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THE LARGEST COMPONENT IN CRITICAL RANDOM INTERSECTION GRAPHS

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Abstract

In this paper, through the coupling and martingale method, we prove the order of the largest component in some critical random intersection graphs is $n^{\frac{2}{3}}$ with high probability and the width of scaling window around the critical probability is $n^{-\frac{1}{3}}$; while in some graphs, the order of the largest component and the width of the scaling window around the critical probability depend on the parameters in the corresponding definition of random intersection graphs. Our results show that there is still an “inside” phase transition in critical random intersection graphs.

Keywords: critical random intersection graph, largest component, scaling window.

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1. Introduction and Main Result

The Erdős-Rényi random graph $G(n,p)$ is obtained by retaining each edge of the complete graph on n vertices independently with probability p. Since it is not accurate to characterize real-world networks by $G(n,p)$, Singer-Cohen [**?**] Karoński *et al.* [6] introduced random intersection graph $G(n,m,p)$ which is defined with V as a set of n vertices and M as a set of m elements. Each vertex $v \in V$, it is assigned a random subset of M, denoted F_v. Each element of M is included in F_v independently with probability p. There is an edge between vertices u and v if and only if $F_u \cap F_v \neq \emptyset$. The graph $G(n,m,p)$ usually is constructed through the bipartite graph $\mathcal{B}(n,m,p)$. $\mathcal{B}(n,m,p)$ is a random bipartite graph with bipartition (V,M). Any vertex in V and any element in M are connected by an edge in $\mathcal{B}(n,m,p)$ independently with probability p. An edge between u and v in $G(n,m,p)$ is present if both u and v are adjacent to some element in $\mathcal{B}(n,m,p)$. Fill *et al.* [4] proved that when $m = n^\alpha$ and $\alpha > 6$, $G(n,m,p)$ and $G(n,p')$ are equivalent in the sense that the total variation distance between the graph-valued random variables has a limit of 0 for some suitable p' as $n \to \infty$. *Here, we note out that when $m > 1$ and is not an integer, $G(n,m,p)$ means $G(n,\lfloor m \rfloor,p)$ with $\lfloor m \rfloor$ being the largest integer less than m.* Rybarczyk [12] extended this result to the case of $\alpha > 3$. However, when $\alpha < 1$, these two graph models seem to have different properties.

Next, denote the largest component in $G(n,m,p)$ by $\mathcal{C}_1\left(G(n,m,p)\right)$ and its size by $|\mathcal{C}_1|$. In $G(n,m,p)$, Behrisch [1] proved that for $m = n^\alpha$ and $mp^2 = \frac{c}{n}$ there is a phase transition in $G(n,m,p)$. When $\alpha > 1$, with high probability[3] (w.h.p.) $|\mathcal{C}_1|$ jumps from the logarithmic order to the linear order in n; when $0 < \alpha < 1$, it jumps from $n^{\frac{1-\alpha}{2}} \log n$ to $n^{\frac{1+\alpha}{2}}$ as c grows. Lagerås and Lindholm [7] extended this result to the case when $m = \beta n$ and $p = \gamma n^{-1}$. $|\mathcal{C}_1|$ exhibits a jump from the logarithmic order to the linear order in n around the point $\beta\gamma^2 = 1$. Note the expected value of the vertex degree is $(n-1)\left(1-\left(1-p^2\right)\right)^m$, which is approximately equal to c in [1] and $\beta\gamma^2$ in [7], respectively. This means that there is a phase transition in $G(n,m,p)$ when the expected value of the vertex degree is close to one, which behaves similarly to the Erdős-Rényi random graph.

It is natural to ask what the order of $|\mathcal{C}_1|$ is in the critical $G(n,m,p)$. The interest of this question lies in looking "inside" the phase transition in the growth of the largest component $\mathcal{C}_1\left(G(n,m,p)\right)$. And the difficulty of studying $G(n,m,p)$ is to deal with the dependencies between the edges, especially in the critical $G(n,m,p)$. Now our main results can be stated in the following way.

[3]Here for a given graph property $\mathcal{A}$, we say that graph G_n possesses $\mathcal{A}$ with high probability if the probability that G_n possesses $\mathcal{A}$ tends to 1 as $n \to \infty$.

Theorem 1. *Let $\epsilon(n)$ be a positive function on n such that $\epsilon(n) \to 0$ and $n^{\frac{1}{3}}\epsilon(n) \to \infty$ as $n \to \infty$. In $G(n,m,p)$, where $m = n^{\alpha}$ and $\alpha > \frac{5}{3}$, the following statements hold.*

(1) *If $mp^2 = \frac{1-\epsilon(n)}{n}$, then there are two positive constants C_1 and C_2 such that w.h.p.*

$$C_1\epsilon^{-2}(n)\log\left\{n\epsilon^3(n)\right\} \le |\mathcal{C}_1| \le C_2\epsilon^{-2}(n)\log\left\{n\epsilon^3(n)\right\}.$$

(2) *If $mp^2 = \frac{1+\lambda n^{-\frac{1}{3}}}{n}$ for some constant λ, then when $\lambda < 0$, there are two positive functions $\omega_1(n)(< \log n)$ and $\omega_2(n)(< \log n)$ on n which tend to infinity as $n \to \infty$ such that w.h.p.*

$$\omega_1^{-1}(n)n^{\frac{2}{3}} \le |\mathcal{C}_1| \le \omega_2(n)n^{\frac{2}{3}},$$

while when $\lambda > 0$, there is a constant $C_3 > 0$ such that w.h.p.

$$\omega_1^{-1}(n)n^{\frac{2}{3}} \le |\mathcal{C}_1| \le C_3 n^{\frac{2}{3}}\log n. \tag{1}$$

(3) *If $mp^2 = \frac{1+\epsilon(n)}{n}$, then there are two positive constants C_4 and C_5 such that w.h.p.*

$$C_4 n\epsilon(n) \le |\mathcal{C}_1| \le C_5 n\epsilon(n)\log n. \tag{2}$$

Theorem 2. *Let $\alpha \in \left(1, \frac{5}{3}\right)$ and $\epsilon(n) > 0$ be a function on n such that $\epsilon(n) \to 0$ and $n^{\frac{\alpha-1}{2}}\epsilon(n) \to \infty$ as $n \to \infty$. In $G(n,m,p)$ with $m = n^{\alpha}$, the following statements hold.*

(4) *If $mp^2 = \frac{1-\epsilon(n)}{n}$, then there are two positive constants C_6 and C_7 such that w.h.p.*

$$C_6\epsilon^{-2}(n)\log\left\{n\epsilon^3(n)\right\} \le |\mathcal{C}_1| \le C_7\epsilon^{-2}(n)\log\left\{n\epsilon^3(n)\right\}.$$

