Discussiones Mathematicae
Graph Theory 39 (2019) 81–93
doi:10.7151/dmgt.2050

FACIAL INCIDENCE COLORINGS OF EMBEDDED MULTIGRAPHS [1]

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Abstract

Let G be a cellular embedding of a multigraph in a 2-manifold. Two distinct edges $e_1, e_2 \in E(G)$ are facially adjacent if they are consecutive on a facial walk of a face $f \in F(G)$. An incidence of the multigraph G is a pair (v, e), where $v \in V(G)$, $e \in E(G)$ and v is incident with e in G. Two distinct incidences (v_1, e_1) and (v_2, e_2) of G are facially adjacent if either $e_1 = e_2$ or e_1, e_2 are facially adjacent and either $v_1 = v_2$ or $v_1 \neq v_2$ and there is $i \in \{1, 2\}$ such that e_i is incident with both v_1, v_2. A facial incidence coloring of G assigns a color to each incidence of G in such a way that facially adjacent incidences get distinct colors. In this note we show that any embedded multigraph has a facial incidence coloring with seven colors. This bound is improved to six for several wide families of plane graphs and to four for plane triangulations.

Keywords: embedded multigraph, incidence, facial incidence coloring.

2010 Mathematics Subject Classification: 05C15, 05C10.

[1]The research was supported by the Slovak grant VEGA 1/0368/16, by the grant APVV-15-0116, and by the Agency of the Ministry of Education, Science, Research and Sport of the Slovak Republic for the Structural Funds of EU under the project ITMS 26220120007.

1. Definitions and Notation

An *incidence* of an undirected multigraph G is a pair (v, e), where $v \in V(G)$, $e \in E(G)$ and $v \sim e$ (v is incident with e). Let $I(G)$ be the set of incidences of G. If $(v, e) \in I(G)$, we say that (v, e) is an incidence around v and over e. Two distinct incidences $(v_1, e_1), (v_2, e_2) \in I(G)$ are *adjacent* if either $e_1 = e_2$ (which implies that v_1 is adjacent to v_2) or e_1 is adjacent to e_2 and either $v_1 = v_2$ or there is $i \in \{1, 2\}$ such that $v_1 \sim e_i \sim v_2$ and $v_1 \neq v_2$. An *incidence coloring* of G is a mapping $\varphi : I(G) \to C$ (C is a set of colors) in which adjacent incidences receive distinct colors. The smallest number of colors in an incidence coloring of G is called the *incidence chromatic number* of G, and is denoted by $\chi_{\mathrm{i}}(G)$.

The notion of incidence coloring was introduced (originally for simple graphs) by Brualdi and Quinn Massey in [5], where they proved that $\chi_{\mathrm{i}}(G) \leq 2\Delta(G)$ for any graph G and found out the incidence chromatic number of trees, complete bipartite graphs and complete graphs. In the pioneering paper it was conjectured that $\chi_{\mathrm{i}}(G) \leq \Delta(G)+2$. The conjecture was disproved by Guiduli [8] who showed (following the ideas of Algor and Alon [1]) that the Paley graph G_p corresponding to a prime $p \equiv 1 \pmod 4$ satisfies $\chi_{\mathrm{i}}(G_p) \geq \Delta(G_p) + \Omega(\log \Delta(G_p))$. The upper bound of $2\Delta(G)$ was improved in [8] for large $\Delta(G)$ to $\Delta(G) + 20 \log \Delta(G) + 84$.

The trivial inequality $\chi_{\mathrm{i}}(G) \geq \Delta(G) + 1$ (provided that $\Delta(G) \geq 1$) follows immediately from the definition. The lower bound is tight and is attained, for instance, by trees and complete graphs. Some necessary conditions for the equality $\chi_{\mathrm{i}}(G) = \Delta(G) + 1$ were found by Shiu and Sun [17] as well as by Wu [20].

There are several papers devoted to the incidence chromatic number for specific families of graphs, see e.g. Shiu, Lam and Chen [16], Wu [20], Maydanskyi [13], Li and Tu [12], where cubic graphs are studied, and Su [18], where regular graphs in general are investigated.

A graph G is said to be *k-degenerate* if each subgraph H of G satisfies $\delta(H) \leq k$. Hosseini Dolama, Sopena and Zhu proved in [10] that $\chi_{\mathrm{i}}(G) \leq \Delta(G)+2k-1$ if G is k-degenerate and $\chi_{\mathrm{i}}(G) \leq \Delta(G)+7$ if G is planar. Better upper bounds were found for specific families of graphs, see Wu [20], Wang and Lih [19], Hosseini Dolama and Sopena [11], Bonamy, Lévêque and Pinlou [4]. The best current general upper bound for planar graphs G with $\Delta(G) \geq 7$, namely $\chi_{\mathrm{i}}(G) \leq \Delta(G)+5$, is due to Yang [21].

