MAKING A DOMINATING SET OF A GRAPH CONNECTED

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Abstract

Let \( G = (V, E) \) be a graph and \( S \subseteq V \). We say that \( S \) is a dominating set of \( G \), if each vertex in \( V \setminus S \) has a neighbor in \( S \). Moreover, we say that \( S \) is a connected (respectively, 2-edge connected or 2-connected) dominating set of \( G \) if \( G[S] \) is connected (respectively, 2-edge connected or 2-connected). The domination (respectively, connected domination, or 2-edge connected domination, or 2-connected domination) number of \( G \) is the cardinality of a minimum dominating (respectively, connected dominating, or 2-edge connected dominating, or 2-connected dominating) set of \( G \), and is denoted \( \gamma(G) \) (respectively \( \gamma_1(G) \), or \( \gamma_2'(G) \), or \( \gamma_2(G) \)). A well-known result of Duchet and Meyniel states that \( \gamma_1(G) \leq 3\gamma(G) - 2 \) for any connected graph \( G \). We show that if \( \gamma(G) \geq 2 \), then \( \gamma_2'(G) \leq 5\gamma(G) - 4 \) when \( G \) is a 2-edge connected graph and \( \gamma_2(G) \leq 11\gamma(G) - 13 \) when \( G \) is a 2-connected triangle-free graph.

Keywords: independent set, dominating set, connected dominating set.

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1. Introduction

In this paper, all graphs considered are finite, undirected graphs. We follow the notation and terminology of Bondy and Murty [3], unless otherwise stated.

Let $G = (V(G), E(G))$ be a graph. The order and the size of $G$ are $|V(G)|$ and $|E(G)|$, respectively. We use $c(G)$ to denote the number of components of $G$. The graph $G$ is trivial if its order is 1, and nontrivial, otherwise. For $D \subseteq V(G)$, the subgraph of $G$ induced by $D$, denoted by $G[D]$, is the graph with $D$ as the vertex set, in which two vertices are adjacent if and only if they are adjacent in $G$. $D$ is an independent set of $G$ if $G[D]$ has no edge. The independence number of $G$, denoted $\alpha(G)$, is the maximum cardinality of an independent set of $G$.

Let $G$ be a nontrivial graph and $x, y \in V(G)$ be two distinct vertices. An $xy$-path is a path joining $x$ and $y$ in $G$. The local connectivity between $x$ and $y$, denoted $\kappa_G(x, y)$, is the maximum number of pairwise internally disjoint $xy$-paths in $G$. For a nonnegative integer $k$, $G$ is $k$-connected if $\kappa_G(x, y) \geq k$ for any two distinct vertices $x$ and $y$. Similarly, the local edge connectivity between $x$ and $y$, denoted $\kappa_G'(x, y)$, is the maximum number of pairwise edge-disjoint $xy$-paths in $G$. For two distinct nonadjacent vertices $x$ and $y$, an $xy$-vertex cut is a subset $S$ of $V(G) \setminus \{x, y\}$ such that $x$ and $y$ belong to different components of $G - S$. We also say that such a subset $S$ separates $x$ and $y$. The minimum size of a vertex cut separating $x$ and $y$ is denoted by $c(x, y)$.

For a nonnegative integer $k$, $G$ is $k$-edge connected if $\kappa_G'(x, y) \geq k$ for any two distinct vertices $x$ and $y$ of $G$. An edge cut $E[X, V(G) \setminus X]$ separates $x$ and $y$ if $x \in X$ and $y \in V(G) \setminus X$. We denote by $c'(x, y)$ the minimum cardinality of such an edge cut. The well-known Menger’s Theorem asserts that $\kappa_G'(x, y) = c'(x, y)$.

In graph theory, the problem concerning domination of graphs (or networks) is a major area that has attracted a large number of researchers and generated a wealth of important achievements in the past few decades. Let $G = (V, E)$ be a graph and $D \subseteq V$. We call $D$ a dominating set of $G$ if every vertex in $V \setminus D$ has a neighbor in $D$. Furthermore, if $G[D]$ is $k$-connected (respectively, $k$-edge connected), $D$ is called a $k$-connected (respectively, $k$-edge connected) dominating set. The $k$-connected domination number (respectively, $k$-edge connected domination number) of a graph $G$, denoted by $\gamma_k(G)$ (respectively, by $\gamma'_k(G)$) is the minimum cardinality of a $k$-connected (respectively, $k$-edge connected) dominating set. Clearly, a graph $G$ has a $k$-connected (respectively, $k$-edge connected) dominating set if $G$ is $k$-connected (respectively, $k$-edge connected). But a graph having a $k$-connected (respectively, $k$-edge connected) dominating set needs not to be $k$-connected (respectively, $k$-edge connected). It is clear that $\gamma_0(G) = \gamma_0(G) = \gamma(G)$ and $\gamma'_1(G) = \gamma_1(G)$.

The theory of connected domination of graphs has important applications in communication and computer networks, especially for its role as a virtual...

An interesting application of the connected domination of graphs is in minor theory. The well-known Hadwiger’s conjecture states that if \( \chi(G) \geq k \), then \( G \) contains a \( K_k \)-minor, where \( \chi(G) \) denotes the chromatic number of \( G \). We use \( \alpha(G) \) to denote the independent number of a graph. Since

\[
\alpha(G) \chi(G) \geq n
\]

for a graph \( G \) on \( n \) vertices, Hadwiger’s conjecture implies that any graph \( G \) on \( n \) vertices has a \( K_{\lceil n/\chi(G) \rceil} \)-minor. Duchet and Meyniel in [8] established the following relation between the connected domination number and the independence number of a connected graph, and by applying this result, they proved that any graph \( G \) on \( n \) vertices has a \( K_{\lfloor n/2 \alpha(G) - 1 \rfloor} \)-minor.

**Theorem 1** (Duchet and Meyniel [8]). *For any connected graph \( G \), \( \gamma_1(G) \leq \min\{2\alpha(G) - 1, 3\gamma_1(G) - 2\} \).*

In some sense, the above theorem of Duchet and Meyniel is related to the following conjecture in combinatorial optimization.

