

## ETERNAL $m$ -SECURITY BONDAGE NUMBERS IN GRAPHS

HAMIDEH ARAM

*Department of Mathematics*  
*Gareziaeddin Center, Khoy Branch*  
*Islamic Azad University, Khoy, Iran*  
**e-mail:** hamideh.aram@gmail.com

MARYAM ATAPOUR

*Department of Mathematics*  
*Faculty of Basic Sciences*  
*University of Bonab, Bonab, I.R. Iran*  
**e-mail:** m.atapour@bonabu.ac.ir

AND

SEYED MAHMOUD SHEIKHOESLAMI

*Department of Mathematics*  
*Azarbaijan Shahid Madani University*  
*Tabriz, I.R. Iran*  
**e-mail:** s.m.sheikholeslami@azaruniv.edu

### Abstract

An eternal  $m$ -secure set of a graph  $G = (V, E)$  is a set  $S_0 \subseteq V$  that can defend against any sequence of single-vertex attacks by means of multiple guard shifts along the edges of  $G$ . The eternal  $m$ -security number  $\sigma_m(G)$  is the minimum cardinality of an eternal  $m$ -secure set in  $G$ . The eternal  $m$ -security bondage number  $b_{\sigma_m}(G)$  of a graph  $G$  is the minimum cardinality of a set of edges of  $G$  whose removal from  $G$  increases the eternal  $m$ -security number of  $G$ . In this paper, we study properties of the eternal  $m$ -security bondage number. In particular, we present some upper bounds on the eternal  $m$ -security bondage number in terms of eternal  $m$ -security number and edge connectivity number, and we show that the eternal  $m$ -security bondage number of trees is at most 2 and we classify all trees attaining this bound.

**Keywords:** eternal  $m$ -secure set, eternal  $m$ -security number, eternal  $m$ -security bondage number.

**2010 Mathematics Subject Classification:** 05C69.

## 1. INTRODUCTION

Throughout this paper,  $G$  is a simple connected graph with vertex set  $V = V(G)$  and edge set  $E = E(G)$  and of order  $n$  and size  $m$ . For every vertex  $v \in V$ , the *open neighborhood* of  $v$  is the set  $N(v) = \{u \in V(G) : uv \in E(G)\}$  and the *closed neighborhood* of  $v$  is the set  $N[v] = N(v) \cup \{v\}$ . The *degree*  $\deg(v)$  of  $v$  is the number of edges incident with  $v$  or, equivalently,  $\deg(v) = |N(v)|$ . The degree sequence of  $G$  is  $(\deg(v_1), \deg(v_2), \dots, \deg(v_n))$ , typically written in nondecreasing order. The minimum and maximum degree of vertices in  $V(G)$  are denoted by  $\delta(G)$  and  $\Delta(G)$ , respectively. Let  $E(A, B)$  denote the set of all edges with one endpoint in  $A$  and the other endpoint in  $B$ ,  $e(A, B)$  be the cardinality of  $E(A, B)$ , and  $E_u$  denote the set of edges incident to  $u$ . A *leaf* of a graph  $G$  is a vertex of degree 1 and a *support vertex* of  $G$  is a vertex adjacent to a leaf. A support vertex is called *strong support vertex* if it is adjacent to at least two leaves. The distance between two vertices  $x$  and  $y$  is denoted by  $d(x, y)$  and the diameter of  $G$  is denoted by  $\text{diam}(G)$ .

A set  $S$  of vertices in a graph  $G$  is called a *dominating set* if every vertex in  $V$  is either an element of  $S$  or is adjacent to an element of  $S$ . The *domination number* of  $G$ , denoted by  $\gamma(G)$ , is the minimum cardinality of a dominating set of  $G$ . A  $\gamma(G)$ -*set* is a dominating set of  $G$  of size  $\gamma(G)$ . For a more thorough treatment of domination parameters and for terminology not presented here see [5, 11]. The bondage number  $b(G)$  of a graph  $G$  is the minimum cardinality of a set of edges of  $G$  whose removal from  $G$  increases the domination number of  $G$ . The bondage number was introduced by Fink *et al.* [2] and was studied by several authors, for example [4, 6, 8–10]. For more information on this topic we refer the reader to the survey article by Xu [12].

An *eternal 1-secure set* of a graph  $G$  is a set  $S_0 \subseteq V$  that can defend against any sequence of single-vertex attacks by means of single-guard shifts along the edges of  $G$ . That is, for any  $k$  and any sequence  $v_1, v_2, \dots, v_k$  of vertices, there exists a sequence of guards  $u_1, u_2, \dots, u_k$  with  $u_i \in S_{i-1}$  and either  $u_i = v_i$  or  $u_i v_i \in E$ , such that each set  $S_i = (S_{i-1} - \{u_i\}) \cup \{v_i\}$  is a dominating set. It follows that each  $S_i$  can be chosen to be an eternal 1-secure set. The *eternal 1-security number* of  $G$ , denoted by  $\sigma_1(G)$ , is the minimum cardinality of an eternal 1-secure set. The eternal 1-security number was introduced by Burger *et al.* [1] using the notation  $\gamma_\infty$ . In order to reduce the number of guards needed in an eternal secure set, Goddard *et al.* [3] considered allowing more guards to move. Suppose that in responding to each attack, every guard may shift along an incident edge. The *eternal  $m$ -security number*  $\sigma_m(G)$  is the minimum number of guards to handle an arbitrary sequence of single attacks using multiple guard shifts. A suitable placement of the guards is called an *eternal  $m$ -secure set* (EmSS). An EmSS of size  $\sigma_m(G)$  is called a  $\sigma_m(G)$ -*set*.

The *eternal  $m$ -security bondage number*  $b_{\sigma_m}(G)$  of a graph  $G$  is the minimum cardinality of a set of edges of  $G$  whose removal from  $G$  increases the eternal  $m$ -security number of  $G$ . Since in the study of eternal  $m$ -security bondage number the assumption  $\sigma_m(G) < n$  is necessary, we always assume that when we discuss  $b_{\sigma_m}(G)$ , all graphs involved satisfy  $\sigma_m(G) < n$ , i.e., all graphs are nonempty. An edge set  $B$  with  $\sigma_m(G - B) > \sigma_m(G)$  is called the *eternal  $m$ -secure bondage set*. A  $b_{\sigma_m}(G)$ -set is an eternal  $m$ -secure bondage set of  $G$  of size  $b_{\sigma_m}(G)$ .

In this paper, we initiate the study of the eternal  $m$ -security bondage number in graphs and we establish some bounds on the eternal  $m$ -security bondage number in terms of vertex degree, eternal  $m$ -security number and edge connectivity number. We also show that the eternal  $m$ -security bondage number of trees is at most 2 and we characterize all trees attaining this bound.

2. PRELIMINARIES AND EXACT VALUES

The proof of the following four results can be found in [3].

**Proposition A.** For any graph  $G$ ,  $\gamma(G) \leq \sigma_m(G)$ .

A set  $P \subseteq V(G)$  is called a  *$k$ -packing* if  $d(u, v) > k$  for each pair of vertices  $u, v \in P$ ,  $u \neq v$ . The  *$k$ -packing number*  $\alpha_k(G)$  is the cardinality of a maximum  $k$ -packing in  $G$ . Note that  $\alpha_1(G) = \alpha(G)$  is the independence number of  $G$ .

**Proposition B.** For any graph  $G$ ,  $\sigma_m(G) \leq \alpha(G)$ .

**Proposition C.** 1.  $\sigma_m(K_n) = 1$ .

2.  $\sigma_m(K_{r,s}) = 2$  for  $r, s \geq 1, r + s \geq 3$ .

3.  $\sigma_m(P_n) = \lceil \frac{n}{2} \rceil$ .

