TOTAL COLORINGS OF EMBEDDED GRAPHS WITH NO 3-CYCLES ADJACENT TO 4-CYCLES

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Abstract

A total-$k$-coloring of a graph $G$ is a coloring of $V \cup E$ using $k$ colors such that no two adjacent or incident elements receive the same color. The total chromatic number $\chi''(G)$ of $G$ is the smallest integer $k$ such that $G$ has a total-$k$-coloring. Let $G$ be a graph embedded in a surface of Euler characteristic $\epsilon \geq 0$. If $G$ contains no 3-cycles adjacent to 4-cycles, that is, no 3-cycle has a common edge with a 4-cycle, then $\chi''(G) \leq \max\{8, \Delta + 1\}$.

Keywords: total coloring, embedded graph, cycle.

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1. Introduction

All graphs considered in this paper are simple, finite and undirected, and we follow [2] for the terminologies and notations not defined here. Let $G$ be a graph.
We use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ (or simply $V$, $E$, $\Delta$ and $\delta$) to denote the vertex set, the edge set, the maximum degree and the minimum degree of $G$, respectively. A total-$k$-coloring of a graph $G$ is a coloring of $V \cup E$ using $k$ colors such that no two adjacent or incident elements receive the same color. The total chromatic number $\chi''(G)$ of $G$ is the smallest integer $k$ such that $G$ has a total-$k$-coloring. Clearly, $\chi''(G) \geq \Delta + 1$. Behzad [1] and Vizing [18] posed independently the following famous conjecture, which is known as the Total Coloring Conjecture (TCC).

**Conjecture.** For any graph $G$, $\chi''(G) \leq \Delta + 2$.

This conjecture was confirmed for all graphs with $\Delta \leq 3$ independently by Vijayaditya and Rosenfeld in 1971, and in [13, 14], Kostochka proved that if $4 \leq \Delta \leq 5$, then $\chi''(G) \leq \Delta + 2$. Later, Kostochka [15] renewed the proof for $\Delta = 5$. We summary these result to the following lemma.

**Lemma 1.** Let $G$ be a graph with $\Delta(G) \leq 5$. Then $\chi''(G) \leq 7$.

But for planar graphs, the famous conjecture was first proved by Borodin [4] for $\Delta \geq 11$ and then for $\Delta \geq 9$ [3], which was extended to $\Delta \geq 8$ by Jensen and Toft [9] and to $\Delta \geq 7$ by Sanders and Zhao [17]. So the only open case is $\Delta = 6$.

Interestingly, planar graphs with high maximum degree allow a stronger assertion, that is, every planar graph with high maximum degree $\Delta$ has a total-$(\Delta + 1)$-coloring. This result was first established in [4] for $\Delta \geq 16$, which was extended to $\Delta \geq 14$ [3], $\Delta \geq 12$ [5], $\Delta \geq 11$ [6], $\Delta \geq 10$ [25] and finally $\Delta \geq 9$ [10]. However, for $\Delta \in \{4, 5, 6, 7, 8\}$, it is not known if the assertion still holds true. Such a study has attracted a considerable amount of attention. Recently, Shen et al. [11] proved that if $G$ is a planar graph with $\Delta = 8$ and $G$ contains no chordal 5-cycles or no chordal 6-cycles, then $\chi''(G) = \Delta + 1$. Wang and Wu [19] proved that if $G$ is a planar graph with $\Delta \geq 7$ and every vertex is incident with at most one triangle, then $\chi''(G) = \Delta + 1$. Wang and Wu [20] proved that if $G$ is a planar graph with $\Delta \geq 7$ with no 4-cycles, then $\chi''(G) = \Delta + 1$ (later, it is extended to $\Delta \geq 6$ by Shen and Wang [12]). Chang et al. [7] proved that if $G$ is a planar graph with $\Delta \geq 7$ and every vertex $v$ has an integer $k_v \in \{3, 4, 5, 6\}$, such that $v$ is not in any $k_v$-cycle, then $\chi''(G) = \Delta + 1$.

Let $G$ be a graph embedded in a surface of Euler characteristic $\varepsilon$, where surfaces in this paper are compact, connected 2-dimensional manifolds without boundary. All embeddings considered in this paper are 2-cell embeddings. Wu and Wang [24] proved that if $\varepsilon < 0$ and $\Delta(G) \geq \sqrt{25 - 24\varepsilon} + 10$, then $\chi'_{list}(G) = \Delta(G)$ and $\chi''_{list}(G) = \Delta(G) + 1$, which extends a result of Borodin, Kostochka and Woodall in [5]. They also proved that $\chi''(G) = \Delta(G) + 1$ if $\varepsilon \geq 0$, $\Delta(G) \geq 9$ and no two triangles have a common edge, or if $\varepsilon \geq 0$, $\Delta(G) \geq 8$ and no two triangles have a common vertex. Wang et al. [22] proved that if $\varepsilon \geq 0$ and $\Delta(G) \geq 7$,
then $\chi''(G) \leq \Delta + 2$. Wang et al. [23] proved that if $\varepsilon \geq 0$ and $\Delta \geq 9$, then $\chi''(G) = \Delta + 1$. In this paper, we shall prove the following result.

