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MAKING A DOMINATING SET OF A GRAPH CONNECTED

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Abstract

Let $G = (V, E)$ be a graph and $S \subseteq V$. We say that S is a dominating set of G, if each vertex in $V \setminus S$ has a neighbor in S. Moreover, we say that S is a connected (respectively, 2-edge connected or 2-connected) dominating set of G if $G[S]$ is connected (respectively, 2-edge connected or 2-connected). The domination (respectively, connected domination, or 2-edge connected domination, or 2-connected domination) number of G is the cardinality of a minimum dominating (respectively, connected dominating, or 2-edge connected dominating, or 2-connected dominating) set of G, and is denoted $\gamma(G)$ (respectively $\gamma_1(G)$, or $\gamma_2'(G)$, or $\gamma_2(G)$). A well-known result of Duchet and Meyniel states that $\gamma_1(G) \leq 3\gamma(G) - 2$ for any connected graph G. We show that if $\gamma(G) \geq 2$, then $\gamma_2'(G) \leq 5\gamma(G) - 4$ when G is a 2-edge connected graph and $\gamma_2(G) \leq 11\gamma(G) - 13$ when G is a 2-connected triangle-free graph.

Keywords: independent set, dominating set, connected dominating set.

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1. Introduction

In this paper, all graphs considered are finite, undirected graphs. We follow the notation and terminology of Bondy and Murty [3], unless otherwise stated.

Let $G = (V(G), E(G))$ be a graph. The *order* and the *size* of G are $|V(G)|$ and $|E(G)|$, respectively. We use $c(G)$ to denote the number of components of G. The graph G is *trivial* if its order is 1, and *nontrivial*, otherwise. For $D \subseteq V(G)$, the subgraph of G induced by D, denoted by $G[D]$, is the graph with D as the vertex set, in which two vertices are adjacent if and only if they are adjacent in G. D is an *independent set* of G if $G[D]$ has no edge. The independence number of G, denoted $\alpha(G)$, is the maximum cardinality of an independent set of G.

Let G be a nontrivial graph and $x, y \in V(G)$ be two distinct vertices. An *xy-path* is a path joining x and y in G. The *local connectivity* between x and y, denoted $\kappa_G(x, y)$, is the maximum number of pairwise internally disjoint xy-paths in G. For a nonnegative integer k, G is *k-connected* if $\kappa_G(x, y) \geq k$ for any two distinct vertices x and y. Similarly, the *local edge connectivity* between x and y, denoted $\kappa'_G(x, y)$, is the maximum number of pairwise edge-disjoint xy-paths in G. For two distinct nonadjacent vertices x and y, an xy-vertex cut is a subset S of $V(G) \setminus \{x, y\}$ such that x and y belong to different components of $G - S$. We also say that such a subset S separates x and y. The minimum size of a vertex cut separating x and y is denoted by $c(x, y)$.

For a nonnegative integer k, G is *k-edge connected* if $\kappa'_G(x, y) \geq k$ for any two distinct vertices x and y of G. An edge cut $E[X, V(G) \setminus X]$ separates x and y if $x \in X$ and $y \in V(G) \setminus X$. We denote by $c'(x, y)$ the minimum cardinality of such an edge cut. The well-known Menger's Theorem asserts that $\kappa'_G(x, y) = c'(x, y)$.

In graph theory, the problem concerning domination of graphs (or networks) is a major area that has attracted a large number of researchers and generated a wealth of important achievements in the past few decades. Let $G = (V, E)$ be a graph and $D \subseteq V$. We call D a *dominating set* of G if every vertex in $V \setminus D$ has a neighbor in D. Furthermore, if $G[D]$ is k-connected (respectively, k-edge connected), D is called a k-connected (respectively, k-edge connected) dominating set. The *k-connected domination number* (respectively, *k-edge connected domination number*) of a graph G, denoted by $\gamma_k(G)$ (respectively, by $\gamma'_k(G)$) is the minimum cardinality of a k-connected (respectively, k-edge connected) dominating set. Clearly, a graph G has a k-connected (respectively, k-edge connected) dominating set if G is k-connected (respectively, k-edge connected). But a graph having a k-connected (respectively, k-edge connected) dominating set needs not to be k-connected (respectively, k-edge connected). It is clear that $\gamma'_0(G) = \gamma_0(G) = \gamma(G)$ and $\gamma'_1(G) = \gamma_1(G)$.

The theory of connected domination of graphs has important applications in communication and computer networks, especially for its role as a virtual backbone in wireless networks, see Du and Wan [6]. Haynes, Hedetniemi and Slater published two monographs [10, 11] concerning domination in graphs, and recently Chellali, Favaron, Hansberg and Volkmann presented a survey paper [4] concerning dominating sets and independent sets. We refer to [1, 2, 5, 13–15, 18] for more results concerning connected dominating sets.

An interesting application of the connected domination of graphs is in minor theory. The well-known Hadwiger's conjecture states that if $\chi(G) \geq k$, then G contains a K_k-minor, where $\chi(G)$ denotes the chromatic number of G. We use $\alpha(G)$ to denote the independent number of a graph. Since

$$\alpha(G)\chi(G) \geq n$$

for a graph G on n vertices, Hadwiger's conjecture implies that any graph G on n vertices has a $K_{\lceil \frac{n}{\alpha(G)} \rceil}$-minor. Duchet and Meyniel in [8] established the following relation between the connected domination number and the independence number of a connected graph, and by applying this result, they proved that any graph G on n vertices has a $K_{\lceil \frac{n}{2\alpha(G)-1} \rceil}$-minor.

Theorem 1 (Duchet and Meyniel [8]). *For any connected graph G, $\gamma_1(G) \leq \min\{2\alpha(G) - 1, 3\gamma(G) - 2\}$.*

In some sense, the above theorem of Duchet and Meyniel is related to the following conjecture in combinatorial optimization.

