BOUNDING THE LOCATING-TOTAL DOMINATION NUMBER OF A TREE IN TERMS OF ITS ANNIHILATION NUMBER

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Abstract

Suppose $G = (V, E)$ is a graph with no isolated vertex. A subset $S$ of $V$ is called a locating-total dominating set of $G$ if every vertex in $V$ is adjacent to a vertex in $S$, and for every pair of distinct vertices $u$ and $v$ in $V - S$, we have $N(u) \cap S \neq N(v) \cap S$. The locating-total domination number of $G$, denoted by $\gamma_L^t(G)$, is the minimum cardinality of a locating-total dominating set of $G$. The annihilation number of $G$, denoted by $a(G)$, is the largest integer $k$ such that the sum of the first $k$ terms of the nondecreasing degree sequence of $G$ is at most the number of edges in $G$. In this paper, we show that for any tree of order $n \geq 2$, $\gamma_L^t(T) \leq a(T) + 1$ and we characterize the trees achieving this bound.

Keywords: total domination, locating-total domination, annihilation number, tree.

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1. Introduction

Given a graph $G = (V(G), E(G))$, we usually use $n$ for the number of vertices and $m$ for the number of edges. For a vertex $v$ in $G$, the set $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ (or $N(v)$) is called the neighborhood of $v$. The degree of $v$ in $G$, denoted by $d_G(v)$ or $d(v)$, is equal to $|N(v)|$. A vertex of degree one is a leaf and a vertex adjacent to a leaf is a support vertex. We will use $l(G)$ to denote the number of leaves of $G$. For arbitrary two vertices $u$ and $v$ in $G$, the distance between $u$ and $v$, denoted by $d(u, v)$, is the number of edges in a shortest path joining $u$ and $v$. If there is no such path, then we define $d(u, v) = \infty$. The diameter of $G$ is the maximum distance among all pairs of vertices of $G$, denoted by $diam(G)$. For a subset $S \subseteq V(G)$, we use $G - S$ to denote the graph obtained from $G$ by deleting the vertices in $S$ and all edges incident with vertices in $S$. If $S = \{v\}$, we simply write $G - v$ rather than $G - \{v\}$. We define

$$\Sigma(S, G) = \sum_{v \in S} d_G(v).$$

Suppose $G = (V, E)$ is a graph with no isolated vertex. A subset $S$ of $V$ is called a total dominating set (TDS) of $G$ if every vertex in $V$ is adjacent to a vertex in $S$. A total dominating set $S$ is called a locating-total dominating set (LTDS) if for every pair of distinct vertices $u$ and $v$ in $V - S$, we have $N(u) \cap S \neq N(v) \cap S$. The locating-total domination number of $G$, denoted by $\gamma^L(G)$, is the minimum cardinality of a locating-total dominating set of $G$. An LTDS of cardinality $\gamma^L(G)$ is called a $\gamma^L(G)$-set. The concept of a locating-total dominating set in a graph was first introduced in [9], since this time many results have been obtained on this parameter (see, for instance, [1–4, 10]).

The annihilation number was first introduced in Pepper’s dissertation [13]. Originally it was defined in terms of a reduction process on the degree sequence akin to the Havel-Hakimi process (see, for example, [8, 14]). In [13], Pepper showed the following equivalent way to define the annihilation number. Let $d_1 \leq d_2 \leq \cdots \leq d_n$ be the nondecreasing degree sequence of a graph $G$ having $n$ vertices and $m$ edges. Then the annihilation number of $G$, denoted by $a(G)$, is the largest integer $k$ such that $\sum_{i=1}^{k} d_i \leq m$ or, equivalently, the largest integer $k$ such that

$$\sum_{i=1}^{k} d_i \leq \sum_{i=k+1}^{n} d_i.$$  

The relation between annihilation number and some graph parameters have been studied by several authors (see for example [5–7, 11–14]).

For a graph $G = (V(G), E(G))$ with $m$ edges, an $a$-set of $G$ is a subset $S$ of $V(G)$ such that $|S| = a(G)$ and $\Sigma(S, G) \leq m$, where $a(G)$ is the annihilation
number of $G$. An $a_{\text{min}}$-set of $G$ is an $a$-set $S$ of $G$ with $\Sigma(S,G)$ minimum. Thus, if $S$ is an $a_{\text{min}}$-set of $G$, then $S$ is a set of vertices (not necessarily unique) corresponding to the first $a(G)$ vertices in the nondecreasing degree sequence of $G$.

