

THE SUPER-CONNECTIVITY OF KNESER GRAPHS

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Abstract

A vertex cut of a connected graph G is a set of vertices whose deletion disconnects G . A connected graph G is super-connected if the deletion of every minimum vertex cut of G isolates a vertex. The super-connectivity is the size of the smallest vertex cut of G such that each resultant component does not have an isolated vertex. The Kneser graph $KG(n, k)$ is the graph whose vertices are the k -subsets of $\{1, 2, \dots, n\}$ and two vertices are adjacent if the k -subsets are disjoint. We use Baranyai's Theorem on the decompositions of complete hypergraphs to show that the Kneser graph $KG(n, 2)$ are super-connected when $n \geq 5$ and that their super-connectivity is $\binom{n}{2} - 6$ when $n \geq 6$.

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1. INTRODUCTION

A *vertex cut* S of a connected graph G is a set of vertices of G whose deletion creates a disconnected graph or a trivial graph. The *connectivity* $\kappa = \kappa(G)$ of a graph G is the minimum size over all vertex cuts of G . If the deletion of every vertex cut of G of size κ isolates a vertex, then G is *super-connected* and such

a vertex cut is a *trivial vertex cut* of G . If G is super-connected, it is generally of interest to determine the size of the smallest vertex cut which is not trivial. This gives the *super-connectivity* $\kappa_1 = \kappa_1(G)$ of G , and such a vertex cut is a *super-vertex cut* of G .

The super-connectivity has been studied for various families of graphs, including circulant graphs [4], hypercubes [15, 16], permutation graphs [1] and products of various graphs (see [12] and [6], and the references therein). It arose from the notion of *conditional connectivity* proposed by Harary [8]. Given a graph G and some graph theoretical property P , Harary asked what is the size of the smallest vertex cut S , if it exists, so that $G - S$ is disconnected and every component of $G - S$ has property P . Super-connectivity yields a better measure of the reliability of a network [3] and establishes the cardinality of the smallest set S such that each one of the resultant components of $G - S$ contains at least one edge. In the current work, we study another specific family of graphs for its super-connectivity, namely Kneser graphs.

Let n and k be integers such that $n \geq k \geq 1$ and let $[n] = \{1, 2, \dots, n\}$. The set of all k -subsets of $[n]$ is denoted by $\binom{[n]}{k}$. The *Kneser graph* $KG(n, k)$ is the graph whose vertex set represents all the k -subsets of $[n]$, and two vertices A_1 and A_2 are adjacent if and only if they correspond to disjoint k -subsets. Thus, the vertex set is $V(KG(n, k)) = \binom{[n]}{k}$ and the edge set is $E(KG(n, k)) = \{\{A_i, A_j\} : A_i, A_j \in V(KG(n, k)) \text{ and } A_i \cap A_j = \emptyset \text{ for } i \neq j\}$. This family of graphs was introduced in 1955 by Kneser [11]. It is well-known that if $n < 2k$, then $KG(n, k)$ is the empty graph (that is, the graph consisting of $\binom{n}{k}$ isolated vertices), while if $n = 2k$, then $KG(n, k)$ consists of disjoint copies of the complete graph on two vertices. The Kneser graph $KG(n, 1)$ is isomorphic to the complete graph on n vertices.

Chen and Lih [7] showed that Kneser graphs are *symmetric*, that is vertex- and edge-transitive. A graph G is *vertex-transitive* when, for every pair of vertices $u, v \in V(G)$, there is an automorphism that maps u to v . Similarly, G is *edge-transitive* when there is an automorphism that maps e_1 to e_2 for every pair of edges $e_1, e_2 \in E(G)$. Symmetric graphs are usually preferred when modelling interconnection networks [9]. Vertex-transitivity permits the implementation of the same routing and communication schemes at each vertex (or node) of the network, whereas edge-transitivity allows recursive constructions to be used. Regularity is also generally sought in networks as this simplifies their study in terms of diameter and diameter vulnerability problems.