In this paper we are interested in a relaxation of the incidence chromatic number for cellular embeddings of (connected) multigraphs into 2-manifolds (without boundary); we shall refer to such embeddings as to embedded multigraphs. Two distinct edges of an embedded multigraph G are said to be *facially adjacent* (see e.g. Fabrici, Jendrol' and Vrbjarová [6]) if they are consecutive in a facial walk of a face of G. Similarly, two distinct incidences (v_1, e_1) and (v_2, e_2) of G are *facially adjacent* if either $e_1 = e_2$ or e_1 is facially adjacent to e_2 and either $v_1 = v_2$ or there is $i \in \{1, 2\}$ such that $v_1 \sim e_i \sim v_2$ and $v_1 \neq v_2$. A *facial incidence coloring* $\tau : I(G) \to C$ assigns a color from a set C to each incidence of G in such a way that facially adjacent incidences get distinct colors. A *facial incidence k-coloring* is a facial incidence coloring $\tau : I(G) \to C$ with $|C| = k$. The smallest k, for which there is a facial incidence k-coloring of G, is called the *facial incidence chromatic number* of G, and is denoted by $\chi_{\mathrm{fi}}(G)$.

We prove that $\chi_{\mathrm{fi}}(G) \leq 7$ for any embedded multigraph G. The upper bound 7 is improved for triangulations with small chromatic number, embedded multigraphs that have an edge coloring with at most three colors, in which facially adjacent edges receive distinct colors, embeddings of bipartite graphs, as well as some families of plane graphs, especially Eulerian graphs and graphs of girth (the length of a shortest cycle) at least five.

2. General Results

For an embedded multigraph G let $C_{\mathrm{fi}}(G)$ be the *facial incidence conflict graph* of G, the graph with $V(C_{\mathrm{fi}}(G)) = I(G)$, in which two distinct incidences are adjacent if and only if they are facially adjacent in G.

Proposition 1. *If G is an embedded multigraph, then $\chi_{\mathrm{fi}}(G) = \chi(C_{\mathrm{fi}}(G))$.*

The number of incidences of G is equal to the sum of degrees of vertices of G.

Proposition 2. *If G is an embedded multigraph, then $|V(C_{\mathrm{fi}}(G))| = |I(G)| = 2|E(G)|$.*

Let G be an embedded multigraph and let $v \in V(G)$. A *v-sequence* is a sequence $\{e_i\}_{i\in\mathbb{Z}}$ of edges of G incident with v as they are encountered when rotating around v, which means that e_i is facially adjacent to e_{i+1} (as well as to e_{i-1}, since the facial adjacency relation is symmetric) for each $i \in \mathbb{Z}$.

Proposition 3. *Let G be an embedded multigraph and let $d = \deg(v) \geq 2$ for a vertex $v \in V(G)$. If d is odd, then $\chi_{\mathrm{fi}}(G) \geq 4$, otherwise $\chi_{\mathrm{fi}}(G) \geq 3$.*

Proof. Consider a v-sequence $\{e_i\}_{i\in\mathbb{Z}}$. The incidences (v, e_i), $i = 1, \ldots, d$, induce in $C_{\mathrm{fi}}(G)$ a cycle of length d. Therefore, if d is odd and a surjection $\tau : I(G) \to C$ is a facial incidence coloring of G, then without loss of generality the incidences (v, e_i), $i = 1, 2, 3$, receive three distinct colors. Suppose that $e_2 \sim v_2$, $v_2 \neq v$. Then the incidence (v_2, e_2) is facially adjacent to each of the incidences (v, e_i), $i = 1, 2, 3$, hence $\tau(v_2, e_2) \neq \tau(v, e_i)$, $i = 1, 2, 3$, and $|C| \geq 4$. If d is even, then the incidences $(v, e_1), (v, e_2)$ and (v_2, e_2) are pairwise facially adjacent in G, and so $\chi_{\mathrm{fi}}(G) \geq 3$. ■

The *degree* of a face f of an embedded graph is the number $\deg(f)$ of edges incident with f in G, where edges incident with f only are counted twice. The face f is *simple* if it is incident with $\deg(f)$ distinct edges. An embedded multigraph T is a *triangulation* if $\deg(f) = 3$ for each $f \in F(T)$.

Proposition 4. *Let G be an embedded multigraph and let $d = \deg(f)$ for a simple face $f \in F(G)$. If $d \not\equiv 0 \pmod 3$, then $\chi_{\text{fi}}(G) \geq 4$.*

Proof. Let $\{e_i : i = 1, \dots, d\}$ be the set of edges incident with f in G. It is easy to see that the $2d$ incidences of G over the edges e_i, $i = 1, \dots, d$, induce in $C_{\text{fi}}(G)$ a supergraph H of a graph C_{2d}^2 (the square of a cycle of length $2d$). If $d \not\equiv 0 \pmod 3$, then $2d \not\equiv 0 \pmod 3$, and, by Proposition 1, $\chi_{\text{fi}}(G) = \chi(C_{\text{fi}}(G)) \geq \chi(H) \geq \chi(C_{2d}^2) = 4$ (a "folklore result"). ■

It is important to realize that a multigraph G can have nonisomorphic embeddings G_1, G_2 with $\chi_{\text{fi}}(G_1) \neq \chi_{\text{fi}}(G_2)$. Indeed, for example, the 3-connected planar graph O of the regular octahedron has an (essentially unique) plane embeding O_0 with $\chi_{\text{fi}}(O_0) = 3$ (as one can easily see) as well as an embedding O_1 into S_1 (a torus) with six simple faces of degree 4, so that, by Proposition 4, $\chi_{\text{fi}}(O_1) \geq 4$.