**Conjecture 1** [20]. *For any connected unit disk graph \( G \), \( \alpha(G) \leq 3\gamma_1(G) + 2 \).*

There are a number of papers devoted to the relation of the independence number and the connected domination number of unit disk graphs, for instance, [12, 17, 19]. Best known result on Conjecture 1 is \( \alpha(G) \leq 3.399\gamma_1(G) + 4.874 \) obtained by Du and Du [7]. So, combining this with Theorem 1, for a connected unit disk graph \( G \),

\[
0.5\gamma_1(G) + 0.5 \leq \alpha(G) \leq 3.399\gamma_1(G) + 4.874.
\]

We refer to [20] for more relevant works concerning domination and packing on wireless networks.

There exist a number of algorithms for constructing maximal independent sets and connected dominating sets. For instance, Vigoda [16] presented a parallel algorithm for constructing a maximal independent set of an input graph on \( n \) vertices, in time polynomial in \( \log n \) and in \( \log n \) using a polynomial in \( n \) processors, Guha and Khuller [9] presented two polynomial time algorithms for constructing a connected dominating set that achieves approximation factors of \( O(h(\Delta)) \), where \( \Delta \) is the maximum degree, and \( h \) is the harmonic function.
We shall get a connected dominating set if we can make a dominating set connected by adding a small vertex set (with respect to the dominating set). In this paper, we generalize Duchet and Meyniel’s theorem by considering the following problems.

Problem 1. Given a connected graph $G$ and a dominating set $S$, what is the least vertex set $T$ such that $G[S \cup T]$ is connected?

Problem 1 was studied in [8] by Duchet and Meyniel. We are mainly concerned with the following two problems.

Problem 2. Given a 2-edge connected graph $G$ and a dominating set $S$, find a vertex set $T$ with minimum $|T|$ such that $G[S \cup T]$ is 2-edge connected.

Problem 3. Given a 2-connected graph $G$ and a dominating set $S$, find a vertex set $T$ with minimum $|T|$ such that $G[S \cup T]$ is 2-connected.

2. Minimum Vertex Set Joining a Given Dominating Set

For two vertices $u, v \in V(G)$, the distance $d_G(u, v)$ between $u$ and $v$ is the number of edges in a shortest path connecting $u$ and $v$ in $G$. In general, for $X \subseteq V(G)$ and $Y \subseteq V(G)$, the distance $d_G(X, Y)$ between $X$ and $Y$ is $\min\{d_G(x, y) : x \in X, \ y \in Y\}$. Thus $d_G(X, Y) = d_G(Y, X)$. If $Y = \{y\}$ for a vertex $y \in V(G)$, we simply write $d_G(X, y)$ instead of $d_G(X, \{y\})$.

2.1. Connected dominating set

The idea of the proof of the following theorem is due to Duchet and Meyniel [8].

Theorem 2. Let $S$ be a dominating set of a connected graph $G$. Then there exists a set $T$ such that $|T| \leq 2|S| - 2$ and $G[S \cup T]$ is connected.

Proof. If $c(G[S]) = 1$, i.e., $S$ is a connected dominating set, then the assertion of the theorem trivially holds by taking $T = \emptyset$. Next we assume that $G[S]$ is disconnected. Since $S$ is a dominating set of $G$, there exist two components of $G[S]$, say $G_1$ and $G_2$, such that $d_G(V(G_1), V(G_2)) \leq 3$. Pick a path $P$ which joins $V(G_1)$ and $V(G_2)$ with $\ell(P) = d_G(V(G_1), V(G_2))$. Hence $S \cup V(P)$ is a dominating set of $G$ with $|S \cup V(P)| \leq |S| + 2$ and $c(G[S \cup V(P)]) \leq c(G[S]) - 1$. If $G[S \cup V(P)]$ is connected, then we are done by letting $T = V(P)$. Otherwise, let $S := S \cup V(P)$, and repeat the above operation until $G[S]$ is connected.

Since $c(G[S]) \leq |S| - 1$, $|S|$ increases by at most two and the number of components decreases by at least one in each iteration of the above operation, we conclude that the desired set $T$ exists.
So the following is immediate from the above theorem.

**Corollary 1.** $\gamma_1(G) \leq 3\gamma(G) - 2$ for any connected graph $G$.

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**Algorithm 1.** An algorithm for constructing a connected dominating set.

**Input:** A connected graph $G$ and a dominating set $S$ of $G$.

**Output:** A set $T$ such that $|T| \leq 2|S| - 2$ and $G[S \cup T]$ is connected.

1. Set $T := \emptyset$, $H := G[S \cup T]$
2. run BFS to get all components of $H$, say $H_1, H_2, \ldots, H_c$, and set $C = \{H_i : 1 \leq i \leq c\}$ and $c = |C|$
3. if $c = 1$, then stop
4. else set $W := V(G) \setminus S$ and $F := E(G[W])$
5. while $W \neq \emptyset$
6. pick a vertex $w \in W$
7. if $N(w) \cap V(H_i) \neq \emptyset$ and $N(w) \cap V(H_j) \neq \emptyset$ for different integers $i$ and $j$, then set $H_i := G[\bigcup_{H \in H} V(H_i) \cup \{w\}], \quad C := (C \setminus \mathcal{H}) \cup \{H_i\}, T := T \cup \{w\}$, $H := G[S \cup T]$, and $k := k - |H| + 1$, where $\mathcal{H} = \{H_i : V(H_i) \cap N_G(w) \neq \emptyset\}$ and $h = |\mathcal{H}|$, go to step 3
8. else $W := W \setminus \{w\}$
9. end if
10. end while
11. while $F \neq \emptyset$, pick $f = uv \in F$
12. pick $f = uv \in F$
13. if $N(u) \cap V(H_i) \neq \emptyset$ and $N(v) \cap V(H_j) \neq \emptyset$ for different integers $i$ and $j$, then set $H_i := G[\bigcup_{H \in H} V(H_i) \cup \{u, v\}], \quad C := (C \setminus \mathcal{H}) \cup \{H_i\}, T := T \cup \{u, v\}$, $H := G[S \cup T]$, and $k := k - |H| + 1$, where $\mathcal{H} = \{H_i : V(H_i) \cap N_G(u) \neq \emptyset\}$ or $V(H_i) \cap N_G(v) \neq \emptyset\}$ and $h = |\mathcal{H}|$, go to step 3
14. else $F := F \setminus \{f\}$.
15. end if
16. end while
17. end if

**Remark 1.** Let $s$, $\Delta$, $n$ and $m$ be the size of a dominating set $S$, the maximum degree, order and size of $G$, respectively. Note that the time complexity of BFS can be expressed as $O(n + m)$. Since the running time of each recursion is at most $\Delta(n + 2m)$ and this algorithm runs at most $s - 1$ recursions, the time complexity of the algorithm is bounded by $O((s - 1)\Delta(n + 2m))$.

