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# BOUNDS ON THE SIGNED ROMAN $k$-DOMINATION NUMBER OF A DIGRAPH

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**Abstract**

Let $k$ be a positive integer. A signed Roman $k$-dominating function (SRkDF) on a digraph $D$ is a function $f : V(D) \to \{-1, 1, 2\}$ satisfying the conditions that (i) $\sum_{x \in N^-[v]} f(x) \geq k$ for each $v \in V(D)$, where $N^-[v]$ is the closed in-neighborhood of $v$, and (ii) each vertex $u$ for which $f(u) = -1$ has an in-neighbor $v$ for which $f(v) = 2$. The weight of an SRkDF $f$ is $\sum_{v \in V(D)} f(v)$. The signed Roman $k$-domination number $\gamma^k_{sR}(D)$ of a digraph $D$ is the minimum weight of an SRkDF on $D$. We determine the exact values of the signed Roman $k$-domination number of some special classes of digraphs and establish some bounds on the signed Roman $k$-domination number of general digraphs. In particular, for an oriented tree $T$ of order

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$n$, we show that $\gamma_{sR}^2(T) \geq (n+3)/2$, and we characterize the oriented trees achieving this lower bound.

**Keywords:** signed Roman $k$-dominating function, signed Roman $k$-domination number, digraph, oriented tree.

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## 1. Introduction

Due to the diversity of its applications to both theoretical and practical problems, domination and its variants have extensively studied recently (see, for example, [2, 4, 5, 7]). Our aim in this paper is to study the signed Roman $k$-domination in digraphs.

Throughout this paper, $D$ denotes a finite simple digraph with vertex set $V(D)$ and arc set $A(D)$. For two vertices $u, v \in V(D)$, we use $(u, v)$ to denote the arc with direction from $u$ to $v$, and we also call $v$ an *out-neighbor* of $u$ and $u$ an *in-neighbor* of $v$. For $v \in V(D)$, the *out-neighborhood* and *in-neighborhood* of $v$, denoted by $N_D^+(v) = N^+(v)$ and $N_D^-(v) = N^-(v)$, are the sets of out-neighbors and in-neighbors of $v$, respectively. The *closed out-neighborhood* and *closed in-neighborhood* of a vertex $v \in V(D)$ are the sets $N_D^+[v] = N^+[v] = N^+(v) \cup \{v\}$ and $N_D^-[v] = N^-[v] = N^-(v) \cup \{v\}$, respectively. The *out-degree* and *in-degree* of a vertex $v \in V(D)$ are defined by $d_D^+(v) = d^+(v) = |N_D^+(v)|$ and $d_D^-(v) = d^-(v) = |N_D^-(v)|$, respectively. The *maximum out-degree* and *minimum in-degree* among the vertices of $D$ are denoted by $\Delta^+(D) = \Delta^+$ and $\delta^-(D) = \delta^-$, respectively. For two vertices $u$ and $v$ of $D$, the *distance* $d_D(u, v) = d(u, v)$ *from* $u$ *to* $v$ is the length of a shortest $u$-$v$ directed path in $D$. If $D$ contains no $u$-$v$ directed path, then $d_D(u, v) = \infty$. For a subdigraph $H$ of $D$ and $v \in V(D)$, the *distance from* $H$ *to* $v$ in $D$ is $d_D(H, v) = d(H, v) = \min\{d_D(u, v) : u \in V(H)\}$. For a real-valued function $f : V(D) \to \mathbb{R}$ and $v \in V(D)$, we define $f[v] = \sum_{x \in N^-[v]} f(x)$.

A *rooted tree* is a connected digraph with a vertex of in-degree 0, called the *root*, such that every vertex different from the root has in-degree 1. A digraph $D$ is *contrafunctional* if each vertex of $D$ has in-degree 1. An *orientation* $D$ of a graph $G$ or *oriented graph* $D$ is a digraph obtained from $G$ by assigning a direction to (that is, orienting) each edge of $G$. In this sense, we also call $G$ the *underlying graph* of $D$.

Let $k$ be a positive integer. A *signed Roman $k$-dominating function* (SRkDF) on a graph $G$ is a function $f : V(G) \to \{-1, 1, 2\}$ satisfying the conditions that (i) $\sum_{x \in N[v]} f(x) \geq k$ for each $v \in V(G)$, where $N[v]$ is the closed neighborhood of $v$, and (ii) each vertex $u$ for which $f(u) = -1$ is adjacent to a vertex $v$ for which $f(v) = 2$. The *weight* of an SRkDF $f$ is $\omega(f) = \sum_{v \in V(G)} f(v)$. The *signed Roman $k$-domination number* $\gamma_{sR}^k(G)$ of a graph $G$ is the minimum weight of an SRkDF on $G$. By definition, $\gamma_{sR}^1(G)$ coincides with $\gamma_{sR}(G)$. The signed Roman $k$-domination in graphs was introduced and investigated by Henning and Volkmann [8, 9]. The special case $k = 1$ was introduced by Ahangar *et al.* [1].