Conjecture 1 [20]. *For any connected unit disk graph G, $\alpha(G) \leq 3\gamma_1(G) + 2$.*

There are a number of papers devoted to the relation of the independence number and the connected domination number of unit disk graphs, for instance, [12, 17, 19]. Best known result on Conjecture 1 is $\alpha(G) \leq 3.399\gamma_1(G) + 4.874$ obtained by Du and Du [7]. So, combining this with Theorem 1, for a connected unit disk graph G,

$$0.5\gamma_1(G) + 0.5 \leq \alpha(G) \leq 3.399\gamma_1(G) + 4.874.$$

We refer to [20] for more relevant works concerning domination and packing on wireless networks.

There exist a number of algorithms for constructing maximal independent sets and connected dominating sets. For instance, Vigoda [16] presented a parallel algorithm for constructing a maximal independent set of an input graph on n vertices, in time polynomial in $\log n$ and in $\log n$ using a polynomial in n processors, Guha and Khuller [9] presented two polynomial time algorithms for constructing a connected dominating set that achieves approximation factors of $O(h(\Delta))$, where Δ is the maximum degree, and h is the harmonic function.

We shall get a connected dominating set if we can make a dominating set connected by adding a small vertex set (with respect to the dominating set). In this paper, we generalize Duchet and Meyniel's theorem by considering the following problems.

Problem 1. Given a connected graph G and a dominating set S, what is the least vertex set T such that $G[S \cup T]$ is connected?

Problem 1 was studied in [8] by Duchet and Meyniel. We are maninly concerned with the following two problems.

Problem 2. Given a 2-edge connected graph G and a dominating set S, find a vertex set T with minimum $|T|$ such that $G[S \cup T]$ is 2-edge connected.

Problem 3. Given a 2-connected graph G and a dominating set S, find a vertex set T with minimum $|T|$ such that $G[S \cup T]$ is 2-connected.

2. Minimum Vertex Set Joining a Given Dominating Set

For two vertices $u, v \in V(G)$, the distance $d_G(u, v)$ between u and v is the number of edges in a shortest path connecting u and v in G. In general, for $X \subseteq V(G)$ and $Y \subseteq V(G)$, the distance $d_G(X, Y)$ between X and Y is $\min\{d_G(x, y) : x \in X,\ y \in Y\}$. Thus $d_G(X, Y) = d_G(Y, X)$. If $Y = \{y\}$ for a vertex $y \in V(G)$, we simply write $d_G(X, y)$ instead of $d_G(X, \{y\})$.

2.1. Connected dominating set

The idea of the proof of the following theorem is due to Duchet and Meyniel [8].

Theorem 2. *Let S be a dominating set of a connected graph G. Then there exists a set T such that $|T| \leq 2|S| - 2$ and $G[S \cup T]$ is connected.*

Proof. If $c(G[S]) = 1$, i.e., S is a connected dominating set, then the assertion of the theorem trivially holds by taking $T = \emptyset$. Next we assume that $G[S]$ is disconnected. Since S is a dominating set of G, there exist two components of $G[S]$, say G_1 and G_2, such that $d_G(V(G_1), V(G_2)) \leq 3$. Pick a path P which joins $V(G_1)$ and $V(G_2)$ with $\ell(P) = d_G(V(G_1), V(G_2))$. Hence $S \cup V(P)$ is a dominating set of G with $|S \cup V(P)| \leq |S| + 2$ and $c(G[S \cup V(P)]) \leq c(G[S]) - 1$. If $G[S \cup V(P)]$ is connected, then we are done by letting $T = V(P)$. Otherwise, let $S := S \cup V(P)$, and repeat the above operation until $G[S]$ is connected.

Since $c(G[S]) \leq |S| - 1$, $|S|$ increases by at most two and the number of components decreases by at least one in each iteration of the above operation, we conclude that the desired set T exists. ■

So the following is immediate from the above theorem.

Corollary 1. $\gamma_1(G) \leq 3\gamma(G) - 2$ *for any connected graph* G.

Algorithm 1. An algorithm for constructing a connected dominating set.
Input: A connected graph G and a dominating set S of G.
Output: A set T such that $|T| \leq 2|S| - 2$ and $G[S \cup T]$ is connected.

1. Set $T := \emptyset$, $H := G[S \cup T]$
2. run BFS to get all components of H, say $H_1, H_2, \ldots, H_c$, and set $\mathcal{C} = \{H_i : 1 \leq i \leq c\}$ and $c = |\mathcal{C}|$
3. **if** $c = 1$, **then** stop
4. **else** set $W := V(G)\backslash S$ and $F := E(G[W])$
5. **while** $W \neq \emptyset$
6. pick a vertex $w \in W$
7. **if** $N(w) \cap V(H_i) \neq \emptyset$ and $N(w) \cap V(H_j) \neq \emptyset$ for different integers i and j, **then** set $H_i := G[\bigcup_{H_i \in \mathcal{H}} V(H_i) \cup \{w\}]$, $\mathcal{C} := (\mathcal{C} \setminus \mathcal{H}) \cup \{H_i\}$, $T := T \cup \{w\}$, $H := G[S \cup T]$, and $k := k - h + 1$, where $\mathcal{H} = \{H_i : V(H_i) \cap N_G(w) \neq \emptyset\}$ and $h = |\mathcal{H}|$, go to step 3
8. **else** $W := W\backslash\{w\}$
9. **end if**
10. **end while**
11. **while** $F \neq \emptyset$, pick $f = uv \in F$
12. pick $f = uv \in F$
13. **if** $N(u) \cap V(H_i) \neq \emptyset$ and $N(v) \cap V(H_j) \neq \emptyset$ for different integers i and j, **then** set $H_i := G[\bigcup_{H_i \in \mathcal{H}} V(H_i) \cup \{u, v\}]$, $\mathcal{C} := (\mathcal{C} \setminus \mathcal{H}) \cup \{H_i\}$, $T := T \cup \{u, v\}$, $H := G[S \cup T]$, and $k := k - |\mathcal{H}| + 1$, where $\mathcal{H} = \{H_i : V(H_i) \cap N_G(u) \neq \emptyset$ or $V(H_i) \cap N_G(v) \neq \emptyset\}$ and $h = |\mathcal{H}|$, go to step 3
14. **else** $F := F\backslash\{f\}$.
15. **end if**
16. **end while**
17. **end if**

Remark 1. Let s, Δ, n and m be the size of a dominating set S, the maximum degree, order and size of G, respectively. Note that the time complexity of BFS can be expressed as $O(n+m)$. Since the running time of each recursion is at most $\Delta(n+2m)$ and this algorithm runs at most $s-1$ recursions, the time complexity of the algorithm is bounded by $O((s-1)\Delta(n+2m))$.