### 2.2. 2-edge connected dominating set

Let $G$ be a connected graph. A subgraph $F \subseteq G$ is called a maximal 2-edge connected subgraph of $G$ if $F$ is trivial or is 2-edge connected, and there exists no other 2-edge connected subgraph $F' \subseteq G$ such that $F \subseteq F'$. It is clear from the
definition that every maximal 2-edge connected subgraph \( F \) of \( G \) is an induced subgraph of \( G \).

For a dominating set \( S \) of \( G \), let \( H = G[S] \). We use \( C_H \) to denote the set of all maximal 2-edge connected subgraphs \( F \) of \( H \) containing at least one vertex of \( S \), and \( c_H = |C_H| \).

Next we assume that \( G \) is a 2-edge connected graph and let \( S \) be a dominating set of \( G \) with \( |S| \geq 2 \), and let \( T \) be an output of Algorithm 1 for \( G \) and \( S \). If \( H = G[S \cup T] \) is 2-edge connected, then \( S \cup T \) is a 2-edge connected dominating set of \( G \). Otherwise, we shall decrease \( c_H \) by at least one by adding at most two vertices, see Lemma 3, Corollary 2, and Lemmas 4–5 for details.

**Lemma 3.** Let \( u_1 \) and \( u_2 \) be two distinct vertices in \( H \). If deleting a cut edge \( e \) separates \( u_1 \) and \( u_2 \) in \( H \), then there exists a vertex \( w \in V(G) \setminus V(H) \) such that \( N_G(w) \cap V(X_e) \neq \emptyset \) and \( N_G(w) \cap V(Y_e) \neq \emptyset \), or an edge \( uv \in E(G - V(H)) \) such that \( N_G(u) \cap V(X_e) \neq \emptyset \) and \( N_G(v) \cap V(Y_e) \neq \emptyset \), where \( X_e \) and \( Y_e \) are two components of \( H \setminus e \).

**Proof.** Without loss of generality, let \( u_1 \in V(X_e) \) and \( u_2 \in V(Y_e) \). Let \( P = x_1x_2\cdots x_k \) be a shortest path joining \( X_e \) and \( Y_e \) in \( G \setminus e \), where \( x_1 \in V(X_e) \) and \( x_k \in V(Y_e) \). If \( k \leq 4 \), then \( P - \{x_1, x_k\} \) is a vertex or an edge, as we desired. If \( k \geq 5 \), we consider \( x_3 \). Since \( S \subseteq V(H) \) is a dominating set of \( G \), \( x_3 \) has a neighbor \( x_3' \in S \) in \( G \). If \( x_3' \in V(X_e) \), then \( x_3'x_3\cdots x_k \) is a shorter path than \( P \) that joins \( X_e \) and \( Y_e \) in \( G \setminus e \), a contradiction; if \( x_3' \in V(Y_e) \), then \( x_1x_2x_3x_3' \) is a shorter path than \( P \) joining \( X_e \) and \( Y_e \) in \( G \setminus e \), a contradiction.

**Corollary 2.** Let \( u_1 \) and \( u_2 \) be two distinct vertices in \( S \). If \( \kappa_H'(u_1, u_2) = 1 \) and \( d_H(u_1, u_2) = 1 \), then there exists a vertex \( w \in V(G) \setminus V(H) \) such that \( c_{H'} \leq c_H - 1 \), where \( H' = G[S \cup T \cup \{w\}] \), or an edge \( e = uv \in E(G - V(H)) \) such that \( c_{H'} \leq c_H - 1 \), where \( H' = G[S \cup T \cup \{u, v\}] \).

**Proof.** Note that \( u_1 \) and \( u_2 \) belong to two distinct maximal 2-edge connected subgraphs of \( H \), while they belong to the same maximal 2-edge connected subgraphs of \( H' \) by Lemma 2. Thus \( c_{H'} \leq c_H - 1 \).

**Lemma 4.** Let \( u_1 \) and \( u_2 \) be two distinct vertices in \( S \) such that \( \kappa_H'(u_1, u_2) = 1 \) and \( d_H(u_1, u_2) \) is as small as possible. If \( d_H(u_1, u_2) = 2 \), then there exists a vertex \( w \in V(G) \setminus V(H) \) such that \( c_{H'} \leq c_H - 1 \), where \( H' = G[S \cup T \cup \{w\}] \), or an edge \( e = uv \in E(G - V(H)) \) such that \( c_{H'} \leq c_H - 1 \), where \( H' = G[S \cup T \cup \{u, v\}] \), or a pair of vertices \( u, v \in V(G) \setminus V(H) \) such that \( c_{H'} \leq c_H - 1 \), where \( H' = G[S \cup T \cup \{u, v\}] \).

**Proof.** Let \( u_1v_1u_2 \) be a path of length 2 in \( H \). By the choice of \( u_1 \) and \( u_2 \), \( v_1 \not\in S \). First, we may suppose that \( u_1v_1 \) is a cut edge of \( H \) and \( u_2v_1 \) is not. Let \( a = u_1v_1 \), and let \( X_a \) and \( Y_a \) be two components of \( H \setminus a \) such that \( u_1 \in V(X_a) \)
and \(v_1 \in V(Y_a)\). By Lemma 3, there exists a vertex \(w \in V(G) \setminus V(H)\) such that \(N_G(w) \cap V(X_a) \neq \emptyset\) and \(N_G(w) \cap V(Y_a) \neq \emptyset\), or an edge \(uv \in E(G - V(H))\) such that \(N_G(u) \cap V(X_a) \neq \emptyset\) and \(N_G(v) \cap V(Y_a) \neq \emptyset\). For the former case, let \(H' = G[S \cup T \cup \{w\}]\). Clearly \(\kappa_{H'}(u_1, u_2) \geq 2\). Thus \(c_{H'} \leq c_H - 1\). For the latter case, let \(H' = G[S \cup T \cup \{u, v\}]\). Clearly \(\kappa_{H'}(u_1, u_2) \geq 2\). Thus \(c_{H'} \leq c_H - 1\).