(5) *If $mp^2 = \frac{1+\lambda n^{-\frac{\alpha-1}{2}}}{n}$ for some constant λ, then when $\lambda < 0$, there is a positive constant C_8 and a positive function $\omega_3(n)(< \log n)$ on n which tends to infinity as $n \to \infty$ such that w.h.p.*

$$C_8 n^{\alpha-1}\log n < |\mathcal{C}_1| \le \omega_3(n)n^{\frac{3-\alpha}{2}}.$$

When $\lambda > 0$, there are two positive constants C_9 and C_{10} such that w.h.p.

$$C_9 n^{\frac{3-\alpha}{2}} < |\mathcal{C}_1| \le C_{10}n^{\frac{3-\alpha}{2}}\log n. \tag{3}$$

(6) *If $mp^2 = \frac{1+\epsilon(n)}{n}$, then there are two positive constants C_{11} and C_{12} such that w.h.p.*

$$C_{11}n\epsilon(n) \le |\mathcal{C}_1| \le C_{12}n\epsilon(n)\log n. \tag{4}$$

Remark 3. (1) Behrisch [1] showed that when $\alpha > 1$ and $m = n^{\alpha}$, the order of $|\mathcal{C}_1|$ in the subcritical and supercritical cases of $G(n, m, p)$ is independent of α. However, from Theorem 2, we can see that it depends on α when $\alpha \in \left(1, \frac{5}{3}\right)$ in critical cases. Theorems 1 and 2 are interesting in that there is still an "inside" phase transition in the critical case.

Due to technical reasons, there is a factor $\log n$ for the upper bound more than the lower bound in (1)–(4), and it is interesting to remove this factor. We fail to show that when $mp^2 = \frac{1+\lambda n^{-\frac{\alpha-1}{2}}}{n}$ for $\lambda < 0$ and $1 < \alpha < \frac{5}{3}$ w.h.p. $|\mathcal{C}_1|$ is at least of order $n^{\frac{3-\alpha}{2}}/\omega_4(n)$, where $\omega_4(n)(< \log n)$ is a positive function on n which tends to infinity as $n \to \infty$.

(2) With the fact that the roles of the vertex and element sets used by [1] can be interchanged, we can study the $|\mathcal{C}_1|$ in critical cases when $0 < \alpha < \frac{3}{5}$ and $\frac{3}{5} < \alpha < 1$. For the following example, we only take the case when $0 < \alpha < \frac{3}{5}$, but the alternative may be of interest to readers as well. Let E_w be the vertex set holding element $w \in M$ which is a clique in $G(n, m, p)$. By the Chernoff inequality,

$$|E_w| = (1 + o(1))np = (1 + o(1))n^{\frac{1-\alpha}{2}}$$

with probability at least $1 - me^{-(np)^{1/2}/3} = 1 - o(1)$, see [1, Lemma 2] for details of the proof. The roles of the vertex sets and element sets can be interchanged in which every two elements are connected if a vertex chooses both of them. In this way $G(n, m, p)$ is dual to $G(m, n, p)$. Now we look for the largest component $C_1^{(e)}$ about the element set in the dual graph, and use $\left|C_1^{(e)}\right|$ to denote its size. By Theorem 1,

$$\mathbf{P}\left\{\left|\mathcal{C}_1^{(e)}\right| \geq \omega_1(m) m^{\frac{2}{3}}\right\} \to 0, \quad \mathbf{P}\left\{\left|\mathcal{C}_1^{(e)}\right| \geq \frac{m^{\frac{2}{3}}}{\omega_2(m)}\right\} \to 1.$$

As $|E_w| = (1 + o(1))np$ with probability $1 - o(1)$, we have

$$\mathbf{P}\left\{|\mathcal{C}_1| \geq \omega_1(m) m^{\frac{2}{3}} (1 + o(1))np\right\} = \mathbf{P}\left\{|\mathcal{C}_1| \geq (1 + o(1))\omega_1(n^{\alpha}) n^{\frac{\alpha+3}{6}}\right\} \to 0,$$

$$\mathbf{P}\left\{|\mathcal{C}_1| \geq \frac{m^{\frac{2}{3}}}{\omega_2(m)}(1 + o(1))np\right\} = \mathbf{P}\left\{|\mathcal{C}_1| \geq (1 + o(1))\frac{n^{\frac{\alpha+3}{6}}}{\omega_2(n^{\alpha})}\right\} \to 1.$$

We will use the following notation. Write $\mathbf{P}(\cdot)$, $\mathbf{E}(\cdot)$ and $\mathbf{Var}(\cdot)$ for the probability, expected value and variance of a random event or a random variable, respectively. For any two positive functions $f(n)$ and $g(n)$ of a natural-valued parameter n, denote $f(n) = O(g(n))$ if there is a positive constant C such that $f(n) \leq Cg(n)$ when n is large enough; $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $g(n) = O(f(n))$; and $f(n) = o(g(n))$ if $\lim_{n\to\infty} f(n)/g(n) = 0$.

The reminder of the paper is organized in the following way. In Section 2, we include some known results on random graphs which will be used in our proofs. In Section 3, we prove Theorem 1 by the coupling method, martingale argument of [10, 11] and the optimal stopping theorem. Since the proof of Theorem 2 is almost the same as Theorem 1, we put this in the Appendix. Finally, in Section 4 we list some questions for critical random intersection graphs.

2. Auxiliary Theorems

First, we mention the result about vertex degree distribution of $G(n,m,p)$. Let $D(n,m,p)$ be the random variable for the vertex degree of $G(n,m,p)$ and $F(n, m,p)$ be the distribution. Stark [14] proved that it has the following probability generating function

$$\mathbf{E}\left[x^{D(n,m,p)}\right] = \sum_{j=0}^{n-1} \binom{n-1}{j} x^j (1-x)^{n-1-j} \left[1-p+p(1-p)^{n-1-j}\right]^m . \tag{5}$$

Hence, from (5) it is easy to see that

$$\mathbf{E}[D(n,m,p)] = (n-1)\left[1-\left(1-p^2\right)^m\right]; \tag{6}$$

$$\begin{aligned}&\mathbf{E}[D(n,m,p)(D(n,m,p)-1)]\\ &= (n-1)(n-2)\left\{1-2\left(1-p^2\right)^m + \left[1-p^2(2-p)\right]^m\right\};\end{aligned} \tag{7}$$

$$\begin{aligned}&\mathbf{E}[D(n,m,p)(D(n,m,p)-1)(D(n,m,p)-2)] = (n-1)(n-2)(n-3)\\ &\cdot\left\{1-3\left(1-p^2\right)^m + 3\left[1-p^2(2-p)\right]^m - \left[1-p^2\left(3-3p+p^2\right)\right]^m\right\}.\end{aligned} \tag{8}$$

Next, Rybarczyk [13] proved the following Theorem 4.

Theorem 4 [13, Theorem 1]**.** *Let* $p' = mp^2\left(1-(n-2)p-\frac{mp^2}{2}\right)$, $mp^2 < 1$, *and* $\mathcal{A}$ *be an increasing graph property. If* $\mathbf{P}\left(G\left(n,p'\right) \in \mathcal{A}\right) \to 1$, *then*

$$\mathbf{P}\left(G\left(n,m,p\right) \in \mathcal{A}\right) \to 1,$$

where $G \in \mathcal{A}$ *denotes that* G *has property* $\mathcal{A}$.