2.2. 2-edge connected dominating set

Let G be a connected graph. A subgraph $F \subseteq G$ is called a *maximal* 2-*edge connected subgraph* of G if F is trivial or is 2-edge connected, and there exists no other 2-edge connected subgraph $F' \subseteq G$ such that $F \subseteq F'$. It is clear from the definition that every maximal 2-edge connected subgraph F of G is an induced subgraph of G.

For a dominating set S of G, let $H = G[S]$. We use $\mathcal{C}_H$ to denote the set of all maximal 2-edge connected subgraphs F of H containing at least one vertex of S, and $c_H = |\mathcal{C}_H|$.

Next we assume that G is a 2-edge connected graph and let S be a dominating set of G with $|S| \geq 2$, and let T be an output of Algorithm 1 for G and S. If $H = G[S \cup T]$ is 2-edge connected, then $S \cup T$ is a 2-edge connected dominating set of G. Otherwise, we shall decrease c_H by at least one by adding at most two vertices, see Lemma 3, Corollary 2, and Lemmas 4–5 for details.

Lemma 3. *Let u_1 and u_2 be two distinct vertices in H. If deleting a cut edge e separates u_1 and u_2 in H, then there exists a vertex $w \in V(G) \setminus V(H)$ such that $N_G(w) \cap V(X_e) \neq \emptyset$ and $N_G(w) \cap V(Y_e) \neq \emptyset$, or an edge $uv \in E(G - V(H))$ such that $N_G(u) \cap V(X_e) \neq \emptyset$ and $N_G(v) \cap V(Y_e) \neq \emptyset$, where X_e and Y_e are two components of $H \setminus e$.*

Proof. Without loss of generality, let $u_1 \in V(X_e)$ and $u_2 \in V(Y_e)$. Let $P = x_1x_2 \cdots x_k$ be a shortest path joining X_e and Y_e in $G \setminus e$, where $x_1 \in V(X_e)$ and $x_k \in V(Y_e)$. If $k \leq 4$, then $P - \{x_1, x_k\}$ is a vertex or an edge, as we desired. If $k \geq 5$, we consider x_3. Since $S \subseteq V(H)$ is a dominating set of G, x_3 has a neighbor $x'_3 \in S$ in G. If $x'_3 \in V(X_e)$, then $x'_3x_3 \cdots x_k$ is a shorter path than P that joins X_e and Y_e in $G \setminus e$, a contradiction; if $x'_3 \in V(Y_e)$, then $x_1x_2x_3x'_3$ is a shorter path than P joining X_e and Y_e in $G \setminus e$, a contradiction. ■

Corollary 2. *Let u_1 and u_2 be two distinct vertices in S. If $\kappa'_H(u_1, u_2) = 1$ and $d_H(u_1, u_2) = 1$, then there exists a vertex $w \in V(G) \setminus V(H)$ such that $c_{H'} \leq c_H - 1$, where $H' = G[S \cup T \cup \{w\}]$, or an edge $e = uv \in E(G - V(H))$ such that $c_{H'} \leq c_H - 1$, where $H' = G[S \cup T \cup \{u, v\}]$.*

Proof. Note that u_1 and u_2 belong to two distinct maximal 2-edge connected subgraphs of H, while they belong to the same maximal 2-edge connected subgraphs of H' by Lemma 2. Thus $c_{H'} \leq c_H - 1$. ■

Lemma 4. *Let u_1 and u_2 be two distinct vertices in S such that $\kappa'_H(u_1, u_2) = 1$ and $d_H(u_1, u_2)$ is as small as possible. If $d_H(u_1, u_2) = 2$, then there exists a vertex $w \in V(G) \setminus V(H)$ such that $c_{H'} \leq c_H - 1$, where $H' = G[S \cup T \cup \{w\}]$, or an edge $e = uv \in E(G - V(H))$ such that $c_{H'} \leq c_H - 1$, where $H' = G[S \cup T \cup \{u, v\}]$, or a pair of vertices $u, v \in V(G) \setminus V(H)$ such that $c_{H'} \leq c_H - 1$, where $H' = G[S \cup T \cup \{u, v\}]$.*

Proof. Let $u_1v_1u_2$ be a path of length 2 in H. By the choice of u_1 and u_2, $v_1 \notin S$. First, we may suppose that u_1v_1 is a cut edge of H and u_2v_1 is not. Let $a = u_1v_1$, and let X_a and Y_a be two components of $H \setminus a$ such that $u_1 \in V(X_a)$

and $v_1 \in V(Y_a)$. By Lemma 3, there exists a vertex $w \in V(G) \setminus V(H)$ such that $N_G(w) \cap V(X_a) \neq \emptyset$ and $N_G(w) \cap V(Y_a) \neq \emptyset$, or an edge $uv \in E(G - V(H))$ such that $N_G(u) \cap V(X_a) \neq \emptyset$ and $N_G(v) \cap V(Y_a) \neq \emptyset$. For the former case, let $H' = G[S \cup T \cup \{w\}]$. Clearly $\kappa'_{H'}(u_1, u_2) \geq 2$. Thus $c_{H'} \leq c_H - 1$. For the latter case, let $H' = G[S \cup T \cup \{u, v\}]$. Clearly $\kappa'_{H'}(u_1, u_2) \geq 2$. Thus $c_{H'} \leq c_H - 1$.

So, we now assume that both u_1v_1 and u_2v_1 are cut edges of H. Let $a = u_1v_1$, and let X_a, Y_a be two components of $H \setminus a$ such that $u_1 \in V(X_a)$ and $v_1 \in V(Y_a)$. We consider the following cases.