So, we now assume that both \(u_1v_1\) and \(u_2v_1\) are cut edges of \(H\). Let \(a = u_1v_1\), and let \(X_a, Y_a\) be two components of \(H \setminus a\) such that \(u_1 \in V(X_a)\) and \(v_1 \in V(Y_a)\). We consider the following cases.

Case 1. There exists a vertex \(w \in V(G) \setminus V(H)\) such that \(N_G(w) \cap V(X_a) \neq \emptyset\) and \(N_G(w) \cap V(Y_a - v_1) \neq \emptyset\). Then \(w\) is the vertex, as we desired.

Case 2. There exists an edge \(uv \in E(G - V(H))\) such that \(N_G(u) \cap V(X_a) \neq \emptyset\) and \(N_G(v) \cap V(Y_a - v_1) \neq \emptyset\). Then \(uv\) is the edge, as we desired.

Case 3. There exists no vertex \(w \in V(G) \setminus V(H)\) such that \(N_G(w) \cap V(X_a) \neq \emptyset\) and \(N_G(w) \cap V(Y_a - v_1) \neq \emptyset\), and no edge \(uv \in E(G - V(H))\) such that \(N_G(u) \cap V(X_a) \neq \emptyset\) and \(N_G(v) \cap V(Y_a - v_1) \neq \emptyset\).

Let \(b = v_1u_2\), and \(X_b\) and \(Y_b\) be two components of \(H \setminus b\) such that \(v_1 \in V(X_b)\) and \(u_2 \in V(Y_b)\). If there exists a vertex \(w \in V(G) \setminus V(H)\) such that \(N_G(w) \cap V(X_a) \neq \emptyset\) and \(N_G(w) \cap V(Y_a) \neq \emptyset\), and a vertex \(w' \in V(G) \setminus V(H)\) such that \(N_G(w') \cap V(X_b) \neq \emptyset\) and \(N_G(w') \cap V(Y_b) \neq \emptyset\), then \(w\) and \(w'\) are a pair of vertices, as we desired.

Next we show that there exist such a pair of vertices in \(H\). Without loss of generality, suppose that there exists no vertex \(w \in V(G) \setminus V(H)\) such that \(N_G(w) \cap V(X_a) \neq \emptyset\) and \(N_G(w) \cap V(Y_a) \neq \emptyset\). By Lemma 3, there exists an edge \(uv \in E(G - V(H))\) such that \(N_G(u) \cap V(X_a) \neq \emptyset\) and \(N_G(v) \cap V(Y_a) \neq \emptyset\).

Since \(v_1 \notin S\), \(S \subseteq V(H)\) and \(S\) is a dominating set of \(G\), it follows that \(v\) has a neighbor \(v' \in S\) which belong to \(V(X_a) \cap S\) or \(V(Y_a - v_1)\). If \(v' \in V(X_a)\), then \(N_G(v) \cap V(X_a) \neq \emptyset\) and \(N_G(v) \cap V(Y_a) \neq \emptyset\), a contradiction. Otherwise, \(v' \in V(Y_a - v_1)\), then \(uv\) is an edge with the specified property in the assumption, a contradiction.

So, the proof is completed. \(\blacksquare\)

**Lemma 5.** Let \(u_1\) and \(u_2\) be two distinct vertices in \(S\) such that \(\kappa_{H'}(u_1, u_2) = 1\) and \(d_H(u_1, u_2)\) is as small as possible. If \(d_H(u_1, u_2) = 3\), then there exists a vertex \(w \in V(G) \setminus V(H)\) such that \(c_{H'} \leq c_H - 1\), where \(H' = G[S \cup T \cup \{w\}]\), or an edge \(e = uv \in E(G - V(H))\) such that \(c_{H'} \leq c_H - 1\), where \(H' = G[S \cup T \cup \{u, v\}]\), or a pair of vertices \(u, v \in V(G) \setminus V(H)\) such that \(c_{H'} \leq c_H - 1\), where \(H' = G[S \cup T \cup \{u, v\}]\).

**Proof.** Let \(P = u_1v_1u_2\) be a path of length 3 in \(H\). By the choice of \(u_1\) and \(u_2\), we have \(v_1 \notin S\) and \(v_2 \notin S\). If exactly one edge of \(P\) is a cut edge of \(H\), then by Lemma 3 the result follows. If exactly two adjacent edges of \(P\) are cut edges,
then by a similar argument to the proof of Lemma 6, we may show the assertion of the lemma. So, we consider the remaining cases.

Case 1. $u_1v_1$ and $v_2u_2$ are cut edges of $H$ and $v_1v_2$ is not. Let $a = u_1v_1$, and let $X_a, Y_a$ be two components of $H \setminus a$ such that $u_1 \in X_a$ and $v_1 \in Y_a$. Similarly, let $b = u_2v_2$, and let $X_b, Y_b$ be two components of $H \setminus b$ such that $v_2 \in V(X_b)$ and $u_2 \in V(Y_b)$.

Subcase 1.1. There exists a vertex $w \in V(G) \setminus V(H)$ such that $N_G(w) \cap V(X_a) \neq \emptyset$ and $N_G(w) \cap V(Y_a) \neq \emptyset$, and a vertex $u' \in V(G) \setminus V(H)$ such that $N_G(u') \cap V(X_b) \neq \emptyset$ and $N_G(u') \cap V(Y_b) \neq \emptyset$. If $w = u'$, then $w$ is a vertex we want, otherwise $w$ and $u'$ are a pair of vertices we want.

Subcase 1.2. There exists no pair of vertices $w$ and $u'$ which satisfies the condition of Subcase 1.1. Without loss of generality, assume that there exists no vertex $w \in V(G) \setminus V(H)$ such that $N_G(w) \cap V(X_a) \neq \emptyset$ and $N_G(w) \cap V(Y_a) \neq \emptyset$. By Lemma 3, there exists an edge $uv \in E(G - V(H))$ such that $N_G(u) \cap V(X_a) \neq \emptyset$ and $N_G(v) \cap V(Y_a) \neq \emptyset$. Since $v_1, v_2 \notin S$, $S \subseteq V(H)$ and $S$ is a dominating set of $G$, we know that $v$ has a neighbor $v' \in X_a$ or $Y_a - \{v_1, v_2\}$. If $v' \in V(X_a)$, this contradicts our assumption that there exists no vertex $w \in V(G) \setminus V(H)$ such that $N_G(w) \cap V(X_a) \neq \emptyset$ and $N_G(w) \cap V(Y_a) \neq \emptyset$. Otherwise, $v' \in V(Y_a - \{v_1, v_2\})$, and the edge $uv$ is an our desired edge.

