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RAINBOW TOTAL-COLORING OF COMPLEMENTARY GRAPHS AND ERDŐS-GALLAI TYPE PROBLEM FOR THE RAINBOW TOTAL-CONNECTION NUMBER

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Abstract

A total-colored graph G is rainbow total-connected if any two vertices of G are connected by a path whose edges and internal vertices have distinct colors. The rainbow total-connection number, denoted by $rtc(G)$, of a graph G is the minimum number of colors needed to make G rainbow total-connected. In this paper, we prove that $rtc(G)$ can be bounded by a constant 7 if the following three cases are excluded: $diam(\overline{G}) = 2$, $diam(\overline{G}) = 3$, $\overline{G}$ contains exactly two connected components and one of them is a trivial graph. An example is given to show that this bound is best possible. We also study Erdős-Gallai type problem for the rainbow total-connection number, and compute the lower bounds and precise values for the function $f(n, k)$,

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where $f(n,k)$ is the minimum value satisfying the following property: if $|E(G)| \geq f(n,k)$, then $rtc(G) \leq k$.

Keywords: Rainbow total-coloring, rainbow total-connection number, complementary graph, Erdős-Gallai type problem.

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1. Introduction

We follow the notations of Bondy and Murty [1], unless otherwise stated. For a graph G, let $V(G)$, $E(G)$, $n(G)$, $m(G)$ and $\overline{G}$, respectively, be the set of vertices, the set of edges, the order, the size and the complement of G. For a set S, we use $|S|$ to denote the number of elements in S.

Let G be a nontrivial connected graph on which an edge-coloring $c : E(G) \rightarrow \{1, 2, \ldots, r\}$, $r \in \mathbb{N}$, is defined, where adjacent edges may be colored the same. A path is *rainbow* if no two edges of it are colored the same. An edge-colored graph G is *rainbow-connected* if any two vertices are connected by a rainbow path. Chartrand *et al.* [3] defined the *rainbow connection number* of a connected graph G, denoted by $rc(G)$, as the smallest number of colors that are needed in order to make G rainbow-connected.

The rainbow connection number is not only a natural combinatorial measure, but also has possible applications in the secure transfer of classified information between agencies [4]. In addition, the rainbow connection number can also be motivated by its interesting interpretation in the area of networking (see [2]). Suppose that G represents a network, we wish to route messages between any two vertices in a pipeline, and require that each link on the route between the vertices (namely, each edge on the path) is assigned a distinct channel. Moreover, we want to minimize the number of distinct channels that we use in our network. This number is precisely $rc(G)$. There are more and more researchers investigating this topic, such as [2–4, 7, 8, 11, 13, 15]. The readers can see [12] for a survey and [16] for a monograph on it.

The concept of rainbow connection number has several interesting variants, including the strong rainbow connection number [3, 17], the rainbow vertex-connection number [6,8,10] and the rainbow total-connection number [18–24]. Let c be an edge-coloring of a connected graph G. For any two vertices u and v of G, a *rainbow $u-v$ geodesic* in G is a rainbow $u-v$ path of length $d(u,v)$, where $d(u,v)$ is the distance between u and v. The graph G is *strongly rainbow-connected* if there exists a rainbow $u-v$ geodesic for any two vertices u and v in G. In this case, the coloring c is called a *strong rainbow coloring* of G. Similarly, we define the *strong rainbow connection number* of a connected graph G, denoted by $src(G)$, as the smallest number of colors that are needed in order to make G strongly rainbow-connected. Clearly, we have $diam(G) \leq rc(G) \leq src(G) \leq m(G)$ where $diam(G)$ denotes the diameter of G. A vertex-colored graph G is *rainbow vertex-connected* if any two vertices are connected by a path whose internal vertices have distinct colors. The *rainbow vertex-connection number* of a connected graph G, denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make G rainbow vertex-connected [10].

A *total-coloring* of a graph G is a coloring $c : V(G) \cup E(G) \to S$, where S is a set of colors. In a total-colored graph $G(V, E)$, a path P connecting two vertices u and v of G is called a *rainbow total-path* between u and v if all elements in $V(P) \cup E(P)$, except for u and v, are assigned distinct colors. The total-colored graph G is *rainbow total-connected* if G has a rainbow total-path between every two vertices in V. Now we define the *rainbow total-connection number*, denoted by $rtc(G)$, as the minimum number of colors needed to make the graph G rainbow total-connected. Note that in the literature, the rainbow total-connection number has also been referred to as the total rainbow connection number (see e.g., [18]). However, in this paper, we will use the term rainbow total-connection number, following the usage of [21].