In order to prove our theorem, we introduce a variation of the annihilation number of a graph defined in [6]. The upper annihilation number of a graph $G$, denoted by $a^*(G)$, is the largest integer $k$ such that the first $k$ terms of the nondecreasing degree sequence of $G$ is at most $|E(G)| + 1$. That is, if $d_1 \leq d_2 \leq \cdots \leq d_n$ is the nondecreasing degree sequence of a graph $G$ with $m$ edges, then $a^*(G)$ is the largest integer $k$ such that $\sum_{i=1}^{k} d_i \leq m + 1$. Similarly, we define an $a_{\text{min}}^*$-set of $G$ to be a set $S$ of vertices in $G$ such that $|S| = a^*(G)$ and $S$ corresponds to the first $a^*(G)$ vertices in the nondecreasing degree sequence of $G$. By the definitions of the annihilation number and the upper annihilation number, we have $a(G) \leq a^*(G) \leq a(G) + 1$.

A path of order $n$ is $P_n$. A star of order $n$ is denoted by $S_n$. A tree is called a double star $S(p,q)$, if it is obtained from $S_{p+2}$ and $S_{q+1}$ by identifying a leaf of $S_{p+2}$ with the center of $S_{q+1}$, where $p,q \geq 1$.

In this paper, we establish an upper bound on the locating-total domination number of a tree in terms of its annihilation number. We show that for any tree of order $n \geq 2$, $\gamma^L_T(T) \leq a(T) + 1$ and we characterize the trees achieving this bound.

2. The Main Result

In order to characterize the trees satisfying $\gamma^L_T(T) = a(T) + 1$, we first introduce a family $\Gamma$ of labeled trees defined in [4].

For each tree $T \in \Gamma$, every vertex $v$ in $T$ has a label $\text{sta}(v) \in \{A,B,C\}$, called its status. Let $\Gamma$ be the family of labeled trees $T = T_k$ that can be obtained as follows. Let $T_0$ be a path $P_6$ in which the two leaves have status $C$, the two support vertices have status $A$ and the remaining two vertices have status $B$. If $k \geq 1$, then $T_k$ can be obtained from $T_{k-1}$ by one of the following operations.

- **Operation $\tau_1$.** For any $y \in V(T_{k-1})$, if $\text{sta}(y) = C$ and $d_{T_{k-1}}(y) = 1$, then add a path $xwyz$ and the edge $xy$. Let $\text{sta}(x) = \text{sta}(w) = B$, $\text{sta}(v) = A$ and $\text{sta}(z) = C$.

- **Operation $\tau_2$.** For any $y \in V(T_{k-1})$, if $\text{sta}(y) = B$, then add a path $xuv$ and the edge $xy$. Let $\text{sta}(x) = B$, $\text{sta}(w) = A$ and $\text{sta}(v) = C$.

Chen and Sohn [4] established the following upper bound of $\gamma^L_T(T)$ of a tree in terms of its order and number of leaves. Moreover, they gave a characterization of the trees achieving this bound.
Theorem 1 [4]. If $T$ is a tree of order $n \geq 3$ with $l$ leaves, then $\gamma^L_{t}(T) \leq \frac{n+l}{2}$.

Theorem 2 [4]. If $T$ is a tree of order $n \geq 3$ with $l$ leaves, then $\gamma^L_{t}(T) = \frac{n+l}{2}$ if and only if $T \in \Gamma$.

For each tree $T \in \Gamma$, we have the following lemma.

Lemma 3. Let $T \in \Gamma$. Then

(1) $\gamma^L_{t}(T) = a(T) + 1 = a^*(T)$.

(2) For any vertex $v \in V(T)$ with $d(v) = 2$, there are an $a_{min}^*$-set $S$ containing $v$ and an $a_{min}^*$-set $S'$ not containing $v$.

(3) For every $a_{min}^*$-set $A$, it contains no vertices of degree larger than two.

Proof. Suppose $T \in \Gamma$ is obtained from $T_0$ by applying $k_1$ $\tau_1$ operations and $k_2$ $\tau_2$ operations. Then $n(T) = 6 + 4k_1 + 3k_2$, $l(T) = 2 + k_2$ and by Theorem 2,

$$\gamma^L_{t}(T) = \frac{n(T) + l(T)}{2} = \frac{(6 + 4k_1 + 3k_2) + (2 + k_2)}{2} = 4 + 2k_1 + 2k_2.$$ 

Note that $V(T)$ consists of $2 + k_2$ leaves with status $C$, $4 + 4k_1 + k_2$ vertices of degree two and $k_2$ vertices with status $B$ and degree larger than two. By simple calculation, we have $a(T) = 3 + 2k_1 + 2k_2$ and $a^*(T) = 4 + 2k_1 + 2k_2$. Thus, (1) holds.

