 + \cdots + k_{n-1})$-regular digraph with $k_1 + k_2 + \cdots + k_{n-1} = \lambda(D_1 \times D_2 \times \cdots \times D_{n-1})$ by Lemma 16. For any integer i with $1 \leq i \leq n-1$, we have $|V(D_1 \times D_2 \times \cdots \times D_{n-1})|k_n = \nu_1\nu_2\cdots\nu_{n-1}k_n \geq (1+k_1)\cdots(1+k_{i-1})\nu_i(1+k_{i+1})\cdots(1+k_{n-1})k_n \geq \nu_i\big(\sum_{j=1}^n k_j - k_i\big)$. By Theorem 13, we see that

$$\begin{aligned}
&\sum_{j=1}^n k_j - 1 \geq S_{\lambda'}\big((D_1 \times D_2 \times \cdots \times D_{n-1}) \times D_n\big)\\
&\geq \min\big\{k_1 + k_2 + \cdots + k_{n-1} + k_n - 1,\\
&\quad |V(D_1 \times D_2 \times \cdots \times D_{n-1})|k_n - 2(k_1 + k_2 + \cdots + k_{n-1}) - 2k_n,\\
&\quad \nu_n(k_1 + k_2 + \cdots + k_{n-1}) - 2(k_1 + k_2 + \cdots + k_{n-1}) - 2k_n\big\}\\
&\geq \min_{1\leq i\leq n-1}\left\{\sum_{j=1}^n k_j - 1, \nu_i\left(\sum_{j=1}^n k_j - k_i\right) - 2\sum_{j=1}^n k_j, \nu_n\left(\sum_{j=1}^n k_j - k_n\right) - 2\sum_{j=1}^n k_j\right\}\\
&= \min_{1\leq i\leq n}\left\{\sum_{j=1}^n k_j - 1, \nu_i(\sum_{j=1}^n k_j - k_i) - 2\sum_{j=1}^n k_j\right\}.
\end{aligned}$$

■

4. Conclusions

In this paper, the concept of the arc fault tolerance $S_{\lambda'}(D)$ of a digraph D on the super-λ' property was presented. The parameter can be used to evaluate the reliability of interconnection networks. We investigate $S_{\lambda'}(D_1 \times D_2)$ for the Cartesian product $D_1 \times D_2$ of regular digraphs D_1 and D_2. We give a necessary and sufficient condition for $D_1 \times D_2$ to be super-λ' and obtain $\min\{k_1 + k_2 - 1, \nu_1 k_2 - 2k_1 - 2k_2, \nu_2 k_1 - 2k_1 - 2k_2\} \leq S_{\lambda'}(D_1 \times D_2) \leq k_1 + k_2 - 1$, where D_i is k_i-regular and $\nu_i = |V(D_i)|$ for $i = 1, 2$. An example shows that the lower and upper bounds are best possible. Moreover, we show that $S_{\lambda'}(D_1 \times D_2) = k_1 + k_2 - 1$ in some cases. The above results show that the arc fault-tolerant capability of the Cartesian product of regular digraphs is nice in terms of the super-λ' property. The lower and upper bounds on $S_{\lambda'}(D_1 \times D_2)$ are also generalized to the Cartesian product of n regular digraphs. The value of $S_{\lambda'}(D)$ will provide a beneficial reference for engineers when designing or selecting interconnection networks to build parallel systems. The determination of the exact value of $S_{\lambda'}(D)$ remains an open problem for the general digraphs.

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