Theorem 5. *If G is an embedded multigraph, then $\chi_{\text{fi}}(G) \leq 7$.*

Proof. An edge $e \in E(G)$ with $u_1 \sim e \sim u_2$, $u_1 \neq u_2$, is facially adjacent to at most $m(u_1) + m(u_2)$ edges of G, where $m(u_i) = \min(2, \deg(u_i) - 1)$, $i = 1, 2$, and, for both $i \in \{1, 2\}$, the incidence (u_i, e) is facially adjacent to at most $1 + 2m(u_i) + m(u_{3-i})$ incidences of G. Namely, if the edge e is facially adjacent in G to edges e_i^j, $i, j = 1, 2$, where $u_i \sim e_i^j \sim v^j$, $v^j \neq u_i$, then, for both $i \in \{1, 2\}$, the incidence (u_i, e) is facially adjacent to the incidences (u_{3-i}, e), (u_i, e_i^j), (v^j, e_i^j) and (u_{3-i}, e_{3-i}^j), $j = 1, 2$. Thus, $\Delta(C_{\text{fi}}(G)) \leq 7$. Moreover, $C_{\text{fi}}(G) \neq K_8$. Indeed, by Proposition 2, the assumption $I(G) = K_8$ means that $|E(G)| = 4$. In such a case, if $\delta(G) \leq 2$ and $\deg(u_1) = \delta(G)$, $u_1 \sim e$, then the incidence (u_1, e) is facially adjacent in G to at most $1 + 2\deg(u_1) \leq 5$ incidences, a contradiction. On the other hand, if $\delta(G) \geq 3$, then $2 \leq |V(G)| \leq \left\lfloor \frac{8}{3} \right\rfloor = 2$, $V(G) = \{u_1, u_2\}$, and $E(G)$ consists of four edges incident with both u_1, u_2. Consider a u_1-sequence $\{e_i\}_{i \in \mathbb{Z}}$. Since e_1 is not facially adjacent to e_3, the incidence (u_1, e_1) is not facially adjacent to the incidence (u_1, e_3), a contradiction again. By Brooks' theorem the graph $C_{\text{fi}}(G)$ is 7-colorable and the statement of our theorem follows from Proposition 1. ■

Theorem 6. *If an embedded multigraph T is a triangulation, then $\chi_{\text{fi}}(T) \leq \chi(T)$.*

Proof. Consider a proper vertex coloring $\varphi : V(T) \to C$ that uses $\chi(T)$ colors. Let $\tau : I(G) \to C$ be the coloring of incidences of T defined as follows: If $e \in E(T)$

and $u \sim e \sim v$, $u \neq v$, then $\tau(u, e) = \varphi(v)$ and $\tau(v, e) = \varphi(u)$. It suffices to show that τ is a facial incidence coloring. For that purpose take two facially adjacent incidences $(v_1, e_1), (v_2, e_2) \in I(T)$. If $e_1 = e_2$, then $v_1 \sim e_1 = e_2 \sim v_2$, $v_1 \neq v_2$, and $\tau(v_1, e_1) = \varphi(v_2) \neq \varphi(v_1) = \tau(v_2, e_2)$. Otherwise, e_1 is facially adjacent to e_2 in T (which means that if $e_j \sim w_j$, and $w_j \neq v_j$, $j = 1, 2$, then $|S| = 3$ for the set $S = \{v_1, w_1\} \cup \{v_2, w_2\}$ and the vertices of S are pairwise adjacent in T) and either (i) $v_1 = v_2$ or (ii) there is $i \in \{1, 2\}$ such that $v_1 \sim e_i \sim v_2$, $v_1 \neq v_2$. If (i) is true, then $w_1 \neq w_2$ and $\tau(v_1, e_1) = \varphi(w_1) \neq \varphi(w_2) = \tau(v_2, e_2)$. On the other hand, if (ii) applies, e_i is incident with v_1, v_2 and w_i, hence there is $j \in \{1, 2\}$ such that $w_i = v_j$; then the vertex w_{3-i} is not incident with e_i (otherwise $|S| = 2$), as a consequence $w_{3-i} \neq w_i$ (w_i *is* incident with e_i), w_{3-i} is adjacent to w_i in T and $\tau(v_i, e_i) = \varphi(w_{3-i}) \neq \varphi(w_i) = \tau(v_{3-i}, e_{3-i})$. ■

Corollary 7. *If T is a plane triangulation, then $\chi_{\mathrm{fi}}(T) \leq 4$.*

Proof. See the Four-Color Theorem (Appel and Haken [2] or else Robertson *et al.* [15]). ■

By Proposition 3 the bound in Corollary 7 is tight and the facial incidence chromatic number of each plane triangulation is either three or four.