Case 1. There exists a vertex $w \in V(G) \setminus V(H)$ such that $N_G(w) \cap V(X_a) \neq \emptyset$ and $N_G(w) \cap V(Y_a - v_1) \neq \emptyset$. Then w is the vertex, as we desired.

Case 2. There exists an edge $uv \in E(G - V(H))$ such that $N_G(u) \cap V(X_a) \neq \emptyset$ and $N_G(v) \cap V(Y_a - v_1) \neq \emptyset$. Then uv is the edge, as we desired.

Case 3. There exists no vertex $w \in V(G) \setminus V(H)$ such that $N_G(w) \cap V(X_a) \neq \emptyset$ and $N_G(w) \cap V(Y_a - v_1) \neq \emptyset$, and no edge $uv \in E(G - V(H))$ such that $N_G(u) \cap V(X_a) \neq \emptyset$ and $N_G(v) \cap V(Y_a - v_1) \neq \emptyset$.

Let $b = v_1u_2$, and X_b and Y_b be two components of $H \setminus b$ such that $v_1 \in V(X_b)$ and $u_2 \in V(Y_b)$. If there exists a vertex $w \in V(G) \setminus V(H)$ such that $N_G(w) \cap V(X_a) \neq \emptyset$ and $N_G(w) \cap V(Y_a) \neq \emptyset$, and a vertex $w' \in V(G) \setminus V(H)$ such that $N_G(w') \cap V(X_b) \neq \emptyset$ and $N_G(w') \cap V(Y_b) \neq \emptyset$, then w and w' are a pair of vertices, as we desired.

Next we show that there exist such a pair of vertices in H. Without loss of generality, suppose that there exists no vertex $w \in V(G) \setminus V(H)$ such that $N_G(w) \cap V(X_a) \neq \emptyset$ and $N_G(w) \cap V(Y_a) \neq \emptyset$. By Lemma 3, there exists an edge $uv \in E(G - V(H))$ such that $N_G(u) \cap V(X_a) \neq \emptyset$ and $N_G(v) \cap V(Y_a) \neq \emptyset$. Since $v_1 \notin S$, $S \subseteq V(H)$ and S is a dominating set of G, it follows that v has a neighbor $v' \in S$ which belong to $V(X_a) \cap S$ or $V(Y_a - v_1)$. If $v' \in V(X_a)$, then $N_G(v) \cap V(X_a) \neq \emptyset$ and $N_G(v) \cap V(Y_a) \neq \emptyset$, a contradiction. Otherwise, $v' \in V(Y_a - v_1)$, then uv is an edge with the specified property in the assumption, a contradiction.

So, the proof is completed. ■

Lemma 5. *Let u_1 and u_2 be two distinct vertices in S such that $\kappa'_H(u_1, u_2) = 1$ and $d_H(u_1, u_2)$ is as small as possible. If $d_H(u_1, u_2) = 3$, then there exists a vertex $w \in V(G) \setminus V(H)$ such that $c_{H'} \leq c_H - 1$, where $H' = G[S \cup T \cup \{w\}]$, or an edge $e = uv \in E(G - V(H))$ such that $c_{H'} \leq c_H - 1$, where $H' = G[S \cup T \cup \{u, v\}]$, or a pair of vertices $u, v \in V(G) \setminus V(H)$ such that $c_{H'} \leq c_H - 1$, where $H' = G[S \cup T \cup \{u, v\}]$.*

Proof. Let $P = u_1v_1v_2u_2$ be a path of length 3 in H. By the choice of u_1 and u_2, we have $v_1 \notin S$ and $v_2 \notin S$. If exactly one edge of P is a cut edge of H, then by Lemma 3 the result follows. If exactly two adjacent edges of P are cut edges, then by a similar argument to the proof of Lemma 6, we may show the assertion of the lemma. So, we consider the remaining cases.

Case 1. u_1v_1 and v_2u_2 are cut edges of H and v_1v_2 is not. Let $a = u_1v_1$, and let X_a, Y_a be two components of $H \setminus a$ such that $u_1 \in X_a$ and $v_1 \in Y_a$. Similarly, let $b = u_2v_2$, and let X_b, Y_b be two components of $H \setminus b$ such that $v_2 \in V(X_b)$ and $u_2 \in V(Y_b)$.

Subcase 1.1. There exists a vertex $w \in V(G) \setminus V(H)$ such that $N_G(w) \cap V(X_a) \neq \emptyset$ and $N_G(w) \cap V(Y_a) \neq \emptyset$, and a vertex $w' \in V(G) \setminus V(H)$ such that $N_G(w') \cap V(X_b) \neq \emptyset$ and $N_G(w') \cap V(Y_b) \neq \emptyset$. If $w = w'$, then w is a vertex we want, otherwise w and w' are a pair of vertices we want.

Subcase 1.2. There exists no pair of vertices w and w' which satisfies the condition of Subcase 1.1. Without loss of generality, assume that there exists no vertex $w \in V(G) \setminus V(H)$ such that $N_G(w) \cap V(X_a) \neq \emptyset$ and $N_G(w) \cap V(Y_a) \neq \emptyset$. By Lemma 3, there exists an edge $uv \in E(G - V(H))$ such that $N_G(u) \cap V(X_a) \neq \emptyset$ and $N_G(v) \cap V(Y_a) \neq \emptyset$. Since $v_1, v_2 \notin S$, $S \subseteq V(H)$ and S is a dominating set of G, we know that v has a neighbor v' in X_a or $Y_a - \{v_1, v_2\}$. If $v' \in V(X_a)$, this contradicts our assumption that there exists no vertex $w \in V(G) \setminus V(H)$ such that $N_G(w) \cap V(X_a) \neq \emptyset$ and $N_G(w) \cap V(Y_a) \neq \emptyset$. Otherwise, $v' \in V(Y_a - \{v_1, v_2\})$, and the edge uv is an our desired edge.