Volkmann [11] extended the concept of signed Roman $k$-domination in graphs to digraphs. Let $k$ be a positive integer. A *signed Roman $k$-dominating function* (SRkDF) on a digraph $D$ is a function $f : V(D) \to \{-1, 1, 2\}$ satisfying the conditions that (i) $f[v] \geq k$ for each $v \in V(D)$, and (ii) each vertex $u$ for which $f(u) = -1$ has an in-neighbor $v$ for which $f(v) = 2$. The *weight* of an SRkDF $f$ is $\omega(f) = \sum_{v\in V(D)} f(v)$. The *signed Roman $k$-domination number* $\gamma_{sR}^k(D)$ of a digraph $D$ is the minimum weight of an SRkDF on $D$. An SRkDF on $D$ with weight $\gamma_{sR}^k(D)$ is called a $\gamma_{sR}^k(D)$-*function.* An SRkDF $f$ on $D$ can be represented by the ordered partition $(V_{-1}, V_1, V_2)$, where $V_i = \{v \in V(D) : f(v) = i\}$ for $i \in \{-1, 1, 2\}$. The signed Roman 1-domination number of a digraph $D$ is usually denoted by $\gamma_{sR}(D)$ and was introduced by Sheikholeslami and Volkmann [10].

As the assumption $\delta^- \geq k/2 - 1$ is clearly necessary, we always assume that when we discuss $\gamma_{sR}^k(D)$, all digraphs involved satisfy $\delta^- \geq k/2 - 1$. For any terminology not given here, the reader is referred to Chartrand and Lesniak [3].

## 2. Special Classes of Digraphs

In this section, we mainly determine the exact values of the signed Roman $k$-domination number of some special classes of digraphs.

The complete bipartite digraph $K_{p,q}^*$ is the digraph obtained from the complete bipartite graph $K_{p,q}$ when each edge $e$ of $K_{p,q}$ is replaced by two oppositely oriented arcs with the same ends as $e$.

**Theorem 1.** *For any positive integers $p$, $q$ and $k$ with $q \geq p \geq k + 2$,*

$$\gamma_{sR}^k(K_{p,q}^*) = 2k + 2.$$

***Proof.*** Let $X = \{x_1, x_2, \ldots, x_p\}$ and $Y = \{y_1, y_2, \ldots, y_q\}$ be the bipartition of $K_{p,q}^*$ and let $f$ be a $\gamma_{sR}^k(K_{p,q}^*)$-function. For each $i \in \{-1, 1, 2\}$, let $X_i = \{x_j \in X : f(x_j) = i\}$ and let $Y_i = \{y_j \in Y : f(y_j) = i\}$. First we claim that $\gamma_{sR}^k(K_{p,q}^*) \geq 2k + 2$. We consider three cases as follows.

*Case* 1. $X_{-1} = Y_{-1} = \emptyset$. We observe that $f(u) = 1$ or $f(u) = 2$ for each $u \in X \cup Y$ and hence

$$\gamma_{sR}^k(K_{p,q}^*) = \omega(f) \geq p + q \geq 2k + 4 \geq 2k + 2.$$

*Case* 2. $X_{-1} \neq \emptyset$ and $Y_{-1} \neq \emptyset$. Without loss of generality, we may assume that $f(x_1) = f(y_1) = -1$. Then by the definition of $\gamma_{sR}^k(K_{p,q}^*)$-function, we have

$f[x_1], f[y_1] \geq k$ and hence

$$\gamma_{sR}^k(K_{p,q}^*) = \omega(f) = f[x_1] + f[y_1] - f(x_1) - f(y_1) \geq 2k + 2.$$

*Case* 3. Exactly one of $X_{-1}$ and $Y_{-1}$ is $\emptyset$. Without loss of generality, we may assume that $X_{-1} = \emptyset$ and $Y_{-1} \neq \emptyset$. Since $Y_{-1} \neq \emptyset$, it follows from the definition of $\gamma_{sR}^k(K_{p,q}^*)$-function that there exists some vertex, say $x_p$, of $X$ such that $f(x_p) = 2$. Then

$$\begin{aligned}\gamma_{sR}^k(K_{p,q}^*) =& \omega(f) = f[x_1] + \sum_{i=2}^{p-1} f(x_i) + f(x_p)\\ &\geq k + (p-2) + 2 \geq k + p \geq 2k + 2.\end{aligned}$$

Therefore, by the above proof, we have $\gamma_{sR}^k(K_{p,q}^*) \geq 2k+2$. To verify that $\gamma_{sR}^k(K_{p,q}^*) \leq 2k+2$, we now provide an SRkDF $g : V(K_{p,q}^*) \to \{-1, 1, 2\}$ as follows. If $p = k + 3t + r$ (respectively, $q = k + 3t + r$), where $1 \leq r \leq 3$, then $g(x_i) = -1$ (respectively, $g(y_i) = -1$) for $1 \leq i \leq 2t + r - 1$, $g(x_i) = 2$ (respectively, $g(y_i) = 2$) for $2t + r \leq i \leq 3t + 2r - 2$ and $g(x_i) = 1$ (respectively, $g(y_i) = 1$) otherwise. Then it is easy to see that $\sum_{i=1}^p g(x_i) = \sum_{i=1}^q g(y_i) = k+1$ and hence

$$\gamma_{sR}^k(K_{p,q}^*) \leq \omega(g) = \sum_{i=1}^p g(x_i) + \sum_{i=1}^q g(y_i) = 2k + 2,$$

which completes our proof. ∎

Note that $N^-_{K_{p,q}^*}[v] = N_{K_{p,q}}[v]$ for each $v \in V(K_{p,q}^*) = V(K_{p,q})$, where $N_{K_{p,q}}[v]$ is the closed neighborhood of $v$ in $K_{p,q}$. Therefore, $\gamma_{sR}^k(K_{p,q}^*) = \gamma_{sR}^k(K_{p,q})$. Hence we have the following immediate consequence of Theorem 1.