Theorem 8. *If an embedded multigraph G is bipartite, then $\chi_{\mathrm{fi}}(G) \leq 6$.*

Proof. Let $\{X, X'\}$ be the bipartition of G. Consider a set of colors $\{a_1, a_2, a_3, b_1, b_2, b_3\}$ and a coloring of incidences of G defined as follows: Suppose that $x \in X$ and $\{e_i\}_{i \in \mathbb{Z}}$ is an x-sequence. Express $p = \deg(x)$ (in a unique way) as $2k + l$ with integers $k \geq 0$ and $l \in \{0, 1\}$. Color the incidences (x, e_i), $i = 1, \ldots, 2k$, alternately with the colors a_1, a_2, and then, if $l = 1$, the incidence $(x, e_{2k+1}) = (x, e_p)$ with the color a_3. If $x' \in X'$, $p' = \deg(x')$ and $\{e'_i\}_{i \in \mathbb{Z}}$ is an x'-sequence, then the incidences (x', e'_i), $i = 1, \ldots, p'$, are colored similarly using the colors b_1, b_2, b_3 and the expression $p' = 2k' + l'$, $k' \geq 0$, $l' \in \{0, 1\}$. Clearly, we have obtained a facial incidence 6-coloring of G. ■

An edge coloring of an embedded multigraph G is *facially proper* if any two facially adjacent edges of G receive distinct colors. If $|V(G)| = 2$ and $|E(G)| = 5$, it is possible that all edges of G are pairwise facially adjacent, and then a facially proper edge coloring of G necessarily uses five colors. Otherwise, the fact that each edge of G is facially adjacent to at most four edges of G allows us to prove (proceeding similarly as in the proof of Theorem 5 with the facial adjacency of edges instead of the facial adjacency of incidences) that there is a facially proper edge coloring of G with four colors.

Theorem 9. *If there is a facially proper edge coloring of an embedded multigraph G with k colors, then $\chi_{\mathrm{fi}}(G) \leq 2k$.*

Proof. Let $\varphi : E(G) \to \{1, \ldots, k\}$ be a facially proper coloring of G. The coloring τ of the incidences of G defined so that if $uv \in E(G)$, then $\tau(u, uv) = (\varphi(uv), 1)$ and $\tau(v, uv) = (\varphi(uv), 2)$, uses $2k$ colors and assigns distinct colors to facially adjacent incidences of G. ■

Let G be a multigraph whose vertices are colored with colors from a set S. Consider a color $s \in S$. By an *s-vertex* we mean any vertex that is colored with the color s and by an *s-component* we understand any component of the submultigraph of G induced by its s-vertices.

Poh [14] and Goddard [7] independently proved that every planar (simple) graph G has a vertex coloring with at most three colors such that each monochromatic subgraph of G is a linear forest (i.e., all its components are paths). We say that an embedded multigraph G has a *strong Poh-Goddard coloring* (an SPG coloring for short) provided that it has a vertex coloring with at most three colors such that each its monochromatic component is a simple graph isomorphic to a path and every trivial (i.e., a single vertex) component is formed by a vertex of an even degree in G.

Theorem 10. *If G is an embedding of a multigraph that has an SPG coloring, then $\chi_{\mathrm{fi}}(G) \leq 6$.*

Proof. Let a, b and c (maybe not all of them are used) be three distinct colors of an SPG coloring of G. We are going to color the incidences of G with colors from the set $C = \{a_1, a_2, b_1, b_2, c_1, c_2\} = \{t_1, t_2 : t \in \{a, b, c\}\}$ of six distinct colors.

Consider an s-vertex v, $s \in \{a, b, c\}$, a v-sequence $\{e_i\}_{i\in\mathbb{Z}}$, the sequence $\{v_i\}_{i\in\mathbb{Z}}$, where $e_i \sim v_i$, $v_i \neq v$, for $i \in \mathbb{Z}$, and a maximal nonempty injective subsequence $(e_k, \ldots, e_l)$ of the sequence $\{e_i\}_{i\in\mathbb{Z}}$ such that each vertex in the sequence $(v_k, \ldots, v_l)$ is colored with a color belonging to the set $\{a, b, c\} - \{s\}$; let us call it an *$\bar{s}$-sequence* corresponding to v. Color the incidences (v, e_i), $i = k, \ldots, l$, alternately with the colors s_1 and s_2. If v forms a trivial s-component of the SPG coloring, then $l + 1 - k = \deg(v) \equiv 0 \pmod 2$. Otherwise, v belongs to a nontrivial s-component H, and the number of neighbors of v colored s is either one or two. Moreover, there are at most two $\bar{s}$-sequences corresponding to the sequence $\{e_i\}_{i\in\mathbb{Z}}$ that have no common terms; note that no $\bar{s}$-sequence corresponding to v exists if $\deg(v) = 2$ and v is an internal vertex of H. Let $\bar{s}(v)$ denote the maximum number of $\bar{s}$-sequences corresponding to v with no common terms. If $\bar{s}(v) = 2$, there is an $\bar{s}$-sequence $(e'_k, \ldots, e'_l)$ corresponding to v such that $e_i \neq e'_j$ for each $i \in \{k, \ldots, l\}$ and $j \in \{k', \ldots, l'\}$, and we color the incidences (v, e'_j), $j = k', \ldots, l'$, in a similar way as the incidences (v, e_i), $i = k, \ldots, l$. Clearly, after doing this for all vertices $v \in V(G)$ we obtain a partial facial incidence coloring φ of G (no conflicts are present among incidences colored so far) using at most 6 colors. We call this first phase of coloring the stage A.