**Theorem 2.** Let $G$ be a graph embedded in a surface of Euler characteristic $\varepsilon \geq 0$. If $G$ contains no 3-cycles adjacent to 4-cycles, then $\chi''(G) \leq \max\{8, \Delta(G)+1\}$.

The theorem shows that if a graph $G$ can be embedded in a surface of Euler characteristic $\varepsilon \geq 0$, and contains no 3-cycles adjacent to 4-cycles, and $\Delta \geq 7$, then $\chi''(G) = \Delta + 1$.

### 2. Proof of Theorem 2

We will introduce some more notations and definitions here for convenience. Let $G = (V, E, F)$ be an embedded graph, where $F$ is the face set of $G$. For a vertex $v \in V$, let $N(v)$ denote the set of vertices adjacent to $v$, and let $d(v) = |N(v)|$ denote the degree of $v$, and for a face $f$, the degree of a face $f$, denoted by $d(f)$, is the number of edges incident with it, where each cut-edge is counted twice. A $k$-vertex, a $k^+$-vertex or a $k^-$-vertex is a vertex of degree $k$, at least $k$ or at most $k$, respectively. Similarly, A $k$-face, a $k^+$-face is a face of degree $k$ or at least $k$, respectively. Let $n_t(v)$ be the number of $t$-vertices adjacent to a vertex $v$, and $f_k(v)$ the number of $k$-faces incident with $v$. Especially, let $f_3(v) = t$.

Let $v_1, v_2, \ldots, v_d$ be neighbors of $v$ in an anticlockwise order. Let $f_i$ be face incident with $v$, $v_i$ and $v_{i+1}$, for all $i$ such that $i \in \{1, 2, \ldots, d\}$. Note that all the subscripts in the paper are taken modulo $d$. For convenience, $(d_1, d_2, \ldots, d_n)$ denotes a cycle (or a face) whose boundary vertices are of degree $d_1, d_2, \ldots, d_n$ in the anticlockwise order. Specially, $(i, j^+, k^+)$-face is a 3-face $uvw$ such that $d(u) = i \leq j \leq d(v) \leq k \leq d(w)$.

**Proof of Theorem 2.** Let $m = \max\{7, \Delta\}$ and $G = (V, E, F)$ be a minimal counterexample to Theorem 2 with $|V| + |E|$ as small as possible. Then every proper subgraph of $G$ has a total-$(m+1)$-coloring, but $G$ itself does not. First we show some known properties of $G$.

(a) Every 3-cycle is not adjacent to a $4^-$-face. It follows that $f_3(v) \leq \left\lfloor \frac{d(v)}{2} \right\rfloor$ for any $v \in V(G)$.

(b) For any edge $uv \in E(G)$, if $\min\{d(u), d(v)\} \leq \left\lfloor \frac{m}{2} \right\rfloor$, then $d(u) + d(v) \geq m+2.$

So all neighbors of any 2-vertex are $7^+$-vertices and all neighbors of any 3-vertex are $6^+$-vertices (see [20]).

(c) The subgraph $G_2$ of $G$ induced by all edges incident with 2-vertices is a forest. So for any component of $G_2$, we root it at a $7^+$-vertex. Then every 2-vertex has exactly one parent and exactly one child (see [3, 6]).
(d) Each 3-face of $G$ is not incident with two $4^-$-vertices (see [16]).
(e) If $v$ is a vertex of $G$ with $n_2(v) \geq 1$, then $n_4^+(v) \geq 1$ (see [7]).

**Lemma 3** [21]. Suppose $v$ is a $d$-vertex of $G$ with $d \geq 5$. Let $v_1, \ldots, v_d$ be the neighbors of $v$ and $f_1, \ldots, f_d$ be the faces incident with $v$ in clockwise order, where $f_i$ is incident with $v_i$ and $v_{i+1}$, $i = 1, 2, \ldots, d$. Note that eventually $v_1$ and $v_{d+1}$ is the same vertex. Then there does not exist an integer $i$ ($2 \leq i \leq d$) such that $d(v_i) = 2$, $d(v_k) = 3$ ($2 \leq k \leq i - 1$) and $d(f_i) = 4$ ($1 \leq t \leq i - 1$).

**Lemma 4.** $G$ contains no subgraph isomorphic to one of the configurations in Figure 1, where the vertices marked by $\bullet$ have no other neighbors in $G$.