**Corollary 2.** *For any positive integers $p$, $q$ and $k$ with $q \geq p \geq k+2$,*

$$\gamma_{sR}^k(K_{p,q}) = 2k + 2.$$

The special case $p = q$ of Corollary 2 can be found in [9].

**Theorem 3.** *For any positive integers $p$ and $q$ with $q \geq p$,*

$$\gamma_{sR}(K_{p,q}^*) = \begin{cases} 1, & \text{if } p = 1 \text{ and } q \neq 2,\\ 2, & \text{if } p = 1 \text{ and } q = 2,\\ 3, & \text{if } p = 2,\\ 4, & \text{if } p \geq 3.\end{cases}$$

***Proof.*** It is easy to verify that $\gamma_{sR}(K^*_{1,1}) = 1$, $\gamma_{sR}(K^*_{1,2}) = 2$ and $\gamma_{sR}(K^*_{2,2}) = 3$. Assume next that $q \geq 3$. Let $X = \{x_1, x_2, \ldots, x_p\}$ and $Y = \{y_1, y_2, \ldots, y_q\}$ be the bipartition of $K^*_{p,q}$ and let $f$ be a $\gamma_{sR}(K^*_{p,q})$-function.

By the definition of $\gamma_{sR}(K^*_{1,q})$-function, we have $\gamma_{sR}(K^*_{1,q}) = \omega(f) = f[x_1] \geq 1$. In order to prove that $\gamma_{sR}(K^*_{1,q}) \leq 1$, we now provide an SR1DF $g : V(K^*_{1,q}) \to \{-1, 1, 2\}$ as follows. If $q = 2l$, where $l \geq 2$ is an integer, then $g(x_1) = g(y_1) = 2$, $g(y_i) = 1$ for $2 \leq i \leq l-1$ and $g(y_i) = -1$ otherwise; if $q = 2l+1$, where $l \geq 1$ is an integer, then $g(x_1) = 2$, $g(y_i) = 1$ for $1 \leq i \leq l$ and $g(y_i) = -1$ otherwise. Then $\gamma_{sR}(K^*_{1,q}) \leq \omega(g) = g(x_1) + \sum_{i=1}^{q} g(y_i) = 1$. As a result, we have $\gamma_{sR}(K^*_{1,q}) = 1$.

If $p \geq 3$, then it follows from Theorem 1 that $\gamma_{sR}(K^*_{p,q}) = 4$.

It remains to show that $\gamma_{sR}(K^*_{2,q}) = 3$. If $f(y_i) \neq -1$ for each $i \in \{1, 2, \ldots, q\}$, then $\gamma_{sR}(K^*_{2,q}) = \omega(f) = f[y_q] + \sum_{i=1}^{q-1} f(y_i) \geq 1 + (q-1) = q \geq 3$. Otherwise, there exists some vertex, say $y_i$, of $Y$ such that $f(y_i) = -1$. Then, by the definition of $\gamma_{sR}(K^*_{2,q})$-function, there exists some vertex, say $x_2$, of $X$ such that $f(x_2) = 2$, implying that

$$\gamma_{sR}(K^*_{2,q}) = \omega(f) = f[x_1] + f(x_2) \geq 1 + 2 = 3.$$

In order to prove that $\gamma_{sR}(K^*_{2,q}) \leq 3$, we now provide an SR1DF $h : V(K^*_{2,q}) \to \{-1, 1, 2\}$ as follows. If $q = 2t$, where $t \geq 2$ is an integer, then $h(x_1) = 1$, $h(x_2) = 2$, $h(y_i) = -1$ for $1 \leq i \leq t$ and $h(y_i) = 1$ otherwise; if $q = 2t+1$, where $t \geq 1$ is an integer, then $h(x_1) = h(x_2) = 2$, $h(y_i) = -1$ for $1 \leq i \leq t+1$ and $h(y_i) = 1$ otherwise. Then $\gamma_{sR}(K^*_{2,q}) \leq \omega(h) = \sum_{i=1}^{2} h(x_i) + \sum_{i=1}^{q} h(y_i) = 3$, which completes our proof. ∎

As an immediate consequence of Theorem 3, we have the following result.

**Corollary 4.** *For any positive integers $p$ and $q$ with $q \geq p$,*

$$\gamma_{sR}(K_{p,q}) = \begin{cases} 1, & \text{if } p = 1 \text{ and } q \neq 2, \\ 2, & \text{if } p = 1 \text{ and } q = 2, \\ 3, & \text{if } p = 2, \\ 4, & \text{if } p \geq 3. \end{cases}$$

The special case $p = 1$ of Corollary 4 can be found in [1] as Observation 5. Note that in the case $q \geq 4$ even, Observation 5 in [1] is not correct.