By the definition of an $a_{min}^*$-set, for any $a_{min}^*$-set $S$, $S$ consists of $2 + k_2$ leaves and $2 + 2k_1 + k_2$ vertices of degree two. Note that $T$ has exactly $4 + 4k_1 + k_2$ vertices of degree two and $k_2$ vertices of degree larger than two. Thus, (2) and (3) hold.

Now we present our main result.

Theorem 4. For a tree $T$ of order $n \geq 2$, the following hold.

(1) $\gamma^L_{t}(T) \leq a^*(T)$.

(2) $\gamma^L_{t}(T) \leq a(T) + 1$.

(3) $\gamma^L_{t}(T) = a(T) + 1$ if and only if $T = P_2$ or $T \in \Gamma$.

Proof. We proceed by induction on the order $n$. If $n = 2$, then $T = P_2$ and $\gamma^L_{t}(T) = 2 = a^*(T) = a(T) + 1$. If $n = 3$, then $T = P_3 \not\in \{P_2\} \cup \Gamma$ and $\gamma^L_{t}(T) = 2 = a^*(T) = a(T)$. This establishes the base cases. Next we assume that every tree $T'$ of order $3 \leq n' < n$ satisfies properties (1)–(3) in the statement of the theorem. Let $T$ be a tree of order $n$.

If $diam(T) = 2$, then $T$ is a star. Obviously, $T \not\in \{P_2\} \cup \Gamma$ and $\gamma^L_{t}(T) = n - 1 = a^*(T) = a(T)$. If $diam(T) = 3$, then $T$ is a double star, i.e., $T \cong S_{p,q}$. 

Note that $T \notin \{P_2\} \cup \Gamma$, $\gamma^L_t(T) = n - 2 = a(T)$, $a^*(T) = n - 1$ if $\min\{p, q\} = 1$ and $a^*(T) = n - 2$ if $\min\{p, q\} \geq 2$. Hence we may assume $\text{diam}(T) \geq 4$.

Let $P = x_0x_1 \cdots x_d$ be a path of length $d = \text{diam}(T)$ in $T$. We root $T$ at $x_d$.

Claim 1. We may assume that $d(x_1) = 2$.

Proof. Suppose $d(x_1) \geq 3$. Then $T \notin \Gamma$. Let $Q = N(x_1) \setminus \{x_2\}$. Then $Q$ is the set of all leaves adjacent to $x_1$. Let $T' = T - Q \cup \{x_1\}$ and $S$ be an $a^*_{\min}$-set of $T'$. Then $|E(T)| = |E(T')| + |Q| + 1$ and $\Sigma(S, T') \leq |E(T')| + 1$. Letting $S_1 = S \cup Q$, we have

$$\Sigma(S_1, T) = \Sigma(S \cup Q, T) = \Sigma(S, T) + |Q|$$

$$\leq \Sigma(S, T') + 1 + |Q|$$

$$\leq |E(T')| + 2 + |Q| = |E(T)| + 1.$$ 

Then $a^*(T) \geq a^*(T') + |Q|$. Note that every LTDS of $T'$ can extend to an LTDS of $T$ by combining it with $(Q \setminus \{x_2\}) \cup \{x_1\}$. Thus, $\gamma^L_t(T) \leq \gamma^L_t(T') + |Q|$. By the inductive hypothesis, we have

$$\gamma^L_t(T) \leq \gamma^L_t(T') + |Q| \leq a^*(T') + |Q| \leq a^*(T)$$

$$\leq a(T) + 1.$$ 

Thus (1) and (2) hold. Next we will show that $\gamma^L_t(T) \leq a(T)$.