Uchizawa *et al.* [24] introduced the concept of total rainbow-connectedness, and obtained some hardness results and algorithmic results for related problems. Recently, Chen *et al.* [5] also studied the hardness problems for the rainbow total-coloring. In [18,20], some basic properties of the rainbow total-connection number along with precise values of the parameter for some special graph classes, including complete graphs, complete bipartite graphs, complete multipartite graphs, trees, cycles and wheels were determined. In particular, it was shown in [18, 20] that $rtc(G) \leq m(G) + n'(G)$, and the equality holds if and only if G is a tree, where $n'(G)$ is the number of internal vertices (that is, vertices of degree at least two) of G. In [21], Sun showed that $rtc(G) \neq m(G)+n'(G)-1, m(G)+n'(G)-2$ and characterized the graphs with $rtc(G) = m(G) + n'(G) - 3$. With this result, the following sharp upper bound holds: for a connected graph G, if G is not a tree, then $rtc(G) \leq m(G) + n'(G) - 3$; moreover, the equality holds if and only if G belongs to five specific graph classes [21]. In the same paper, Sun also investigated Nordhaus-Gaddum-type lower bounds for the rainbow total-connection number of a graph and derived that if G is a connected graph of order $n \geq 8$, then $rtc(G) + rtc(\overline{G}) \geq 6$ and $rtc(G)rtc(\overline{G}) \geq 9$. An example is given to show that both of these bounds are sharp. Note that Ma [19] proved the same lower bound for $rtc(G) + rtc(\overline{G})$. In addition, he obtained an upper bound for $rtc(G)+rtc(\overline{G})$. In [22], Sun compared $rtc(G)$ with two other parameters of rainbow coloring, $rc(G)$ and $rvc(G)$. For an integer $k \geq 3$, he determined sufficient conditions that guarantee $rtc(G) \leq k$ in terms of the minimum degree. Among the results, Sun also proved the sharp threshold function for a random graph to have $rtc(G) \leq 3$.

In this paper, we continue the research on this topic and investigate the rainbow total-connection number of a graph G under some constraints on its complement $\overline{G}$. Three examples will be given to show that $rtc(G)$ can be arbitrarily large if one of the three situations happens: $diam(\overline{G}) = 2$, $diam(\overline{G}) = 3$, $\overline{G}$ contains exactly two connected components and one of them is K_1 (see Examples 1–3). However, the parameter $rtc(G)$ can be bounded by a small constant if these three cases are excluded (Theorem 9). Our argument is similar to that of [14], where the rainbow connection number was discussed. Note that for the case that $diam(\overline{G}) = 3$, Ma [19] recently showed that for a triangle-free graph $\overline{G}$ with $diam(\overline{G}) = 3$, if G is connected, then $rtc(G) \leq 5$, and this bound is tight.

Recall that in [22], Sun determined some sufficient conditions that guarantee $rtc(G) \leq k$, and all of these conditions are related to the minimum degree. In this paper, we will find some sufficient conditions that guarantee $rtc(G) \leq k$ in terms of the size of G. Hence, we study the following Erdős-Gallai type problem.

Problem 1.1 For every k with $k \geq 1$, compute the minimum value for $f(n, k)$ with the following property: if $|E(G)| \geq f(n, k)$, then $rtc(G) \leq k$.

By definition, we clearly have $f(n, k) \geq n - 1$. In this paper, we will compute the lower bounds and precise values for the function $f(n, k)$ (Theorem 20).

The rest of this paper is organized as follows. In Section 2 we will give the proof of Theorem 9 which consists of Lemmas 7 and 8. In order to prove these lemmas we need a few preliminary results and terminology that will be given in the section. In Section 3, we will prove that $rtc(G) \leq 2n - 3$ and characterize those graphs G with $rtc(G) = 2n - 3, 2n - 4$, respectively (Theorem 13). Based on Theorem 13 and other results, we will obtain Theorem 20 in Section 3.

2. Complementary Graphs

Let c be a total-coloring of G. We use $c(e)$ to denote the color of an edge e and $c(v)$ to denote the color of a vertex v. For a subset X of $V(G)$, the subgraph of G induced by X is denoted by $G[X]$. For two disjoint sets X and Y of $V(G)$, we use $E[X, Y]$ to denote the set of edges with one end in X and another end in Y. The *eccentricity* of a vertex x in G is defined as $ecc_G(x) = \max_{v \in V(G)} d(v, x)$. In a connected graph G, let $N_G^i(x) = \{v \mid d(x, v) = i\}$, where $x \in V(G)$.

The following observation is clear.

Observation 1. *For a connected graph G, if H is a connected spanning subgraph of G, then $rtc(G) \leq rtc(H)$.*

Proposition 2 [18, 20]**.** *For a connected graph G, we have*

(i) *$rtc(G) = 1$ if and only if G is a complete graph;*

(ii) *$rtc(G) \neq 2$ for any graph G;*

(iii) *$rtc(G) = m + n'$ if and only if G is a tree, where n' is the number of internal vertices of G.*

We now give three examples which show that $rtc(G)$ can be arbitrarily large if one of the three situations happens: $diam(\overline{G}) = 2$, $diam(\overline{G}) = 3$, $\overline{G}$ contains exactly two connected components and one of them is K_1.