By Theorem 4, to give a lower bound for $|\mathcal{C}_1|$, the following theorem on the largest component in the critical Erdős-Rényi random graph is useful.

Theorem 5 [5, Theorem 5.23], [8]**.** *Let* $|\mathcal{C}_1'|$ *denote the size of the largest component in Erdős-Rényi random graph* $G\left(n,p'\right)$. *For any* $\omega(n)$ $\left(< n^{\frac{1}{6}}\right)$ *which tends to infinity as* $n\to\infty$, *the following statements hold.*

(a) *In* $G\left(n, \frac{1-\epsilon(n)}{n}\right)$, *if* $\omega(n)n^{-\frac{1}{3}} \le \epsilon(n) \le \frac{1}{\omega(n)}$, *then w.h.p.*

$$\left|\mathcal{C}_1'\right| = (2+o(1))\epsilon^{-2}(n)\log\left\{n\epsilon^3(n)\right\}.$$

(b) *In* $G\left(n, \frac{1+\epsilon(n)}{n}\right)$, *if* $\omega(n)n^{-\frac{1}{3}} \le \epsilon(n) \le \frac{n}{\omega(n)}$, *then w.h.p.*

$$\left|\mathcal{C}_1'\right| = (2+o(1))n\epsilon(n).$$

(c) *In* $G\left(n, \frac{1\pm\epsilon(n)}{n}\right)$, *if* $0 \le \epsilon(n) \le \omega(n)n^{-\frac{1}{3}}$, *then w.h.p.*

$$\frac{n^{\frac{2}{3}}}{\omega^2(n)} \le \left|\mathcal{C}_1'\right| \le \omega(n)n^{\frac{2}{3}}.$$

Finally, we present the standard exploration process to explore components of random intersection graphs, which is the same as Erdős-Rényi random graphs.

Exploration process $\{Y_t\}_{t\ge 0}$. Let $\mathcal{C}(v)$ be the component containing vertex v in $G(n,m,p)$. In this procedure, the vertices will be active, inactive or neutral. In the beginning, assuming that all the vertices are neutral, we choose a vertex v_0 uniformly and make it active. At each time $t \ge 1$, we choose a vertex v_t uniformly from the active vertices and check the pairs $v_t v'$, where v' runs over all the neutral vertices. If $v_t v' \in G(n,m,p)$, then make v' active, otherwise keep it neutral. After checking all the neutral vertices, let v_t be inactive. When there is no active vertex, the component $\mathcal{C}(v_0)$, which is the set of inactive vertices, is explored. Then, we choose a neutral vertex uniformly from the rest of the neutral ones and proceed on.

Let Z_t be the number of vertices which become active due to the exploration of active vertex v_t, and Y_t be the total number of active vertices at step $t \in \{0,1,\ldots,n\}$, where $Y_0 = 1$. It is easy to see that for any $t \ge 1$,

$$Y_t = \begin{cases} Y_{t-1} + Z_t - 1, & \text{if } Y_{t-1} > 0, \\ Z_t, & \text{if } Y_{t-1} = 0. \end{cases}$$

Define T to be the least t for which $Y_t = 0$, i.e.,

$$T = \min\{t\colon Y_t = 0\} = \min\{t : Z_1 + Z_2 + \cdots + Z_t = t-1\}.$$

Then, at the time T, the set of explored vertices is precisely $\mathcal{C}(v_0)$, which means $|\mathcal{C}(v_0)| = T$.

3. PROOF OF THEOREM 1

Note that when $\alpha > \frac{5}{3}$ and $mp^2 = \frac{1\pm\epsilon(n)}{n}$,

$$p' = mp^2\left(1-(n-2)p - \frac{mp^2}{2}\right) = \frac{1\pm\epsilon(n)}{n} + o\left(\frac{\epsilon(n)}{n}\right).$$

Keeping Theorems 4 and 5 in mind, we can determine the lower bounds directly for the subcritical, critical and supercritical phases. Therefore, in the rest of this section, we only need to prove the upper bounds.

3.1. Below the critical window

To overcome the difficulty from dependencies between edges, in this subsection we use a coupling method to prove the upper bound.

Outline of the proof. We first use the exploration process to explore the components of $G(n,m,p)$. Then, for the upper bound on $|\mathcal{C}_1|$, we only need to bound the stopping time T as $|\mathcal{C}(v_0)| = T$. For this, we will define a random walk and a stopping time τ, which is stochastically larger[4] than T of the random walk, then bound the stopping time τ to determine the desired result similar to [11, Proposition 1]. We need the following lemmas.

Lemma 6. *Let X and Y be two random variables with distributions $F(n_1,m_1,p)$ and $F(n_2,m_2,p)$, respectively. Suppose $n_1 \le n_2$ and $m_1 \le m_2$. Then we have $X \preceq Y$.*

Proof. Note that $G(n,m,p)$ can be constructed through the random bipartite graph $\mathcal{B}(n,m,p)$ with partition(V,M). Any vertex in V and any element in M are connected by an edge in $\mathcal{B}(n,m,p)$ independently with probability p. An edge between u and v in $G(n,m,p)$ is present if both u and v are adjacent to some element in $\mathcal{B}(n,m,p)$.

Now, suppose $V_1 = \{v_1,\dots,v_{n_1}\}$ and $V_2 = \{v_1,\dots,v_{n_1},\dots,v_{n_2}\} \supseteq V_1$ are two sets of vertices, and $\mathcal{E}_1 = \{e_1,\dots,e_{m_1}\}$ and $\mathcal{E}_2 = \{e_1,\dots,e_{m_1},\dots,e_{m_2}\} \supseteq \mathcal{E}_1$ are two sets of elements. It is easy to see that Y is the degree of a vertex v in the random intersection graph $G(n_2,m_2,p)$ which is derived from the bipartite graph $\mathcal{B}(n_2,m_2,p)$ with vertex set V_2 and element set $\mathcal{E}_2$, i.e., Y is distributed as $F(n_2,m_2,p)$. Given $\mathcal{B}(n_2,m_2,p)$, $\mathcal{B}(n_1,m_1,p)$ is defined to be the bipartite subgraph of $\mathcal{B}(n_2,m_2,p)$ induced by the vertex set V_1 and element set $\mathcal{E}_1$. Then we can determine the corresponding random intersection graph $G(n_1,m_1,p)$. By letting X be the degree of the same vertex v in $G(n_1,m_1,p)$, we can get that X is distributed as $F(n_1,m_1,p)$ and $X \preceq Y$, since removing vertices $v_{n_1+1},\dots,v_{n_2}$ and elements $e_{m_1+1},\dots,e_{m_2}$ cannot make the degree of v increase. ∎