Suppose $\gamma^L_t(T) = a(T) + 1$. Then equalities hold throughout the above inequalities, that is, $\gamma^L_t(T) = \gamma^L_t(T') + |Q|$, $\gamma^L_t(T') = a^*(T')$ and $a^*(T) = a^*(T') + |Q| = a(T) + 1$. Let $A$ be an $a^*_{\min}$-set of $T'$ and $A_1 = A \cup Q$. Then

$$\Sigma(A_1, T) = \Sigma(A \cup Q, T) = \Sigma(A, T) + |Q|$$

$$\leq \Sigma(A, T') + 1 + |Q|$$

$$\leq |E(T')| + 1 + |Q| = |E(T)|.$$ 

Hence $a(T) \geq a(T') + |Q|$. If $\gamma^L_t(T') \leq a(T')$, then $\gamma^L_t(T) = \gamma^L_t(T') + |Q| \leq a(T') + |Q| \leq a(T)$, a contradiction to the assumption of $\gamma^L_t(T) = a(T) + 1$. Thus, $\gamma^L_t(T') = a(T') + 1$. Since $d \geq 4$, we have $n(T') \geq 3$. By the inductive hypothesis, we have $T' \in \Gamma$. Thus, $\gamma^L_t(T') = (n(T') + l(T'))/2$ by Theorem 2.

If $d_{T'}(x_2) \geq 2$, then there is an $a^*_{\min}$-set $B$ not containing $x_2$ by Lemma 3 (2) and (3). Let $B_1 = B \cup Q$. Then

$$\Sigma(B_1, T) = \Sigma(B \cup Q, T) = \Sigma(B, T) + |Q|$$

$$= \Sigma(B, T') + |Q|$$

$$\leq |E(T')| + 1 + |Q| = |E(T)|.$$
and so $a(T) \geq a^*(T') + |Q|$, a contradiction to $a^*(T') + |Q| = a(T) + 1$. Thus, $d_{T'}(x_2) = 1$. Now, we have

$$
\gamma^L_i(T) = \gamma^L_i(T') + |Q| = (n(T') + l(T'))/2 + |Q|
$$

$$
= ((n(T) - |Q| - 1) + (l(T) - |Q| + 1))/2 + |Q|
$$

$$
= (n(T) + l(T))/2.
$$

By Theorem 2, $T \notin \Gamma$, a contradiction. □

By Claim 1, $d(x_1) = 2$. Let $Y = \{y_1, \ldots, y_l\}$ be the children of $x_2$, where $y_1 = x_1$. By Claim 1, we may assume $1 \leq d(y_i) \leq 2$ for all $y_i \in Y \setminus \{y_1\}$.

**Claim 2.** We may assume that $d_T(y) = 1$ for any $y \in Y \setminus \{y_1\}$.

**Proof.** Suppose there is a vertex, say $y_2 \in Y \setminus \{y_1\}$, such that $d_T(y_2) = 2$. Then $T \notin \Gamma$. Let $y_2$ be the leaf adjacent to $y_2$. Let $T' = T - \{x_0, x_1\}$ and $S$ be an $a_{\min}^*$-set of $T'$. Then $|E(T)| = |E(T')| + 2$ and $\Sigma(S, T') \leq |E(T')| + 1$. Since $d \geq 4$, $d_T(x_2) \geq 3$ and $d_{T'}(x_2) \geq 2$.

If $x_2 \in S$, by letting $S_2 = (S \cup \{x_0, x_1\}) \setminus \{x_2\}$, we have

$$
\Sigma(S_2, T) = \Sigma(S \setminus \{x_2\}, T) + 3 = \Sigma(S \setminus \{x_2\}, T') + 3
$$

$$
\leq \Sigma(S, T') - d_{T'}(x_2) + 3 \leq \Sigma(S, T') - 2 + 3
$$

$$
\leq |E(T')| + 1 + 1 = |E(T)|.
$$

If $x_2 \notin S$, by letting $S_2 = S \cup \{x_0\}$, we have

$$
\Sigma(S_2, T) = \Sigma(S, T) + 1 = \Sigma(S, T') + 1
$$

$$
\leq |E(T')| + 1 + 1 = |E(T)|.
$$

In both cases, we have $\Sigma(S_2, T) \leq |E(T)|$ which implies $a(T) \geq a^*(T') + 1$.

Let $D$ be a $\gamma^L_i(T')$-set of $T'$ that contains a minimum number of leaves. Then $\{x_2, y_2\} \subseteq D$, and so $D \cup \{x_1\}$ is an LTDS of $T$. Thus, $\gamma^L_i(T) \leq \gamma^L_i(T') + 1$. By the inductive hypothesis, we have

$$
\gamma^L_i(T) \leq \gamma^L_i(T') + 1 \leq a^*(T') + 1 \leq a(T) \leq a^*(T)
$$