What remains is to color in the stage B incidences over edges belonging to components created by the SPG coloring under consideration. We do that step by step using an arbitrary ordering of those components. Let x_i, $i = 1, \dots, p$, $p \geq 2$, be consecutive vertices of the first component P, and let P be an s-component, $s \in \{a, b, c\}$. Since P is a simple graph, its edges are $x_i x_{i+1}$, $i = 1, \dots, p-1$. The incidences over these edges can be ordered to form the natural sequence

$$\xi = ((x_1, x_1x_2), (x_2, x_1x_2), \dots, (x_{p-1}, x_{p-1}x_p), (x_p, x_{p-1}x_p)).$$

We shall color the incidences of ξ using mostly the colors of the set $T = \{t_1, t_2 : t \in \{a, b, c\} - \{s\}\}$. Consider a subsequence $X_{q,r} = (x_q, \dots, x_r)$ of the sequence $X_{1,p} = (x_1, \dots, x_p)$. The sequence $X_{q,r}$ is a 1-*sequence* if it is a maximal subsequence of $X_{1,p}$ such that $\bar{s}(x_i) = 1$ for each $i \in \{q+1, \dots, r-1\}$. The sequence $X_{q,r}$ is a $(0, 2)$-*sequence* if it is a maximal subsequence of $X_{1,p}$ such that $\bar{s}(x_i) \in \{0, 2\}$ for each $i \in \{q, \dots, r\} - \{1, p\}$. Clearly, there is a (uniquely determined) subsequence $(p_1, \dots, p_t)$ of the sequence $(1, \dots, p)$ such that $p_1 = 1$, $p_t = p$ and $X_{p_i, p_{i+1}}$ is either a 1-sequence or a $(0, 2)$-sequence for each $i \in \{1, \dots, t-1\}$; moreover, 1-sequences and $(0, 2)$-sequences alternate in the sequence $\left(X_{p_1,p_2}, X_{p_2,p_3} \dots, X_{p_{t-1},p_t}\right)$.

For each $(q, r) \in \{(p_1, p_2), (p_2, p_3), \dots, (p_{t-1}, p_t)\}$ we color the incidences of the subsequence

$$\xi_{q,r} = ((x_q, x_qx_{q+1}), (x_{q+1}, x_qx_{q+1}), \dots, (x_{r-1}, x_{r-1}x_r), (x_r, x_{r-1}x_r))$$

of the sequence ξ separately step by step in the order given by the sequence $((p_1, p_2), (p_2, p_3), \dots, (p_{t-1}, p_t))$. When jumping from the incidence (p_{j-1}, p_j) to $(q, r) = (p_j, p_{j+1})$, $j \in \{2, t-1\}$, we only have to take care if $\bar{s}(x_q) = 0$, since then colors of $(x_{q-1}, x_{q-1}x_q)$ and $(x_q, x_{q-1}x_q)$ intervene in coloring of (x_q, x_qx_{q+1}) and (x_{q+1}, x_qx_{q+1}). Let φ_j, $j \in \{1, \dots, t-1\}$, be a continuation of φ (a partial facial incidence coloring of G), in which the last colored incidence is $(x_{p_{j+1}}, x_{p_{j+1}-1}x_{p_{j+1}})$. We begin with $\varphi_0 = \varphi$ and we define φ_j with $j \geq 1$ as a continuation of the coloring φ_{j-1}.

Suppose first that $X_{q,r}$ is a $(0, 2)$-sequence. We color the involved incidences with colors from T in the following order:

$$(x_{q+1}, x_qx_{q+1}), (x_q, x_qx_{q+1}), \dots, (x_r, x_{r-1}x_r), (x_{r-1}, x_{r-1}x_r).$$

It is sufficient to show that if $i \in \{q, \dots, r-1\}$, then, for both (x_{i+1}, x_ix_{i+1}) and (x_i, x_ix_{i+1}), in the moment when the incidence under consideration is colored, at most three colors from T are forbidden for it. In the case of the incidence (x_{i+1}, x_ix_{i+1}) those forbidden colors are the color of $(x_i, x_{i-1}x_i)$ (if $i \geq 2$ and $\bar{s}(x_i) = 0$) and at most two colors assigned by φ to incidences around neighbors of x_{i+1} outside $V(P)$. Further, when coloring (x_i, x_ix_{i+1}), we cannot use the color of $(x_{i+1}, x_i x_{i+1})$ and either the colors of $(x_{i-1}, x_{i-1}x_i)$ and $(x_i, x_{i-1}x_i)$ (if $i \geq 2$ and $\bar{s}(x_i) = 0$) or at most two colors assigned by φ to incidences around neighbors of x_i outside $V(P)$ (if $i = 1$ or $\bar{s}(x_i) = 2$).