**Proof.** The proof that $G$ contains no subgraph isomorphic to one of the configurations in Figure 1(1)–(4) can be found in [8]. It remains to prove that $G$ has no configurations depicted in Figure 1(5)–(13).

By the minimality of $G$, every proper subgraph of $G$ has a total-$(m+1)$-coloring $\varphi$ with the color set $C = \{1, 2, \ldots, m + 1\}$. Erase the colors on all $3^-$-vertices. Let $C(v) = \{\varphi(uw) : u \in N(v)\} \cup \{\varphi(v)\}$.

Suppose that $G$ contains a configuration depicted in Figure 1(5). Then $G' = G - vv_6$ has a total-8-coloring $\varphi$. If $\varphi(v_6x_5) \in C(v)$ or $\varphi(v_6x_6) \in C(v)$, then the forbidden colors for $vv_6$ is at most 7, so $vv_6$ can be properly colored. By recoloring the erased vertices, we obtain a total-8-coloring of $G$, a contradiction. So we can assume that $\varphi(v_6x_5) \notin C(v)$ and $\varphi(v_6x_6) \notin C(v)$. Without loss of generality, assume that $\varphi(v) = 6$, $\varphi(v_6x_5) = 7$, $\varphi(v_6x_6) = 8$, and $\varphi(vv_j) = j$ for $j \in \{1, 2, \ldots, 5\}$. Then we recolor $v$ with 7 or 8, and color $vv_6$ with 6. By recoloring the erased vertices, we obtain a total-8-coloring of $G$, a contradiction.

Suppose that $G$ contains a configuration depicted in Figure 1(6)–(13). Then $G' = G - vv_7$ has a total-8-coloring $\varphi$. If $\varphi(vv_7x) \in C(v)$, then the forbidden colors for $vv_7$ is at most 7, so $vv_7$ can be properly colored. By recoloring the erased vertices, we obtain a total-8-coloring of $G$, a contradiction. So we can assume that $\varphi(vv_7) \notin C(v)$. Without loss of generality, assume that $\varphi(v) = 8$, $\varphi(vv_7x) = 7$, and $\varphi(vv_j) = j$ for $j \in \{1, 2, \ldots, 6\}$. Thus, for each $3^-$-vertex $v_k$ ($1 \leq k \leq 7$), there is an edge incident with $v_k$ colored 7, otherwise we can recolor $vv_k$ with 7, and color $vv_7$ with $k$ to obtain a total-8-coloring of $G$, a contradiction.

For each 4-vertex $v_i$ ($1 \leq i \leq 6$), suppose its adjacent vertices are $v, x_{i-1}, x_i, x_j$. If $\varphi(v_i) \neq 7$ ($1 \leq i \leq 6$), then recolor $v$ with 7, and color $vv_7$ with 8. By recoloring the erased vertices, we obtain a total-8-coloring of $G$, a contradiction. Otherwise, there is at least one 4-vertex colored with 7. Suppose $v$ is adjacent to only one 4-vertex $v_i$ colored with 7. If $|C(v_i)| < 8$, then we recolor $v_i$ with a color in $C\setminus C(v_i)$, recolor $v$ with 7, and color $vv_7$ with 8. Otherwise, $|C(v_i)| = 8$. If $i \not\in \{\varphi(x_{i-1}), \varphi(x_i), \varphi(x_j)\}$, then we recolor $v_i$ with $i$, recolor $vv_i$ with 7, and color
Figure 1. Reducible configurations.
$vv_7$ with $i$. Otherwise, $i \in \{\varphi(x_{i-1}), \varphi(x_i), \varphi(x_j)\}$. Without loss of generality, $\varphi(x_i) = i$, then $8 \notin \{C(v_i) \setminus \varphi(v)\}$. Therefore, we recolor $v_i$ with 8, recolor $v$ with 7, and color $vv_7$ with 8. Finally, we recolor the erased vertices, we obtain a total-8-coloring of $G$, a contradiction. Otherwise, $v$ is adjacent to two or three 4-vertices colored with 7, then we take the same operations as above, respectively. Thus we can also obtain a total-8-coloring of $G$, a contradiction. 

Let $G = (V, E, F)$ be a graph which is embedded in a surface of nonnegative Euler characteristic. By Euler’s formula $|V| - |E| + |F| = \varepsilon$, we have

$$\sum_{v \in V}(2d(v) - 6) + \sum_{f \in F}(d(f) - 6) = -6(|V| - |E| + |F|) = -6\varepsilon \leq 0.$$ 

Now we define the initial charge function $ch(x)$ of $x \in V \cup F$ to be $ch(v) = 2d(v) - 6$ if $v \in V$ and $ch(f) = d(f) - 6$ if $f \in F$. It follows that $\sum_{x \in V \cup F}ch(x) \leq 0$. Now we design appropriate discharging rules and redistribute weights accordingly. Note that any discharging procedure preserves the total charge of $G$. If we can define suitable discharging rules to charge the initial charge function $ch$ to the final charge function $ch'$ on $V \cup F$ such that $\sum_{x \in V \cup F}ch'(x) > 0$, then we get an obvious contradiction.