Case 2. All edges of $P$ are cut edges in $H$. Let $a = v_1v_2$, and let $X_a, Y_a$ be two components of $H \setminus a$ such that $v_1 \in V(X_a)$ and $v_2 \in V(Y_a)$. Consider the following three subcases.

Subcase 2.1. There exists a vertex $w \in V(G) \setminus V(H)$ such that $N_G(w) \cap V(X_a - v_1) \neq \emptyset$ and $N_G(w) \cap V(Y_a - v_2) \neq \emptyset$. Then $w$ is a vertex, as we desired.

Subcase 2.2. There exists an edge $uv \in V(G) \setminus V(H)$ such that $N_G(u) \cap (X_a - v_1) \neq \emptyset$ and $N_G(v) \cap (Y_a - v_2) \neq \emptyset$. Then $uv$ is an edge, as we desired.

Subcase 2.3. There exists no such vertex satisfying the condition of Subcase 2.1, and no such edge satisfying the condition of Subcase 2.2. We shall show that there exists a pair of vertices which satisfies the assertion of this lemma.

Claim 1. There exists a vertex $w \in V(G) \setminus V(H)$ such that $N_G(w) \cap V(X_a) \neq \emptyset$ and $N_G(w) \cap V(Y_a - v_2) \neq \emptyset$, or $N_G(w) \cap V(Y_a) \neq \emptyset$, and $N_G(w) \cap V(X_a - v_1) \neq \emptyset$.

Proof. Assume that there exists a vertex $w$ satisfying $N_G(w) \cap V(X_a) \neq \emptyset$ and $N_G(w) \cap V(Y_a) \neq \emptyset$. If $N_G(w) \cap V(X_a) = \{v_1\}$ and $N_G(w) \cap Y_a = \{v_2\}$, then it contradicts the assumption that $S$ is a dominating set of $G$. Thus, $w$ is a vertex, as we want.

Assume that there does not exist a vertex $w$ satisfying $N_G(w) \cap V(X_a) \neq \emptyset$ and $N_G(w) \cap V(Y_a) \neq \emptyset$. By Lemma 3, there exists an edge $uv$ satisfying $N_G(u) \cap
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Since \( v_1 \notin S \), \( v_2 \notin S \), \( S \subseteq V(H) \) and \( S \) is a dominating set of \( G \), we know that \( u \) has a neighbor \( u' \in S \) which belong to \( X_a - v_1 \) or \( Y_a - v_2 \), and \( v \) has a neighbor \( v' \in S \) which belong to \( X_a - v_1 \) or \( Y_a - v_2 \). If \( u' \) and \( v' \) belong to different components of \( H \setminus a \), then \( uv \) is an edge which contradicts the assumption of Subcase 2.3. Thus \( u' \) and \( v' \) belong to the same component of \( H \setminus a \). We may suppose that \( u', v' \in V(X_a - v_1) \). Then \( v \) is the vertex, as we want. This proves the claim.

By Claim 1, we may assume that there exists a vertex \( w \in V(G) \setminus V(H) \) such that \( N_G(w) \cap V(X_a - v_1) \neq \emptyset \) and \( N_G(w) \cap V(Y_a) \neq \emptyset \).

Let \( b = v_2u_2 \), and \( X_b \) and \( Y_b \) be two components of \( H \setminus b \) such that \( v_2 \in V(X_b) \) and \( u_2 \in V(Y_b) \). If there exists a vertex \( u' \in V(G) \setminus V(H) \) such that \( N_G(u') \cap V(X_b) \neq \emptyset \) and \( N_G(u') \cap V(Y_b) \neq \emptyset \), then \( w \) and \( u' \) are a pair of vertices, as we desired. If this is not the case, then by Lemma 3, there exists an edge \( uv \in E(G - V(H)) \) such that \( N_G(u) \cap V(X_b) \neq \emptyset \) and \( N_G(v) \cap V(Y_b) \neq \emptyset \). Since \( v_1 \notin S \), \( v_2 \notin S \), \( S \subseteq V(H) \) and \( S \) is a dominating set of \( G \), it follows that \( u \) has a neighbor \( u' \in S \) which belongs to \( X_b \setminus \{v_1, v_2\} \) or \( Y_b \). If \( u' \in V(X_b \setminus \{v_1, v_2\}) \), then \( uv \) is an edge that contradicts the assumption of Subcase 2.3. So, \( u' \in V(Y_b) \), which implies that \( N_G(u) \cap V(X_b) \neq \emptyset \) and \( N_G(v) \cap V(Y_b) \neq \emptyset \). Hence \( w \) and \( u \) are a pair of vertices, as we desired.

**Theorem 6.** Let \( G \) be 2-edge connected graph. If \( S \) is a dominating set of \( G \) with \( |S| \geq 2 \), then there exists a set \( T \subseteq V(G) \) such that \( |T| \leq 4|S| - 4 \) and \( G[S \cup T] \) is 2-edge connected.

**Proof.** For \( G \) and \( S \), let \( T \) be an output of Algorithm 1 and \( H = G[S \cup T] \). We may suppose that \( c_H \geq 2 \) and pick a pair of vertices \( u_1 \in S \) and \( u_2 \in S \) such that \( \kappa'_H(u_1, u_2) = 1 \) and \( d_H(u_1, u_2) \) is as small as possible.

**Claim 2.** \( d_H(u_1, u_2) \leq 3 \).

**Proof.** Suppose that the claim is not true, and let \( P = x_1x_2 \cdots x_k \) be a shortest path joining \( u_1 \) and \( u_2 \) in \( H \), where \( k \geq 5 \), \( x_1 = u_1 \) and \( x_k = u_2 \). We consider \( x_3 \). Since \( S \) is a dominating set of \( H \), \( x_3 \) has a neighbor \( x'_3 \in S \) in \( H \).