Case 2. All edges of P are cut edges in H. Let $a = v_1v_2$, and let X_a, Y_a be two components of $H \setminus a$ such that $v_1 \in V(X_a)$ and $v_2 \in V(Y_a)$. Consider the following three subcases.

Subcase 2.1. There exists a vertex $w \in V(G) \setminus V(H)$ such that $N_G(w) \cap V(X_a - v_1) \neq \emptyset$ and $N_G(w) \cap V(Y_a - v_2) \neq \emptyset$. Then w is a vertex, as we desired.

Subcase 2.2. There exists an edge $uv \in V(G) \setminus V(H)$ such that $N_G(u) \cap (X_a - v_1) \neq \emptyset$ and $N_G(v) \cap (Y_a - v_2) \neq \emptyset$. Then uv is an edge, as we desired.

Subcase 2.3. There exists no such vertex satisfying the condition of Subcase 2.1, and no such edge satisfying the condition of Subcase 2.2. We shall show that there exists a pair of vertices which satisfies the assertion of this lemma.

Claim 1. *There exists a vertex $w \in V(G) \setminus V(H)$ such that $N_G(w) \cap V(X_a) \neq \emptyset$ and $N_G(w) \cap V(Y_a - v_2) \neq \emptyset$, or $N_G(w) \cap V(Y_a) \neq \emptyset$, and $N_G(w) \cap V(X_a - v_1) \neq \emptyset$.*

Proof. Assume that there exists a vertex w satisfying $N_G(w) \cap V(X_a) \neq \emptyset$ and $N_G(w) \cap V(Y_a) \neq \emptyset$. If $N_G(w) \cap V(X_a) = \{v_1\}$ and $N_G(w) \cap Y_a = \{v_2\}$, then it contradicts the assumption that S is a dominating set of G. Thus, w is a vertex, as we want.

Assume that there does not exist a vertex w satisfying $N_G(w) \cap V(X_a) \neq \emptyset$ and $N_G(w) \cap V(Y_a) \neq \emptyset$. By Lemma 3, there exists an edge uv satisfying $N_G(u) \cap$

$V(X_a) \neq \emptyset$ and $N_G(v) \cap V(Y_a) \neq \emptyset$. Since $v_1 \notin S$, $v_2 \notin S$, $S \subseteq V(H)$ and S is a dominating set of G, we know that u has an neighbor $u' \in S$ which belong to $X_a - v_1$ or $Y_a - v_2$, and v has an neighbor $v' \in S$ which belong to $X_a - v_1$ or $Y_a - v_2$. If u' and v' belong to different components of $H \setminus a$, then uv is an edge which contradicts the assumption of Subcase 2.3. Thus u' and v' belong to the same component of $H \setminus a$. We may suppose that $u', v' \in V(X_a - v_1)$. Then v is the vertex, as we want. This proves the claim. □

By Claim 1, we may assume that there exists a vertex $w \in V(G) \setminus V(H)$ such that $N_G(w) \cap V(X_a - v_1) \neq \emptyset$ and $N_G(w) \cap V(Y_a) \neq \emptyset$.

Let $b = v_2u_2$, and X_b and Y_b be two components of $H \setminus b$ such that $v_2 \in V(X_b)$ and $u_2 \in V(Y_b)$. If there exists a vertex $w' \in V(G) \setminus V(H)$ such that $N_G(w') \cap V(X_b) \neq \emptyset$ and $N_G(w') \cap V(Y_b) \neq \emptyset$, then w and w' are a pair of vertices, as we desired. If this is not the case, then by Lemma 3, there is an edge $uv \in E(G - V(H))$ such that $N_G(u) \cap V(X_b) \neq \emptyset$ and $N_G(v) \cap V(Y_b) \neq \emptyset$. Since $v_1 \notin S$, $v_2 \notin S$, $S \subseteq V(H)$ and S is a dominating set of G, it follows that u has a neighbor $u' \in S$ which belongs to $X_b - \{v_1, v_2\}$ or Y_b. If $u' \in V(X_b - \{v_1, v_2\})$, then uv is an edge that contradicts the assumption of Subcase 2.3. So, $u' \in V(Y_b)$, which implies that $N_G(u) \cap V(X_b) \neq \emptyset$ and $N_G(u) \cap V(Y_b) \neq \emptyset$. Hence w and u are a pair of vertices, as we desired. ■

Theorem 6. *Let G be 2-edge connected graph. If S is a dominating set of G with $|S| \geq 2$, then there exists a set $T \subseteq V(G)$ such that $|T| \leq 4|S| - 4$ and $G[S \cup T]$ is 2-edge connected.*

Proof. For G and S, let T be an output of Algorithm 1 and $H = G[S \cup T]$. We may suppose that $c_H \geq 2$ and pick a pair of vertices $u_1 \in S$ and $u_2 \in S$ such that $\kappa'_H(u_1, u_2) = 1$ and $d_H(u_1, u_2)$ is as small as possible.

Claim 2. $d_H(u_1, u_2) \leq 3$.

Proof. Suppose that the claim is not true, and let $P = x_1x_2 \cdots x_k$ be a shortest path joining u_1 and u_2 in H, where $k \geq 5$, $x_1 = u_1$ and $x_k = u_2$. We consider x_3. Since S is a dominating set of H, x_3 has a neighbor $x'_3 \in S$ in H.

If at least one of x_1x_2 and x_2x_3 is a cut edge of H, then u_1 and x'_3 are a pair of vertices such that $\kappa'_H(u_1, x'_3) = 1$ and $d_H(u_1, x'_3) < d_H(u_1, u_2)$, a contradiction; otherwise, at least one edge of the path $x_3x_4 \cdots x_k$ is a cut edge of H. Thus u_2 and x'_3 are a pair of vertices of S such that $\kappa'_H(x'_3, u_2) = 1$ and $d_H(x'_3, u_2) < d_H(u_1, u_2)$, a contradiction. Thus $d_H(u_1, u_2) \leq 3$. □

By Lemmas 3, 4 and 5, there exists a vertex set T' such that $|T'| \leq 2$ and $c_{H'} \leq c_H - 1$, where $H' = G[S \cup T \cup T']$. If H' is 2-edge connected, then we are done by letting $T := T \cup T'$. Otherwise, let $T := T \cup T'$, and repeat the above operation until $G[S \cup T]$ is 2-edge connected.