4.  $\sigma_m(C_n) = \lceil \frac{n}{3} \rceil$ .

**Proposition D.** For any graph  $G$ ,  $\sigma_m(G) \geq (\text{diam}(G) + 1)/2$ .

Next results are immediate consequences of Propositions C and D.

**Corollary 1.** For any graph  $G$ ,  $\sigma_m(G) = 1$  if and only if  $G \simeq K_n$ .

**Corollary 2.** For  $n \geq 2$ , we have  $b_{\sigma_m}(K_n) = 1$ .

**Corollary 3.** For  $n \geq 5$ ,  $b_{\sigma_m}(C_n) = 1$ .

**Corollary 4.** For  $n \geq 3$ ,  $b_{\sigma_m}(P_n) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 2 & \text{if } n \text{ is odd.} \end{cases}$

**Proposition E** [7]. For any graph  $G$ ,  $\alpha_2(G) \leq \gamma(G)$ .

**Corollary 5.** *For any graph  $G$ ,  $\alpha_2(G) \leq \sigma_m(G)$ .*

**Observation 6.** *Let  $G$  be a graph and  $H$  be a spanning subgraph of  $G$  such that  $\sigma_m(H) = \sigma_m(G)$ . If  $K = E(G) \setminus E(H)$ , then  $b_{\sigma_m}(H) \leq b_{\sigma_m}(G) \leq b_{\sigma_m}(H) + |K|$ .*

**Proof.** Let  $F$  be a  $b_{\sigma_m}(H)$ -set. Then  $\sigma_m(G) = \sigma_m(H) < \sigma_m(H - F) = \sigma_m(G - (K \cup F))$ , which implies that

$$b_{\sigma_m}(G) \leq |K \cup F| = |K| + |F| = b_{\sigma_m}(H) + |K|.$$

Let now  $T$  be a  $b_{\sigma_m}(G)$ -set. Then we have  $\sigma_m(H) = \sigma_m(G) < \sigma_m(G - T) \leq \sigma_m(H - T)$ . Thus  $b_{\sigma_m}(H) \leq |T| = b_{\sigma_m}(G)$  and the proof is complete. ■

**Proposition 7.** *If  $G$  contains a vertex adjacent to at least three leaves, then  $b_{\sigma_m}(G) = 1$ .*

**Proof.** Let  $u$  be adjacent to the leaves  $u_1, u_2, u_3$ . Consider the graph  $G'$  obtained from  $G$  by deleting the edge  $uu_1$ . Let  $S$  be a  $\sigma_m(G')$ -set which contains  $u$  (we may assume that  $S$  is a response to an attack on  $u$ ). Obviously  $u_1 \in S$  and  $S \setminus \{u_1\}$  is an EmSS of  $G$  and so  $\sigma_m(G) \leq \sigma_m(G') - 1$ . Hence,  $b_{\sigma_m}(G) = 1$ . ■

Next we determine the eternal  $m$ -security bondage number of complete bipartite graphs.

**Proposition 8.** *For  $m \geq n \geq 2$ ,  $b_{\sigma_m}(K_{m,n}) = 2$ .*

**Proof.** By Proposition C,  $\sigma_m(K_{m,n}) = 2$ . If  $m = n = 2$ , then clearly  $b_{\sigma_m}(K_{2,2}) = 2$ . Assume that  $m \geq 3$ . It is not hard to see that for any edge  $e = uv \in E(G)$ , the set  $S = \{u, v\}$  is an EmSS of  $K_{m,n} - e$  and so  $b_{\sigma_m}(K_{m,n}) \geq 2$ .

Now we show that  $b_{\sigma_m}(K_{m,n}) \leq 2$ . Suppose that  $X = \{u_1, \dots, u_m\}$  and  $Y = \{v_1, \dots, v_n\}$  be the partite sets of  $K_{m,n}$  and let  $F = \{v_1u_1, v_1u_2\}$ . Let  $S$  be a  $\sigma_m(K_{m,n} - F)$ -set which contains  $u_1$ . To dominate  $v_1$ , we have  $v_1 \in S$  or  $u_i \in S$  for some  $i \geq 3$ . If  $v_1 \in S$ , then  $u_2$  is not dominated by  $\{u_1, v_1\}$  and so  $|S| \geq 3$ . Let  $v_1 \notin S$ . Assume without loss of generality that  $u_3 \in S$ . Then  $u_2$  is not dominated by  $\{u_1, u_3\}$  and this implies that  $|S| \geq 3$ . Hence,  $b_{\sigma_m}(K_{m,n}) \leq 2$  and the proof is complete. ■

### 3. BOUNDS ON THE ETERNAL $m$ -SECURITY BONDAGE NUMBER

In this section, we present various bounds on the eternal  $m$ -security bondage number. We start with an observation.

**Observation 9.** *Let  $G$  be a connected graph. If  $\sigma_m(G - v) \geq \sigma_m(G)$  for some vertex  $v \in V(G)$ , then  $b_{\sigma_m}(G) \leq \deg(v)$ .*

**Proof.** First, note that  $\sigma_m(G - E_v) \geq \sigma_m(G)$ . If  $\sigma_m(G - E_v) > \sigma_m(G)$ , then we are done. Suppose  $\sigma_m(G - E_v) = \sigma_m(G)$  and let  $S$  be a  $\sigma_m(G - E_v)$ -set. Clearly,  $v \in S$  and  $S \setminus \{v\}$  is an EmSS of  $G - v$ . It follows that

$$\sigma_m(G - E_v) - 1 \geq \sigma_m(G - v) \geq \sigma_m(G),$$

and the proof is complete. ■

**Theorem 10.** *Let  $G$  be a connected graph and  $uv \in E(G)$ . Then*

$$b_{\sigma_m}(G) \leq \deg(u) + \deg(v) - 1 - |N(u) \cap N(v)|.$$

**Proof.** Let  $X$  be the set consisting of all edges incident with  $u$  and  $v$  with exception of the edges  $E(v, N(u))$ . Then  $|X| = \deg(u) + \deg(v) - 1 - |N(u) \cap N(v)|$ ,  $u$  is an isolated vertex in  $G - X$  and  $v$  is only adjacent to the vertices of  $N_G(u) \cap N_G(v)$ . Let  $S$  be a  $\sigma_m(G - X)$ -set which contains  $v$  (we may assume a response to an attack on  $v$ ). It is easy to verify that  $S \setminus \{u\}$  is an EmSS of  $G$  and hence  $\sigma_m(G) \leq \sigma_m(G - X) - 1$ . This completes the proof. ■

**Corollary 11.** *For any nonempty graph  $G$ ,  $b_{\sigma_m}(G) \leq \delta(G) + \Delta(G) - 1$ .*

**Theorem 12.** *Let  $G$  be a connected graph with degree sequence  $(d_1, d_2, \dots, d_n)$ . Then*

$$b_{\sigma_m}(G) \leq d_\alpha + d_{\alpha+1} - 1,$$

where  $\alpha$  is the independence number of  $G$ .