Volkmann [11] established the lower and upper bounds on the signed Roman 1-domination number of rooted trees and cotrafunctional digraphs. We will supplement these results for $k \in \{2, 3, 4\}$.

**Theorem 5.** *For any rooted tree $T$ of order $n$, $\gamma^2_{sR}(T) = n + 1$.*

***Proof.*** Let $f$ be a $\gamma^2_{sR}(T)$-function and let $r$ be the root of $T$. Note that $d^-(x) = 1$ for each $x \in V(T)\backslash\{r\}$. Therefore, if there exists some vertex, say $u$, of $V(T)\backslash\{r\}$ such that $f(u) = -1$, then $f[u] \leq 1$, a contradiction. Thus, $f(x) = 1$ or $f(x) = 2$ for each $x \in V(T)\backslash\{r\}$. Moreover, since $d^-(r) = 0$, $f(r) = 2$. Therefore, $\gamma^2_{sR}(T) = \omega(f) \geq n + 1$. On the other hand, it is easy to see that $g = (\emptyset, V(T)\backslash\{r\}, \{r\})$ is an SR2DF on $T$ and hence $\gamma^2_{sR}(T) \leq \omega(g) = n + 1$. Then the desired result holds. ∎

Harary *et al.* [6] showed that every connected contrafunctional digraph has a unique directed cycle and the removal of any arc of the directed cycle results in a rooted tree. We define the *height* of a connected contrafunctional digraph $D$, denoted by $h(D)$, to be the maximum distance from its unique directed cycle $C$ to all vertices of $D$, i.e., $h(D) = \max\{d_D(C, v) : v \in V(D)\}$. In particular, the height of a directed cycle is exactly equal to 0.

**Lemma 6.** *Let $D$ be a connected contrafunctional digraph of order $n$ with $h(D) = 1$. Then*
$$\gamma^3_{sR}(D) \leq 3n/2.$$

***Proof.*** Let $C$ be the unique directed cycle of $D$, $v_{i_j}$ be the vertex of $C$ such that $v_{i_j}$ has at least one out-neighbor not in $C$ for $1 \leq j \leq l$ and let $V'$ be the set of out-neighbors of $v_{i_j}$ not in $C$ for $1 \leq j \leq l$. Define the function $f : V(D) \to \{-1, 1, 2\}$ by $f(v_{i_j}) = 2$ for $1 \leq j \leq l$ and $f(x) = 1$ for each $x \in V'$ and hence
$$\sum_{x\in N^+[v_{i_j}]\backslash V(C)} f(x) = |N^+[v_{i_j}]\backslash V(C)| + 1 \leq 3|N^+[v_{i_j}]\backslash V(C)|/2.$$

We observe that $D' = D - (\{v_{i_1}, v_{i_2}, \ldots, v_{i_l}\} \cup V')$ is empty or consists of some directed paths. If $v_1v_2 \cdots v_k$ is such a directed path of $D'$, then we define $f(v_i) = 1$ if $i$ is odd and $f(v_i) = 2$ if $i$ is even and hence $\sum_{i=1}^{k} f(v_i) = \lfloor 3k/2 \rfloor$. Altogether, it is easy to verify that $f$ is an SR3DF on $D$ with $\omega(f) \leq 3n/2$. Therefore, $\gamma^3_{sR}(D) \leq \omega(f) \leq 3n/2$. ∎

**Theorem 7.** *Let $D$ be a connected contrafunctional digraph of order $n$. Then*

(*a*) $\gamma^2_{sR}(D) = n$;

(*b*) $n + k/2 \leq \gamma^3_{sR}(D) \leq (3n+1)/2$, *where $k$ is the length of the unique directed cycle of $D$. In particular, if $k$ is even, then $\gamma^3_{sR}(D) \leq 3n/2$;*

(*c*) $\gamma^4_{sR}(D) = 2n$.

***Proof.*** (a) Let $f$ be a $\gamma^2_{sR}(D)$-function. Note that $d^-(x) = 1$ for each $x \in V(D)$. Therefore, if there exists some vertex, say $u$, of $D$ such that $f(u) = -1$, then

$f[u] = -1 + 2 = 1$, a contradiction. This implies that $f(x) = 1$ or $f(x) = 2$ for each $x \in V(D)$ and hence $\gamma^2_{sR}(D) = \omega(f) \geq n$. On the other hand, $g = (\emptyset, V(D), \emptyset)$ is an SR2DF on $D$ and hence $\gamma^2_{sR}(D) \leq \omega(g) = n$. Therefore, $\gamma^2_{sR}(D) = n$.

(b) Let $f$ be a $\gamma^3_{sR}(D)$-function. Then by the similar method to (a), we have $f(x) = 1$ or $f(x) = 2$ for each $x \in V(D)$. Let $C = v_1v_2\cdots v_kv_1$ be the unique directed cycle of $D$.