Since $c_H \leq |S|$, $|T|$ increases by at most two and c_H decreases by at least one in each iteration of the above operation, we conclude that the desired set T exists. ■

Corollary 3. *For a 2-edge connected graph G, if $\gamma(G) \geq 2$, then $\gamma_2'(G) \leq 5\gamma(G) - 4$.*

Algorithm 2. An algorithm for constructing a 2-edge connected dominating set.
Input: A 2-edge connected graph G, a dominating set S with at least 2 vertices.
Output: A set T such that $|T| \leq 4|S| - 4$ and $G[S \cup T]$ is 2-edge connected.

I. run Algorithm 1 to get set T

II. 1. for $G[S \cup T]$, run DFS to get all blocks, say $B_1, B_2, \ldots, B_k$, and all cut vertices, say $w_1, w_2, \ldots, w_\ell$
2. set $H := G[S \cup T]$, $W = \{w_1, w_2, \ldots, w_\ell\}$, and $\mathcal{B}$ the set of blocks B_i in H such that $|V(B_i)| \geq 3$
3. **if** $W = \emptyset$, **then** stop
4. **else** pick $w \in W$
5. **if** $B_{i_1}, B_{i_2}, \ldots, B_{i_r}$ are blocks in G such that $w \in V(B_{i_1}) \cap V(B_{i_2}) \cap \cdots \cap V(B_{i_r})$, **then** set $B_{i_1} = B_{i_1} \cup B_{i_2} \cup \cdots \cup B_{i_r}$, $W = W \setminus \{w\}$, go to Step 3
6. **else** $W = W \setminus \{w\}$
7. **end if**
8. **end if**

III. 1. set $\mathcal{B} = \mathcal{B} \cup \big(S \setminus \bigcup_{B_i \in \mathcal{B}} V(B_i)\big)$, $b := |\mathcal{B}|$
2. **if** $b = 1$, **then** stop
3. **else** set $W := V \setminus V(H)$, $F := E(G[W])$, and $R := W \times W$
4. **while** $F \neq \emptyset$
5. pick $f = uv \in F$
6. **if** $N_G(u) \cap V(B_i) \neq \emptyset$ and $N_G(u) \cap V(B_j) \neq \emptyset$ for different integers i and j, **then** set $B_i := G[\bigcup_{B_i \in \mathcal{H}} V(B_i) \cup \{u, v\}]$, $\mathcal{B} := (\mathcal{B} \setminus \mathcal{H}) \cup \{B_i\}$, $T := T \cup \{u, v\}$, $H := G[S \cup T]$, and $b := b - h + 1$, where $\mathcal{H} = \{H_i : N_G(B_i) \cap N_G(u) \neq \emptyset$ or $N_G(B_i) \cap N_G(v) \neq \emptyset\}$ and $h = |\mathcal{H}|$, go to Step 2
7. **else** $F := F \setminus \{f\}$
8. **end if**
9. **end while**
11. **while** $W \neq \emptyset$,
12. pick $w \in W$
13. **if** $N_G(w) \cap V(B_i) \neq \emptyset$ and $N_G(w) \cap V(B_j) \neq \emptyset$ for different integers i and j, **then** set $B_i := G[\bigcup_{B_i \in \mathcal{H}} V(B_i) \cup \{w\}]$, $\mathcal{B} := (\mathcal{B} \setminus \mathcal{H}) \cup \{B_i\}$, $T := T \cup \{w\}$, $H := G[S \cup T]$, and $b := b - h + 1$, where $\mathcal{H} = \{B_i : N_G(B_i) \cap N_G(w) \neq \emptyset\}$ and $h = |\mathcal{H}|$, go to Step 2
14. **else** $W := W \setminus \{w\}$
15. **end if**
16. **end while**
17. **while** $R \neq \emptyset$,
18. pick $r = (u, v) \in R$