Now let $X_{q,r}$ be a 1-sequence (note that then $r \geq q + 2$). We are going to color the involved incidences according to the order (from the left to the right) given by the sequence $\xi_{q,r}$. For an incidence (v, e) of this sequence let $C(v, e)$ be the *set of candidate colors* containing those colors of the set C that are not assigned by φ_{j-1} to incidences adjacent to (v, e) in the graph $C_{\mathrm{fi}}(G)$.

If $q = 1$ or $\bar{s}(x_q) = 2$, then $|C(x_q, x_q x_{q+1}) \cap T| \geq 2$, since at most two colors from T, used for incidences around neighbors of x_q, are forbidden for $(x_q, x_q x_{q+1})$. If $q \geq 2$ and $\bar{s}(x_q) = 0$, then we have $C(x_q, x_q x_{q+1}) = C - \{\varphi_{j-1}(x_{q-1}, x_{q-1}x_q), \varphi_{j-1}(x_q, x_{q-1}x_q)\}$, and $|C(x_q, x_q x_{q+1}) \cap T| \geq 2$ is true, too.

For each $i \in \{q+1, \ldots, r-1\}$ we have $\bar{s}(x_i) = 1$, and, consequently, there is exactly one incidence $\left(v_i^{i-1}, e_i^{i-1}\right)$ around a neighbor of x_i out of $V(P)$ that is adjacent to $(x_i, x_{i-1}x_i)$ as well as exactly one incidence $\left(v_i^{i+1}, e_i^{i+1}\right)$ around a neighbor of x_i out of $V(P)$ that is adjacent to $(x_i, x_i x_{i+1})$; evidently, if $\deg(x_i) = 3$, then $\left(v_i^{i+1}, e_i^{i+1}\right) = \left(v_i^{i-1}, e_i^{i-1}\right)$. Thus, $|C(x_{q+1}, x_q x_{q+1})| \geq 3$ and $|C(x_{q+1}, x_q x_{q+1}) \cap T| \geq 2$.

Let $i \in \{q+1, \ldots, r-2\}$. Without loss of generality we may suppose that $\varphi\left(x_{i+1}, e_{i+1}^i\right) = \varphi\left(x_i, e_i^{i+1}\right)$. Indeed, during the phase A we can color incidences around the vertices $x_{q+1}, \ldots, x_{r-1}$ that are not over edges of P with colors from $\{s_1, s_2\}$ starting with incidences around x_{q+1}, and, if incidences around x_k, $k \in \{q+1, \ldots, r-2\}$, are already colored, then we color incidences around x_{k+1} in such a way that first the incidence $\left(x_{k+1}, e_{k+1}^k\right)$ is colored the same as the incidence $\left(x_k, e_k^{k+1}\right)$, and then remaining incidences around x_{k+1} are colored alternately with the colors s_1 and s_2; the last colored incidence is $\left(x_{k+1}, e_{k+1}^{k+2}\right)$. Therefore, $|C(x_i, x_i x_{i+1})| = |C(x_{i+1}, x_i x_{i+1})| = 4$, and, if $\varphi\left(x_i, e_i^{i+1}\right) = s_l$, $l \in \{1, 2\}$, then $C(x_i, x_i x_{i+1}) \cap \{s_1, s_2\} = C(x_{i+1}, x_i x_{i+1}) \cap \{s_1, s_2\} = \{s_{3-l}\}$.

We have $|C(x_{r-1}, x_{r-1}x_r) \cap T| \geq 3$. Finally, for the incidence $(x_r, x_{r-1}x_r)$ it holds either $|C(x_r, x_{r-1}x_r) \cap T| \geq 2$ (in the case $\bar{s}(x_r) = 2$ or $r = p$) or $|C(x_r, x_{r-1}x_r)| = 5$ (in the case $\bar{s}(x_r) = 0$ and $r \leq p - 1$), which implies the inequality $|C(x_r, x_{r-1}x_r) \cap T| \geq 2$.

To find a required continuation φ_j of φ_{j-1} it suffices to choose for each incidence (v, e) a color from $C(v, e)$ in such a way that these new colors are not in conflict with each other.

If $r = q + 2$, recall that both $C(x_q, x_q x_{q+1})$ and $C(x_{q+2}, x_{q+1}x_{q+2})$ have at least two colors and both $C(x_{q+1}, x_q x_{q+1})$ and $C(x_{q+1}, x_{q+1}x_{q+2})$ have at least three colors. Provided that there is $\alpha \in C(x_q, x_q x_{q+1}) \cap C(x_{q+2}, x_{q+1}x_{q+2})$, we set $\varphi_j(x_q, x_q x_{q+1}) = \varphi_j(x_{q+2}, x_{q+1}x_{q+2}) = \alpha$, and then find $\varphi_j(x_{q+1}, x_q x_{q+1})$ and $\varphi_j(x_{q+1}, x_{q+1}x_{q+2})$. On the other hand, if the sets $C(x_q, x_q x_{q+1})$ and $C(x_{q+2},$

$x_{q+1}x_{q+2}$) are disjoint, then, because of Hall's theorem (see [9]), the family of sets $\{C(x_q, x_q x_{q+1}), C(x_{q+1}, x_q x_{q+1}), C(x_{q+1}, x_{q+1}x_{q+2}), C(x_{q+2}, x_{q+1}x_{q+2})\}$ has a system of distinct representatives, which yields a desired coloring φ_j.