Our discharging rules are defined as follows.

**R1.** Every 2-vertex receives $\frac{3}{2}$ from its child and $\frac{1}{2}$ from its parent.

**R2.** Let $f$ be a 3-face. If $f$ is incident with a 3$^-$-vertex, then it gets $\frac{5}{6}$ from each of its incident 6$^+$-vertices. If $f$ is incident with a 4-vertex, then it gets $\frac{1}{2}$ from the 4-vertex and gets $\frac{5}{6}$ from each of its incident 5$^+$-vertices. If $f$ is not incident with any 4$^-$-vertex, then it gets 1 from each of its incident 5$^+$-vertices.

**R3.** Let $f$ be a 4-face. If $f$ is incident with two 3$^-$-vertices, then it gets 1 from each of its two incident 6$^+$-vertices. If $f$ is incident with only one 3$^-$-vertex and one 4-vertex, then it gets $\frac{1}{2}$ from the incident 4-vertex and gets $\frac{3}{4}$ from each of its two incident 6$^+$-vertices. If $f$ is incident with only one 3$^-$-vertex and no 4-vertex, then it gets $\frac{2}{3}$ from each of its incident 5$^+$-vertices. If $f$ is not incident with any 3$^-$-vertex, then it gets $\frac{1}{2}$ from each of its incident vertices.

**R4.** Every 5-face gets $\frac{1}{3}$ from each of its incident 4$^+$-vertices.

First, we begin to check $ch'(x) \geq 0$ for all $x \in V \cup F$. By our discharging rules, it is easy to check that $ch'(f) \geq 0$ for all $f \in F$ and $ch'(v) \geq 0$ for all 2-vertices $v \in V$. If $d(v) = 3$, then $ch'(v) = ch(v) = 0$. So it suffices to check that $ch'(v) \geq 0$ for all 4$^+$-vertices $G$.

Let $v$ be a 4$^+$-vertex of $G$. If $d(v) = 4$, then $v$ sends at most $\frac{1}{2}$ to each of its incident faces by R2 and R3, and it follows that $ch'(v) \geq ch(v) - \frac{1}{2} \times 4 = 0$. Suppose $d(v) = 5$. Then $v$ sends at most $\frac{2}{3}$ to each of its incident 3-faces by
R2, at most $\frac{2}{3}$ to each of its incident 4+-faces by R3, at most $\frac{1}{3}$ to each of its incident 5-faces by R4. By (a), $f_3(v) \leq 2$. If $f_3(v) = 2$, then $v$ is incident with at least three 5+-faces, that is, $f_{3+}(v) \geq 3$, and it follows that $ch'(v) \geq ch(v) - \frac{2}{3} \times 2 - \frac{1}{3} \times 3 = \frac{1}{2} > 0$. If $f_3(v) = 1$, then $f_{3+}(v) \geq 2$ and $f_{4+}(v) \leq 2$, and it follows that $ch'(v) \geq ch(v) - \frac{2}{3} \times 2 - \frac{2}{3} \times 2 = \frac{3}{4} > 0$. If $f_3(v) = 0$, then $ch'(v) \geq ch(v) - \frac{2}{3} \times 5 = \frac{2}{3} > 0$. Suppose $d(v) = 6$. Then $f_3(v) \leq 3$ and $v$ sends at most $\frac{2}{3}$ to each of its incident 3-faces by R2, at most 1 to each of its incident 4+-faces by R3, at most $\frac{1}{3}$ to each of its incident 5-faces by R4. Thus, if $1 \leq f_3(v) \leq 3$, then by the similar argument as above, we have $ch'(v) \geq ch(v) - \max \left\{ \frac{2}{3} \times 3 + \frac{1}{3} \times 3, \frac{2}{3} \times 2 + \frac{1}{3} \times 3 + 1, \frac{3}{2} + \frac{1}{3} \times 2 + 1 \times 3 \right\} = \frac{1}{2} > 0$. Otherwise, $v$ is adjacent to at least one 4+-vertex or incident with at least one 5+-face by Lemma 4, configuration (5). If $v$ is incident to at least one 4+-vertex, then $ch'(v) \geq ch(v) - 2 \times \frac{2}{3} - 4 = \frac{1}{2} > 0$ by R3. If $v$ is incident with at least one 5+-face, then $ch'(v) \geq ch(v) - \frac{5}{3} - 5 = \frac{2}{3} > 0$ by R4.