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19.       if $N_G(u) \cap V(B_i) \neq \emptyset$, $N_G(u) \cap N_H(B_j) \neq \emptyset$, $N_G(v) \cap V(B_i) \neq \emptyset$, and $N_G(v) \cap N_H(B_i) \neq \emptyset$ for different integers $i$ and $j$, then set $B_i := G[\bigcup_{B_i \in \mathcal{H}} V(B_i) \cup \{u, v\}]$, $\mathcal{B} := (\mathcal{B} \setminus \mathcal{H}) \cup B_i$, $T := T \cup \{u, v\}$, $H := G[S \cup T]$, and $b := b - h + 1$, where $\mathcal{H} = \{H_i : N_G(H_i) \cap N_G(u) \neq \emptyset$ or $N_G(H_i) \cap N_G(v) \neq \emptyset\}$ and $h = |\mathcal{H}|$, go to Step 2
20.        else $R := R \setminus \{r\}$
21.        end if
22.    end while
23. end if
```

Remark 2. Let s, Δ, n and m be the size of a dominating set S, the maximum degree, order and size of G, respectively. Note that the time complexity of stage I can be expressed as $O((s-1)\Delta(n+2m))$, and the time complexity of II can be expressed as $O(m + k\ell)$. In III, since the running time of each recursion is at most $\Delta(n + 2m + n^2)$ and III runs at most $s - 1$ recursions. Thus the time complexity of this algorithm is bounded by $O((s-1)\Delta(m + n^2))$.

2.3. 2-connected dominating set

Let G be a connected graph which is not complete, let X be a vertex cut of G, and let Y be the vertex set of a component of $G - X$. The subgraph H of G induced by $X \cup Y$ is called an *X-component* of G. We simply write *x-component* if $X = \{x\}$.

Lemma 7. *Let S be a dominating set of a 2-edge connected graph G with $|S| \geq 2$. If T is an output of Algorithm 2 for G and S, and $T' \subseteq T$ is an output of stage I of Algorithm 2 for G and S, then the following is true for $H = G[S \cup T]$:*

(i) *if u is a cut vertex in H, then $u \in S \cup T'$,*

(ii) *$b(H) \leq 2|S| - 2$, where $b(H)$ is the number of blocks in H.*

Proof. To show (i), it suffices to show that each vertex $u \in T \setminus T'$ is not a cut vertex of H. Since T' is an output of stage I of Algorithm 2 for G and S, $S \cup T'$ is a connected dominating set of G, and thus $S \cup T'$ is also a connected dominating set of H. Therefore $H - u$ is connected, i.e., u is not a cut vertex of H.

Suppose that (ii) is not true, and G is a graph of minimum order satisfying the conditions of this lemma but $b(H) > 2|S| - 2 \geq 2$. If $|S| = 2$, then $b(H) \leq 2$, and thus $b(H) = 2 \leq 2|S| - 2$, a contradiction. So, $|S| \geq 3$. Let u be a cut vertex of H. We consider the following two cases according to (i).

Case 1. $u \in S$. Let $H_1, H_2, \ldots, H_k$ be the u-components of H. Clearly H_i is 2-edge connected. Let $S_i = V(H_i) \cap S$ and $T_i = V(H_i) \setminus S_i$ for $i = 1, 2, \ldots, k$. Since T_i is a possible output of Algorithm 2 for H_i and S_i, we have $b(H_i) =$

$b(G[S_i \cup T_i]) \leq 2|S_i| - 2$ by the minimality of G. Thus $b(H) = \sum_{i=1}^{k} b(H_i) \leq \sum_{i=1}^{k}(2|S_i|-2) \leq 2\sum_{i=1}^{k}|S_i|-2k = 2(|S|+k-1)-2k = 2|S|-2$, a contradiction.

Case 2. $u \in T'$. Let $H_1, H_2, \ldots, H_k$ be the u-components of H. Clearly H_i is 2-edge connected. Let $S_i = V(H_i) \cap S$ and $T_i = V(H_i) \setminus S_i$ for $i = 1, 2, \ldots, k$. Without loss of generality, let $N_{H_i}(u) \cap S_i \neq \emptyset$ for $1 \leq i \leq r$ for an integer r and $N_{H_j}(u) \cap S_j = \emptyset$, $r < j \leq k$. Since S is a dominating set of H, $r \geq 1$.

When $1 \leq i \leq r$, since T_i is a possible output of Algorithm 2 for H_i and S_i, we have $b(H_i) = b(G[S_i \cup T_i]) \leq 2|S_i| - 2$ by the minimality of G.

When $r < j \leq k$, let $S'_j = S_j \cup \{u\}$ and $T'_j = (T_j \setminus u)$. Since T'_j is a possible output of Algorithm 2 for H_j and S'_j, we have $b(H_j) = b(G[S_j \cup T_j]) \leq 2|S'_j| - 2 = 2(|S_j| + 1) - 2$ by the choice of G.

Thus $b(H) = \sum_{i=1}^{k} b(H_i) = \sum_{i=1}^{r}(2|S_i| - 2) + \sum_{j=r+1}^{k}(2(|S_j| + 1) - 2) \leq \sum_{i=1}^{r}(2|S_i| - 2) + \sum_{j=r+1}^{k}(2|S_j|) \leq 2|S| - 2r \leq 2|S| - 2$, a contradiction. This shows (ii). ■

Theorem 8. *Let G be a 2-connected triangle-free graph G. If S is a dominating set of G with $|S| \geq 2$, then there exists a set $T \subseteq V(G)$ such that $|T| \leq 10|S| - 13$ and $G[S \cup T]$ is 2-connected.*

Proof. Let T be an output of Algorithm 2 for G and S. We may suppose that $G[S \cup T]$ is not 2-connected and let $b(H)$ be the number of blocks in $H = G[S \cup T]$. Since H is 2-edge connected, each block of H is 2-edge connected. Let u be a cut vertex in H, let B_1 and B_2 be two blocks of H such that $u \in V(B_1) \cap V(B_2)$, and let H_1 and H_2 be u-components such that $B_i \subseteq H_i$ for $i = 1, 2$.

Let $P = x_1 x_2 \cdots x_k$ be a shortest path connecting $V(H_1)$ and $V(H_2)$ in $G \setminus u$ where $x_1 \in V(H_1)$, $x_k \in V(H_2)$ and $x_2, x_3, \ldots, x_{k-1} \notin V(H_1) \cup V(H_2)$. Suppose $k \geq 6$. Then $ux_3, ux_4 \in E(G)$ since $S \subseteq V(H)$ is a dominating set of G, and P is a shortest path connecting $V(H_1)$ and $V(H_2)$ in $G \setminus u$. Thus ux_3x_4 is a triangle, a contradiction. Thus $k \leq 5$. Let $T' = V(P) \setminus \{x_1, x_k\}$. Then $|T'| \leq 3$.