**Claim 1.** $\sum_{i=1}^{k} f(v_i) \geq 3k/2$.

***Proof.*** If $f(v_i)=2$ for each $i \in \{1, 2, \ldots, k\}$, then clearly $\sum_{i=1}^{k} f(v_i) = 2k > 3k/2$. Otherwise, there exists some vertex, say $v_1$, of $C$ such that $f(v_1) = 1$. Since $v_k$ is the unique in-neighbor of $v_1$ in $D$, $f(v_k) = 2$. Thus, we have that if $k$ is even, then $\sum_{i=1}^{k} f(v_i) = \sum_{j=1}^{k/2} f[v_{2j}] \geq 3k/2$; and if $k$ is odd, then $\sum_{i=1}^{k} f(v_i) = \sum_{j=1}^{(k-1)/2} f[v_{2j}] + f(v_k) \geq 3(k-1)/2 + 2 \geq 3k/2$. So, this claim is true. $\square$

Note that $f(x) = 1$ or $f(x) = 2$ for each $x \in V(D)$. Therefore, by Claim 1, we have

$$\gamma^3_{sR}(D) = \omega(f) = \sum_{i=1}^{k} f(v_i) + \sum_{x \in V(D)\setminus V(C)} f(x) \geq 3k/2 + (n-k) = n + k/2,$$

establishing the desired lower bound.

We proceed to show the upper bound by induction on $n$. If $n = 3$, then the assertion is trivial. Hence we may assume that $n \geq 4$. If $D$ is a directed even cycle (respectively, a directed odd cycle), then it is easy to verify that $\gamma^3_{sR}(D) = 3n/2$ (respectively, $\gamma^3_{sR}(D) = (3n+1)/2$). If $h(D) = 1$, then by Lemma 6, $\gamma^3_{sR}(D) \leq 3n/2$. Assume now that $h(D) \geq 2$. Let $y \in V(D)$ such that $d(C, y) = h(D)$, $x$ be the unique in-neighbor of $y$, $D' = D - N^+[x]$ and let $g$ be a $\gamma^3_{sR}(D')$-function. Define the function $g' : V(D) \to \{-1, 1, 2\}$ by $g'(v) = g(v)$ for $v \in V(D')$, $g'(x) = 2$ and $g'(v) = 1$ for $v \in N^+(x)$. By the similar method to (a), we have $g'(z) = g(z) = 1$ or $g'(z) = g(z) = 2$ for the unique in-neighbor $z$ of $x$. Therefore, $g'$ is an SR3DF on $D$ with $\omega(g') = \omega(g) + g'[x] = \omega(g) + (|N^+[x]| + 1)$. Moreover, it is easy to see that $D'$ is also a connected contrafunctional digraph, which has the same length of the unique directed cycle as $D$. Thus, if $k$ is odd, then by the induction hypothesis,

$$\begin{aligned}\gamma^3_{sR}(D) &\leq \omega(g') = \omega(g) + (|N^+[x]| + 1) \leq \frac{3(n - |N^+[x]|) + 1}{2} + (|N^+[x]| + 1)\\ &= \frac{3n+1}{2} - \frac{|N^+[x]| - 2}{2} \leq \frac{3n+1}{2}.\end{aligned}$$

The discussion for the case when $k$ is even is analogous, which establish the desired upper bound.

(c) Let $f$ be a $\gamma^4_{sR}(D)$-function. Note that each vertex of $D$ has in-degree 1. So by the definition of $\gamma^4_{sR}(D)$-function, $f(x) = 2$ for each $x \in V(D)$. Therefore, $\gamma^4_{sR}(D) = \omega(f) = 2n$. ■

Note that a contrafunctional digraph is a disjoint union of connected contrafunctional digraphs. Therefore, as an immediate consequence of Theorem 7, we have the following result.

**Corollary 8.** *Let $D$ be a contrafunctional digraph of order $n$. Then*
(a) $\gamma^2_{sR}(D) = n$*;*
(b) *If the length of the unique directed cycle of every connected component of $D$ is even, then $\gamma^3_{sR}(D) \le 3n/2$;*
(c) $\gamma^4_{sR}(D) = 2n$.

**Theorem 9.** *Let $D$ be a connected contrafunctional digraph of order $n$. Then $\gamma^3_{sR}(D) = n + 1$ if and only if $D$ is a directed cycle of length 2 or $D$ consists of a directed cycle $C_2$ of length 2 and exactly one of the vertices of $C_2$ having $n - 2$ out-neighbors not in $C_2$.*

***Proof.*** Clearly, if $D$ is a directed cycle of length 2 or $D$ consists of a directed cycle $C_2$ of length 2 and exactly one of the vertices of $C_2$ having $n - 2$ out-neighbors not in $C_2$, then $\gamma^3_{sR}(D) = n + 1$.

Conversely, assume that $\gamma^3_{sR}(D) = n + 1$. Then Theorem 7(b) shows that the unique directed cycle of $D$ has length 2. Let $C_2 = v_1v_2v_1$ be the unique directed cycle of $D$, and let $f$ be a $\gamma^3_{sR}(D)$-function. By the similar method to (a) of Theorem 7, we have $f(x) = 1$ or $f(x) = 2$ for each $x \in V(D)$. If $v_1$ has an out-neighbor $w_1$ not in $C_2$ and $v_2$ has an out-neighbor $w_2$ not in $C_2$, then $f(v_1) + f(w_1) \ge 3$ and $f(v_2) + f(w_2) \ge 3$, and we obtain the contradiction $\gamma^3_{sR}(D) \ge n + 2$. So assume that, without loss of generality, only $v_1$ has an out-neighbor. Assume next that $h(D) \ge 2$. Let $y \in V(D)$ such that $d(C_2, y) = h(D)$, and let $x$ be the unique in-neighbor of $y$. Then $f(x) + f(y) \ge 3$ and $f(v_1) + f(v_2) \ge 3$, and we therefore arrive at the contradiction $\gamma^3_{sR}(D) \ge n + 2$. Consequently, $h(D) = 0$ or $h(D) = 1$ such that only $v_1$ or $v_2$ has out-neighbors. This completes the proof.