**Proof.** Let  $V(G) = \{v_1, v_2, \dots, v_n\}$  and let  $\deg(v_i) = d_i$  for each  $i$ . Since the set  $\{v_1, \dots, v_{\alpha+1}\}$  is not independent, there is an edge  $v_i v_j$  for some  $1 \leq i < j \leq \alpha + 1$ . It follows from Theorem 10 that

$$b_{\sigma_m}(G) \leq \deg(v_i) + \deg(v_j) - 1 \leq \deg(v_\alpha) + \deg(v_{\alpha+1}) - 1,$$

and the proof is complete. ■

**Theorem 13.** *Let  $G$  be a connected graph and  $u, v$  be two vertices of  $G$  with  $d(u, v) = 2$ . Then*

$$b_{\sigma_m}(G) \leq \deg(u) + \deg(v).$$

**Proof.** Let  $w$  be a common neighbor of  $u$  and  $v$  and let  $X$  be the set consisting of all edges incident with  $u$  and  $v$ . Then  $|X| = \deg(u) + \deg(v)$  and  $u, v$  are isolated vertices in  $G - X$ . Let  $S$  be a  $\sigma_m(G - X)$ -set which contains  $w$  (we may assume a response to an attack on  $w$ ). Obviously  $u, v \in S$  and we can easily check that  $S \setminus \{u\}$  is an EmSS of  $G$  and so  $\sigma_m(G) < \sigma_m(G - |X|)$ . Thus  $b_{\sigma_m}(G) \leq |X| = \deg(u) + \deg(v)$  as desired. ■

**Corollary 14.** *Let  $G$  be a connected graph of order  $n$  with degree sequence  $(\deg(v_1), \deg(v_2), \dots, \deg(v_n))$ . Then  $b_{\sigma_m}(G) \leq \deg(v_{\alpha_2}) + \deg(v_{\alpha_2+1})$ .*

**Proof.** Clearly, the set  $\{v_1, \dots, v_{\alpha_2+1}\}$  is not a 2-packing. Hence,  $d(v_i, v_j) \leq 2$  for some  $1 \leq i \neq j \leq \alpha_2 + 1$  and the result follows by Theorems 10 and 13. ■

Next result is an immediate consequence of Corollaries 5 and 14.

**Corollary 15.** *If  $G$  is a connected graph with degree sequence  $(\deg(v_1), \deg(v_2), \dots, \deg(v_n))$ , then  $b_{\sigma_m}(G) \leq \deg(v_{\sigma_m}) + d(v_{\sigma_m+1})$ .*

**Theorem 16.** *For any connected graph  $G$ ,  $b_{\sigma_m}(G) \leq (\sigma_m(G) - \alpha_2(G) + 1)\Delta(G)$ .*

**Proof.** By Corollary 5,  $\alpha_2(G) \leq \sigma_m(G)$ . Let  $s = \sigma_m(G) - \alpha_2(G) + 1$  and  $U = \{u_1, \dots, u_{\alpha_2}\}$  be a maximum 2-packing in  $G$ . Clearly,  $U \neq V(G)$ . Let  $T$  be a subset of  $V(G) - U$  of size  $s$  and let  $G'$  be the graph obtained from  $G$  by removing all edges incident to the vertices in  $T$ . Obviously,  $|E(G)| - |E(G')| \leq \Delta s$ . Now we have

$$\sigma_m(G') \geq \alpha_2(G') \geq \alpha_2(G) + s = \alpha_2(G) + \sigma_m(G) - \alpha_2(G) + 1 = \sigma_m(G) + 1 > \sigma_m(G)$$

and the proof is complete. ■

The next result is an immediate consequence of Theorem 16.

**Corollary 17.** *If  $\sigma_m(G) = \alpha_2(G)$ , then  $b_{\sigma_m}(G) \leq \Delta(G)$ .*

The edge connectivity number  $\kappa'(G)$  of a connected graph  $G$  is the minimum number of edges that have to be removed out of  $G$  to decompose  $G$  in two components. The inequality  $\kappa'(G) \leq \delta(G)$  is immediate. Next result is an improvement of Corollary 11.

**Theorem 18.** *If  $G$  is a nontrivial connected graph, then*

$$b_{\sigma_m}(G) \leq \Delta(G) + \kappa'(G) - 1.$$

**Proof.** Let  $K$  be a set of edges such that  $\kappa'(G) = |K|$  and  $G - K$  is disconnected. Assume that  $G_1$  and  $G_2$  are the components of  $G - K$ . It is easy to see that  $\sigma_m(G) \leq \sigma_m(G_1) + \sigma_m(G_2) = \sigma_m(G - K)$ . If  $\sigma_m(G) < \sigma_m(G_1) + \sigma_m(G_2)$ , then  $b_{\sigma_m}(G) \leq \kappa'(G)$  and we are done. Let  $\sigma_m(G) = \sigma_m(G_1) + \sigma_m(G_2)$ . We claim that there is a vertex  $v \in V(G_i)$  such that  $v$  is incident to an edge of  $K$  and  $\sigma_m(G_i - E_v) > \sigma_m(G_i)$  for some  $i$ . In this case we have  $\sigma_m(G - K - E_v) > \sigma_m(G)$ , which implies that

$$b_{\sigma_m}(G) \leq \deg(v) + \kappa'(G) - 1 \leq \Delta(G) + \kappa'(G) - 1.$$

Assume, to the contrary, that  $\sigma_m(G_1 - E_v) = \sigma_m(G_1)$  for each vertex  $v \in V(G_1)$  incident to an edge of  $K$  and  $\sigma_m(G_2 - E_v) = \sigma_m(G_2)$  for every vertex  $v \in V(G_2)$  incident to an edge of  $K$ . Let  $u_1u_2 \in K$  where  $u_i \in V(G_i)$  for  $i = 1, 2$ . Let  $S_i$  be a  $\sigma_m(G_i - E_{u_i})$ -set for  $i = 1, 2$ . Clearly,  $u_1 \in S_1$  and  $u_2 \in S_2$ . It is easy to verify that  $S = S_1 \cup S_2 \setminus \{u_1\}$  is an eternal  $m$ -secure set of  $G$  which implies that

$$\sigma_m(G) \leq \sigma_m(G_1 - E_u) + \sigma_m(G_2 - E_v) - 1 = \sigma_m(G_1) + \sigma_m(G_2) - 1 = \sigma_m(G) - 1,$$

a contradiction. This completes the proof. ■

**Proposition 19.** *If  $\sigma_m(G) = 2$ , then  $b_{\sigma_m}(G) \leq \delta(G) + 1$ .*

**Proof.** Let  $u \in V(G)$  be a vertex of minimum degree. If  $\sigma_m(G - u) \geq \sigma_m(G)$ , then the result follows by Observation 9. Let  $\sigma_m(G - u) \leq \sigma_m(G) - 1$ . Then obviously  $\sigma_m(G - u) = 1$  and so  $G - u$  is a complete graph. By Corollary 2, we have  $b_{\sigma_m}(G) \leq b_{\sigma_m}(G - u) + \delta(G) = \delta(G) + 1$ . ■

#### 4. COMPLETE MULTIPARTITE GRAPHS

In this section we determine the eternal  $m$ -security bondage number of complete multipartite graphs yielding that the eternal  $m$ -security bondage number can be arbitrary large.

**Theorem 20.** *Let  $t \geq 3$  and  $G = K_{n_1, n_2, \dots, n_t}$  be the complete  $t$ -partite graph with  $n_1, n_2, \dots, n_t \geq 2$ . Then  $b_{\sigma_m}(G) = \lceil \frac{3(t-1)}{2} \rceil$ .*

**Proof.** Let  $X_1, X_2, \dots, X_t$  be the partite sets of  $G$  and let  $X_i = \{x_1^i, \dots, x_{n_i}^i\}$  for  $1 \leq i \leq t$ . Clearly  $\sigma_m(G) = 2$ . Assume

$$X = \left\{ x_1^1 x_1^j, x_1^{2s} x_1^{2s+1} : 2 \leq j \leq t, 1 \leq s \leq \frac{t-1}{2} \right\}$$

if  $t$  is odd and

$$X = \left\{ x_1^1 x_1^j, x_1^{2s} x_1^{2s+1}, x_1^1 x_2^t : 2 \leq j \leq t, 1 \leq s \leq \frac{t-2}{2} \right\}$$

when  $t$  is even. Obviously,  $|X| = \lceil \frac{3(t-1)}{2} \rceil$ . It is easy to see that for any eternal  $m$ -secure set  $S$  of  $G - X$  containing  $x_1^1$ , we have  $|S| \geq 3$  and so  $b_{\sigma_m}(G) \leq \lceil \frac{3(t-1)}{2} \rceil$ .