Lemma 7 [11, Lemma 8]**.** *Let β be an integer-valued random variable with $\mathbf{E}(\beta^2) < \infty$ such that for any integer $h \ge 2$, $\mathbf{P}(\beta \in h\mathbb{Z}) < 1$. Let $\{\beta_i\}_{i=1}^{\infty}$ be*

[4]For two random variables X and Y, we say X is stochastically larger than Y, denote it by $X \succeq Y$, if and only if there exists a coupling $(\hat{X},\hat{Y})$ of X and Y such that $\mathbf{P}(\hat{X} \ge \hat{Y}) = 1$.

i.i.d. random variables distributed as β *and*

$$W_t = W_0 + \sum_{i=1}^{t} \beta_i, \ t \in \{0, 1, 2, \ldots\},$$

where W_0 *is an integer constant. Define* τ *to be its hitting time of* 0, *i.e.,*

$$\tau = \min\{t \colon W_t = 0\}.$$

If $\theta_0 > 0$ *satisfies*

$$\mathbf{E}\left(\beta e^{\theta_0 \beta}\right) = 0,$$

then for any integer $\ell \geq 1$,

$$\mathbf{P}(\tau = \ell) = \Theta\left(\ell^{-3/2} \phi(\theta_0)^{\ell}\right),$$

where $\phi(\theta) = \mathbf{E}\left(e^{\theta\beta}\right)$ *and the constants in* Θ *depend only on* β *and* W_0*, but not on* ℓ.

Proof of the upper bound. To make use of Lemma 7, let $\{\xi_t\}_{t=1}^{\infty}$ be i.i.d. random variables distributed as $F(n, m, p)$. Set

$$S_t = 1 + \sum_{i=1}^{t} (\xi_i - 1).$$

Take $\beta = \xi_1 - 1$. By (6)–(7) and the following inequality

$$\begin{aligned} &\frac{1-\epsilon(n)}{n} - \frac{(1-\epsilon(n))^2}{2n^2} + \frac{(1-\epsilon(n))^3}{7n^3} \\ &< 1 - \left(1-p^2\right)^m < \frac{1-\epsilon(n)}{n} - \frac{(1-\epsilon(n))^2}{2n^2} + \frac{(1-\epsilon(n))^3}{5n^3}, \end{aligned}$$

it is easy to check that, noting $nmp^2 = 1 - \epsilon(n)$, where $\alpha > \frac{5}{3}$, $\epsilon(n) \to 0$ and $n^{1/3}\epsilon(n) \to \infty$ as $n \to \infty$,

$$\begin{aligned} (9) \qquad \mathbf{E}[\beta] &= \mathbf{E}(\xi_i - 1) = (n-1)\left(1 - \left(1-p^2\right)^m\right) - 1 \\ &= \epsilon(n) - \frac{3(1-\epsilon(n))}{2n} + O\left(\frac{\epsilon(n)}{n}\right), \\ (10) \quad \mathbf{E}\left[\beta^2\right] &= \mathbf{E}\left[(\xi_i - 1)^2\right] = \mathbf{E}\left[\xi_i(\xi_i - 1)\right] - \mathbf{E}(\xi_i - 1) \\ &= (n-1)(n-2)\left\{1 - 2\left(1-p^2\right)^m + \left[1 - p^2(2-p)\right]^m\right\} \\ &- (n-1)\left[1 - \left(1-p^2\right)^m\right] + 1 \\ &= (n-1)(n-2)\left[1 - 2\left(1 - mp^2 + \frac{m(m-1)p^4}{2}\right) + 1 + \epsilon(n)\right. \end{aligned}$$

$$
\begin{aligned}
&- \, mp^2(2-p) + \frac{m(m-1)p^4(2-p)^2}{2} + O\left(m^3p^6\right)\Big] + O\left(\frac{1}{n}\right) \\
&= 1-\epsilon(n)+\epsilon^2(n) + \frac{(1-\epsilon(n))^{3/2}}{n^{\frac{\alpha-1}{2}}} + o\left(\frac{1}{n^{\frac{\alpha-1}{2}}}\right) + O\left(\frac{1}{n}\right),
\end{aligned}
$$

$$
\begin{aligned}
(11)\quad \mathbf{E}\left[\beta^3\right] &= \mathbf{E}\left[(\xi_i-1)^3\right] = \mathbf{E}[\xi_i(\xi_i-1)(\xi_i-2)] + \mathbf{E}(\xi_i-1) \\
&= -\epsilon(n) - \frac{3(1-\epsilon(n))}{2n} + O\left(\frac{\epsilon(n)}{n}\right) \\
&+ \, (n-1)(n-2)(n-3)\left\{1-3\left(1-p^2\right)^m\right. \\
&+ \, 3\left[1-p^2(2-p)\right]^m - \left.\left[1-p^2\left(3-3p+p^2\right)\right]^m\right\} \\
&= (1-\epsilon(n))^3 + \frac{3\left(1-\epsilon(n)\right)^{5/2}}{n^{\frac{\alpha-1}{2}}} + o\left(\frac{1}{n^{\frac{\alpha-1}{2}}}\right) - \epsilon(n) + O\left(\frac{1}{n}\right).
\end{aligned}
$$

Define

$$
F(\theta) = \mathbf{E}\left[\beta e^{\theta\beta}\right] \text{ for } \theta \in \left[\frac{\epsilon(n)}{2}, \epsilon(n) + \frac{\epsilon(n)}{2\log n}\right].
$$

Next, we will prove that there is a θ_0 satisfying

$$
\frac{\epsilon(n)}{2} < \theta_0 < \epsilon(n) + \frac{\epsilon(n)}{2\log n} \quad \text{and} \quad F(\theta_0) = \mathbf{E}\left(\beta e^{\theta_0\beta}\right) = 0.
$$

In fact, first, note $\beta \in \{-1, 0, 1, \ldots, n-2\}$ and set $\mathbf{P}(\beta = i) = p_i$ for $i \in \{-1, 0, 1, \ldots, n-2\}$. By these notations, rewrite $\mathbf{E}[\beta]$ and $\mathbf{E}\left[\beta^2\right]$ as

$$
(12)\qquad \mathbf{E}[\beta] = \sum_{i=-1}^{n-2} ip_i = \sum_{i=1}^{n-2} ip_i - p_{-1} = -\epsilon(n) - \frac{3(1-\epsilon(n))}{2n} + O\left(\frac{\epsilon(n)}{n}\right),
$$

$$
(13)\qquad \mathbf{E}\left[\beta^2\right] = \sum_{i=-1}^{n-2} i^2p_i = \sum_{i=1}^{n-2} i^2p_i + p_{-1} = 1-\epsilon(n) + o\left(\epsilon(n)\right).
$$