■

## 3. General Digraphs

Our aim in this section is to establish some bounds on the signed Roman $k$-domination number of general digraphs.

For a positive integer $k$, a *$k$-dominating set* of a digraph $D$ is a subset $S$ of the vertex set of $D$ such that every vertex not in $S$ has at least $k$ in-neighbors in $S$. The *$k$-domination number* of a digraph $D$, denoted by $\gamma_k(D)$, is the minimum cardinality of a $k$-dominating set of $D$.

**Theorem 10.** *For any digraph $D$ of order $n$ with $\Delta^- \geq 2$,*

$$\gamma_{sR}^2(D) \geq 2\gamma_2(D) + 1 - n.$$

***Proof.*** Let $f = (V_{-1}, V_1, V_2)$ be a $\gamma_{sR}^2(D)$-function. Assume that $V_2 = \emptyset$. Then clearly $V_{-1} = \emptyset$ and $V_1 = V(D)$, implying that $\gamma_{sR}^2(D) = |V_1| = n$. On the other hand, it is easy to see that $V(D)\backslash\{v\}$ is a 2-dominating set of $D$, where $v$ is a vertex of $D$ with $d^-(v) = \Delta^-$, and hence $\gamma_2(D) \leq |V(D)\backslash\{v\}| = n - 1$. Therefore, $\gamma_{sR}^2(D) = n > 2(n-1)+1-n \geq 2\gamma_2(D)+1-n$. Hence we may assume that $V_2 \neq \emptyset$. Note that $|V_{-1}| = n - |V_1| - |V_2|$ and $V_1 \cup V_2$ is a 2-dominating set of $D$. Therefore,

$$\begin{aligned}\gamma_{sR}^2(D) &= \omega(f) = |V_1| + 2|V_2| - |V_{-1}| = 2|V_1| + 3|V_2| - n\\ &= 2(|V_1| + |V_2|) + |V_2| - n \geq 2\gamma_2(D) + 1 - n,\end{aligned}$$

which completes our proof. ■

Let $G$ be a bipartite graph with bipartition $(\mathcal{L}, \mathcal{R})$ (standing for left and right). A subset $S$ of vertices in $\mathcal{R}$ is a *left dominating set* of $G$ if every vertex of $\mathcal{L}$ is adjacent to a vertex in $S$. The *left domination number*, denoted by $\gamma_{\mathcal{L}}(G)$, is the minimum cardinality of a left dominating set of $G$. A left dominating set of $G$ of cardinality $\gamma_{\mathcal{L}}(G)$ is called a *$\gamma_{\mathcal{L}}(G)$-set*. Let $\delta_{\mathcal{L}}(G)$ denote the minimum degree of a vertex of $\mathcal{L}$ in $G$. Ahangar *et al.* [1] established the following upper bound on the left domination number of a bipartite graph in terms of its order.

**Theorem 11** [1]**.** *Let $G$ be a bipartite graph of order $n$ with bipartition $(\mathcal{L}, \mathcal{R})$. If $\delta_{\mathcal{L}}(G) \geq 2$, then $\gamma_{\mathcal{L}}(G) \leq n/3$.*

For a positive integer $k$, a function $f : V(D) \to \{-1, 1\}$ is called a *signed $k$-dominating function* (SkDF) on a digraph $D$ if $f[v] \geq k$ for each vertex $v \in V(D)$. The *weight* of an SkDF $f$ is $\omega(f) = \sum_{v \in V(D)} f(v)$. The *signed $k$-domination number* $\gamma_{kS}(D)$ of a digraph $D$ is the minimum weight of an SkDF on $D$. An SkDF on $D$ with weight $\gamma_{kS}(D)$ is called a *$\gamma_{kS}(D)$-function.* The special case $k = 1$ was introduced and investigated by Zelinka [12]. Using Theorem 11, we can derive the following result.

**Theorem 12.** *For any digraph $D$ of order $n$ and positive integer $k$ with $\delta^- \geq k - 1$,*

$$\gamma_{sR}^k(D) \leq \gamma_{kS}(D) + n/3.$$

***Proof.*** Let $f$ be a $\gamma_{kS}(D)$-function and let $\mathcal{L}$ and $\mathcal{R}$ denote the sets of those vertices in $D$ which are assigned under $f$ the values $-1$ and 1, respectively. Then $|\mathcal{L}| + |\mathcal{R}| = n$ and $\gamma_{kS}(D) = \omega(f) = |\mathcal{R}| - |\mathcal{L}|$.