Now we show that  $b_{\sigma_m}(G) \geq \lceil \frac{3(t-1)}{2} \rceil$ . Let  $F$  be a set of edges of size at most  $\lceil \frac{3(t-1)}{2} \rceil - 1$  and let  $G_2 = G - F$ .

**Claim.** For each  $x \in V(G_2)$ , there exists a vertex  $x'$  such that  $N_{G_2}[x] \cup N_{G_2}[x'] = V(G_2)$ .

**Proof.** Assume, to the contrary, that there exists a vertex  $x \in V(G_2)$  such that  $N_{G_2}[x] \cup N_{G_2}[v] \neq V(G_2)$  for each  $v \in V(G_2)$ . Without loss of generality we may assume that  $x = x_1^1$ . For  $2 \leq i \leq t$ , let  $F_i = F \cap \{x_1^1 x_j^i : 1 \leq j \leq n_i\}$ . Let first  $F_i = \emptyset$  for each  $2 \leq i \leq t$ . Since  $N_{G_2}[x_1^1] \cup N_{G_2}[v] \neq V(G_2)$  for each  $v \in V(G_2)$ , we have  $x_j^1 v \in F$  for every  $v \in V(G_2) \setminus X_1$  and for some  $2 \leq j \leq n_1$ . This implies that  $|F| \geq |V(G_2) \setminus X_1| \geq 2t - 2$ , a contradiction. Assume that  $F_i \neq \emptyset$  for some  $2 \leq i \leq t$ . We consider two cases.

*Case 1.*  $|F_i| \leq 1$  for each  $2 \leq i \leq t$ . Let  $I \subseteq \{2, \dots, t\}$  be the set of all elements such that  $|F_i| = 1$  for each  $i \in I$  and let  $J = \{2, \dots, t\} \setminus I$ . Without loss of generality, assume that  $\{x_1^1 x_1^i : i \in I\} \subseteq F$ . We estimate the number of edges in  $F$  as follows. Since  $N_{G_2}[x_1^1] \cup N_{G_2}[x_1^i] \neq V(G_2)$  for  $i \in I$ , there exists a vertex  $z^i$  such that  $z^i x_1^i, z^i x_1^1 \notin E(G_2)$ . Obviously,  $z^i \notin X_i \cup (\bigcup_{j \in J} X_j)$ . If  $z^i \in X_1$ , then  $E_i = \{x_1^1 x_1^i, x_1^1 z^i\} \subseteq F$ , and if  $z^i \in X_\ell$  for some  $\ell \in I - \{i\}$ , then  $z^i = x_1^\ell$  and  $E_i = \{x_1^1 x_1^i, x_1^i x_1^\ell, x_1^1 x_1^\ell\} \subseteq F$ . Since  $N_{G_2}[x_1^1] \cup N_{G_2}[x_s^j] \neq V(G_2)$  for  $j \in J$  and  $1 \leq s \leq n_j$ , there exists a vertex  $z_s^j$  such that  $z_s^j x_s^j, z_s^j x_1^1 \notin E(G_2)$ . We note that

$$(1) \quad z_s^j \in X_1 \cup \left( \bigcup_{i \in I} X_i \right)$$

for  $j \in J$  and  $1 \leq s \leq n_j$ . If  $z_s^j \in X_1$ , then  $x_s^j z_s^j \in F \setminus (\bigcup_{i \in I} E_i)$ , and if  $z_s^j \in X_i$  for some  $i \in I$ , then  $z_s^j = x_1^i$  and  $x_s^j z_s^j \in F \setminus (\bigcup_{i \in I} E_i)$  again. Since  $n_j \geq 2$ , we conclude that  $|F \cap \{z_s^j x_s^j : 1 \leq s \leq n_j\}| \geq 2$  for each  $j \in J$ . By (1) we have  $\{z_s^j x_s^j : 1 \leq s \leq n_j\} \cap \{z_{s'}^{j'} x_{s'}^{j'} : 1 \leq s' \leq n_{j'}\} = \emptyset$  for  $j \neq j'$ . Hence, we have

$$\begin{aligned} |F| &\geq \left| \bigcup_{i \in I} E_i \right| + \left| \bigcup_{j \in J} \left( F \cap \{z_s^j x_s^j : 1 \leq s \leq n_j\} \right) \right| \\ &\geq \frac{3|I|}{2} + 2|J| \geq \left\lceil \frac{3}{2}(|I| + |J|) \right\rceil = \left\lceil \frac{3(t-1)}{2} \right\rceil, \end{aligned}$$

which is a contradiction.

*Case 2.*  $|F_i| \geq 2$  for some  $2 \leq i \leq t$ . Let  $I \subseteq \{2, \dots, t\}$  be the set of all elements  $i$  such that  $|F_i| \geq 2$ ,  $J \subseteq \{2, \dots, t\}$  be the set of all elements  $j$  such that  $|F_j| = 1$  and  $R = \{2, \dots, t\} \setminus (I \cup J)$ . Without loss of generality, assume that  $\{x_1^1 x_1^i, x_1^1 x_2^i, x_1^1 x_1^j : i \in I, j \in J\} \subseteq F$ . We estimate the number of edges in  $F$  as follows. Obviously,  $|\bigcup_{i \in I} F_i| \geq 2|I|$ . Since  $N_{G_2}[x_1^1] \cup N_{G_2}[x_1^j] \neq V(G_2)$  for each  $j \in J$ , there exists a vertex  $z^j$  such that  $z^j x_1^j, z^j x_1^1 \notin E(G_2)$ . Obviously,  $z^j \notin X_j \cup (\bigcup_{r \in R} X_r)$ . If  $z^j \in X_i$ , for some  $i \in I$ , then  $E_j = \{x_1^1 x_1^j, x_1^1 z^j\} \subseteq F$ , and if  $z^j \in X_\ell$  for some  $\ell \in J - \{j\}$ , then  $z^j = x_1^\ell$  and  $E_j = \{x_1^1 x_1^j, x_1^j x_1^\ell, x_1^1 x_1^\ell\} \subseteq F$ .



As in Case 1, we can see that  $|F \cap \{z_s^r x_s^r : 1 \leq s \leq n_r\}| \geq 2$  for each  $r \in R$ , and  $\{z_s^r x_s^r : 1 \leq s \leq n_r\} \cap \{z_s^{r'} x_s^{r'} : 1 \leq s \leq n_{r'}\} = \emptyset$  for  $r \neq r'$ .

Hence, we have

$$\begin{aligned} |F| &\geq \left| \bigcup_{i \in I} F_i \right| + \left| \bigcup_{j \in J} E_j \right| + \left| \bigcup_{j \in J} \left( F \cap \{z_s^j x_s^j : 1 \leq s \leq n_j\} \right) \right| \\ &\geq 2|I| + \frac{3|J|}{2} + 2|R| \geq \left\lceil \frac{3}{2}(|I| + |J| + |R|) \right\rceil = \left\lceil \frac{3(t-1)}{2} \right\rceil, \end{aligned}$$

which is a contradiction. □

Now, for each  $v \in V(G_2)$ , let  $x_v \in V(G_2)$  be a vertex such that  $N_{G_2}[v] \cup N_{G_2}[x_v] = V(G_2)$ . We show that the set  $S_v = \{v, x_v\}$  is an EmSS of  $G_2$ . Obviously,  $S_v$  is a dominating set of  $G_2$ . Consider an attack on a vertex  $u$  of  $V(G_2)$ . Then one of  $v$  or  $x_v$  is adjacent to  $u$ . Let  $uv \in E(G_2)$ . If  $x_u$  is adjacent to  $x_v$ , then we can shift guards from  $v$  and  $x_v$  to  $u$  and  $x_u$ , respectively. Let  $x_u x_v \notin E(G_2)$ . Then  $x_u v, u x_v \in E(G_2)$  and we can shift guards from  $v$  and  $x_v$  to  $x_u$  and  $u$ , respectively. Therefore,  $\sigma_m(G_2) = 2$  and this implies that  $b_{\sigma_m}(G) \geq \left\lceil \frac{3(t-1)}{2} \right\rceil$ .