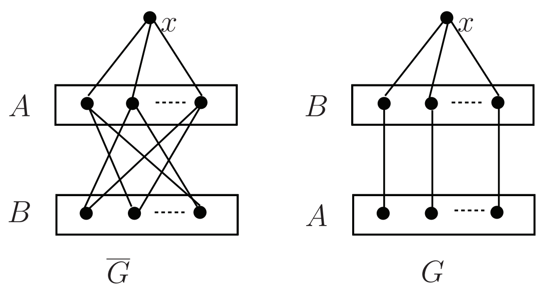

Figure 1. The graph of Example 1.

Example 1. As shown in Figure 1, in the graph $\overline{G}$, let $A = N^1_{\overline{G}}(x) = \{u_i \mid 1 \leq i \leq k\}$, $B = N^2_{\overline{G}}(x) = \{v_j \mid 1 \leq j \leq k\}$ where $k \geq 3$. Furthermore, let $E(\overline{G}) = \{xu_i \mid 1 \leq i \leq k\} \cup \{u_{i_1}u_{i_2} \mid 1 \leq i_1, i_2 \leq k\} \cup \{v_{j_1}v_{j_2} \mid 1 \leq j_1, j_2 \leq k\} \cup (\{u_iv_j \mid 1 \leq i, j \leq k\} \setminus \{u_iv_i \mid 1 \leq i \leq k\})$. Clearly, $diam(\overline{G}) = 2$ and G is a tree. By Proposition 2, $rtc(G) = m + n' = 2k + (k + 1) = 3k + 1$. Thus, in this case, $rtc(G)$ can be made arbitrarily large by increasing k.

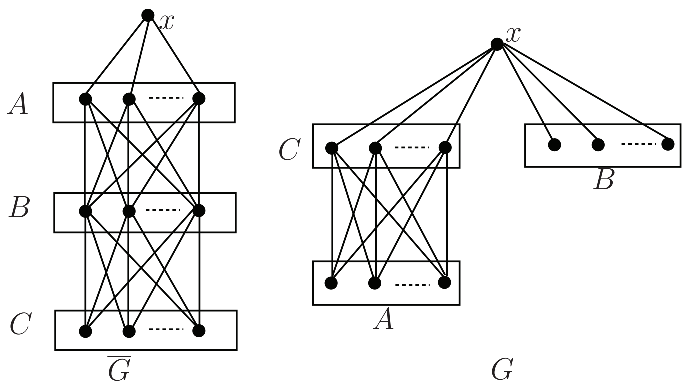

Figure 2. The graph of Example 2.

Example 2. As shown in Figure 2, in the graph $\overline{G}$, let $A = N^1_{\overline{G}}(x)$, $B = N^2_{\overline{G}}(x)$, $C = N^3_{\overline{G}}(x)$. Furthermore, let B be a clique of $\overline{G}$, each vertex in A be adjacent to all vertices of B and each vertex in B be adjacent to all vertices of C. Thus, $diam(\overline{G}) = 3$ and G is connected. Clearly, B is a stable set and each edge between x and B is a pendant edge in G, and $rtc(G) \geq |B| + 1$. Thus, in this case, $rtc(G)$ can be made arbitrarily large by increasing $|B|$.

Example 3. Let $\overline{G}$ be a graph with two components $\overline{G}_1, \overline{G}_2$ where $\overline{G}_1$ is trivial and $\overline{G}_2$ is a clique of order k. Clearly, G is a star of order $k+1$. By Proposition 2, $rtc(G) = k + 1$. Thus, in this case, $rtc(G)$ can be made arbitrarily large by increasing k.

In [3], the authors determined the precise values of rainbow connection numbers of complete bipartite graphs and complete multipartite graphs.

Theorem 3 [3]**.** *For integers s and t with $2 \leq s \leq t$, $rc(K_{s,t}) = \min\left\{\lceil \sqrt[s]{t}\rceil, 4\right\}$.*

Theorem 4 [3]**.** *Let $G = K_{n_1,n_2,\ldots,n_k}$ be a complete k-partite graph, where $k \geq 3$ and $n_1 \leq n_2 \leq \cdots \leq n_k$ such that $s = \sum_{i=1}^{k-1} n_i$ and $t = n_k$. Then*

$$rc(G) = \begin{cases} 1 & \text{if } n_k = 1, \\ 2 & \text{if } n_k \geq 2 \text{ and } s > t, \\ \min\left\{\lceil \sqrt[s]{t}\rceil, 3\right\} & \text{if } s \leq t. \end{cases}$$

Motivated by these two results, the authors of [18] determined the precise values of the rainbow total-connection numbers for these two graph classes.