In this work, we consider the class of Kneser graphs with the smallest value of k for which the super-connectivity is not known, namely $KG(n, k)$ when $k = 2$ and $n \geq 2k + 1$. It has been argued by many that a network is more reliable if it is super-connected (see, for example, [10]). In Section 3, we prove that $KG(n, 2)$ is super-connected for $n \geq 5$. In Section 4, we determine that $\kappa_1(KG(n, 2))$ is

$\binom{n}{2} - 6$ for $n \geq 6$, and 4 for $n = 5$. We end by presenting an open problem on the super-connectivity of $KG(n, k)$.

2. PRELIMINARIES

Let H be a hypergraph $(V(H), E(H))$, where the vertex set $V(H)$ is a finite set and $E(H)$ is a multiset of subsets of $V(H)$. The number of edges of H containing a vertex v is the *degree* of v , denoted by $d_H(v)$. The hypergraph H is *almost regular* if $|d_H(v) - d_H(w)| \leq 1$ for all vertices $v, w \in V(H)$.

The fundamental tool that we use in this work is Baranyai's Theorem stated hereunder in Theorem 1.

Theorem 1 [2]. *Let a_1, \dots, a_s be positive integers such that $\sum_{j=1}^s a_j = \binom{n}{k}$. Then the edges of the k -uniform hypergraph with vertex set $[n]$ and edge set $\binom{[n]}{k}$ can be partitioned into almost regular hypergraphs with vertex set $[n]$ and edge set E_j where $|E_j| = a_j$, for $j = 1, \dots, s$.*

In particular, for the case $k = 2$, a *Baranyai Partition* of $\binom{[n]}{2}$ is a family of $m \in \mathbb{Z}^+$ partitions \mathcal{F}_i of $[n]$ (for $1 \leq i \leq m$), where $\mathcal{F}_i = A_i^1 \cup A_i^2 \cup \dots \cup A_i^p$, such that, for any given i

- (i) $|A_i^1| = |A_i^2| = \dots = |A_i^p| = 2$;
- (ii) $\mathcal{F}_i = [n]$ when n is even and $\mathcal{F}_i = [n] - \{x_i\}$ when n is odd, where $x_i \in \{1, 2, \dots, n\}$ and $x_i \neq x_j$ for $i \neq j$;
- (iii) each 2-subset of $[n]$ occurs exactly once among the A_i^j 's,

where

$$p = \left\lfloor \frac{n}{2} \right\rfloor$$

and

$$m = \frac{\binom{n}{2}}{p} = \begin{cases} n-1 & \text{when } n \text{ is even,} \\ n & \text{when } n \text{ is odd.} \end{cases}$$

Such a partition for $k = 2$ was already known to exist in the nineteenth century, was discovered for $k = 3$ by Peltsohn and for $k = 4$ by Bermond, and was generalized by Baranyai for all k . For a more detailed historical overview and an interesting exposition about this problem, the reader is referred to Chapter 38 of [13].

3. $KG(n, 2)$ IS SUPER-CONNECTED

In his beautiful paper investigating the conditions imposed on the connectivity of graphs by vertex-transitivity and edge-transitivity, Watkins [14] proved that

if a connected simple graph G is edge-transitive and all vertices have degree at least r , then $\kappa(G) = r$. Thus, the connectivity of the Kneser graphs $KG(n, k)$ for $n > 2k$ follows immediately, since $KG(n, k)$ is regular of degree $\binom{n-k}{k}$.

Theorem 2. *The connectivity of the Kneser graph $G = KG(n, k)$ for $n > 2k$ is $\binom{n-k}{k}$.*

Henceforth, we consider the case when $k = 2$ and $n \geq 5$. Let $G = KG(n, 2)$ and let

$$p = \left\lfloor \frac{n}{2} \right\rfloor \quad \text{and} \quad m = \frac{\binom{n}{2}}{p}.$$

Each vertex of $G = KG(n, 2)$ is represented by A_i^j , for $1 \leq i \leq m$ and $1 \leq j \leq p$. Note that each partition \mathcal{F}_i represents a clique in G .

To show that G is super-connected, we need to prove that every vertex cut of cardinality $\kappa(G)$ isolates a vertex. First we prove an important result which we need in the proof of Theorem 4.