If $\mathcal{L} = \emptyset$, that is, if $\mathcal{R} = V(D)$, then define the function $g : V(D) \to \{-1, 1, 2\}$ by $g(x) = f(x) = 1$ for each $x \in V(D)$. We observe that $g$ is an SRkDF on $D$, implying that

$$\gamma^k_{sR}(D) \le \omega(g) = \omega(f) = \gamma_{kS}(D) < \gamma_{kS}(D) + n/3.$$

Hence we may assume that $\mathcal{L} \neq \emptyset$. In this case, it follows from the definition of $\gamma_{kS}(D)$-function that $\mathcal{R} \neq \emptyset$. Let $D'$ be the bipartite spanning subdigraph of $D$ with bipartition $(\mathcal{L}, \mathcal{R})$, where $A(D') = \{(u, v) \in A(D) : u \in \mathcal{R} \text{ and } v \in \mathcal{L}\}$. Since $f$ is a $\gamma_{kS}(D)$-function, each vertex of $\mathcal{L}$ has at least $k + 1$ in-neighbors in $\mathcal{R}$ in $D'$ and hence $\delta^-_{\mathcal{L}}(D') \ge k + 1 \ge 2$, where $\delta^-_{\mathcal{L}}(D') = \min\{d^-_{D'}(v) : v \in \mathcal{L}\}$. Let $H$ be the graph obtained from $D'$ by replacing any arc with an edge and let $\mathcal{R}_2$ be a $\gamma_{\mathcal{L}}(H)$-set. Then clearly $\delta_{\mathcal{L}}(H) = \delta_{\mathcal{L}}(D') \ge 2$ and hence by Theorem 11, $\gamma_{\mathcal{L}}(H) \le n/3$, implying that $|\mathcal{R}_2| = \gamma_{\mathcal{L}}(H) \le n/3$. Moreover, since $\mathcal{R}_2$ is a $\gamma_{\mathcal{L}}(H)$-set, any vertex in $\mathcal{L}$ is adjacent to some vertex in $\mathcal{R}_2$ in $H$ and hence any vertex in $\mathcal{L}$ is adjacent from some vertex in $\mathcal{R}_2$ in $D'$ and so in $D$. Let $\mathcal{R}_1 = \mathcal{R} \backslash \mathcal{R}_2$. Define the function $h : V(D) \to \{-1, 1, 2\}$ by

$$h(x) = \begin{cases} f(x) = -1, & \text{if } x \in \mathcal{L}, \\ f(x) = 1, & \text{if } x \in \mathcal{R}_1, \\ 2, & \text{if } x \in \mathcal{R}_2. \end{cases}$$

Note that $f$ is a $\gamma_{kS}(D)$-function. Therefore, $h$ is an SR$k$DF on $D$ and hence

$$\begin{aligned} \gamma^k_{sR}(D) &\le \omega(h) = |\mathcal{R}_1| + 2|\mathcal{R}_2| - |\mathcal{L}| = (|\mathcal{R}| - |\mathcal{R}_2|) + 2|\mathcal{R}_2| - |\mathcal{L}| \\ &= (|\mathcal{R}| - |\mathcal{L}|) + |\mathcal{R}_2| = \gamma_{kS}(D) + |\mathcal{R}_2| \le \gamma_{kS}(D) + n/3, \end{aligned}$$

which completes our proof. ■

## 4. Oriented Trees

In this section, we establish a lower bound on the signed Roman 2-domination number of an oriented tree in terms of its order and characterize the oriented trees achieving the lower bound. For this purpose, we first give some definitions and properties.

Let $\mathcal{P}$ denote the family consisting of all oriented paths $P$ of odd order, where

(a) $d^+(v) \cdot d^-(v) = 0$ for each vertex $v$ of $P$;
(b) $d^+(v) = 1$ and $d^-(v) = 0$ for each vertex $v$ of $P$ with $d^+(v) + d^-(v) = 1$.

The complete bipartite graph $K_{1,n-1}$ is called a *star* of order $n$.

**Lemma 13.** *For any oriented star $S_n$ of order $n \geq 2$,*

$$\gamma_{sR}^2(S_n) \geq (n+3)/2,$$

*with equality if and only if $n = 3$ and $S_n \in \mathcal{P}$.*

***Proof.*** Let $V(S_n) = \{v_0, v_1, v_2, \ldots, v_{n-1}\}$, where $v_0$ is the vertex of $S_n$ such that $d^+(v_i) + d^-(v_i)$ is maximum. If $n = 2$, then clearly $S_n \notin \mathcal{P}$ and $\gamma_{sR}^2(S_n) = 3 > (n+3)/2$. Hence we may assume that $n \geq 3$. If $d^-(v_0) = 0$, then clearly $S_n \notin \mathcal{P}$ and $f = (\emptyset, V\backslash\{v_0\}, \{v_0\})$ is a $\gamma_{sR}^2(S_n)$-function, implying that $\gamma_{sR}^2(S_n) = \omega(f) = (n-1) + 2 = n + 1 > (n+3)/2$.

If $1 \leq d^-(v_0) \leq n-2$, then $S_n \notin \mathcal{P}$ and $g = (\emptyset, V(S_n)\backslash N^-(v_0), N^-(v_0))$ is a $\gamma_{sR}^2(S_n)$-function, implying that $\gamma_{sR}^2(S_n) = \omega(g) = (n - |N^-(v_0)|) + 2|N^-(v_0)| = n + |N^-(v_0)| = n + d^-(v_0) \geq n + 1 > (n+3)/2$.