Theorem 5 [18]**.** *For integers s and t with $2 \leq s \leq t$, we have*

$$rtc(K_{s,t}) = \min\{\lceil \sqrt[s]{t}\rceil + 1, 7\}.$$

Theorem 6 [18]**.** *Let $G = K_{n_1,n_2,\ldots,n_k}$ be a complete k-partite graph, where $k \geq 3$ and $n_1 \leq n_2 \leq \cdots \leq n_k$ such that $s = \sum_{i=1}^{k-1} n_i$ and $t = n_k$. Then*

$$rtc(G) = \begin{cases} 1 & \text{if } n_k = 1, \\ 3 & \text{if } n_k \geq 2 \text{ and } s > t, \\ \min\left\{\lceil \sqrt[s]{t}\rceil + 1, 5\right\} & \text{if } s \leq t. \end{cases}$$

We remark in Theorem 6 that if the graph G is not complete, then there is a rainbow total-connected coloring of G using $rtc(G)$ colors such that the sets of colors for the vertex set $V(G)$ and the edge set $E(G)$ are disjoint.

We now investigate the rainbow total-connection number for the complement of a graph with diameter at least 4.

Lemma 7. *If G is a connected graph with $diam(G) \geq 4$, then $rtc(\overline{G}) \leq 7$.*

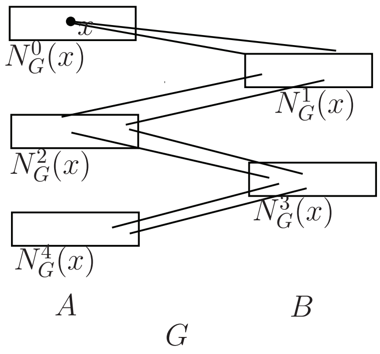

Figure 3. The graph for the example with $d = 4$.

Proof. Observe that $\overline{G}$ must be connected, since otherwise $diam(G) \leq 2$, contradicting the condition. Choose a vertex x with $ecc_G(x) = diam(G) = d \geq 4$. Let $N_G^i(x) = \{v \mid d(x,v) = i\}$ be the i-distance neighborhood of x, where $0 \leq i \leq d$. Then $N_G^0(x) = \{x\}, N_G^1(x) = N_G(x)$ as usual and $\bigcup_{0 \leq i \leq d} N_G^i(x)$ is a vertex partition of $V(G)$ with $|N_G^i(x)| = n_i$. Let $A = \bigcup_{i \ is \ even} N_G^i(x)$, $B = \bigcup_{i \ is \ odd} N_G^i(x)$.

By the definition of complementary graphs, we know that $\overline{G}[A]$ ($\overline{G}[B]$) contains a spanning complete k_1-partite subgraph (complete k_2-partite subgraph) where $k_1 = \left\lceil \frac{d+1}{2} \right\rceil$ $\left(k_2 = \left\lceil \frac{d}{2} \right\rceil\right)$. For example, see Figure 3 for a graph G with diameter four. The subgraph $\overline{G}[A]$ contains a spanning complete tripartite subgraph K_{n_0,n_2,n_4} and $\overline{G}[B]$ contains a spanning complete bipartite subgraph K_{n_1,n_3}.

Case 1. $d \geq 5$. Then $k_1, k_2 \geq 3$ and we have $rtc(\overline{G}[A]), rtc(\overline{G}[B]) \leq 5$ by Theorem 6. Now we provide a total-coloring of $\overline{G}$ as follows.

By Theorem 6, we first provide the subgraph $\overline{G}[A]$ with a rainbow total-coloring by using five colors and then provide the subgraph $\overline{G}[B]$ with a rainbow total-coloring by using the same set of colors as that of $\overline{G}[A]$; finally we assign a fresh color to all elements of $E[A,B]$. By the remark after Theorem 6, we can color $V(\overline{G}) \cup E(\overline{G})$ with six colors such that the sets of colors for A and $E(\overline{G}[A])$ are disjoint, and similarly for B and $E(\overline{G}[B])$.

For $u, v \in \overline{G}[A]$ or $u, v \in \overline{G}[B]$, there exists a rainbow total-path connecting u and v. Now, let $u \in \overline{G}[A]$ and $v \in \overline{G}[B]$, with $u \in N_G^i(x)$ and $v \in N_G^j(x)$. We use $|i-j|$ to denote the range of i and j in the i,j-distance neighborhoods. If $|i-j| \geq 2$, then u and v are adjacent in $\overline{G}$. Otherwise, $|i-j| = 1$. Choose $u_1 \in N_G^{\ell}(x) \subset A$ such that $\ell \neq i$ and $|j-\ell| \geq 2$. Then uu_1v is a rainbow total-path connecting u and v in $\overline{G}$. Thus, $rtc(\overline{G}) \leq 6$ in this case.

Case 2. $d = 4$, that is, $A = N_G^0(x) \cup N_G^2(x) \cup N_G^4(x)$ and $B = N_G^1(x) \cup N_G^3(x)$. Then $\overline{G}[A]$ ($\overline{G}[B]$) contains a spanning complete tripartite subgraph K_{n_0,n_2,n_4} (complete bipartite subgraph K_{n_1,n_3}).