Thus  $b_{\sigma_m}(G) = \left\lceil \frac{3(t-1)}{2} \right\rceil$  and the proof is complete. ■

### 5. TREES

In this section, we first prove that for any nontrivial tree  $T$ ,  $b_{\sigma_m}(T) \leq 2$  and then we characterize all trees attaining this bound.

**Theorem 21.** *For any tree  $T$  of order  $n \geq 2$ ,  $b_{\sigma_m}(T) \leq 2$ .*

**Proof.** If  $\text{diam}(T) \leq 2$ , then  $T$  is a star and the result is immediate. Let  $\text{diam}(T) \geq 3$ . Suppose  $P := v_1 v_2 \cdots v_k$  is a diametral path in  $T$  and root  $T$  at  $v_k$ . Obviously,  $k \geq 4$ . If  $\text{deg}(v_2) = 2$ , then  $b_{\sigma_m}(T) \leq 2$  by Theorem 10. Let  $\text{deg}(v_2) \geq 3$ . Then  $v_2$  is adjacent to a leaf  $v'$  other than  $v_1$ . Since  $d(v_1, v') = 2$ , Theorem 13 implies that  $b_{\sigma_m}(T) \leq 2$ . This completes the proof. ■

Next, we provide a constructive characterization of all trees attaining the bound of Theorem 21. For this purpose, we describe a procedure to build a family  $\mathfrak{T}$  of trees as follows. Let  $\mathfrak{T}$  be the family of trees such that a path  $P_3$  is a tree in  $\mathfrak{T}$  and if  $T$  is a tree in  $\mathfrak{T}$ , then the tree  $T'$  obtained from  $T$  by the following four operations which extend the tree  $T$  by attaching a tree to a vertex  $v \in V(T)$ , called an *attacher*, is also a tree in  $\mathfrak{T}$  (see Figure 1).

**Operation  $\mathfrak{T}_1$ .** If  $v \in V(T)$ , then  $\mathfrak{T}_1$  adds a path  $vxy$  to  $T$ .

**Operation  $\mathfrak{T}_2$ .** If  $v \in V(T)$ , then  $\mathfrak{T}_2$  adds a star  $K_{1,3}$  with central vertex  $y$  and leaves  $x, w, z$  and joins  $x$  to  $v$ .

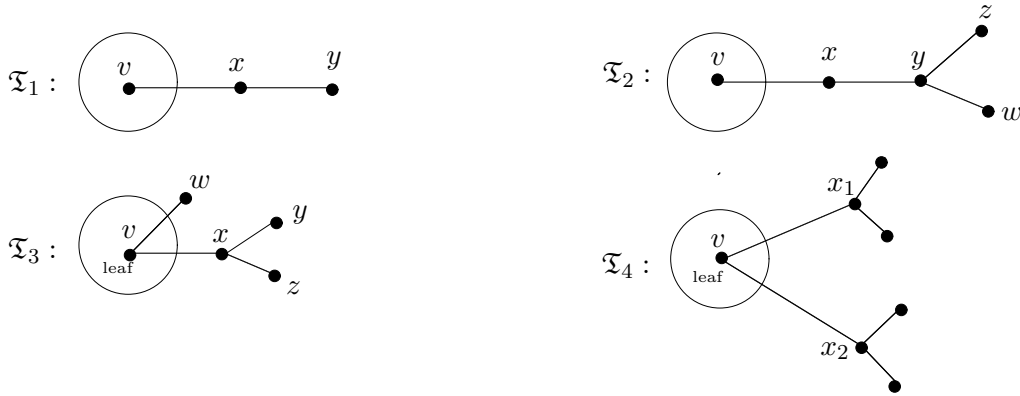


Figure 1. The four operations.

**Operation  $\mathfrak{T}_3$ .** If  $v \in V(T)$  is a leaf, then  $\mathfrak{T}_3$  adds a pendant edge  $vw$  and a star  $K_{1,2}$  with central vertex  $x$  and leaves  $y, z$  and joins  $x$  to  $v$ .

**Operation  $\mathfrak{T}_4$ .** If  $v \in V(T)$  is a leaf, then  $\mathfrak{T}_4$  adds two new stars  $K_{1,2}$  centered at  $x_1$  and  $x_2$ , and joins  $v$  to  $x_1$  and  $x_2$ .

We start with some lemmas.

**Lemma 22.** Let  $G$  be a graph and  $v \in V(G)$ . If  $G'$  is the graph obtained from  $G$  by attaching a path  $vxy$ , then  $\sigma_m(G') = \sigma_m(G) + 1$ . In particular,  $b_{\sigma_m}(G') \leq b_{\sigma_m}(G)$ .

**Proof.** Clearly, adding  $x$  to any  $\sigma_m(G)$ -set yields an  $EmSS$  of  $G'$  and so  $\sigma_m(G') \leq \sigma_m(G) + 1$ . Let now  $S'$  be a  $\sigma_m(G')$ -set containing  $y$  (we may assume a response to an attack on  $y$ ). If  $x \in S'$ , then the set  $(S' \setminus \{x, y\}) \cup \{w\}$ , where  $w \in N_G[v] \setminus S'$ , is an  $EmSS$  of  $G$ . If  $x \notin S'$ , then  $S' \setminus \{y\}$  is an  $EmSS$  of  $G$ . Thus  $\sigma_m(G) \leq \sigma_m(G') - 1$  and so  $\sigma_m(G') = \sigma_m(G) + 1$ . ■

**Lemma 23.** Let  $G$  be a graph and  $v \in V(G)$ . If  $G'$  is the graph obtained from  $G$  by adding a star  $K_{1,3}$  with central vertex  $y$  and leaves  $x, w, z$  and joining  $x$  to  $v$ , then  $\sigma_m(G') = \sigma_m(G) + 2$ . In particular,  $b_{\sigma_m}(G') \leq b_{\sigma_m}(G)$ .

**Proof.** Clearly, adding  $x$  and  $y$  to any  $\sigma_m(G)$ -set yields an  $EmSS$  of  $G'$  and so  $\sigma_m(G') \leq \sigma_m(G) + 2$ . Suppose now  $S'$  is a  $\sigma_m(G')$ -set containing  $z$  (we may assume a response to an attack on  $z$ ). Since  $S'$  is a dominating set, we must have  $|S' \cap \{y, w\}| \geq 1$ . If  $x \in S'$  then the set  $(S' \setminus \{x, y, z, w\}) \cup \{u\}$ , where  $u \in N_G[v] \setminus S'$  is an  $EmSS$  of  $G$ , and if  $x \notin S'$  then the set  $S' \setminus \{x, y, z, w\}$  is an  $EmSS$  of  $G$ . Hence  $\sigma_m(G) \leq \sigma_m(G') - 2$  and this implies that  $\sigma_m(G') = \sigma_m(G) + 2$ . ■

**Lemma 24.** Let  $G$  be a graph and let  $v \in V(G)$ . If  $G'$  is the graph obtained from  $G$  by adding a pendant edge  $vw$  and a star  $K_{1,2}$  with central vertex  $x$

and leaves  $y, z$  and joining  $x$  to  $v$ , then  $\sigma_m(G') = \sigma_m(G) + 2$ . In particular,  $b_{\sigma_m}(G') \leq b_{\sigma_m}(G)$ .