If at least one of \( x_1x_2 \) and \( x_2x_3 \) is a cut edge of \( H \), then \( u_1 \) and \( x'_3 \) are a pair of vertices such that \( \kappa'_H(u_1, x'_3) = 1 \) and \( d_H(u_1, x'_3) < d_H(u_1, u_2) \), a contradiction; otherwise, at least one edge of the path \( x_3x_4 \cdots x_k \) is a cut edge of \( H \). Thus \( u_2 \) and \( x'_3 \) are a pair of vertices of \( S \) such that \( \kappa'_H(x'_3, u_2) = 1 \) and \( d_H(x'_3, u_2) < d_H(u_1, u_2) \), a contradiction. Thus \( d_H(u_1, u_2) \leq 3 \). □

By Lemmas 3, 4 and 5, there exists a vertex set \( T' \) such that \( |T'| \leq 2 \) and \( c_{H'} \leq c_H - 1 \), where \( H' = G[S \cup T \cup T'] \). If \( H' \) is 2-edge connected, then we are done by letting \( T := T \cup T' \). Otherwise, let \( T := T \cup T' \), and repeat the above operation until \( G[S \cup T] \) is 2-edge connected.
Since \( c_H \leq |S| \), \( |T| \) increases by at most two and \( c_H \) decreases by at least one in each iteration of the above operation, we conclude that the desired set \( T \) exists.

**Corollary 3.** For a 2-edge connected graph \( G \), if \( \gamma(G) \geq 2 \), then \( \gamma'_2(G) \leq 5\gamma(G) - 4 \).

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**Algorithm 2.** An algorithm for constructing a 2-edge connected dominating set.

**Input:** A 2-edge connected graph \( G \), a dominating set \( S \) with at least 2 vertices.

**Output:** A set \( T \) such that \( |T| \leq 4|S| - 4 \) and \( G[S \cup T] \) is 2-edge connected.

**I.** run Algorithm 1 to get set \( T \)

**II.** 1. for \( G[S \cup T] \), run DFS to get all blocks, say \( B_1, B_2, \ldots, B_k \), and all cut vertices, say \( w_1, w_2, \ldots, w_f \)

2. set \( H := G[S \cup T] \), \( W = \{w_1, w_2, \ldots, w_f\} \), and \( \mathcal{B} \) the set of blocks \( B_i \) in \( H \) such that \( |V(B_i)| \geq 3 \)

3. if \( W = \emptyset \), then stop

4. else pick \( w \in W \)

5. if \( B_{i_1}, B_{i_2}, \ldots, B_{i_v} \) are blocks in \( G \) such that \( w \in V(B_{i_1}) \cap V(B_{i_2}) \cap \cdots \cap V(B_{i_v}) \), then set \( B_1 = B_{i_1} \cup B_{i_2} \cup \cdots \cup B_{i_v} \), \( W = W \setminus \{w\} \), go to Step 3

6. else \( W = W \setminus \{w\} \)

7. end if

8. end if

**III.** 1. set \( \mathcal{B} = \mathcal{B} \cup (S \cup \{B \in \mathcal{B} \mid V(B) \}) \), \( b := |\mathcal{B}| \)

2. if \( b = 1 \), then stop

3. else set \( W := V(H) \), \( F := E(G[W]) \), and \( R := W \times W \)

4. while \( F \neq \emptyset \)

5. pick \( f = uv \in F \)

6. if \( N_G(u) \cap V(B_i) \neq \emptyset \) and \( N_G(u) \cap V(B_j) \neq \emptyset \) for different integers \( i \) and \( j \), then set \( B_i := G[\bigcup_{B \in \mathcal{B}} V(B_i) \cup \{u, v\}], B_j := G[\bigcup_{B \in \mathcal{B}} V(B_j) \cup \{u, v\}] \), \( H := G[S \cup T] \), and \( b := b - h + 1 \), where \( \mathcal{H} = \{H_i : N_G(B_i) \cap N_G(u) \neq \emptyset \} \) or \( N_G(B_i) \cap N_G(v) \neq \emptyset \) and \( h = |\mathcal{H}| \), go to Step 2

7. else \( F := F \setminus \{f\} \)

8. end if

9. end while

10. while \( W \neq \emptyset \),

11. pick \( w \in W \)

12. if \( N_G(w) \cap V(B_i) \neq \emptyset \) and \( N_G(w) \cap V(B_j) \neq \emptyset \) for different integers \( i \) and \( j \), then set \( B_i := G[\bigcup_{B \in \mathcal{B}} V(B_i) \cup \{w\}], B_j := G[\bigcup_{B \in \mathcal{B}} V(B_j) \cup \{w\}] \), \( H := G[S \cup T] \), and \( b := b - h + 1 \), where \( \mathcal{H} = \{B_i : N_G(B_i) \cap N_G(w) \neq \emptyset \} \) and \( h = |\mathcal{H}| \), go to Step 2

13. else \( W := W \setminus \{w\} \)

14. end if

15. end while

16. while \( R \neq \emptyset \),

17. pick \( r = (u, v) \in R \)
19. if \( N_G(u) \cap V(B_i) \neq \emptyset, N_G(u) \cap N_H(B_j) \neq \emptyset, N_G(v) \cap V(B_i) \neq \emptyset, \) and \( N_G(v) \cap N_H(B_j) \neq \emptyset \) for different integers \( i \) and \( j \), then set \( B_i := G[\bigcup_{B \in H} V(B) \cup \{u, v\}], B := (B \setminus H) \cup B_i, T := T \cup \{u, v\}, H := G[S \cup T], \) and \( b := b - h + 1, \) where \( H = \{H_i : N_G(H_i) \cap N_G(u) \neq \emptyset \text{ or } N_G(H_i) \cap N_G(v) \neq \emptyset\} \) and \( h = |H|, \) go to Step 2
20. else \( R := R \setminus \{r\} \)
21. end if
22. end while
23. end if

Remark 2. Let \( s, \Delta, n \) and \( m \) be the size of a dominating set \( S \), the maximum degree, order and size of \( G \), respectively. Note that the time complexity of stage I can be expressed as \( O((s - 1)\Delta(n + 2m)) \), and the time complexity of II can be expressed as \( O(m + k\ell) \). In III, since the running time of each recursion is at most \( \Delta(n + 2m + n^2) \) and III runs at most \( s - 1 \) recursions. Thus the time complexity of this algorithm is bounded by \( O((s - 1)\Delta(m + n^2)) \).