Suppose $d(v) = d \geq 7$. Let $N(v) = \{v_1, v_2, \ldots, v_d\}$ and $f_1, f_2, \ldots, f_d$ be faces incident with $v$ in the clockwise order, where $f_i$ is incident with $v_i$ and $v_{i+1}$, for $i \in \{1, 2, \ldots, d\}$, where all the subscripts here are taken modulo $d$. If $n_2(v) \geq 1$, then $v$ sends at most $\frac{n_2(v) + 2}{2}$ to all its adjacent 2-vertices by R1, at most $\frac{2}{3}$ to each of its incident 3-faces by R2, at most 1 to each of its incident 4+-faces by R3, at most $\frac{1}{3}$ to each of its incident 5-faces by R4.

**Lemma 5.** Suppose that $d(v_i) = d(v_k) = 2$ and $d(v_j) \geq 3$ for all $j = i + 1, \ldots, k - 1$. If $f_i, f_{i+1}, \ldots, f_{k-1}$ are 4+-faces, then $v$ sends at most $\frac{3}{2} + (k - 3)$ (in total) to $f_i, f_{i+1}, \ldots, f_{k-1}$.

**Proof.** By Lemma 3, $\max\{d(v_{i+1}), \ldots, d(v_{k-1})\} \geq 4$ or $\max\{d(f_1), \ldots, d(f_{k-1})\} \geq 5$. If $\max\{d(v_{i+1}), \ldots, d(v_{k-1})\} \geq 4$, then $v$ sends at most $2 \times \frac{3}{4} + (k - 1 - 2)$ to $f_i, f_{i+1}, \ldots, f_{k-1}$ by R2. If $\max\{d(f_1), \ldots, d(f_{k-1})\} \geq 5$, then $v$ sends at most $\frac{1}{3} + (k - 1 - 1)$ to $f_i, f_{i+1}, \ldots, f_{k-1}$ by R3 and R4. Since $2 \times \frac{3}{4} + 1 + \frac{3}{4} \geq 1 + \frac{3}{4}$, $v$ sends at most $\frac{3}{2} + (k - 3)$ to $f_i, f_{i+1}, \ldots, f_{k-1}$.

**Case 1.** $\Delta(G) = 7$. Let $v$ be a 7-vertex. Then $ch(v) = 2 \times 7 - 6 = 8$. We consider the following cases.

**Subcase 1.1.** $n_2(v) = 6$. By (c), any 2–vertex is not incident with a 4-face. Moreover, $t = f_3(v) = 0$ and $f_6(v) \geq 5$ by Lemma 4, configurations (1) and (4). So $ch'(v) \geq ch(v) - \frac{6+2}{2} - 2 = 2 > 0$.

**Subcase 1.2.** $n_2(v) = 5$. Then $t \leq 1$ by Lemma 4, configurations (1) and (4). If $t = 0$, then $f_6(v) \geq 3$ and $f_4(v) \leq 4$ by Lemma 4, configuration (4), and it follows that $ch'(v) \geq ch(v) - \frac{5+2}{2} - 4 \times 1 = \frac{1}{2} > 0$. Otherwise, $f_6(v) \geq 4$ and $f_4(v) \leq 2$. Thus $ch'(v) \geq ch(v) - \frac{5+2}{2} - \frac{3}{2} - 2 \times 1 = 1 > 0$.

**Subcase 1.3.** $n_2(v) = 4$. There are four possible configurations as shown in Figure 2.
For Figure 2(a), $t \leq 1$ and $f_{6^+}(v) \geq 3$ by Lemma 4, configurations (1) and (4). If $t = 1$, then $f_{5^+}(v) \geq 2$, and it follows that $ch'(v) \geq ch(v) - \frac{4+2}{2} - 2 \times \frac{1}{2} - 1 = 1.4$. Otherwise, $ch'(v) \geq ch(v) - \frac{4+2}{2} - 2 = \frac{2}{2} > 0$ by Lemma 5.

For Figure 2(b) and (c), $t \leq 1$ and $f_{6^+}(v) \geq 2$ by Lemma 4, configurations (1) and (4). If $t = 1$, then $f_{5^+}(v) \geq 2$, and it follows that $ch'(v) \geq ch(v) - \frac{4+2}{2} - 3 \times \frac{3}{2} = \frac{1}{2} > 0$ by Lemma 5. Otherwise, $ch'(v) \geq ch(v) - \frac{4+2}{2} - \frac{3}{2} - 3 - 1 = 1 > 0$ by Lemma 5.

For Figure 2(d), $t = 0$ and $f_{6^+}(v) \geq 1$ by Lemma 4, configurations (1) and (4). Then $ch'(v) \geq ch(v) - \frac{4+2}{2} - 3 \times \frac{3}{2} = \frac{1}{2} > 0$ by Lemma 5.