Second, define $b(n) := \epsilon(n) + \frac{\epsilon(n)}{2\log n}$. Note that for any natural number i and $x \in (0,\infty)$, $e^{ix} - e^{-x} > (i+1)x$. Then when n is large enough we can obtain

$$
\begin{aligned}
\mathbf{E}\left[\beta e^{b(n)\beta}\right] &= -e^{-b(n)}p_{-1} + e^{b(n)}p_1 + 2e^{2b(n)}p_2 + \cdots + (n-2)e^{(n-2)b(n)}p_{n-2} \\
&\overset{(12)}{\geq} e^{-b(n)}\left\{-\epsilon(n) - \sum_{i=1}^{n-2} ip_i - 2n^{-1}\right\} + \sum_{i=1}^{n-2} ie^{ib(n)}p_i
\end{aligned}
$$

$$
\begin{aligned}
&= \quad -\epsilon(n)e^{-b(n)} + \sum_{i=1}^{n-2} ip_i\left(e^{ib(n)} - e^{-b(n)}\right) - 2n^{-1}e^{-b(n)} \\
&\geq \quad -\epsilon(n)e^{-b(n)} + b(n)\sum_{i=1}^{n-2} i(i+1)p_i - 2n^{-1}e^{-b(n)} \\
(14) \quad &\overset{(12)(13)}{=} \quad -\epsilon(n)e^{-b(n)} - 2n^{-1}e^{-b(n)} + b(n)\left[\mathbf{E}\left(\beta^2\right) - p_{-1} + \mathbf{E}(\beta) + p_{-1}\right] \\
&\overset{(9)(10)}{\geq} \quad -\epsilon(n)\left(1 - b(n) + b^2(n)\right) - o\left(\epsilon^2(n)\right) + b(n)\left(1 - 2\epsilon(n) + o(\epsilon(n))\right) \\
&\geq \quad \frac{2\epsilon(n)}{\log n} - \epsilon^2(n) + o\left(\epsilon^2(n)\right) > 0.
\end{aligned}
$$

By (9)–(11) and the inequality $e^{\frac{x}{2}} < 1 + \frac{2x}{3}$ for $x = o(1)$, when n is large enough we have

$$
\begin{aligned}
\mathbf{E}\left[\beta e^{\frac{\epsilon(n)}{2}\beta}\right] &= \sum_{i=-1}^{n-2}\left(e^{\frac{\epsilon(n)}{2}}\right)^i ip_i \leq \sum_{i=-1}^{n-2}\left(1 + \frac{2\epsilon(n)}{3}\right)^i ip_i \\
&\leq \sum_{i=-1}^{n-2}\left(1 + \frac{2i\epsilon(n)}{3} + \frac{4i^2\epsilon^2(n)}{9}\right) ip_i \\
&= \sum_{i=-1}^{n-2} kp_k + \frac{2\epsilon(n)}{3}\sum_{i=-1}^{n-2} i^2p_i + \frac{4\epsilon^2(n)}{9}\sum_{i=-1}^{n-2} i^3p_i \\
&= \mathbf{E}(\beta) + \frac{2\epsilon(n)}{3}\mathbf{E}\left(\beta^2\right) + \frac{4\epsilon^2(n)}{9}\mathbf{E}\left(\beta^3\right) = -\frac{\epsilon(n)}{3} + o(\epsilon(n)) < 0.
\end{aligned} \tag{15}
$$

Third, it is easy to check that $F(\theta)$ is continuous in θ when $\theta \in \left[\frac{\epsilon(n)}{2}, \epsilon(n) + \frac{\epsilon(n)}{2\log n}\right]$. Hence, with the fact that $F\left(\epsilon(n) + \frac{\epsilon(n)}{2\log n}\right) \overset{(14)}{>} 0$ and $F\left(\frac{\epsilon(n)}{2}\right) \overset{(15)}{<} 0$, we have a constant $\theta_0 \in \left(\frac{\epsilon(n)}{2}, \epsilon(n) + \frac{\epsilon(n)}{2\log n}\right)$ such that $F(\theta_0) = 0$. So far by (9)–(10) and the inequalities $e^x < 1 + x + x^2$, $(1+x)^k < 1 + kx + \frac{2k^2x^2}{3}$ for any $x \in (0, 2\epsilon(n))$, when n is large enough we can deduce that

$$
\begin{aligned}
\mathbf{E}\left[e^{\theta_0\beta}\right] &= \sum_{k=-1}^{n-2}\left(e^{\theta_0}\right)^k p_k \leq \sum_{k=-1}^{n-2}\left(1 + \theta_0 + \theta_0^2\right)^k p_k \\
&\leq \sum_{k=-1}^{n-2}\left(1 + k\left(\theta_0 + \theta_0^2\right) + \frac{2k^2}{3}\left(\theta_0 + \theta_0^2\right)^2\right) p_k
\end{aligned}
$$

$$= \sum_{k=-1}^{n-2} p_k + \left(\theta_0 + \theta_0^2\right) \sum_{k=-1}^{n-2} k p_k + \frac{2\left(\theta_0 + \theta_0^2\right)^2}{3} \sum_{k=-1}^{n-2} k^2 p_k$$
$$= 1 + \left(\theta_0 + \theta_0^2\right) \mathbf{E}(\beta) + \frac{2\left(\theta_0 + \theta_0^2\right)^2}{3} \mathbf{E}\left(\beta^2\right)$$
$$= 1 - \frac{\epsilon^2(n)}{3} + o\left(\epsilon^2(n)\right).$$

Now, we are in the position to explore components of $G(n, m, p)$. It is helpful to keep the bipartite graph $\mathcal{B}(n, m, p)$ in mind, so that we can keep track not only of explored vertices, but also explored elements. When we explore a vertex, we also say its elements have been explored. Suppose that k vertices have been explored at step i in the exploration process and ℓ elements have been explored. When we are in a position of exploration through a vertex, say v, the number of newly explored vertices Z_i through v has the distribution $F(n-k, m-\ell, p)$. Therefore, by Lemma 6, $Z_i \preceq \xi_i$, where ξ_i is distributed as $F(n, m, p)$. That is $\sum_{i=1}^k Z_i$ is dominated above by $\sum_{i=1}^k \xi_i$ for all $k \leq T$, where T is defined to be $\min\{t\colon Y_t = 0\}$ in the exploration process. Let $\mathcal{C}_t(v_0)$ denote total inactive (explored) vertices which are explored from time 0 to t by the exploration process on $G(n, m, p)$ starting from vertex v_0 which is chosen uniformly at random from vertex set V. Define $\tau = \min\{t\colon S_t = 0, t \leq n\}$. Notice that

$$|\mathcal{C}_t(v_0)| = 1 + \sum_{i=1}^{t} (Z_i - 1) \text{ and } S_t = 1 + \sum_{i=1}^{t} (\xi_i - 1).$$