Lemma 3. *Let S be a vertex set of $G = KG(n, 2)$ such that $|S| \leq \binom{n-2}{2}$. Then there are at least two partitions \mathcal{F}_i and \mathcal{F}_j ($i \neq j$) each having at least two of their elements in $G - S$.*

Proof. Suppose not, then either m or $m - 1$ of the partitions contribute at most one element to $G - S$. In either case, we have $|S| \geq (m - 1)(p - 1)$, and

- if n is even, then $|S| \geq (n - 2) \left(\frac{n}{2} - 1 \right) = \frac{1}{2}(n - 2)^2 > \frac{1}{2}(n - 2)(n - 3)$,
- if n is odd, then $|S| \geq (n - 1) \left(\frac{n-1}{2} - 1 \right) = \frac{1}{2}(n - 1)(n - 3) > \frac{1}{2}(n - 2)(n - 3)$.

Thus, $|S| > \binom{n-2}{2}$, a contradiction. ■

Theorem 4. *For $n \geq 5$, the Kneser graph $G = KG(n, 2)$ is super-connected.*

Proof. Suppose that there is a vertex cut S such that $|S| = \binom{n-2}{2}$ and the resulting graph $G - S$ does not have an isolated vertex.

By Lemma 3, there are at least two partitions, say \mathcal{F}_1 and \mathcal{F}_2 , each having at least two of their elements in $G - S$. Without loss of generality, let these elements in $G - S$ be $\{A_1^1, A_1^2\} \subseteq \mathcal{F}_1$ and $\{A_2^1, A_2^2\} \subseteq \mathcal{F}_2$. Since $G - S$ is disconnected, then it has at least two components, say C_1 and C_2 . Note that, for any i , the vertices of G corresponding to elements of \mathcal{F}_i which are not in S are either all in C_1 or all in C_2 , otherwise C_1 and C_2 are linked by some edge in $G - S$. There are two cases to consider.

Case 1. $\{A_1^1, A_1^2\} \subseteq C_1$ and $\{A_2^1, A_2^2\} \subseteq C_2$. Since A_1^1 and A_2^1 are in different components, one of their entries is the same. Hence, let $A_1^1 = \{a, b\}$ and $A_2^1 = \{a, c\}$ for $b \neq c$. The only elements of \mathcal{F}_1 that are in $G - S$ are those of the form

$\{c, x\}$ for $x \in [n]$, $x \notin \{a, b, c\}$, while the only elements of \mathcal{F}_2 that are in $G - S$ are those of the form $\{b, y\}$ for $y \in [n]$, $y \notin \{a, b, c\}$. However, since A_1^1 and A_2^2 are in different components, we have that $x = y = d$, for some $d \in [n]$, $d \notin \{a, b, c\}$. Now, out of all the remaining vertices of G , the only ones that are in $G - S$ are $\{a, d\}$ and $\{b, c\}$, because all the others are adjacent to at least a vertex in C_1 and a vertex in C_2 . Thus $|S| \geq \binom{n}{2} - 6 = \frac{1}{2}(n^2 - n - 12) = \binom{n-2}{2} + 2n - 9 > \binom{n-2}{2}$ since $2n - 9 \geq 1$, giving a contradiction.

Case 2. $\{A_1^1, A_1^2, A_2^1, A_2^2\} \subseteq C_1$. Since C_2 cannot be empty, without loss of generality, let $A_3^1 \in C_2$. Note that A_3^1 has common entries with both $A_1^1 = \{a, b\}$ and $A_1^2 = \{c, d\}$, since otherwise it is adjacent to them. Let $A_3^1 = \{a, c\}$. Similarly, A_3^1 has common entries with both A_2^1 and A_2^2 . Thus, let $A_2^1 = \{a, x\}$ for $x \in [n]$, $x \notin \{a, b, c\}$, and let $A_2^2 = \{c, y\}$ for $y \in [n]$, $y \notin \{a, c, d\}$, where $x \neq y$. Now, A_3^1 cannot be an isolated vertex in C_2 . Let $A_r^s = \{z_1, z_2\}$ be adjacent to A_3^1 , for $z_1 \neq z_2$. Thus $z_1 \notin \{a, c\}$ and $z_2 \notin \{a, c\}$. Since A_r^s is not adjacent to any vertex in C_1 , we have that $\{z_1, z_2\}$ is equal to $\{b, d, x, y\}$ and this is only possible when $x = d$ and $y = b$. This implies that $A_r^s = \{b, d\}$ and that the vertices in C_1 form two disjoint edges. Thus there is at least another vertex A_t^u in C_1 which is adjacent to at least one of $\{A_1^1, A_1^2, A_2^1, A_2^2\}$. Without loss of generality, assume A_t^u is adjacent to A_1^1 and thus, either