**Proof.** Clearly, adding  $x, y$  to any  $\sigma_m(G)$ -set containing  $v$  yields an EmSS of  $G'$  and so  $\sigma_m(G') \leq \sigma_m(G) + 2$ . Assume now that  $S'$  is a  $\sigma_m(G')$ -set. As in the proof of Lemma 23, we may assume that  $y \in S'$  and  $|S' \cap \{x, z\}| \geq 1$ . Since  $S'$  is a dominating set, we must have  $|S' \cap \{v, w\}| \geq 1$ . If  $|S' \cap \{x, y, z, w\}| \geq 3$ , then let  $S'' = (S' \setminus \{x, y, z, w\}) \cup \{u\}$  where  $u \in N_G[v] \setminus S'$ , and if  $|S' \cap \{x, y, z, w\}| = 2$ , then let  $S'' = S' \setminus \{x, y, z, w\}$ . Clearly,  $S''$  is an EmSS of  $G$  and hence  $\sigma_m(G) \leq \sigma_m(G') - 2$ . Thus  $\sigma_m(G') = \sigma_m(G) + 2$ . ■

**Lemma 25.** Let  $G$  be a graph and let  $v \in V(G)$ . If  $G'$  is the graph obtained from  $G$  by adding two new stars  $K_{1,2}$  centered at  $x_1, x_2$  and joining  $v$  to  $x_1, x_2$ , then  $\sigma_m(G') = \sigma_m(G) + 3$ . In particular,  $b_{\sigma_m}(G') \leq b_{\sigma_m}(G)$ .

**Proof.** Let  $y_i, z_i$  be the leaves adjacent to  $x_i$  for  $i = 1, 2$ . Clearly, adding  $x_1, x_2, y_1$  to any  $\sigma_m(G)$ -set containing  $v$  yields an EmSS of  $G'$  and so  $\sigma_m(G') \leq \sigma_m(G) + 3$ . Let now  $S'$  be a  $\sigma_m(G')$ -set. As above we may assume that  $y_1 \in S'$ ,  $|S' \cap \{x_1, z_1\}| \geq 1$  and  $|S' \cap \{x_2, y_2, z_2\}| \geq 1$ . It is easy to see that  $|S' \cap \{x_2, y_2, z_2, v\}| \geq 2$ . If  $|S' \cap \{x_2, y_2, z_2\}| = 2$ , then let  $S'' = (S' - \{x_1, y_1, z_1, x_2, y_2, z_2\}) \cup \{u\}$  where  $u \in N_G[v] \setminus S'$ , and if  $|S' \cap \{x_2, y_2, z_2\}| = 1$ , then let  $S'' = S' \setminus \{x_1, y_1, z_1, x_2, y_2, z_2\}$ . Clearly,  $S''$  is an EmSS of  $G$  and hence  $\sigma_m(G) \leq \sigma_m(G') - 3$ . Thus  $\sigma_m(G') = \sigma_m(G) + 3$ . ■

**Lemma 26.** Let  $T \in \mathfrak{T}$  and  $u \in V(T)$ . If  $T'$  is a tree obtained from  $T$  by adding a pendant edge  $uu'$ , then  $\sigma_m(T') = \sigma_m(T)$ .

**Proof.** Let  $T'$  be a tree obtained from  $T$  by adding the pendant edge  $uu'$ . If  $S$  is a  $\sigma_m(T')$ -set, then let  $S' = S$  if  $u' \notin S$  and  $S' = (S - \{u'\}) \cup \{w\}$ , where  $w \in N_T[u] \setminus S$ , when  $u' \in S$ . Clearly,  $S'$  is an EmSS for  $G$  and so  $\sigma_m(T) \leq \sigma_m(T')$ .

Now we show that  $\sigma_m(T') \leq \sigma_m(T)$ . Let  $P_3 = v_1v_2v_3$  and let  $T$  be obtained from  $P_3$  by successive operations  $\mathfrak{T}^1, \dots, \mathfrak{T}^m$ , respectively, where  $\mathfrak{T}^i \in \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4\}$  for  $1 \leq i \leq m$ , if  $m \geq 1$ , and  $T = P_3$  if  $m = 0$ . The proof is by induction on  $m$ . If  $m = 0$ , then clearly the statement is true. Assume  $m \geq 1$  and that the statement holds for all trees which are obtained from  $P_3$  by applying at most  $m - 1$  operations. Suppose  $T_{m-1}$  is a tree obtained by applying the first  $m - 1$  operations  $\mathfrak{T}^1, \dots, \mathfrak{T}^{m-1}$  and let  $T$  be obtained from  $T_{m-1}$  by adding a new part to the attacher  $v$ . Assume that  $T'_{m-1}$  is obtained from  $T_{m-1}$  by adding a pendant edge  $uu'$  when  $u \in V(T_{m-1})$ . We consider four cases.

*Case 1.*  $\mathfrak{T}^m = \mathfrak{T}_1$ . Then  $T$  is obtained from  $T_{m-1}$  by attaching a path  $vxy$  to  $v \in V(T_{m-1})$ . If  $u \in V(T_{m-1})$ , then by the inductive hypothesis,  $\sigma_m(T_{m-1}) = \sigma_m(T'_{m-1})$  and by Lemma 22 we have  $\sigma_m(T') = \sigma_m(T)$ . Suppose  $u \in \{x, y\}$ . Let  $T^* = T' - \{y, u'\}$ . Then, obviously,  $T^*$  is obtained from  $T_{m-1}$  by adding the

pendant edge  $xv$ . By the inductive hypothesis,  $\sigma_m(T^*) = \sigma_m(T_{m-1})$ . Let  $S$  be a  $\sigma_m(T^*)$ -set containing  $x$ . Then  $S \cup \{y\}$  is an EmSS of  $T'$  and by Lemma 22 we have

$$\sigma_m(T') \leq \sigma_m(T^*) + 1 = \sigma_m(T_{m-1}) + 1 = \sigma_m(T).$$

*Case 2.*  $\mathfrak{T}^m = \mathfrak{T}_2$ . Then  $T$  is obtained from  $T_{m-1}$  by adding a star  $K_{1,3}$  with central vertex  $y$  and leaves  $x, w, z$  and joining  $x$  to  $v$ . If  $u \in V(T_{m-1})$ , then the result follows from the induction hypothesis and Lemma 23. Assume that  $u \in \{x, y, z, w\}$ . Let  $T^* = T - \{y, z, w\}$ . By the induction hypothesis, we have  $\sigma_m(T^*) = \sigma_m(T_{m-1})$ . Let  $S$  be a  $\sigma_m(T^*)$ -set containing  $x$ . Then the set  $S \cup \{y, z\}$  if  $u \neq w$  and the set  $S \cup \{y, w\}$  if  $u = w$ , is an EmSS of  $T'$  and so  $\sigma_m(T') \leq \sigma_m(T^*) + 2$ . By Lemma 23, we obtain