Hence, there is a coupling $\left(\left(\left|\widehat{\mathcal{C}}_t(v_0)\right|\right)_t, \left(\widehat{S}_t\right)_t\right)$ of the processes $(|\mathcal{C}_t(v_0)|)_t$ and $(S_t)_t$ such that

$$\left|\widehat{\mathcal{C}}_t(v_0)\right| \leq \widehat{S}_t, \ t \leq \min\left\{s\colon \left|\widehat{\mathcal{C}}_s(v_0)\right| = 0\right\},$$

which means τ is stochastically larger than T. Therefore, by Lemma 7, we have that

$$\begin{aligned}\mathbf{P}\left(|\mathcal{C}(v_0)| \geq \eta\right) &= \mathbf{P}\left(T \geq \eta\right) \leq \mathbf{P}(\tau \geq \eta) = \sum_{\ell \geq \eta} \mathbf{P}(\tau = \ell) \\ &= \sum_{\ell \geq \eta} O\left(\ell^{-3/2}\left(1 - \frac{\epsilon^2(n)}{3} + o\left(\epsilon^2(n)\right)\right)^{\ell}\right).\end{aligned} \tag{16}$$

For a positive constant $C_2 > 3$, set $\eta = C_2 \epsilon^{-2}(n) \log\left(n\epsilon^3(n)\right)$. Then, by the inequality $1 - x \leq e^{-x}$, $x \geq 0$, we obtain that

$$\begin{aligned}\mathbf{P}(\tau \geq \eta) &= O\left(\eta^{-3/2}\right) \sum_{\ell \geq \eta} \left(1 - \frac{\epsilon^2(n)}{3} + o\left(\epsilon^2(n)\right)\right)^{\ell} \\ &= O\left(\epsilon^{-2}(n)\eta^{-3/2}\right) \left(1 - \frac{\epsilon^2(n)}{3} + o\left(\epsilon^2(n)\right)\right)^{\eta} \\ &= O\left(\epsilon^{-2}(n)\eta^{-3/2}\right) \exp\left\{-\frac{\eta\epsilon^2(n)}{3} + o\left(\eta\epsilon^2(n)\right)\right\} \\ &= O\left(\epsilon(n)\left(\log n\epsilon^3(n)\right)^{-3/2} \left(n\epsilon^3(n)\right)^{-C_2/3+o(1)}\right).\end{aligned}$$

Denote $Z_{\geq k} = \sum_{v \in V} 1_{\{|\mathcal{C}(v)| \geq k\}}$. Then

$$\begin{aligned}\mathbf{P}(|\mathcal{C}_1| \geq \eta) &= \mathbf{P}(Z_{\geq \eta} \geq \eta) \leq \frac{\mathbf{E}\left(Z_{\geq \eta}\right)}{\eta} = \frac{\sum_{v \in V} \mathbf{E}\left(1_{\{|\mathcal{C}(v)| \geq \eta\}}\right)}{\eta} \\ &= \frac{n\mathbf{P}(|\mathcal{C}(v_0)| \geq \eta)}{\eta} \leq \frac{n\mathbf{P}(\tau \geq \eta)}{\eta} \\ &= O\left(\left(n\epsilon^3(n)\right)^{-(C_2/3-1-o(1))} \left(\log n\epsilon^3(n)\right)^{-5/2}\right) = o(1).\end{aligned}$$

■

3.2. Inside the critical window

3.2.1. The case of $\boldsymbol{\lambda < 0}$

In this subsection, when proving the upper bound we make use of the coupling method and the martingale arguments of [10, 11] to handle the difficulty from dependencies between the edges. For this we need the following lemma.

Lemma 8. *Let $m = n^{\alpha}\left(\alpha > \frac{5}{3}\right)$, p be such that $nmp^2 = 1 + \epsilon(n)$ where $\epsilon(n) = \lambda n^{-\frac{1}{3}}(\lambda < 0)$ and $\{\xi_t\}_{t \geq 1}$ be i.i.d. random variables distributed as $F(n, m, p)$, where $F(n, m, p)$ is the vertex degree distribution of $G(n, m, p)$. Let $W_t = 1 + \sum_{i=1}^{t}(\xi_i - 1)$ for $t \geq 0$. Define*

$$\gamma = \min\left\{t > 1 \colon W_t \geq n^{\frac{1}{3}} \text{ or } W_t = 0\right\}.$$

Then

$$\mathbf{P}\left(W_{\gamma} \geq n^{\frac{1}{3}}\right) = O\left(n^{-\frac{1}{3}}\right) \text{ and } \mathbf{E}(\gamma) = O\left(n^{\frac{1}{3}}\right).$$

Proof. When n is large enough,

$$\frac{1 + \epsilon(n)}{n} - \frac{(1 + \epsilon(n))^2}{2n^2} < 1 - \left(1 - p^2\right)^m < \frac{1 + \epsilon(n)}{n} - \frac{(1 + \epsilon(n))^2}{3n^2}.$$

Similarly to (9)–(10), we can deduce that

$$1+\epsilon(n)-2n^{-1} \le \mathbf{E}\left(\xi_1\right)=(n-1)\left(1-\left(1-p^2\right)^m\right) \le 1+\epsilon(n)-n^{-1}, \tag{17}$$

$$\mathbf{E}\left[\left(\xi_1-1\right)^2\right]=\mathbf{E}\left[\xi_1(\xi_1-1)\right]-\mathbf{E}\left[\xi_1-1\right]=1+\epsilon(n)+o\left(\epsilon(n)\right). \tag{18}$$

Set $a := 36(|\lambda|+1)n^{-\frac{1}{3}}$. Then, by (5) and the inequality that for $x=o(1)$,

$$1-nx+\frac{n^2x^2}{3} \le (1-x)^n \le 1-nx+\frac{n^2x^2}{2},$$

when n is large enough we can determine that

$$\begin{aligned}
&\mathbf{E}\left[e^{-a\xi_t}\right] \\
&=\sum_{j=0}^{n-1}\binom{n-1}{j}\left(e^{-a}\right)^j\left(1-e^{-a}\right)^{n-1-j}\left[1-p\left(1-(1-p)^{n-1-j}\right)\right]^m \\
&\ge \sum_{j=0}^{n-1}\binom{n-1}{j}\left(e^{-a}\right)^j\left(1-e^{-a}\right)^{n-1-j} \\
&-mp\sum_{j=0}^{n-1}\binom{n-1}{j}\left(e^{-a}\right)^j\left(1-e^{-a}\right)^{n-1-j}\left[1-(1-p)^{n-1-j}\right] \\
&+\frac{m^2p^2}{3}\sum_{j=0}^{n-1}\binom{n-1}{j}\left(e^{-a}\right)^j\left(1-e^{-a}\right)^{n-1-j}\left[1-(1-p)^{n-1-j}\right]^2 \\
&=1-mp+mp\left(1-p+pe^{-a}\right)^{n-1} \\
&+\frac{m^2p^2}{3}\left\{1-2\left[1-p+pe^{-a}\right]^{n-1}+\left[1-p(2-p)\left(1-e^{-a}\right)\right]^{n-1}\right\} \\
&\ge 1-mp+mp\left[1-(n-1)p\left(1-e^{-a}\right)+\frac{(n-1)^2p^2\left(1-e^{-a}\right)^2}{3}\right] \\
&+\frac{m^2p^2}{3}\left\{1-2\left[1-(n-1)p\left(1-e^{-a}\right)+\frac{(n-1)^2p^2}{2}\left(1-e^{-a}\right)^2\right]\right\} \\
&+\frac{m^2p^2}{3}\left\{1-(n-1)p(2-p)\left(1-e^{-a}\right)+\frac{(n-1)^2p^2(2-p)^2\left(1-e^{-a}\right)^2}{3}\right\} \\
&\ge e^{-a}-\epsilon(n)\left(1-e^{-a}\right)+\frac{n^2m^2p^4\left(1-e^{-a}\right)^2}{18} \\
&=e^{-a}-\epsilon(n)\left(1-e^{-a}\right)+\frac{(1+\epsilon(n))^2\left(1-e^{-a}\right)^2}{18} \\
&\ge e^{-a}-\epsilon(n)\left(1-e^{-a}\right)+\frac{\left(1-e^{-a}\right)^2}{18} \ge e^{-a}.
\end{aligned}$$