2.3. 2-connected dominating set

Let \( G \) be a connected graph which is not complete, let \( X \) be a vertex cut of \( G \), and let \( Y \) be the vertex set of a component of \( G - X \). The subgraph \( H \) of \( G \) induced by \( X \cup Y \) is called an \( X \)-component of \( G \). We simply write \( x \)-component if \( X = \{x\} \).

Lemma 7. Let \( S \) be a dominating set of a 2-edge connected graph \( G \) with \(|S| \geq 2\). If \( T \) is an output of Algorithm 2 for \( G \) and \( S \), and \( T' \subseteq T \) is an output of stage I of Algorithm 2 for \( G \) and \( S \), then the following is true for \( H = G[S \cup T'] \):
(i) if \( u \) is a cut vertex in \( H \), then \( u \in S \cup T' \),
(ii) \( b(H) \leq 2|S| - 2 \), where \( b(H) \) is the number of blocks in \( H \).

Proof. To show (i), it suffices to show that each vertex \( u \in T \setminus T' \) is not a cut vertex of \( H \). Since \( T' \) is an output of stage I of Algorithm 2 for \( G \) and \( S \), \( S \cup T' \) is a connected dominating set of \( G \), and thus \( S \cup T' \) is also a connected dominating set of \( H \). Therefore \( H - u \) is connected, i.e., \( u \) is not a cut vertex of \( H \).

Suppose that (ii) is not true, and \( G \) is a graph of minimum order satisfying the conditions of this lemma but \( b(H) > 2|S| - 2 \geq 2 \). If \(|S| = 2\), then \( b(H) \leq 2 \), and thus \( b(H) = 2 \leq 2|S| - 2 \), a contradiction. So, \(|S| \geq 3 \). Let \( u \) be a cut vertex of \( H \). We consider the following two cases according to (i).

Case 1. \( u \in S \). Let \( H_1, H_2, \ldots, H_k \) be the \( u \)-components of \( H \). Clearly \( H_i \) is 2-edge connected. Let \( S_i = V(H_i) \cap S \) and \( T_i = V(H_i) \setminus S_i \) for \( i = 1, 2, \ldots, k \).

Since \( T_i \) is a possible output of Algorithm 2 for \( H_i \) and \( S_i \), we have \( b(H_i) = \)
\[b(G[S_i \cup T_i]) \leq 2|S_i| - 2\] 
by the minimality of \(G\). Thus \(b(H) = \sum_{i=1}^{k} b(H_i) \leq \sum_{i=1}^{k} (2|S_i| - 2) \leq 2 \sum_{i=1}^{k} |S_i| - 2k = 2|S| - 2k = 2|S| - 2\), a contradiction.

**Case 2.** \(u \in T'\). Let \(H_1, H_2, \ldots, H_k\) be the \(u\)-components of \(H\). Clearly \(H_i\) is 2-edge connected. Let \(S_i = V(H_i) \cap S\) and \(T_i = V(H_i) \setminus S_i\) for \(i = 1, 2, \ldots, k\).

Without loss of generality, let \(N_{H_i}(u) \cap S_i \neq \emptyset\) for \(1 \leq i \leq r\) for an integer \(r\) and \(N_{H_j}(u) \cap S_j = \emptyset\), \(r < j \leq k\). Since \(S\) is a dominating set of \(H\), \(r \geq 1\).

When \(1 \leq i \leq r\), since \(T_i\) is a possible output of Algorithm 2 for \(H_i\) and \(S_i\), we have \(b(H_i) = b(G[S_i \cup T_i]) \leq 2|S_i| - 2\) by the minimality of \(G\).

When \(r < j \leq k\), let \(S'_j = S_j \cup \{u\}\) and \(T'_j = (T_j \setminus u)\). Since \(T'_j\) is a possible output of Algorithm 2 for \(H_j\) and \(S'_j\), we have \(b(H_j) = b(G[S_j \cup T_j]) \leq 2|S'_j| - 2 = 2(|S_j| + 1) - 2\) by the choice of \(G\).

Thus \(b(H) = \sum_{i=1}^{k} b(H_i) = \sum_{i=1}^{r} (2|S_i| - 2) + \sum_{j=r+1}^{k} (2(|S_j| + 1) - 2) \leq \sum_{i=1}^{r} (2|S_i| - 2) + \sum_{j=r+1}^{k} (2|S_j|) \leq 2|S| - 2r \leq 2|S| - 2\), a contradiction. This shows (ii).

**Theorem 8.** Let \(G\) be a 2-connected triangle-free graph \(G\). If \(S\) is a dominating set of \(G\) with \(|S| \geq 2\), then there exists a set \(T \subseteq V(G)\) such that \(|T| \leq 10|S| - 13\) and \(G[S \cup T]\) is 2-connected.

**Proof.** Let \(T\) be an output of Algorithm 2 for \(G\) and \(S\). We may suppose that \(G[S \cup T]\) is not 2-connected and let \(b(H)\) be the number of blocks in \(H = G[S \cup T]\).

Since \(H\) is 2-edge connected, each block of \(H\) is 2-edge connected. Let \(u\) be a cut vertex in \(H\), let \(B_1\) and \(B_2\) be two blocks of \(H\) such that \(u \in V(B_1) \cap V(B_2)\), and let \(H_1\) and \(H_2\) be \(u\)-components such that \(B_i \subseteq H_i\) for \(i = 1, 2\).

Let \(P = x_1 x_2 \cdots x_k\) be a shortest path connecting \(V(H_1)\) and \(V(H_2)\) in \(G \setminus u\) where \(x_1 \in V(H_1)\), \(x_k \in V(H_2)\) and \(x_2, x_3, \ldots, x_{k-1} \notin V(H_1) \cup V(H_2)\). Suppose \(k \geq 6\). Then \(u x_3, u x_4 \in E(G)\) since \(S \subseteq V(H)\) is a dominating set of \(G\), and \(P\) is a shortest path connecting \(V(H_1)\) and \(V(H_2)\) in \(G \setminus u\). Thus \(u x_3, u x_4\) is a triangle, a contradiction. Thus \(k \leq 5\). Let \(T' = V(P) \setminus \{x_1, x_k\}\). Then \(|T'| \leq 3\).

Hence \(S \cup T \cup T'\) is a 2-edge connected dominating set of \(G\) with \(|T' \cup T'| \leq |T| + 2 \leq 4|S| - 4 + 3\) and \(b(G[S \cup T \cup T']) \leq b(G[S \cup T]) - 1 = b(H) - 1\). If \(G[S \cup T \cup T']\) is 2-connected, then we are done by letting \(T := T \cup T'\). Otherwise, let \(T := T \cup T'\), and repeat the above operation until \(G[S \cup T]\) is 2-connected.