$$\sigma_m(T') \leq \sigma_m(T^*) + 2 = \sigma_m(T_{m-1}) + 2 = \sigma_m(T).$$

*Case 3.*  $\mathfrak{T}^m = \mathfrak{T}_3$ . Then  $T$  is obtained from  $T_{m-1}$  by attaching a pendant edge  $vw$  at  $v$  and adding a star  $K_{1,2}$  with central vertex  $x$  and leaves  $y, z$  and joining  $x$  to  $v$ . If  $u \in V(T_{m-1})$ , then we deduce from the induction hypothesis and Lemma 24 that  $\sigma_m(T') = \sigma_m(T)$ . If  $u = x$  or  $u = y$  (the case  $u = z$  is similar), then let  $T^* = T' - \{u', x, y, z\}$ . Obviously,  $T^*$  is obtained from  $T_{m-1}$  by adding the pendant edge  $vw$  at  $v$ . By the induction hypothesis, we have  $\sigma_m(T^*) = \sigma_m(T_{m-1})$ . Clearly, adding  $x, y$  to any  $\sigma_m(T^*)$ -set yields an EmSS of  $T'$  and so

$$\sigma_m(T') \leq \sigma_m(T^*) + 2 = \sigma_m(T_{m-1}) + 2 = \sigma_m(T).$$

If  $u = w$ , then let  $T^* = T' - \{u', x, y, z\}$ . Obviously,  $T^*$  is obtained from  $T_{m-1}$  by adding the pendant edge  $vw$  at  $v$ . By the inductive hypothesis,  $\sigma_m(T^*) = \sigma_m(T_{m-1})$ . Let  $S$  be a  $\sigma_m(T^*)$ -set containing  $w$ . Then  $S \cup \{v, x\}$  if  $v \notin S$  and  $S \cup \{x, y\}$  if  $v \in S$ , is an EmSS of  $T'$  and so

$$\sigma_m(T') \leq \sigma_m(T^*) + 2 = \sigma_m(T_{m-1}) + 2 = \sigma_m(T).$$

*Case 4.*  $\mathfrak{T}^m = \mathfrak{T}_4$ . Then  $T$  is obtained from  $T_{m-1}$  by adding two stars  $K_{1,2}$  with central vertices  $x_1$  and  $x_2$  and joining  $x_1, x_2$  to  $v \in V(T_{m-1})$ . Let  $y_i, z_i$  be the leaves adjacent to  $x_i$  for  $i = 1, 2$ . If  $u \in V(T_{m-1})$ , then the result follows from the induction hypothesis and Lemma 25. If  $u = x_1$  (the case  $u = x_2$  is similar), then adding  $x_1, y_1, x_2$  to any  $\sigma_m(T_{m-1})$ -set containing  $v$  yields an EmSS of  $T'$  and we deduce from Lemma 25 that

$$\sigma_m(T') \leq \sigma_m(T_{m-1}) + 3 = \sigma_m(T).$$

Assume that  $u = y_1$  (the cases  $u = z_1, u = y_2, u = z_2$  are similar). Let  $T^* = T' - \{x_1, y_1, z_1, u', y_2, z_2\}$ . Obviously,  $T^*$  is obtained from  $T_{m-1}$  by adding pendant edge  $vx_2$  at  $v$ . By the inductive hypothesis, we have  $\sigma_m(T^*) = \sigma_m(T_{m-1})$ .

Clearly, adding  $x_1, y_1, y_2$  to any  $\sigma_m(T^*)$ -set containing  $x_2$ , yields an EmSS of  $T'$  and this implies that

$$\sigma_m(T') \leq \sigma_m(T^*) + 3 = \sigma_m(T_{m-1}) + 3 = \sigma_m(T).$$

Hence  $\sigma_m(T') \leq \sigma_m(T)$ . Thus  $\sigma_m(T') = \sigma_m(T)$  and the proof is complete. ■

**Theorem 27.** If  $T \in \mathfrak{T}$ , then  $b_{\sigma_m}(T) = 2$ .

*Proof.* Let  $T \in \mathfrak{T}$ ,  $e \in E(T)$  and  $T' = T - e$ . Clearly  $\sigma_m(T') \geq \sigma_m(T)$ . Now we show that  $\sigma_m(T') \leq \sigma_m(T)$ . Let  $P_3 := v_1v_2v_3$  and let  $T$  be obtained from  $P_3$  by successive operations  $\mathfrak{T}^1, \dots, \mathfrak{T}^m$ , respectively, where  $\mathfrak{T}^i \in \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4\}$  for  $1 \leq i \leq m$  if  $m \geq 1$  and  $T = P_3$  if  $m = 0$ . The proof is by induction on  $m$ . If  $m = 0$ , then the statement is true by Corollary 4. Assume  $m \geq 1$  and that the statement holds for all trees obtained from  $P_3$  by applying at most  $m - 1$  operations. Suppose  $T_{m-1}$  is a tree obtained by applying the first  $m - 1$  operations  $\mathfrak{T}^1, \dots, \mathfrak{T}^{m-1}$ . We consider four cases.

*Case 1.*  $\mathfrak{T}^m = \mathfrak{T}_1$ . Then  $T$  is obtained from  $T_{m-1}$  by attaching a path  $vxy$  at  $v \in V(T_{m-1})$ . If  $e \in E(T_{m-1})$ , then we deduce from the induction hypothesis and Lemma 22 that

$$\sigma_m(T') = \sigma_m(T_{m-1} - e) + 1 = \sigma_m(T_{m-1}) + 1 = \sigma_m(T).$$

If  $e = vx$ , then clearly  $\sigma_m(T') = \sigma_m(T_{m-1}) + 1 = \sigma_m(T)$ . Assume that  $e = xy$ . Let  $T^* = T' - \{y\}$ . Then  $T^*$  is obtained from  $T_{m-1}$  by adding a pendant path  $vx$  at  $v$ . Clearly  $\sigma_m(T') = \sigma_m(T^*) + 1$ . It follows from Lemmas 26 and 22 that

$$\sigma_m(T') = \sigma_m(T^*) + 1 = \sigma_m(T_{m-1}) + 1 = \sigma_m(T).$$

*Case 2.*  $\mathfrak{T}^m = \mathfrak{T}_2$ . Then  $T$  is obtained from  $T_{m-1}$  by adding a star  $K_{1,3}$  with central vertex  $y$  and leaves  $x, w, z$  and joining  $x$  to  $v$ . If  $e \in E(T_{m-1})$ , then by the inductive hypothesis and Lemma 23 we have

$$\sigma_m(T') = \sigma_m(T_{m-1} - e) + 2 = \sigma_m(T_{m-1}) + 2 = \sigma_m(T).$$

If  $e = vx$ , then clearly  $\sigma_m(T') = \sigma_m(T_{m-1}) + 2 = \sigma_m(T)$ . If  $e = xy$ , then let  $T^* = T - \{y, z, w\}$ . By Lemma 26, we have  $\sigma_m(T^*) = \sigma_m(T_{m-1})$ . Clearly  $\sigma_m(T') = \sigma_m(T^*) + 2$  and by Lemma 23 we have

$$\sigma_m(T') = \sigma_m(T^*) + 2 = \sigma_m(T_{m-1}) + 2 = \sigma_m(T).$$

Assume that  $e = yz$ . Let  $T^* = T' - \{z, w\}$ . Then  $T^*$  is obtained from  $T_{m-1}$  by Operation  $\mathfrak{T}_1$  and so  $T^* \in \mathfrak{T}$  and  $\sigma_m(T^*) = \sigma_m(T_{m-1}) + 1$ . By Lemma 26, we have

$\sigma_m(T^* + yw) = \sigma_m(T^*)$ . Now it is easy to check that  $\sigma_m(T') \leq \sigma_m(T^* + yw) + 1$  and by Lemma 23 we have

$$\sigma_m(T') \leq \sigma_m(T^*) + 1 = \sigma_m(T_{m-1}) + 2 = \sigma_m(T).$$

*Case 3.*  $\mathfrak{T}^m = \mathfrak{T}_3$ . Then  $T$  is obtained from  $T_{m-1}$  by adding a pendant edge  $vw$  at a leaf  $v \in V(T_{m-1})$  and adding a star  $K_{1,2}$  with central vertex  $x$  and leaves  $y, z$  and joining  $x$  to  $v$ . If  $e \in E(T_{m-1})$ , then we conclude from the induction hypothesis and Lemma 24 that

$$\sigma_m(T') = \sigma_m(T_{m-1} - e) + 2 = \sigma_m(T_{m-1}) + 2 = \sigma_m(T).$$