So $\mathbf{E}\left(e^{-a(\xi_t-1)}\right) \geq 1$. Therefore, for any $t \leq \gamma$,

$$\begin{aligned}\mathbf{E}\left(e^{-aW_t} \middle| W_1,\ldots,W_{t-1}\right) &= \mathbf{E}\left(e^{-aW_{t-1}-a(\xi_t-1)} \middle| W_1,\ldots,W_{t-1}\right)\\ &= e^{-aW_{t-1}}\mathbf{E}\left(e^{-a(\xi_t-1)}\right) \geq e^{-aW_{t-1}}.\end{aligned}$$

That is, $\left\{e^{-aW_t}\right\}_{t\geq 0}$ is a submartingale for a large enough n. By the optional stopping theorem (see [2, Theorem 5.7.4]), we obtain that

$$\begin{aligned}&e^{-a} \leq \mathbf{E}\left(e^{-aW_\gamma}\right)\\ &= \mathbf{E}\left(e^{-aW_\gamma} \middle| W_\gamma \geq n^{\frac{1}{3}}\right)\mathbf{P}\left(W_\gamma \geq n^{\frac{1}{3}}\right) + \mathbf{E}\left(e^{-aW_\gamma} \middle| W_\gamma < n^{\frac{1}{3}}\right)\mathbf{P}\left(W_\gamma < n^{\frac{1}{3}}\right)\\ &\leq e^{-an^{\frac{1}{3}}}\mathbf{P}\left(W_\gamma \geq n^{\frac{1}{3}}\right) + 1 - \mathbf{P}\left(W_\gamma \geq n^{\frac{1}{3}}\right),\end{aligned}$$

which means that

$$\mathbf{P}\left(W_\gamma \geq n^{\frac{1}{3}}\right) \leq \frac{1-e^{-a}}{1-e^{-an^{\frac{1}{3}}}} = O\left(n^{-\frac{1}{3}}\right). \tag{19}$$

Now set

$$\gamma_1 = \min\{t : W_t = 0 \text{ and } t \leq n\} \quad \text{and} \quad \gamma_2 = \min\left\{t : W_t \geq n^{1/3} \text{ and } t \leq n\right\}.$$

So $\gamma = \min\{\gamma_1, \gamma_2\}$,

$$\mathbf{E}\gamma \leq \min\{\mathbf{E}(\gamma_1),\ \mathbf{E}(\gamma_2)\}.$$

It is easy to see that $W_t - t\mathbf{E}(\xi_1 - 1)$ is a martingale with respect to the filtration generated by $\{\xi_i\}_{i\geq 1}$. Hence, when $\lambda < 0$,

$$\mathbf{E}(\gamma) \leq \mathbf{E}(\gamma_1) = \frac{1}{-\mathbf{E}(\xi_1 - 1)} = O\left(n^{\frac{1}{3}}\right). \tag{20}$$

■

Now we come to the position of proving the upper bound in the critical case by the above lemma.

Proof of the upper bound. Now fix a vertex v. To analyze the component of v in $G(n,m,p)$, due to the same reason as the subcritical case, we can couple the sequence $\{Z_t\}_{t\geq 1}$ to a sequence of i.i.d. random variables $\{\xi_t\}_{t\geq 1}$ with distribution $F(n,m,p)$ so that $\sum_{i=1}^t \xi_i \geq \sum_{i=1}^t Z_i$ for $t \leq \min\{\gamma, n\}$. This means that $W_t \geq Y_t$ for all t, where W_t is defined in Lemma 8 and Y_t is defined in the exploration process. Define γ as in Lemma 8. Let $\kappa = \gamma \wedge n^{\frac{2}{3}}\omega(n)$, where $\omega(n)$ tends to infinity as $n \to \infty$ and is of the order less than $\log n$. By Lemma 8, we have

$$\begin{aligned}\mathbf{P}\left(W_\kappa > 0\right) &= \mathbf{P}\left(W_\gamma > 0 \middle| \gamma < n^{\frac{2}{3}}\omega(n)\right)\mathbf{P}\left(\gamma < n^{\frac{2}{3}}\omega(n)\right) \\ &\quad + \mathbf{P}\left(W_{n^{\frac{2}{3}}\omega(n)} > 0 \middle| \gamma \geq n^{\frac{2}{3}}\omega(n)\right)\mathbf{P}\left(\gamma \geq n^{\frac{2}{3}}\omega(n)\right) \\ &\leq \mathbf{P}\left(W_\gamma \geq n^{\frac{1}{3}}\right) + \mathbf{P}\left(\gamma \geq n^{\frac{2}{3}}\omega(n)\right) \leq \mathbf{P}\left(W_\gamma \geq n^{\frac{1}{3}}\right) + \frac{\mathbf{E}\left(\gamma\right)}{n^{\frac{2}{3}}\omega(n)} \\ &\leq 2n^{-\frac{1}{3}} + O\left(n^{-\frac{1}{3}}\omega^{-1}(n)\right) = O\left(n^{-\frac{1}{3}}\right).\end{aligned}$$

Note that when $|\mathcal{C}(v)| \geq n^{\frac{2}{3}}\omega(n)$, then $W_\kappa > 0$. So

$$\mathbf{P}\left(|\mathcal{C}(v)| \geq n^{\frac{2}{3}}\omega(n)\right) \leq \mathbf{P}\left(W_\kappa > 0\right) = O\left(n^{-\frac{1}{3}}\right).$$

Denote $Z_{\geq k} = \sum_{v \in V} 1_{\{|\mathcal{C}(v)| \geq k\}}$. Then we have

$$\begin{aligned}&\mathbf{P}\left(|\mathcal{C}_1| \geq n^{\frac{2}{3}}\omega(n)\right) == \mathbf{P}\left(Z_{\geq n^{\frac{2}{3}}\omega(n)} \geq n^{\frac{2}{3}}\omega(n)\right) \\ &\leq \frac{\mathbf{E}\left(Z_{\geq n^{\frac{2}{3}}\omega(n)}\right)}{n^{\frac{2}{3}}\omega(n)} = \frac{\sum_{v \in V}\mathbf{E}\left(1_{\left\{|\mathcal{C}(v)| \geq n^{\frac{2}{3}}\omega(n)\right\}}\right)}{n^{\frac{2}{3}}\omega(n)} \leq \frac{n\mathbf{P}\left(|\mathcal{C}(v)| \geq n^{\frac{2}{3}}\omega(n)\right)}{n^{\frac{2}{3}}\omega(n)} \to 0.\end{aligned}$$

■

3.2.2. The case of $\boldsymbol{\lambda > 0}$

When $\lambda > 0$, we fail to show that $\mathbf{E}(\gamma)$ is of the order at most $n^{\frac{1}{3}}$ likely to Lemma 8. We appeal to the optimal stopping theorem for the upper bound to overcome the difficulty from the dependencies between edges.