Now we provide a total-coloring c of $\overline{G}$ as follows. Let $c(x) = 1$, $c(v) = 2$ for $v \in N_G^4(x)$, $c(v) = 3$ for $v \in V \setminus (N_G^4(x) \cup \{x\})$ and

$$c(e) = \begin{cases} 4 & \text{if } \ e \in E\left[N_G^0(x), N_G^4(x)\right], \\ 5 & \text{if } \ e \in E\left[N_G^0(x), N_G^2(x)\right] \cup E\left[N_G^0(x), N_G^3(x)\right], \\ 6 & \text{if } \ e \in E\left[N_G^2(x), N_G^4(x)\right] \cup E\left[N_G^1(x), N_G^3(x)\right], \\ 7 & \text{if } \ e \in E\left[N_G^1(x), N_G^4(x)\right]. \end{cases}$$

We only show that there is a rainbow total-path connecting two vertices u and v where $u \in N_G^2(x), v \in N_G^3(x)$, since the arguments for the remaining cases are similar. Let $P := u, x, x_1, x_2, v$ where $x_1 \in N_G^4(x)$ and $x_2 \in N_G^1(x)$. Clearly, we have that P is a rainbow $u - v$ path. Thus, $rtc(\overline{G}) \leq 7$ in this case. ■

If G is a graph with $h \geq 2$ connected components, then $\overline{G}$ contains a complete h-partite spanning subgraph. Then, by Observation 1, the following statement holds.

Lemma 8. *If G is a graph with $h \geq 2$ connected components G_i and $n'_i = n(G_i)$ $(1 \leq i \leq h)$, then $rtc(\overline{G}) \leq rtc\big(K_{n'_1,\ldots,n'_h}\big)$.*

Now we can prove our first main result.

Theorem 9. *For a connected graph G, if $\overline{G}$ does not belong to the following two cases:*

(i) *$diam(\overline{G}) \in \{2,3\}$,*

(ii) *$\overline{G}$ contains exactly two connected components and one of them is K_1, then $rtc(G) \leq 7$. Moreover, the bound is best possible.*

Proof. If $\overline{G}$ is connected, since $diam(\overline{G}) \notin \{2,3\}$ and clearly $diam(\overline{G}) \neq 1$, we have $rtc(G) \leq 7$ by Lemma 7. If $\overline{G}$ is disconnected, by the assumption, it has either at least three connected components or exactly two nontrivial components, then $rtc(G) \leq 7$ by Theorems 5, 6 and Lemma 8.

For the sharpness of the bound, we consider the following graph G. Let $\overline{G}$ contain two connected components, one of which is a clique with $s \geq 2$ vertices and the other is a clique with $t \geq 6^s + 1$ vertices. We have $G = K_{s,t}$, and $rtc(G) = 7$ by Theorem 5. Thus the upper bound is best possible. ■

3. Lower Bounds and Precise Values for $f(n,k)$

Liu *et al.* [18,20] determined the precise values for $rtc(C_n)$.

Theorem 10 [18, 20]**.**

$$rtc(C_n) = \begin{cases} n-2, & \text{if } n \in \{3,5\}, \\ n-1, & \text{if } n \in \{4,6,7,8,9,10,12\}, \\ n, & \text{if } n = 11 \text{ or } n \geq 13. \end{cases}$$

Let G be a connected unicyclic graph with girth ℓ and C be the cycle of G such that $V(C) = \{u_i \mid 1 \leq i \leq \ell\}$ and $E(C) = \{u_iu_{i+1} \mid 1 \leq i \leq \ell\}$ where $u_{\ell+1} = u_1$. Let $\mathcal{T}_G = \{T_i \mid 1 \leq i \leq \ell\}$, where T_i denotes the component containing u_i in the subgraph $G \setminus E(C)$. Clearly, each T_i is a tree rooted at u_i for $1 \leq i \leq \ell$. We say that T_i and T_j are *adjacent (nonadjacent)* if u_i and u_j are adjacent (nonadjacent) in the cycle C.

We now consider the rainbow total-connection number for unicyclic graphs which are not cycles. Recall that $n'(G)$ denotes the number of internal vertices, that is, vertices of degree at least two of G. We need the following result from [21].

Theorem 11 [21]**.** *Let G be a connected unicyclic graph which is not a cycle. Let $\ell \geq 3$ be the length of the unique cycle in G.*

(i) *For the case $\ell \geq 5$, we have $rtc(G) \leq m(G) + n'(G) - 4$.*

(ii) *For the case $\ell \in \{3,4\}$, we have $rtc(G) \leq m(G) + n'(G) - 3$.*

Based on Theorem 11, we can deduce the following result.