If  $e = vw$ , then let  $T^* = T - \{y, z, w\}$ . Then we have  $\sigma_m(T^*) = \sigma_m(T_{m-1})$  by Lemma 26. On the other hand, adding  $y, w$  to any  $\sigma_m(T^*)$ -set containing  $x$ , yields an EmSS of  $T'$  and we deduce from Lemma 24 that

$$\sigma_m(T') \leq \sigma_m(T^*) + 2 = \sigma_m(T_{m-1}) + 2 = \sigma_m(T).$$

If  $e \in \{xv, xy, xz\}$ , then let  $T^* = T' - \{x, y, z\}$ . Then  $T^*$  is obtained from  $T_{m-1}$  by attaching a pendant edge  $vw$ . By Lemma 26, we have  $\sigma_m(T^*) = \sigma_m(T_{m-1})$ . On the other hand, adding  $x, y$  to any  $\sigma_m(T^*)$ -set yields an EmSS of  $T'$  and it follows from Lemma 24 that

$$\sigma_m(T') \leq \sigma_m(T^*) + 2 = \sigma_m(T_{m-1}) + 2 = \sigma_m(T).$$

*Case 4.*  $\mathfrak{T}^m = \mathfrak{T}_4$ . Then  $T$  is obtained from  $T_{m-1}$  by adding two stars  $K_{1,2}$  with central vertices  $x_1$  and  $x_2$  and joining  $x_1, x_2$  to a leaf  $v$ . Let  $y_i, z_i$  be the leaves adjacent to  $x_i$  for  $i = 1, 2$ . If  $e \in E(T_{m-1})$ , then by the inductive hypothesis and Lemma 25 we have

$$\sigma_m(T') = \sigma_m(T_{m-1} - e) + 3 = \sigma_m(T_{m-1}) + 3 = \sigma_m(T).$$

If  $e = x_1v$  or  $e = x_1y_1$ , then let  $T^* = T' - \{x_1, y_1, z_1, y_2, z_2\}$ . Then  $T^*$  is obtained from  $T_{m-1}$  by attaching a pendant edge  $vx_2$  at  $v$ . By Lemma 26, we have  $\sigma_m(T^*) = \sigma_m(T_{m-1})$ . On the other hand, adding  $x_1, y_1, y_2$  to any  $\sigma_m(T^*)$ -set containing  $x_2$  yields an EmSS of  $T'$  and it follows from Lemma 25 that

$$\sigma_m(T') \leq \sigma_m(T^*) + 3 = \sigma_m(T_{m-1}) + 3 = \sigma_m(T).$$

In the other cases, we can see that  $\sigma_m(T') \leq \sigma_m(T)$  as above. Hence  $\sigma_m(T') \leq \sigma_m(T)$ . Thus  $\sigma_m(T') = \sigma_m(T)$  and this implies that  $b_{\sigma_m}(T) \geq 2$ . Now the result follows from Theorem 21.  $\blacksquare$

Now we are ready to prove the main theorem of this section.

**Theorem 28.** Let  $T$  be a tree of order  $n \geq 3$ . Then  $b_{\sigma_m}(T) = 2$  if and only if  $T \in \mathfrak{T}$ .

*Proof.* According to Theorem 27, we only need to prove the necessity. We proceed by the induction on  $n$ . If  $n = 3$ , then the result is trivial. Assume that  $n \geq 4$  and the statement holds for all trees  $T$  of order less than  $n$ . Let  $T$  be a tree of order  $n$  with  $b_{\sigma_m}(T) = 2$ . Since  $b_{\sigma_m}(K_{1,n-1}) = 1$ , we have  $\text{diam}(T) \geq 3$ . Suppose  $P := v_1 \cdots v_k$  is a diametral path in  $T$  such that  $\deg(v_2)$  is as small as possible and root  $T$  at  $v_k$ . If  $\deg(v_2) = 2$ , then let  $T' = T - \{v_1, v_2\}$ . By Lemma 22, we have  $\sigma_m(T) = \sigma_m(T') + 1$  and  $b_{\sigma_m}(T') = 2$ . It follows from the induction hypothesis that  $T' \in \mathfrak{T}$ . Now  $T$  can be obtained from  $T'$  by Operation  $\mathfrak{T}_1$  and hence  $T \in \mathfrak{T}$ . Let  $\deg(v_2) \geq 3$ . We conclude from Proposition 7 that  $\deg(v_2) = 3$ . Let  $w \neq v_1$  be a leaf adjacent to  $v_2$ . If  $\deg(v_3) = 2$ , then let  $T' = T - \{v_1, v_2, v_3, w\}$ . By Lemma 23, we have  $\sigma_m(T) = \sigma_m(T') + 2$  and  $b_{\sigma_m}(T') = 2$ . By the induction hypothesis, we obtain  $T' \in \mathfrak{T}$ . Now  $T$  can be obtained from  $T'$  by Operation  $\mathfrak{T}_2$  and so  $T \in \mathfrak{T}$ . Let  $\deg(v_3) \geq 3$ . We consider the following cases.

*Case 1.* There exists a path  $v_3xy$  in  $T$  such that  $x \notin \{v_2, v_4\}$ . By the choice of diametral path and Proposition 7, we have  $\deg(x) = 3$ . If  $v_3$  is a support vertex and  $u$  is a leaf adjacent to  $v_3$ , then it is not hard to see that deleting the edge  $v_3u$  increases the eternal  $m$ -security number which leads to a contradiction. Suppose  $v_3$  is not a support vertex. If  $v_3$  is adjacent to a support vertex  $w$  other than  $x, v_2, v_4$ , then as above we may assume that  $\deg(w) = 3$ . It is easy to see that deleting the edge  $v_3w$  increases the eternal  $m$ -security number which leads to a contradiction. Hence,  $\deg(v_3) = 3$ . Let  $T' = T - \{v_1, v_2, w, x, y, z\}$  where  $y$  and  $z$  are the leaves adjacent to  $x$ . Then  $\sigma_m(T) = \sigma_m(T') + 3$  and  $b_{\sigma_m}(T') = 2$  by Lemma 25. We deduce from the induction hypothesis that  $T' \in \mathfrak{T}$  and so  $T$  can be obtained from  $T'$  by Operation  $\mathfrak{T}_4$ . Hence  $T \in \mathcal{T}$ .

*Case 2.* Any neighbor of  $v_3$ , except  $v_2, v_4$ , is a leaf. Let  $u$  be a leaf adjacent to  $v_3$ . If  $\deg(v_3) \geq 4$ , then it is easy to see that deleting the edge  $v_3u$  increases the eternal  $m$ -security number and so  $b_{\sigma_m}(T) = 1$ , a contradiction. Thus  $\deg(v_3) = 3$ . Let  $T' = T - \{v_1, v_2, u, w\}$ . By Lemma 24 we have  $\sigma_m(T) = \sigma_m(T') + 2$  and  $b_{\sigma_m}(T') = 2$ . It follows from the inductive hypothesis that  $T' \in \mathfrak{T}$ . By Operation  $\mathfrak{T}_3$ ,  $T$  can be obtained from  $T'$  and so  $T \in \mathcal{T}$ . This completes the proof. ■

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Received 12 September 2016

Revised 3 April 2017

Accepted 3 April 2017