and we are done. □

**Claim 3.** We may assume that $Y = \{x_1\}$, i.e., $l = 1$.

**Proof.** Suppose $l \geq 2$. Then $T \notin \Gamma$ and every vertex in $Y \setminus \{y_1\}$ is a leaf in $T$ by Claim 2.
Let \( T' = T - \{x_0\} \cup (Y \setminus \{y_1\}) \) and \( S \) be an \( a_{\min}^* \)-set of \( T' \). Then \(|E(T)| = |E(T')| + l\) and \( \Sigma(S, T') \leq |E(T')| + 1 \). Since \( d_{T'}(x_1) = 1 \) and \( d_{T'}(x_2) = 2 \), we can choose \( S \) so that \( \{x_1, x_2\} \subseteq S \). Let \( S_3 = (S \setminus \{x_2\}) \cup \{x_0\} \cup (Y \setminus \{y_1\}) \). Then
\[
\Sigma(S_3, T) = \Sigma(S \setminus \{x_2\}, T) + l = \Sigma(S \setminus \{x_2\}, T') + 1 + l = \Sigma(S, T') - d_{T'}(x_2) + 1 + l = \Sigma(S, T') + l - 1 \leq |E(T')| + l = |E(T)|,
\]
and so \( a(T) \geq a^*(T') + l - 1 \). Let \( D \) be a \( \gamma_{\ell}^l(T') \)-set of \( T' \) that contains a minimum number of leaves. Then \( x_2 \in D \). Thus, \( D \cup (Y \setminus \{y_1\}) \) is an LTDS of \( T \) and \( \gamma_{\ell}^l(T) \leq \gamma_{\ell}^l(T') + l - 1 \). By the inductive hypothesis, we have
\[
\gamma_{\ell}^l(T) \leq \gamma_{\ell}^l(T') + l - 1 \leq a^*(T') + l - 1 \leq a(T) \leq a^*(T)
\]
and we are done. \( \square \)

By Claim 3, we have \( Y = \{x_1\} \). Since \( d \geq 4 \), \( d_{T}(x_3) \geq 2 \). We will finish the proof by considering the following two cases.

**Case 1.** \( d_{T}(x_3) \geq 3 \). Let \( T'' = T - \{x_0, x_1, x_2\} \) and \( S \) be an \( a_{\min}^* \)-set of \( T'' \). Then \(|E(T)| = |E(T'')| + 3\), \( d_{T''}(x_3) \geq 2 \) and \( \Sigma(S, T') \leq |E(T')| + 1 \).

If \( x_3 \notin S \), by letting \( S_4 = S \cup \{x_0, x_1\} \), we have
\[
\Sigma(S_4, T) = \Sigma(S, T) + 3 = \Sigma(S, T') + 3 \leq |E(T')| + 1 + 3 = |E(T)| + 1.
\]
If \( x_3 \in S \), by letting \( S_4 = (S \setminus \{x_3\}) \cup \{x_0, x_1, x_2\} \), we have
\[
\Sigma(S_4, T) = \Sigma(S \setminus \{x_3\}, T') + 5 = \Sigma(S, T') - d_{T'}(x_3) + 5 \leq \Sigma(S, T') - 2 + 5 \leq |E(T')| + 1 + 3 = |E(T)| + 1.
\]
In both cases, we have \( \Sigma(S_4, T) \leq |E(T)| + 1 \) which implies \( a^*(T) \geq a^*(T') + 2 \). Note that every LTDS of \( T'' \) can extend to an LTDS of \( T \) by combining it with \( \{x_1, x_2\} \). Thus, \( \gamma_{\ell}^l(T) \leq \gamma_{\ell}^l(T') + 2 \). By the inductive hypothesis, we have
\[
\gamma_{\ell}^l(T) \leq \gamma_{\ell}^l(T') + 2 \leq a^*(T') + 2 \leq a^*(T) \leq a(T) + 1.
\]

It remains to show that \( T \) satisfies property (3). By Lemma 3, if \( T \in \Gamma \), then \( \gamma_{\ell}^l(T) = a(T) + 1 \), as desired. Suppose now \( \gamma_{\ell}^l(T) = a(T) + 1 \). Then equalities hold throughout the above inequalities, that is,
\[
\gamma_{\ell}^l(T) = \gamma_{\ell}^l(T') + 2, \quad \gamma_{\ell}^l(T') = \gamma_{\ell}^l(T'') + 2
\]
\[ a^*(T') \text{ and } a^*(T) = a^*(T') + 2 = a(T) + 1. \text{ Since } d_T(x_3) \geq 3, x_3 \text{ is not a leaf of } T'. \text{ Thus, by Theorem 2, we have} \]
\[ \gamma^L_i(T) = \gamma^L_i(T') + 2 = \frac{(n(T') + l(T'))}{2} + 2 \]
\[ = \frac{(n(T) - 3 + l(T) - 1) + 2 = \frac{(n(T) + l(T))}{2},} \]
and then \( T \in \Gamma \).