Now suppose that $d^-(v_0) = n-1$. Then $h = (\{v_0\}, \emptyset, V\backslash\{v_0\})$ is a $\gamma_{sR}^2(S_n)$-function, implying that $\gamma_{sR}^2(S_n) = \omega(h) = -1 + 2(n-1) = 2n-3$. Therefore, if $n = 3$, then clearly $S_n \in \mathcal{P}$ and $\gamma_{sR}^2(S_n) = 2n - 3 = (n+3)/2$; otherwise, $n \geq 4$, implying that $S_n \notin \mathcal{P}$ and $\gamma_{sR}^2(S_n) = 2n - 3 > (n+3)/2$. ∎

**Lemma 14.** *Let $P \in \mathcal{P}$ be an oriented path of odd order $n \geq 1$. Then*

$$\gamma_{sR}^2(P) = (n+3)/2.$$

***Proof.*** Since $P \in \mathcal{P}$, we may assume that the underlying graph of $P$ is the path $v_1 v_2 \cdots v_n$ of odd order $n$, $d_P^+(v_i) \cdot d_P^-(v_i) = 0$ for each $i$ and $d_P^-(v_1) = d_P^-(v_n) = 0$. Let $f$ be a $\gamma_{sR}^2(P)$-function. Since $d_P^-(v_i) = 0$ for each odd integer $i$, it follows from the definition of $\gamma_{sR}^2(P)$-function that $f(v_i) = 2$. Moreover, we observe that $f(v_i) \geq -1$ for each even integer $i$. Therefore, we have

$$\begin{aligned}\gamma_{sR}^2(P) = {} & \omega(f) = \sum_{k=1}^{(n+1)/2} f(v_{2k-1}) + \sum_{k=1}^{(n-1)/2} f(v_{2k}) \\ & \geq (n+1) - (n-1)/2 = (n+3)/2.\end{aligned}$$

Note that $g = (V_{-1}, \emptyset, V_2)$ is an SR2DF on $P$, where $V_{-1} = \{v_{2k} : k = 1, 2, \ldots, (n-1)/2\}$ and $V_2 = \{v_{2k-1} : k = 1, 2, \ldots, (n+1)/2\}$, implying that $\gamma_{sR}^2(P) \leq \omega(g) = (n+3)/2$. Then the desired result holds. ∎

**Theorem 15.** *For any oriented tree $T$ of order $n \geq 1$,*

$$\gamma_{sR}^2(T) \geq (n+3)/2,$$

*with equality if and only if $T \in \mathcal{P}$.*

***Proof.*** We proceed by induction on $n$. If $n \in \{1, 2, 3\}$, then by Lemma 13 or 14, the assertion is trivial. Hence we may assume that $n \geq 4$. Let $f$ be a $\gamma^2_{sR}(T)$-function. Then there exists some vertex, say $v_0$, of $T$ such that $f(v_0) = -1$, for otherwise $\gamma^2_{sR}(T) = \omega(f) \geq n > (n+3)/2$, a contradiction. Let $T_1, T_2, \ldots, T_k$ be the connected components of $T - v_0$ and let $f_i$ be the restriction of $f$ on $T_i$. Since $f(v_0) = -1$ and $f[v_0] \geq 2$, $d^-(v_0) \geq 2$. This implies that $k \geq 2$. Moreover, clearly $f_i$ is an SR2DF on $T_i$ for each $i \in \{1, 2, \ldots, k\}$.

Now suppose that $k=2$. In this case, we note that $d^-(v_0)=2$ and $d^+(v_0) = 0$. Therefore, we may assume, without loss of generality, that $v_1 \in V(T_1), v_2 \in V(T_2)$ such that $(v_1, v_0), (v_2, v_0) \in A(T)$. If $T_1, T_2 \in \mathcal{P}$, then clearly $T \in \mathcal{P}$ and hence by Lemma 14, $\gamma^2_{sR}(T) = (n+3)/2$. Otherwise, assume, without loss of generality, that $T_1 \notin \mathcal{P}$. Then $T \notin \mathcal{P}$ and by the induction hypothesis, we have $\omega(f_1) \geq \gamma^2_{sR}(T_1) > (|V(T_1)| + 3)/2$ and $\omega(f_2) \geq \gamma^2_{sR}(T_2) \geq (|V(T_2)| + 3)/2$, and hence

$$\gamma^2_{sR}(T) = \omega(f) = f(v_0) + \sum_{i=1}^{2} \omega(f_i) > -1 + \sum_{i=1}^{2} (|V(T_i)| + 3)/2 = (n+3)/2.$$

If $k \geq 3$, then $T \notin \mathcal{P}$ and by the induction hypothesis, we have that for each $i$, $\omega(f_i) \geq \gamma^2_{sR}(T_i) \geq (|V(T_i)| + 3)/2$ and hence

$$\begin{aligned}\gamma^2_{sR}(T) &= f(v_0) + \sum_{i=1}^{k} \omega(f_i) \geq -1 + \sum_{i=1}^{k} (|V(T_i)| + 3)/2 \\ &= -1 + (n - 1 + 3k)/2 = (n + 3k - 3)/2 > (n+3)/2,\end{aligned}$$

which completes our proof. ∎

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