Lemma 12. *Let G be a connected unicyclic graph with order n which is not a cycle and let C be the unique cycle in G with length $\ell \geq 3$. Then we have $rtc(G) \leq 2n - 5$. Moreover, $rtc(G) = 2n - 5$ if and only if G satisfies one of the following.*

(i) *$\ell \in \{3,4\}$, and G is a graph with a non-trivial pendant path attached to C.*

(ii) *$\ell = 3$, and G is a graph with two non-trivial pendant paths attached to two distinct vertices of C.*

(iii) *$\ell = 4$, and G is a graph with two non-trivial pendant paths attached to two opposite vertices of C.*

Proof. For the case $\ell \geq 5$, since $n'(G) \leq n - 1$, the bound $rtc(G) \leq 2n - 5$ clearly holds by Theorem 11. Moreover, we would require $n'(G) = n - 1$ to possibly achieve $rtc(G) = 2n - 5$. In this case, G must be the cycle C with a non-trivial pendant path P attached to a vertex of C, say u_1. We provide a total-coloring of G as follows. First, give C a rainbow total-coloring with $rtc(C_\ell)$ colors. Then, recolor u_1 with a new color, and give the edges and internal vertices of P further new colors. Clearly, this is a rainbow total-coloring for G with $rtc(C_\ell) + 2m(P) = rtc(C_\ell) + 2(n - \ell)$ colors. It is easy to check, using

Theorem 10, that $rtc(G) \leq rtc(C_\ell) + 2(n - \ell) \leq 2n - 7$ for $\ell \geq 5$. Thus, $rtc(G) = 2n - 5$ does not hold.

Now let $\ell \in \{3, 4\}$. If $n'(G) \leq n-3$, then $rtc(G) \leq m(G)+n'(G)-3 \leq 2n-6$ by Theorem 11. In particular, this holds if $\mathcal{T}_G$ has at least three non-trivial elements. Thus, assume that $n'(G) \in \{n-2, n-1\}$.

Suppose that $\mathcal{T}_G$ has exactly two non-trivial elements and $n'(G) = n - 2$. Then similarly, we have $rtc(G) \leq 2n - 5$. Note that in this case, G must be the cycle C with two non-trivial pendant paths attached to two distinct vertices of C. If G is a graph in the form of (ii) or (iii), then $diam(G) = n - 2$, so that $rtc(G) \geq 2(n-2) - 1 = 2n - 5$, and we have $rtc(G) = 2n - 5$. Otherwise, we must have $\ell = 4$, and G is the cycle C with two non-trivial pendant paths attached to adjacent vertices of C, say u_1, u_2. We provide a total-coloring c of G by letting $c(u_1) = c(u_3) = 1$, $c(u_1u_2) = c(u_3u_4) = 2, c(u_2) = c(u_4) = 3, c(u_2u_3) = c(u_1u_4) = 4$, and all other edges and internal vertices of G are given further distinct colors. Then c is a rainbow total-coloring for G with $2n - 6$ colors, and thus $rtc(G) \leq 2n - 6$.

Finally, suppose that $\mathcal{T}_G$ has exactly one non-trivial element, say T_1 is attached to C at u_1. We provide a total-coloring c of G as follows. If $\ell = 4$, then we use the same coloring as before. If $\ell = 3$, then we let $c(u_2) = c(u_3) = c(u_1u_2) = c(u_2u_3) = c(u_1u_3) = 1$, and all other edges and internal vertices of G are given further distinct colors. Then in each case, c is a rainbow total-coloring with $m(G) + n'(G) - 4$ colors. If $n'(G) = n - 2$, then we have $rtc(G) \leq 2n - 6$. If $n'(G) = n - 1$, then G is a graph in the form of (i), and $rtc(G) \leq 2n - 5$. In this case, we also have $diam(G) = n - 2$, so that $rtc(G) \geq 2n - 5$ as before, and therefore, $rtc(G) = 2n - 5$.

This concludes the proof of the lemma. ■

Next, we have the following result.

Theorem 13. *For a connected graph G with order n, we have $rtc(G) \leq 2n - 3$. Moreover, $rtc(G) = 2n-3$ if and only if G is a path; $rtc(G) = 2n-4$ if and only if G is a subdivided $K_{1,3}$.*

Proof. For the upper bound, we choose a spanning tree T of G. By Observation 1 and Proposition 2, we have $rtc(G) \leq rtc(T) = n'(T)+m(T) = 2n-(p+1) \leq 2n-3$ since $p \geq 2$, where p is the number of vertices in T with degree one.

If G is not a tree, then G contains at least one cycle. Let G' be a connected unicyclic spanning subgraph of G. Then $rtc(G) \leq rtc(G') \leq 2n - 5$ by Observation 1, Theorem 10 and Lemma 12. Recall that for a tree T, $rtc(T) = 2n-(p+1)$, where p is the number of vertices with degree one. Thus, $rtc(G) = 2n - 3$ if and only if G is a path; $rtc(G) = 2n - 4$ if and only if G is a tree with exactly three vertices of degree one, and it is easy to see that such a tree is precisely a subdivided $K_{1,3}$. ■

Kemnitz and Schiermeyer [9] obtained the following sufficient condition which guarantees that $rc(G) = 2$.