Hence $S \cup T \cup T'$ is a 2-edge connected dominating set of G with $|T \cup T'| \leq |T| + 2 \leq 4|S| - 4 + 3$ and $b(G[S \cup T \cup T']) \leq b(G[S \cup T]) - 1 = b(H) - 1$. If $G[S \cup T \cup T']$ is 2-connected, then we are done by letting $T := T \cup T'$. Otherwise, let $T := T \cup T'$, and repeat the above operation until $G[S \cup T]$ is 2-connected.

Since $|T| \leq 4|S| - 4$, $b(H) \leq 2|S| - 2$, $|T|$ increases by at most three and $b(H)$ decreases by at least one in each iteration of the above operation, we conclude that the desired set T exists since $|T| \leq 4|S| - 4 + 3(b(H) - 1) = 10|S| - 13$. ■

Corollary 4. *For a 2-connected triangle-free graph G, if $\gamma(G) \geq 2$, then $\gamma_2(G) \leq 11\gamma(G) - 13$.*

Remark 3. For a graph with triangle, Theorem 8 does not holds. For example, let G be the graph in Figure 1. Since $\{u, v, w\}$ is a smallest dominating set and any proper subgraph of G is not 2-connected, we have that $\gamma(G) = 3$ but $\gamma_2(G) = V(G)$, that is, there is not a constant k such that $\gamma_2(G) \leq k\gamma(G)$ for graphs with triangle. So the condition that G is triangle-free is indispensable.

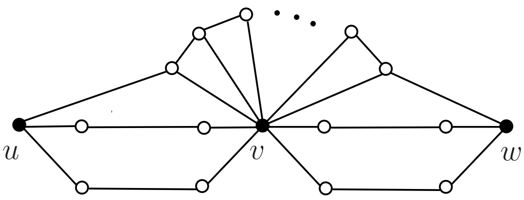

Figure 1. A graph with $\gamma(G) = 3$ but $\gamma_2 = V(G)$.

Algorithm 3. An algorithm for constructing a 2-connected dominating set.
Input: A 2-connected graph G, a dominating set S with at least 2 vertices.
Output: A set T such that $|T| \leq 10|S| - 13$ and $G[S \cup T]$ is 2-connected.

I. run Algorithm 2.
II. run DFS to get all blocks of $G[S \cup T]$, say $B_1, B_2, \ldots, B_k$
III. 1. set $H := G[S \cup T]$, $\mathcal{B} = \{B_i : 1 \leq i \leq k\}$, $b := |\mathcal{B}|$
2. **if** $b = 1$, **then** stop
3. **else** set $W := V \backslash V(H)$ and $F := E(G[W])$
4. **while** $F \neq \emptyset$
5. pick $f = uv \in F$
6. **if** $N_G(u) \cap V(B_i) \neq \emptyset$ and $w \in N_G(v) \cap N_G(B_j) \neq \emptyset$, **then** set $B_i := G[\bigcup_{B_i \in \mathcal{H}} V(B_i) \cup \{u, v\}]$, $\mathcal{B} := (\mathcal{B} \backslash \mathcal{H}) \cup \{B_i\}$, $T := T \cup \{u, v, w\}$, $H := G[S \cup T]$, $b := b - h + 1$, where $\mathcal{H} = \{B_i : V(B_i) \cap N_G(u) \neq \emptyset$ or $V(B_i) \cap N_G(v) \neq \emptyset\}$, and $h = |\mathcal{H}|$, go to Step 2
7. **else** $F := F \backslash \{f\}$
8. **end if**
9. **end while**
10. **while** $W \neq \emptyset$
11. pick $w \in W$
12. **if** $N_G(w) \cap V(B_i) \neq \emptyset$ and $N_G(w) \cap V(B_j) \neq \emptyset$, **then** set $B_i := G[\bigcup_{B_i \in \mathcal{H}} V(B_i) \cup \{w\}]$, $\mathcal{B} := (\mathcal{B} \setminus \mathcal{H}) \cup \{B_i\}$, $T := T \cup \{w\}$, $H := G[S \cup T]$, $b := b - h + 1$, where $\mathcal{H} = \{B_i : V(B_i) \cap N_G(w) \neq \emptyset\}$, and $h = |\mathcal{H}|$, go to Step 2
13. **else** $F := F \backslash \{f\}$
14. **end if**
15. **end while**
16. **end if**

Remark 4. Let s, Δ, n and m be the size of a dominating set S, the maximum degree, order and size of G, respectively. Note the time complexity of stage I is $O((s-1)\Delta(m+n^2))$, and the time complexity of II is $O(m)$. In III, since the running time of each recursion is at most $2\Delta n^2$ and III implements at most $s-1$ recursions. Thus the time complexity of the algorithm is bounded by $O((s-1)\Delta(m+n^2))$.

3. Concluding Remarks

Let $P = u_0u_1 \cdots u_{3k}$ and $Q = v_0v_1 \cdots v_{3k}$ be two path of length $3k$. The symbol G denotes the graph obtained from P and Q by identifying u_{3i} and v_{3i} (denote the resulting vertex by w_{3i}), where $0 \leq i \leq n$. It is easy to check that G is 2-edge connected and $S = \{w_{3i} : 0 \leq i \leq n\}$ is a dominating set. Note that $T = \{u_{3i+1}, u_{3i+2} : 0 \leq i \leq n-1\}$ and $T' = \{v_{3i+1}, v_{3i+2} : 0 \leq i \leq n-1\}$ are minimum sets of vertices such that $G[S \cup T]$ and $G[S \cup T']$ are connected, and $Q = T \cup T'$ is the unique set of vertices such that $G[S \cup Q]$ is 2-edge connected. Thus the bounds given in Theorem 2, 6 and Corollary 3 are sharp.

We suspect that the bound of Theorem 8 is not sharp and the best possible bound might be the following.

Conjecture 2. *For a dominating set S of a 2-connected triangle-free graph G with $|S| \geq 2$, there exists a vertex set $T \subseteq V(G)$ with $|T| \leq 5|S|$ such that $G[S \cup T]$ is 2-connected.*

Inspired by Corollaries 1, 3 and 4, one may ask the following two problems.

Problem 4. Does there exist an absolute constant c'_k for a given integer $k \geq 1$ such that $\gamma'_k(G) \leq c'_k\gamma(G)$ for any k-edge connected graph G?

Problem 5. Does there exist an absolute constant c_k for a given integer $k \geq 1$ such that $\gamma_k(G) \leq c_k\gamma(G)$ for any k-connected graph G?

By our main results, c'_k and c_k exist for $1 \leq k \leq 2$. But, c'_k and c_k do not exist for an integer $k \geq 3$. Let C_n and K_{k-2} be the cycle of order n and the complete graph of order $k-2$. Let $G_{n,k} = C_n \vee K_{k-2}$, be the graph obtained from C_n and K_{k-2} by joining every vertex of C_n to all vertices of K_{k-2}. It is clear that $G_{n,k}$ is k-connected, and thus k-edge connected. But, $\gamma(G_{n,k}) = 1$ and $\gamma'_k(G_{n,k}) = \gamma_k(G_{n,k}) = n + k$.

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