Subcase 1.4. $n_2(v) = 3$. There are four possible configurations as shown in Figure 3.
\[ \frac{3}{2} - 4 \times \frac{1}{3} = \frac{7}{6} > 0. \] If \( t = 1 \), then \( f_{5+}(v) \geq 2 \), and it follows that \( ch'(v) \geq ch(v) - \frac{3+2}{2} - \frac{3}{2} - 2 \times \frac{1}{3} - \frac{3}{2} = \frac{1}{3} > 0 \) by Lemma 5. Otherwise, \( ch'(v) \geq ch(v) - \frac{3+2}{2} - 2 \times \left( \frac{2}{3} + 1 \right) = \frac{1}{2} > 0 \) by Lemma 5.

For Figure 3(d), \( t \leq 1 \) by Lemma 4, configuration (1). If \( t = 1 \), then \( f_{5+}(v) \geq 2 \), and it follows that \( ch'(v) \geq ch(v) - \frac{3+2}{2} - \frac{3}{2} - 2 \times \frac{1}{3} - \frac{3}{2} = \frac{1}{3} > 0 \) by Lemma 5. Otherwise, \( t = 0 \). If \( v \) is adjacent to at least four \( 4^+ \)-vertices, then \( ch'(v) \geq ch(v) - \frac{3+2}{2} - \frac{3}{2} - 2 \times \frac{1}{3} - \frac{3}{2} = \frac{1}{3} > 0 \) by Lemma 5. Otherwise, \( v \) is adjacent to at least one \( 5^+ \)-vertex or incident with at least one \( 5^+ \)-face by Lemma 4, configuration (6), and Lemma 3. If \( v \) is adjacent to at least one \( 5^+ \)-vertex, then \( ch'(v) \geq ch(v) - \frac{3+2}{2} - \max \left\{ 2 \times \frac{1}{3} + 1 + 2 \times \frac{3}{3} + 1 + 2 \times \frac{3}{3} + \frac{3}{3} \right\} = \frac{1}{6} > 0 \) by R3 and Lemma 5. If \( v \) is incident with at least one \( 5^+ \)-face, then \( ch'(v) \geq ch(v) - \frac{3+2}{2} - \max \left\{ \frac{1}{3} + 2 + 2 \times \frac{3}{3} + 1 + \frac{1}{3} + \frac{3}{3} + 1 \right\} = \frac{1}{6} > 0 \) by R4 and Lemma 5.

**Subcase 1.5.** \( n_2(v) = 2 \). There are three possible configurations as shown in Figure 4.

![Figure 4](image-url)

**Figure 4.** \( n_2(v) = 2 \).

For Figure 4(a), we have that the face \( f_6 \) is a \( 5^+ \)-face, and it follows that \( f_6 \) receives at most \( \frac{1}{4} \) from \( v \) by R4. So \( ch(v) - \frac{2+2}{2} - \frac{1}{3} = 8 - \frac{2}{2} = \frac{17}{7} > 0 \). Moreover, \( t \leq 2 \) by Lemma 4, configuration (1), and (a). If \( 1 \leq t \leq 2 \), then \( f_{5+}(v) \geq (t+1) \) (except \( f_6 \)), and it follows that \( ch'(v) \geq ch(v) - \frac{2+2}{2} - t \times \frac{1}{3} - \frac{1}{3} = \frac{2t+1}{6} > 0 \). Otherwise, \( ch'(v) \geq \frac{2t}{3} - \left( \frac{3}{2} + 4 \right) = \frac{1}{6} > 0 \).

For Figure 4(b), \( t \leq 2 \) by Lemma 4, configuration (1), and (a). If \( t = 2 \), then \( f_{5+}(v) \geq 3 \), it follows that \( ch'(v) \geq ch(v) - \frac{2+2}{2} - 2 \times \frac{3}{3} - 3 \times \frac{1}{3} - \frac{3}{2} = \frac{1}{2} > 0 \). If \( t = 1 \), then \( f_{5+}(v) \geq 2 \), and it follows that \( ch'(v) \geq ch(v) - \frac{2+2}{2} - 2 \times \frac{3}{3} - 2 \times \frac{1}{3} - \frac{3}{2} = \frac{1}{3} > 0 \). Suppose \( t = 0 \). If \( v \) is adjacent to at least three \( 4^+ \)-vertices, then \( ch'(v) \geq ch(v) - \frac{2+2}{2} - 3 \times \frac{3}{3} - \frac{2}{2} = \frac{1}{4} > 0 \) by Lemma 5. Otherwise, \( v \) is adjacent to at least one \( 5^+ \)-vertex or incident with at least one \( 5^+ \)-face by Lemma 4, configurations (7)–(8). By the same argument as above, we can obtain \( ch'(v) \geq ch(v) - \frac{2+2}{2} - \max \left\{ \frac{3}{3} + 2 + 2 \times \frac{3}{3} + 4 + \frac{1}{3} \right\} = \frac{1}{6} > 0 \).