Case 2. \( d_T(x_3) = 2 \). Let \( T'' = T - \{x_0, x_1, x_2, x_3\} \). Then \( |E(T)| = |E(T'')| + 4 \).
If \( n(T') = 1 \), then \( T = P_5 \notin \{P_2\} \cup \Gamma \) and \( \gamma^L_i(T) = 3 = a(T) = a^*(T) \). Thus, we may assume that \( n(T') \geq 2 \). Let \( S \) be an \( a^*_{\min} \)-set of \( T' \). Then \( \Sigma(S, T') \leq |E(T'')| + 1 \).
If \( x_4 \notin S \), by letting \( S_5 = S \cup \{x_0, x_1\} \), we have \( \Sigma(S_5, T) = \Sigma(S, T') + 3 \leq |E(T'')| + 4 = |E(T)| \), and so \( a(T) \geq a^*(T') + 2 \). Note that every LTDS of \( T' \) can extend to an LTDS of \( T \) by combining it with \( \{x_1, x_2\} \). Thus, \( \gamma^L_i(T) \leq \gamma^L_i(T') + 2 \).
By the inductive hypothesis, we have
\[ \gamma^L_i(T) \leq \gamma^L_i(T') + 2 \leq a^*(T') + 2 \leq a(T) \leq a^*(T). \]

Suppose now \( x_4 \in S \). Let \( S_6 = (S \setminus \{x_4\}) \cup \{x_0, x_1, x_2\} \). Then we have
\[ \Sigma(S_6, T) = \Sigma(S \setminus \{x_4\}, T') + 5 = \Sigma(S, T') - d_{T'}(x_4) + 5 \]
\[ \leq \Sigma(S, T') + 4 \leq |E(T')| + 5 \]
\[ = |E(T)| + 1 \]
which implies that \( a^*(T) \geq a^*(T') + 2 \). By the inductive hypothesis, we have
\[ \gamma^L_i(T) \leq \gamma^L_i(T') + 2 \leq a^*(T') + 2 \leq a^*(T) \leq a(T) + 1. \]

It remains to show that \( T \) satisfies property (3). By Lemma 3, if \( T \in \Gamma \), then \( \gamma^L_i(T) = a(T) + 1 \), as desired. Suppose now \( \gamma^L_i(T) = a(T) + 1 \).

Obviously, we have \( \gamma^L_i(T) = \gamma^L_i(T') + 2, \gamma^L_i(T') = a^*(T') \) and \( a^*(T) = a^*(T') + 2 = a(T) + 1 \). If \( d_{T'}(x_4) \geq 2 \), then we have
\[ \Sigma(S_6, T) = \Sigma(S, T') - d_{T'}(x_4) + 5 \leq \Sigma(S, T') + 3 \leq |E(T)|, \]
implying that \( a(T) \geq a^*(T') + 2 \), a contradiction to \( a^*(T) = a^*(T') + 2 = a(T) + 1 \). Thus, \( d_{T'}(x_4) = 1 \). If \( T' = P_2 \), then \( T = P_5 \in \Gamma \). If \( n(T') \geq 3 \), then
\[ \gamma^L_i(T) = \gamma^L_i(T') + 2 \]
\[ = \frac{(n(T') + l(T'))}{2} + 2 \]
\[ = \frac{(n(T) - 4 + l(T))}{2} + 2 \]
and then \( T \in \Gamma \) by Theorem 2. ■
3. Corollaries

Since $\gamma_t(T) \leq \gamma^L_t(T)$ for any tree $T$ of order $n \geq 2$ and $\gamma_t(T_0) = \gamma^L_t(T_0)$ for any tree $T_0 \in \Gamma$ (see [4]), by Theorems 2 and 4, we easily obtain the following corollaries which are stated as main theorems in [5].

**Corollary 5** [5]. If $T$ is a nontrivial tree, then $\gamma_t(T) \leq a(T) + 1$, and this bound is sharp.

**Corollary 6** [5]. Let $T$ be a nontrivial tree of order $n$ with $n_1$ vertices of degree 1. Then, $\gamma_t(T) = a(T) + 1$ if and only if $\gamma_t(T) = (n + n_1)/2$.

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