If $r \geq q+3$, we are able to find a continuation φ'_j of φ_{j-1}, in which only the incidences $(x_{r-1}, x_{r-1}x_r)$ and $(x_r, x_{r-1}x_r)$ of the sequence $\xi_{q,r}$ are not colored, and which uses a color from the set $\{s_1, s_2\}$ just for the incidence $(x_{r-1}, x_{r-2}x_{r-1})$. We determine colors of involved incidences in the order given by (the subsequence of) the sequence $\xi_{q,r}$. The first two incidences of $\xi_{q,r}$ can be colored in an appropriate way, since their corresponding candidate sets have at least two colors from T. If $i \in \{q+1, \ldots, r-2\}$, we use for $\varphi'_j(x_i, x_i x_{i+1})$ a color from the (nonempty) set $(C(x_i, x_i x_{i+1}) \cap T) - \{\varphi'_j(x_{i-1}, x_{i-1}x_i), \varphi'_j(x_i, x_{i-1}x_i)\}$. Moreover, if $i \in \{q+1, \ldots, r-3\}$, we use for $\varphi'_j(x_{i+1}, x_i x_{i+1})$ a color from the (nonempty) set $(C(x_{i+1}, x_i x_{i+1}) \cap T) - \{\varphi'_j(x_i, x_{i-1}x_i), \varphi'_j(x_i, x_i x_{i+1})\}$. Then a color $\varphi'_j(x_{r-1}, x_{r-2}x_{r-1})$ can be chosen (in a unique way) so that it is one of the colors s_1, s_2.

Finally, we define φ_j as a continuation of φ'_j. Since $|C(x_{r-1}, x_{r-1}x_r)| \geq 3$, there is a color $\varphi_j(x_{r-1}, x_{r-1}x_r) \in C(x_{r-1}, x_{r-1}x_r)$ that is distinct from both $\varphi'_j(x_{r-2}, x_{r-2}x_{r-1})$ and $\varphi'_j(x_{r-1}, x_{r-2}x_{r-1})$. Further, from the fact that $\varphi'_j(x_{r-1}, x_{r-2}x_{r-1}) \in \{s_1, s_2\}$ and from the inequality $|C(x_r, x_{r-1}x_r) \cap T| \geq 2$ it follows that we are done by taking $\varphi_j(x_r, x_{r-1}x_r) \in C(x_r, x_{r-1}x_r)$ so that this color is distinct from both $\varphi'_j(x_{r-1}, x_{r-2}x_{r-1})$ (a color outside T) and $\varphi_j(x_{r-1}, x_{r-1}x_r)$.

We proceed similarly as above with all remaining components created by our SPG coloring (so that, for example, when working with the second component, φ_p plays the role of φ). After finishing the stage B we obtain a facial incidence 6-coloring of G. ■

Theorem 11. *If G is a plane graph of girth at least 6, then $\chi_{\mathrm{fi}}(G) \leq 6$.*

Proof. By a theorem of Axenovich, Ueckerdt and Weiner [3] the graph G has a vertex coloring using the colors a and b such that for both $s \in \{a, b\}$ each s-component is a path (moreover, of length at most 14). Color the incidences of G as in the proof of Theorem 10 and set $a_3 = c_1$, $b_3 = c_2$. If there is a trivial s-component, $s \in \{a, b\}$, that is formed by a vertex v of an odd degree $2k+1$, let $\{e_i\}_{i \in \mathbb{Z}}$ be a v-sequence. Color the incidences (v, e_i), $i = 1, \ldots, 2k$, alternately with the colors s_1 and s_2, and, finally, color the incidence (v, e_{2k+1}) with the color s_3. Evidently, we have obtained a facial incidence 6-coloring of G. ■

3. WHEELS

Let $n \geq 3$ be an integer and let K_1, C_n be vertex disjoint graphs with $V(K_1) = \{u\}$, $V(C_n) = \{v_i : i = 1, \ldots, v_n\}$ and $E(C_n) = \{v_i v_{i+1} : i = 1, \ldots, n\}$ (with indices counted modulo n). The *wheel* W_n is the join $C_n + K_1$, i.e., the graph with $V(W_n) = V(K_1) \cup V(C_n)$ and $E(W_n) = E(C_n) \cup \{uv_i : i = 1, \ldots, n\}$. The graph W_n is 3-connected, hence it has an essentially unique plane embedding. In what follows we shall consider a plane embedding $\tilde{W}_n$ of W_n in which the unbounded face is incident with the edges v_iv_{i+1}, $i = 1, \ldots, n$.