For Figure 4(c), \( t \leq 2 \) by Lemma 4, configuration (1), and (a). If \( t = 2 \), then \( f_{5+}(v) \geq 4 \), it follows that \( ch'(v) \geq ch(v) - \frac{2+2}{2} - 2 \times \frac{3}{3} - 4 \times \frac{1}{3} - 1 = \frac{2}{3} > 0 \). If \( t = 1 \), then \( f_{5+}(v) \geq 2 \), and it follows that \( ch'(v) \geq ch(v) - \frac{2+2}{2} -
\[
\max \{ \frac{3}{4} + \frac{1}{3} \times 2 + \frac{3}{2} + 2, \ \frac{3}{2} + \frac{1}{3} \times 2 + 1 + \frac{3}{2} + 1 \} = \frac{1}{4} > 0. \] Suppose \( t = 0 \). If \( v \) is adjacent to at least three 4*-vertices, then \( ch'(v) \geq ch(v) - \frac{4+2}{2} - \frac{3}{2} - \frac{3 \times \frac{3}{4} - 2}{\frac{1}{4}} > 0 \) by Lemma 5. Otherwise, \( v \) is adjacent to at least one 5*-vertex or incident with at least one 5*-face by Lemma 4, configurations (9)–(10). By the same argument as above, we can obtain \( ch'(v) \geq ch(v) - \frac{4+2}{2} - \max \{ \frac{3}{2} + 3 \times 2 + \frac{3}{2} + 4 + \frac{1}{4} \} = \frac{1}{6} > 0. \)

Subcase 1.6. \( n_2(v) = 1 \). Note that \( n_4(v) \geq 1 \) by (e) and \( t \leq 3 \) by (a). Suppose \( t = 0 \). If \( v \) is adjacent to at least two 4*-vertices, then \( ch'(v) \geq ch(v) - \frac{1+2}{2} - \frac{3 \times 3}{4} - 4 = \frac{1}{4} > 0 \). Otherwise, \( v \) is adjacent to at least one 5*-vertex or incident with at least one 5*-face by Lemma 4, configurations (11)–(13), then \( ch'(v) \geq ch(v) - \frac{1+2}{2} - 2 \times \frac{3}{2} - 5 = \frac{1}{6} > 0 \) by R3 or \( ch'(v) \geq ch(v) - \frac{1+2}{2} - \frac{1}{3} - 6 = \frac{1}{6} > 0 \) by R4. Suppose \( 1 \leq t \leq 3 \). If \( v \) is incident with a \((2,7,t)\)-face, then the other face incident with the 2-vertex is a \( 6^+ \)-face. Moreover, the other \( 3 \)-faces incident with \( v \) are \((4,5^+5^+)\)-faces by Lemma 4, configuration (3), and \( v \) is incident with at least \( t 5^+\)-faces. Then we can obtain that \( ch'(v) \geq ch(v) - \frac{1+2}{2} - \frac{3}{2} - (t - 1) \times \frac{3}{2} - \frac{1}{3} \times t - (7 - t - t - t) = \frac{2+5+2}{2} > 0 \), where \( v \) sends at most \( \frac{5}{3} \) to each of its incident \((4,5^+,5^+)\)-faces by R2. Otherwise, 2-vertex is not incident with any \( 3 \)-face. Then \( v \) is incident with at least \( t+1 \) \( 5^+\)-faces, and it follows that \( ch'(v) \geq ch(v) - \frac{1+2}{2} - t \times \frac{3}{2} - \frac{1}{3} \times (t + 1) - (7 - t - t - 1) - \frac{3+2}{2} > 0 \).

Subcase 1.7. \( n_2(v) = 0 \). Note that \( t \leq 3 \) by (a). If \( t = 0 \), then \( ch'(v) \geq ch(v) - 7 \times 1 = 1 > 0 \). Otherwise, by the same argument as above, we can obtain that \( ch'(v) \geq ch(v) - t \times \frac{3}{2} - \frac{1}{3} \times (t + 1) - (7 - t - t - 1) - \frac{10+4}{6} > 0 \).

Case 2. \( \Delta(G) \geq 8 \). In [23], Theorem 1 was established for \( \Delta \geq 9 \). So we assume that \( \Delta = 8 \). Then \( ch(v) = 2 \times 8 - 6 = 10 \). By the same argument as above, we consider the following cases.

Subcase 2.1. \( n_2(v) = 7 \). Then \( t = 0 \) and \( f_{6+}(v) \geq 6 \) by Lemma 4, configurations (1) and (4). So \( ch'(v) \geq ch(v) - \frac{7+2}{2} - 2 = \frac{3}{2} > 0 \).