Theorem 14 [9]**.** *Let G be a connected graph of order n and size m. If $\binom{n-1}{2}+1 \leq m \leq \binom{n}{2} - 1$, then $rc(G) = 2$.*

Proposition 15 [22]**.** *For a connected graph G, if $rc(G) = 2$, then $rtc(G) = 3$.*

By Theorem 14 and Proposition 15, the following result clearly holds.

Proposition 16. *Let G be a connected graph of order n and size m. If $\binom{n-1}{2}+1 \leq m \leq \binom{n}{2} - 1$, then $rtc(G) = 3$.*

Recall that we define the function $f(n, k)$ as the minimum value satisfying the following property: if $|E(G)| \geq f(n, k)$, then $rtc(G) \leq k$. Then we clearly have the following result which concerns the monotonicity of $f(n, k)$.

Observation 17. *For any two positive integers k, ℓ with $k \leq \ell$, we have $f(n, k) \geq f(n, \ell)$.*

The following result is about a lower bound of the function $f(n, k)$ for the case that $1 \leq k \leq 2n - 4$.

Lemma 18. *Let $1 \leq k \leq 2n - 4$. If k is even, then*

$$f(n, k) \geq \binom{n + 1 - \frac{k}{2}}{2} + \frac{k-2}{2}.$$

If k is odd, then

$$f(n, k) \geq \binom{n + \frac{1-k}{2}}{2} + \frac{k-1}{2}.$$

Proof. For the case that k is even, we consider the following graph G whose construction is due to Kemnitz and Schiermeyer [9]. Let $V(G) = \big\{u_1, \ldots, u_{n+1-\frac{k}{2}}, v_2, \ldots, v_{\frac{k}{2}}\big\}$ such that $G\big[\big\{u_2, v_2, \ldots, v_{\frac{k}{2}}\big\}\big]$ is a path of order $\frac{k}{2}$ and $G\big[\big\{u_1, \ldots, u_{n+1-\frac{k}{2}}\big\}\big] \cong K_{n+1-\frac{k}{2}} \setminus e$ with $u_1u_2 \notin E(G)$. Clearly, $|E(G)| = \binom{n+1-\frac{k}{2}}{2} + (\frac{k}{2} - 2)$ and furthermore, we have $rtc(G) \geq 2\,diam(G) - 1 = k + 1$. Thus, $f(n, k) \geq |E(G)| + 1 = \binom{n+1-\frac{k}{2}}{2} + \frac{k-2}{2}$.

For the case that k is odd, by Observation 17, we have $f(n, k) \geq f(n, k+1) \geq \binom{n+\frac{1-k}{2}}{2} + \frac{k-1}{2}$ since $k + 1$ is even. ■

In the following result, we will give precise values of $f(n, k)$ for some special cases, and this result means that the bound in Lemma 18 is sharp for $k \in \{1, 2, 3, 2n - 5, 2n - 4\}$.

Proposition 19. *The following assertions hold.*

(i) $f(n,1) = f(n,2) = \binom{n}{2}$.

(ii) $f(n,3) = \binom{n-1}{2} + 1$.

(iii) $f(n,2n-5) = f(n,2n-4) = n$.

(iv) $f(n,k) = n-1$ *for* $k \geq 2n-3$.

Proof. By Proposition 2, we clearly have $f(n,2) = f(n,1) = \binom{n}{2}$. By Proposition 16, we have that $f(n,3) \leq \binom{n-1}{2} + 1$. Furthermore, by Lemma 18, we have $f(n,3) \geq \binom{n-1}{2} + 1$, and so $f(n,3) = \binom{n-1}{2} + 1$. The assertion (iv) holds from Observation 1, Theorem 13 and the fact that $f(n,k) \geq n-1$.

We now prove (iii). By Observation 17 and Lemma 18, we have $f(n,2n-5) \geq f(n,2n-4) \geq n$. By Theorem 13, we know that $rtc(G) \leq 2n-5$ if and only if G is neither a path nor a subdivided $K_{1,3}$. This means that $rtc(G) \leq 2n-5$ if $m(G) \geq n$, so $f(n,2n-5) \leq n$. Therefore, we have $f(n,2n-5) = f(n,2n-4) = n$. ∎

By combining Lemma 18 and Proposition 19, we have the following result.

Theorem 20. *The following assertions hold.*

(i) $f(n,1) = f(n,2) = \binom{n}{2}$.

(ii) $f(n,3) = \binom{n-1}{2} + 1$.

(iii) *Let* $4 \leq k \leq 2n-6$. *If* k *is even, then*

$$f(n,k) \geq \binom{n+1-\frac{k}{2}}{2} + \frac{k-2}{2}.$$

If k *is odd, then*

$$f(n,k) \geq \binom{n+\frac{1-k}{2}}{2} + \frac{k-1}{2}.$$

(iv) $f(n,2n-5) = f(n,2n-4) = n$.