Theorem 12. *Let $n \geq 3$ be an integer. If $n = 5$, then $\chi_{\mathrm{fi}}(\tilde{W}_n) = 5$, otherwise $\chi_{\mathrm{fi}}(\tilde{W}_n) = 4$.*

Proof. Since $\deg(v_1) = 3$, by Proposition 3 we know that $\chi_{\mathrm{fi}}(\tilde{W}_n) \geq 4$. Let

$$I_{j,k} = \{(u, uv_i), (v_i, uv_i), (v_i, v_iv_{i+1}), (v_{i+1}, v_iv_{i+1}) : i = j, \ldots, j + k - 1\}$$

and let $\varphi_{j,k} : I_{j,k} \to \{1, 2, 3, 4\}$ be the mapping determined by Figure 1 ($k = 3$) or Figure 2 ($k = 4$), where small full circles are auxiliary "vertices" that "cut" an edge joining two ordinary (empty circle) vertices x and y into two "halfedges" representing the incidences (x, xy) and (y, xy).

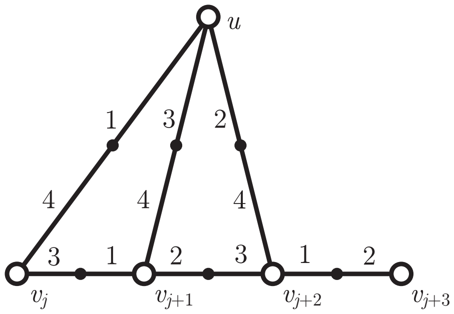

Figure 1. Coloring of twelve incidences of the wheel $\tilde{W}_n$.

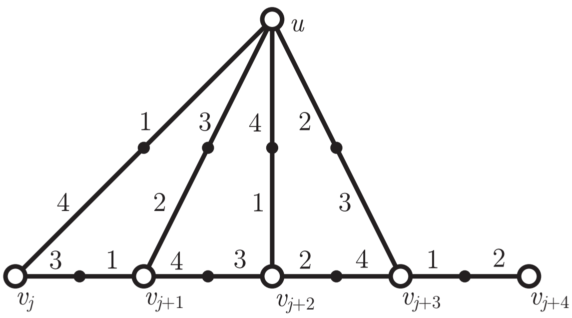

Figure 2. Coloring of sixteen incidences of the wheel $\tilde{W}_n$.

If $n \neq 5$, then $n = 3p + 4q$ with integers $p \geq 0$ and $q \in \{0,1,2\}$. Let $\varphi : I(\tilde{W}_n) \to \{1,2,3,4\}$ be the mapping which is created from the mappings $\varphi_{3l+1,3}$, $l = 0,\dots,p-1$ and $\varphi_{3p+4l+1,4}$, $l = 0,\dots,q-1$ (the restriction of φ to $I_{j,k}$ is $\varphi_{j,k}$ for all pairs (j,k)). It is straightforward to see that φ is a facial incidence 4-coloring of $\tilde{W}_n$. For that purpose it is useful to realize that, given a pair (j_1,k_1), there are at most three pairs (j_2,k_2) such that there is an incidence in I_{j_1,k_1} facially adjacent to an incidence in I_{j_2,k_2} (including the case $(j_1,k_1) = (j_2,k_2)$).

Suppose now that there is a facial incidence 4-coloring ψ of $\tilde{W}_5$. Then there exists $i \in \{1,2,3,4,5\}$ such that $\psi(u, uv_{i-1}) = c = \psi(u, uv_{i+1})$ (with indices counted modulo 5). For both $j \in \{i-1, i+1\}$ each of the incidences (u, uv_j), $(v_{j-1}, v_{j-1}v_j)$, (v_{j+1}, v_jv_{j+1}) is facially adjacent to all the incidences in the set $\{(v_j, v_{j-1}v_j), (v_j, uv_j), (v_j, v_jv_{j+1})\}$ of pairwise facially adjacent incidences. Therefore, all the six mentioned incidences must be colored with the color c. This, however, is a contradiction, since the incidences $(v_i, v_{i-1}v_i)$ and (v_i, v_iv_{i+1}) are facially adjacent in $\tilde{W}_5$.

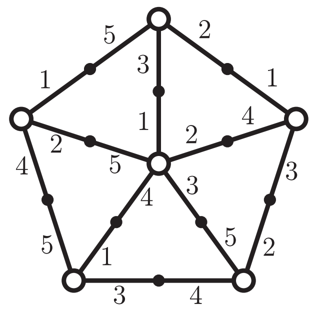

Figure 3. A facial incidence 5-coloring of the wheel $\tilde{W}_5$.

Thus, to finish the proof it is sufficient to check that Figure 3 determines a facial incidence 5-coloring of $\tilde{W}_5$. ∎

4. Concluding Remarks

A plane embedding of the planar multigraph with two vertices and three (parallel) edges has six incidences that are pairwise facially adjacent so that its facial incidence chromatic number is six. For the moment we do not know any other embedded multigraph with facial incidence chromatic number six.

Problem 1. Does there exist an embedded simple graph G with $\chi_{\mathrm{fi}}(G) = 6$?

Problem 2. Does there exist an embedded multigraph G with $\chi_{\mathrm{fi}}(G) = 7$?

We strongly believe that the following could be proved.

Conjecture 1. *If G is a plane embedding of a planar multigraph, then $\chi_{\mathrm{fi}}(G) \leq 6$.*

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Received 12 December 2016
Revised 17 May 2017
Accepted 17 May 2017