Subcase 2.2. \( n_2(v) = 6 \). Then \( t \leq 1 \) and \( f_{6+}(v) \geq 4 \) by Lemma 4, configurations (1) and (4). If \( t = 0 \), then \( ch'(v) \geq ch(v) - \frac{6+2}{2} - 4 = 2 > 0 \). Otherwise, \( ch'(v) \geq ch(v) - \frac{6+2}{2} - \frac{3}{2} - 2 \times \frac{1}{3} = \frac{23}{6} > 0 \).

Subcase 2.3. \( n_2(v) = 5 \). Then \( t \leq 1 \) and \( f_{6+}(v) \geq 2 \) by Lemma 4, configurations (1) and (4). If \( t = 0 \), then \( ch'(v) \geq ch(v) - \frac{5+2}{2} - 6 = \frac{1}{2} > 0 \). Otherwise, \( f_{5+}(v) \geq 2 \). Thus \( ch'(v) \geq ch(v) - \frac{5+2}{2} - \frac{3}{2} - 2 \times \frac{1}{3} - 3 = \frac{4}{3} > 0 \).

Subcase 2.4. \( n_2(v) = 4 \). Then \( t \leq 2 \) by Lemma 4, configuration (1). If \( 1 \geq t \geq 2 \), then \( f_{6+}(v) \geq 1 \) and \( f_{5+}(v) \geq t + 1 \), and it follows that \( ch'(v) \geq ch(v) - \frac{4+2}{2} - t \times \frac{3}{2} - (t + 1) \times \frac{1}{3} - (8 - 1 - t - t - 1) = \frac{4+4}{6} > 0 \). Otherwise, \( ch'(v) \geq ch(v) - \frac{4+2}{2} - \max \{ \frac{3}{2} + 3, \frac{3}{2} \times 2 + 2, \frac{3}{2} \times 3 + 1, \frac{3}{2} \times 4 \} = 1 > 0 \).

Subcase 2.5. \( n_2(v) = 3 \). Then \( t \leq 2 \) by Lemma 4 configuration (1). If \( 1 \geq t \geq 2 \), then \( f_{5+}(v) \geq t + 1 \), and it follows that \( ch'(v) \geq ch(v) - \frac{3+2}{2} - t \times \frac{3}{2} -
\[(t+1)\times \frac{1}{3}-(8-t-t-1) = \frac{1+4}{6} > 0. \text{ Otherwise, } ch'(v) \geq ch(v)-\frac{2+2}{2} \max\{\frac{3}{2}+4, \frac{3}{2} \times 2 + 3, \frac{3}{2} \times 3 + 2\} = 1 > 0.\]

Subcase 2.6. \(n_2(v) = 0\). Then \(t \leq 2\) by Lemma 4 configuration (1). If \(1 \geq t \geq 2\), then \(f_{5^+}(v) \geq t+1\), and it follows that \(ch'(v) \geq ch(v)-\frac{2+2}{2} \times t \times \frac{3}{2} - (t+1) \times \frac{1}{3} - (8-t-t-1) = \frac{4+4}{6} > 0. \) Otherwise, \(ch'(v) \geq ch(v)-\frac{2+2}{2} \max\{\frac{3}{2}+5, \frac{3}{2} \times 2 + 4\} = 1 > 0.\)

Subcase 2.7. \(n_2(v) = 1\). Then \(t \leq 4\) by (a). If \(1 \geq t \geq 4\), then \(f_{5^+}(v) \geq t\), and it follows that \(ch'(v) \geq ch(v)-\frac{1+2}{2} - t \times \frac{3}{2} - t \times \frac{1}{3} - (8-t-t) = \frac{3+4}{6} > 0. \) Otherwise, \(ch'(v) \geq ch(v)-\frac{1+2}{2} - 8 = \frac{1}{2} > 0.\)

Subcase 2.8. \(n_2(v) = 0\). Then \(t \leq 4\) by (a). If \(1 \geq t \geq 4\), then \(f_{5^+}(v) \geq t\), and it follows that \(ch'(v) \geq ch(v)-t \times \frac{3}{2} - t \times \frac{1}{3} - (8-t-t) = \frac{12+4}{6} > 0. \) Otherwise, \(ch'(v) \geq ch(v)-8 = 2 > 0.\)

Finally, according to the above argument, we have checked \(ch'(x) \geq 0\) for all \(x \in V \cup F\) and \(ch'(x) > 0\) for any \(5^+\)-vertex \(x \in V\). By Lemma 1, we have \(\Delta(G) \geq 6\). So \(\sum_{x \in V \cup F} ch'(x) > 0\). Hence we complete the proof of Theorem 2.

References


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