- $A_t^u = \{c, z_3\}$ for $z_3 \in [n]$, $z_3 \notin \{a, b, c, d\}$; or
- $A_t^u = \{d, z_3\}$ for $z_3 \in [n]$, $z_3 \notin \{a, b, c, d\}$; or
- $A_t^u = \{z_3, z_4\}$ for $z_3 \in [n]$, $z_3 \notin \{a, b, c, d\}$, and $z_4 \in [n]$, $z_4 \notin \{a, b, c, d\}$, where $z_3 \neq z_4$.

In either case, A_t^u is also adjacent to at least a vertex in C_2 , a contradiction. ■

4. SUPER-CONNECTIVITY OF $KG(n, 2)$

In this section we discuss the super-connectivity of $G = KG(n, 2)$. First we note that $KG(5, 2)$ is the Petersen graph having super-connectivity four [5]. In the next theorem we determine the super-connectivity of $KG(n, 2)$ for $n \geq 6$.

Theorem 5. *The super-connectivity of the Kneser graph $G = KG(n, 2)$ for $n \geq 6$ is $\binom{n}{2} - 6$.*

Proof. Let $S \subseteq V(G)$ be a super-vertex cut of G and let C_1, C_2, \dots, C_s be the components of the resulting graph $G - S$, where $n \geq 6$ and $s \geq 2$. Since S is a super-vertex cut, each component C_i contains at least two vertices, where $i \in \{1, 2, \dots, s\}$. Suppose, for contradiction, that $|S| < \binom{n}{2} - 6$. Since S is a super-vertex cut, there are two adjacent vertices in C_1 . These two vertices do not share any common entries, so let $\{\{a, b\}, \{c, d\}\} \subseteq C_1$. Similarly, there

are two adjacent vertices in C_2 and they share at least one common entry with every vertex in C_1 , otherwise there will be an edge between C_1 and C_2 . Thus $\{\{a, c\}, \{b, d\}\} \subseteq C_2$. Vertex cut S contains the common neighbors of the vertices in the different components. Note that any vertex A_t^u given by either

- $A_t^u = \{x, y\}$ for $x \in \{a, b, c, d\}$ and $y \in [n]$, $y \notin \{a, b, c, d\}$, or
- $A_t^u = \{x, y\}$ for $x \in [n]$, $x \notin \{a, b, c, d\}$ and $y \in [n]$, $y \notin \{a, b, c, d\}$, where $x \neq y$,

must belong to S , otherwise components C_1 and C_2 of $G - S$ are not different. There are $\binom{n-4}{2} + 4(n-4)$ possible candidates for the vertex A_t^u described above. Hence, $|S| \geq \binom{n-4}{2} + 4(n-4) = \binom{n}{2} - 6$.

Finally, to see that the super-connectivity of G is equal to $\binom{n}{2} - 6$, consider any four elements $a, b, c, d \in [n]$. The vertex set $W = \{\{x, y\} : x \in \{a, b, c, d\} \text{ and } y \in \{a, b, c, d\}, \text{ where } x \neq y\}$ has cardinality $\binom{4}{2}$ and induces a graph consisting of three disjoint copies of the complete graph on two vertices. As a consequence, $V(G) \setminus W$ forms a super-vertex cut of G of cardinality $\binom{n}{2} - 6$. ■

5. OPEN PROBLEM

We envisage that the technique used to prove our results in the case $k = 2$ can be extended to values of $k \geq 3$, although the exact way forward is still elusive. This is the main motivation why we mentioned Baranyai's Theorem instead of describing only the partitions for the particular case $k = 2$.

Conjecture 6. *Let $G = KG(n, k)$ for $n \geq 2k + 1$. Then the super-connectivity $\kappa_1 = \kappa_1(G)$ is given by*

$$\kappa_1 = \begin{cases} 2 \binom{n-k}{k} - 1 & \text{if } 2k + 1 \leq n < 3k, \\ 2 \binom{n-k}{k} - 1 - \binom{n-2k}{k} & \text{if } n \geq 3k. \end{cases}$$

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