(v) $f(n,k) = n-1$ *for* $k \geq 2n-3$.

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REFERENCES

[1] J.A. Bondy and U.S.R. Murty, Graph Theory, GTM 244 (Springer, Berlin, 2008).

[2] S. Chakraborty, E. Fischer, A. Matsliah and R. Yuster, *Hardness and algorithms for rainbow connection*, J. Comb. Optim. **21** (2011) 330–347.
doi:10.1007/s10878-009-9250-9

[3] G. Chartrand, G.L. Johns, K.A. McKeon and P. Zhang, *Rainbow connection in graphs*, Math. Bohem. **133** (2008) 85–98.

[4] G. Chartrand, G.L. Johns, K.A. McKeon and P. Zhang, *The rainbow connectivity of a graph*, Networks **54(2)** (2009) 75–81.
doi:10.1002/net.20296

[5] L. Chen, B. Huo and Y. Ma, *Hardness results for total rainbow connection of graphs*, Discuss. Math. Graph Theory **36** (2016) 355–362.
doi:10.7151/dmgt.1856

[6] L. Chen, X. Li and Y. Shi, *The complexity of determining the rainbow vertex-connection of a graph*, Theoret. Comput. Sci. **412** (2011) 4531–4535.
doi:10.1016/j.tcs.2011.04.032

[7] X. Huang, H. Li, X. Li and Y. Sun, *Oriented diameter and rainbow connection number of a graph*, Discrete Math. Theor. Comput. Sci. **16(3)** (2014) 51–60.

[8] X. Huang, X. Li, Y. Shi, J. Yue and Y. Zhao, *Rainbow connections for outerplanar graphs with diameter* 2 *and* 3, Appl. Math. Comput. **242** (2014) 277–280.
doi:10.1016/j.amc.2014.05.066

[9] A. Kemnitz and I. Schiermeyer, *Graphs with rainbow connection number two*, Discuss. Math. Graph Theory **31** (2011) 313–320.
doi:10.7151/dmgt.1547

[10] M. Krivelevich and R. Yuster, *The rainbow connection of a graph is* (*at most*) *reciprocal to its minimum degree*, J. Graph Theory **63** (2010) 185–191.
doi:10.1002/jgt.20418

[11] H. Li, X. Li, Y. Sun and Y. Zhao, *Note on minimally d-rainbow connected graphs*, Graphs Combin. **30** (2014) 949–955.
doi:/10.1007/s00373-013-1309-9

[12] X. Li, Y. Shi and Y. Sun, *Rainbow connections of graphs: A survey*, Graphs Combin. **29** (2013) 1–38.
doi:10.1007/s00373-012-1243-2

[13] X. Li and Y. Sun, *Rainbow connection numbers of line graphs*, Ars Combin. **100** (2011) 449–463.

[14] X. Li and Y. Sun, *Rainbow connection numbers of complementary graphs*, Util. Math. **86** (2011) 23–31.

[15] X. Li and Y. Sun, *Upper bounds for the rainbow connection numbers of line graphs*, Graphs Combin. **28** (2012) 251–263.
doi:10.1007/s00373-011-1034-1

[16] X. Li and Y. Sun, Rainbow Connections of Graphs, SpringerBriefs in Math. (Springer, New York, 2012).
doi:10.1007/978-1-4614-3119-0

[17] X. Li and Y. Sun, *On the strong rainbow connection of a graph*, Bull. Malays. Math. Sci. Soc. **36** (2013) 299–311.

[18] H. Liu, Â. Mestre and T. Sousa, *Total rainbow k-connection in graphs*, Discrete Appl. Math. **174** (2014) 92–101.
doi:10.1016/j.dam.2014.04.012

[19] Y. Ma, *Total rainbow connection number and complementary graph*, Results Math. **70** (2016) 173–182.
doi:10.1007/s00025-015-0469-8

[20] Y. Sun, *On two variants of rainbow connection*, WSEAS Trans. Math. **12** (2013) 266–276.

[21] Y. Sun, *On rainbow total-coloring of a graph*, Discrete Appl. Math. **194** (2015) 171–177.
doi:10.1016/j.dam.2015.05.012

[22] Y. Sun, *On the total rainbow connection of a graph*, Acta Math. Appl. Sin. Engl. Ser., accepted.

[23] Y. Sun, Z. Jin and F. Li, *On total rainbow k-connected graphs*, Appl. Math. Comput. **311** (2017) 223–227.
doi:10.1016/j.amc.2017.05.020

[24] K. Uchizawa, T. Aoki, T. Ito, A. Suzuki and X. Zhou, *On the rainbow connectivity of graphs: complexity and FPT algorithms*, Algorithmica **67** (2013) 161–179.
doi:10.1007/s00453-012-9689-4

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