Since \(|T| \leq 4|S| - 4\), \(b(H) \leq 2|S| - 2\), \(|T|\) increases by at most three and \(b(H)\) decreases by at least one in each iteration of the above operation, we conclude that the desired set \(T\) exists since \(|T| \leq 4|S| - 4 + 3(b(H) - 1) = 10|S| - 13\).

**Corollary 4.** For a 2-connected triangle-free graph \(G\), if \(\gamma(G) \geq 2\), then \(\gamma_2(G) \leq 11\gamma(G) - 13\).

**Remark 3.** For a graph with triangle, Theorem 8 does not holds. For example, let \(G\) be the graph in Figure 1. Since \{\(u, v, w\}\} is a smallest dominating set
and any proper subgraph of G is not 2-connected, we have that \( \gamma(G) = 3 \) but \( \gamma_2(G) = V(G) \), that is, there is not a constant \( k \) such that \( \gamma_2(G) \leq k\gamma(G) \) for graphs with triangle. So the condition that G is triangle-free is indispensable.

![Figure 1. A graph with \( \gamma(G) = 3 \) but \( \gamma_2 = V(G) \).](image)

**Algorithm 3.** An algorithm for constructing a 2-connected dominating set.

**Input:** A 2-connected graph \( G \), a dominating set \( S \) with at least 2 vertices.

**Output:** A set \( T \) such that \( |T| \leq 10|S| - 13 \) and \( G[S \cup T] \) is 2-connected.

1. run Algorithm 2.
2. run DFS to get all blocks of \( G[S \cup T] \), say \( B_1, B_2, \ldots, B_k \).
3. if \( b = 1 \), then stop
4. while \( F \neq \emptyset \)
5. pick \( f = uv \in F \)
6. if \( N_G(u) \cap V(B_i) \neq \emptyset \) and \( w \in N_G(v) \cap N_G(B_j) \neq \emptyset \), then set \( B_i := G[B_i \cup \{u, v\}] \), \( B := (B \setminus H) \cup \{B_i\} \), \( T := T \cup \{u, v\} \), \( H := G[S \cup T] \), \( b := b - h + 1 \), where \( H = \{B_i : V(B_i) \cap N_G(u) \neq \emptyset \} \) or \( V(B_i) \cap N_G(v) \neq \emptyset \), and \( h = |H| \), go to Step 2
7. else \( F := F \setminus \{f\} \)
8. end if
9. end while
10. while \( W \neq \emptyset \)
11. pick \( w \in W \)
12. if \( N_G(w) \cap V(B_i) \neq \emptyset \) and \( N_G(w) \cap V(B_i) \neq \emptyset \), then set \( B_i := G[B_i \cup \{w\}] \), \( B := (B \setminus H) \cup \{B_i\} \), \( T := T \cup \{w\} \), \( H := G[S \cup T] \), \( b := b - h + 1 \), where \( H = \{B_i : V(B_i) \cap N_G(w) \neq \emptyset \} \), and \( h = |H| \), go to Step 2
13. else \( F := F \setminus \{f\} \)
14. end if
15. end while
16. end if
Remark 4. Let $s$, $\Delta$, $n$ and $m$ be the size of a dominating set $S$, the maximum degree, order and size of $G$, respectively. Note the time complexity of stage I is $O((s-1)\Delta(m+n^2))$, and the time complexity of II is $O(m)$. In III, since the running time of each recursion is at most $2\Delta n^2$ and III implements at most $s-1$ recursions. Thus the time complexity of the algorithm is bounded by $O((s-1)\Delta(m+n^2))$.

3. Concluding Remarks

Let $P = u_0u_1 \cdots u_{3k}$ and $Q = v_0v_1 \cdots v_{3k}$ be two path of length $3k$. The symbol $G$ denotes the graph obtained from $P$ and $Q$ by identifying $u_{3i}$ and $v_{3i}$ (denote the resulting vertex by $w_{3i}$), where $0 \leq i \leq n$. It is easy to check that $G$ is 2-edge connected and $S = \{w_{3i} : 0 \leq i \leq n\}$ is a dominating set. Note that $T = \{u_{3i+1}, u_{3i+2} : 0 \leq i \leq n-1\}$ and $T' = \{v_{3i+1}, v_{3i+2} : 0 \leq i \leq n-1\}$ are minimum sets of vertices such that $G[S \cup T]$ and $G[S \cup T']$ are connected, and $Q = T \cup T'$ is the unique set of vertices such that $G[S \cup Q]$ is 2-edge connected. Thus the bounds given in Theorem 2, 6 and Corollary 3 are sharp.

We suspect that the bound of Theorem 8 is not sharp and the best possible bound might be the following.

Conjecture 2. For a dominating set $S$ of a 2-connected triangle-free graph $G$ with $|S| \geq 2$, there exists a vertex set $T \subseteq V(G)$ with $|T| \leq 5|S|$ such that $G[S \cup T]$ is 2-connected.

 Inspired by Corollaries 1, 3 and 4, one may ask the following two problems.

Problem 4. Does there exist an absolute constant $c'_k$ for a given integer $k \geq 1$ such that $\gamma_k'(G) \leq c'_k \gamma(G)$ for any $k$-edge connected graph $G$?

Problem 5. Does there exist an absolute constant $c_k$ for a given integer $k \geq 1$ such that $\gamma_k(G) \leq c_k \gamma(G)$ for any $k$-connected graph $G$?

By our main results, $c_k'$ and $c_k$ exist for $1 \leq k \leq 2$. But, $c_k'$ and $c_k$ do not exist for an integer $k \geq 3$. Let $C_n$ and $K_{k-2}$ be the cycle of order $n$ and the complete graph of order $k-2$. Let $G_{n,k} = C_n \lor K_{k-2}$, be the graph obtained from $C_n$ and $K_{k-2}$ by joining every vertex of $C_n$ to all vertices of $K_{k-2}$. It is clear that $G_{n,k}$ is $k$-connected, and thus $k$-edge connected. But, $\gamma(G_{n,k}) = 1$ and $\gamma_k'(G_{n,k}) = \gamma_k(G_{n,k}) = n+k$.

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