Proof. Recall $\{\xi_i\}_{i=1}^{\infty}$ is a sequence of the i.i.d. random variables distributed as $F(n, m, p)$, and

$$S_t = 1 + \sum_{i=1}^{t}(\xi_i - 1) \text{ and } \tau = \min\{t\colon S_t = 0,\ t \leq n\}.$$

For any $\theta \in \mathbb{R}$, define

$$\phi(\theta) = \mathbf{E}\left[e^{\theta(\xi_i - 1)}\right] = e^{-\theta}\mathbf{E}\left[e^{\theta\xi_i}\right] \text{ and } \psi(\theta) = \log\phi(\theta).$$

Let $X_t := X_t(\theta) = \exp\left(-\theta S_t - t\psi(-\theta)\right)$. Then it is easy to check that X_t is a martingale with $X_0 = e^{-\theta}$. By the optimal stopping theorem, we have $\mathbf{E}[X_\tau] =$

$\mathbf{E}[X_0] = e^{-\theta}$. To give an upper bound for the stopping time τ, we need to show that $\psi(-\epsilon(n)) < 0$. To this end, by (5), we obtain that

$$
\begin{aligned}
&\mathbf{E}\left[e^{-\epsilon(n)\xi_t}\right] \\
&= \sum_{j=0}^{n-1} \binom{n-1}{j} \left(e^{-\epsilon(n)}\right)^j \left(1-e^{-\epsilon(n)}\right)^{n-1-j} \left[1-p\left(1-(1-p)^{n-1-j}\right)\right]^m \\
&\le \sum_{j=0}^{n-1} \binom{n-1}{j} \left(e^{-\epsilon(n)}\right)^j \left(1-e^{-\epsilon(n)}\right)^{n-1-j} \\
&- mp \sum_{j=0}^{n-1} \binom{n-1}{j} \left(e^{-\epsilon(n)}\right)^j \left(1-e^{-\epsilon(n)}\right)^{n-1-j} \left[1-(1-p)^{n-1-j}\right] \\
&+ \frac{m^2p^2}{2} \sum_{j=0}^{n-1} \binom{n-1}{j} \left(e^{-\epsilon(n)}\right)^j \left(1-e^{-\epsilon(n)}\right)^{n-1-j} \left[1-(1-p)^{n-1-j}\right]^2 \\
&= 1 - mp + mp\left(1-p+pe^{-\epsilon(n)}\right)^{n-1} \\
&+ \frac{m^2p^2}{2}\left\{1 - 2\left[1-p+pe^{-\epsilon(n)}\right]^{n-1} + \left[1-p(2-p)\left(1-e^{-\epsilon(n)}\right)\right]^{n-1}\right\} \\
&\le\ 1 - mp + mp\left[1-(n-1)p\left(1-e^{-\epsilon(n)}\right) + \frac{(n-1)^2p^2\left(1-e^{-\epsilon(n)}\right)^2}{2}\right] \\
&+ \frac{m^2p^2}{2}\left\{1-2\left[1-np\left(1-e^{-\epsilon(n)}\right)\right.\right. \\
&+ \left.\left.\frac{n^2p^2\left(1-e^{-\epsilon(n)}\right)^2}{2} - \frac{n^3p^3\left(1-e^{-\epsilon(n)}\right)^3}{6}\right]\right\} \\
&+ \frac{m^2p^2}{2}\left\{1-(n-1)p(2-p)\left(1-e^{-\epsilon(n)}\right)\right. \\
&+ \left.\frac{(n-1)^2p^2(2-p)^2\left(1-e^{-\epsilon(n)}\right)^2}{2}\right\} \\
&\le\ e^{-\epsilon(n)} - \epsilon(n)\left(1-e^{-\epsilon(n)}\right) + \frac{(1+\epsilon(n))^2\left(1-e^{-\epsilon(n)}\right)^2}{2} \\
&+ \frac{\left(1-e^{-\epsilon(n)}\right)^2}{n^{(\alpha-1)/2}} + \frac{\left(1-e^{-\epsilon(n)}\right)^3}{n^{(\alpha-1)/2}} + \frac{1-e^{-\epsilon(n)}}{n}(1+o(1)),
\end{aligned}
$$

which means that

$$
\begin{aligned}
\phi(-\epsilon(n)) \ &\le\ 1 - \epsilon(n)\left(e^{\epsilon(n)}-1\right) + \frac{(1+\epsilon(n))^2\left(e^{\epsilon(n)}-1\right)\left(1-e^{-\epsilon(n)}\right)}{2} \\
&+ \frac{2\left(e^{\epsilon(n)}-1\right)\left(1-e^{-\epsilon(n)}\right)}{n^{(\alpha-1)/2}} + \frac{e^{\epsilon(n)}-1}{n} = 1 - \frac{\epsilon^2(n)}{2} + o\left(\epsilon^2(n)\right),
\end{aligned}
$$

and $\psi(-\epsilon(n)) < 0$ for a large enough n. So for any positive constant $C_3 > \frac{2}{3\lambda^2}$, we have that

$$\begin{aligned}\mathbf{P}\left(\tau \geq C_3 n^{\frac{2}{3}} \log n\right) &\leq \mathbf{P}\left(e^{-\psi(-\epsilon(n))\tau} \geq e^{-C_3\psi(-\epsilon(n))n^{\frac{2}{3}}\log n}\right)\\ &\leq \frac{\mathbf{E}[X_\tau]}{e^{-C_3\psi(-\epsilon(n))n^{\frac{2}{3}}\log n}} = e^{-\epsilon(n)+C_3\psi(-\epsilon(n))n^{\frac{2}{3}}\log n}\\ &\leq e^{-\epsilon(n)}\left(1-\frac{\epsilon^2(n)}{2}+o\left(\epsilon^2(n)\right)\right)^{C_3 n^{\frac{2}{3}}\log n}\\ &\leq O\left(e^{-\frac{C_3\lambda^2}{2}\log n}\right) = O\left(n^{-\frac{C_3\lambda^2}{2}}\right).\end{aligned}$$

As in the proof of (16), we can determine that

$$\mathbf{P}\left(|\mathcal{C}(v_0)| \geq C_3 n^{\frac{2}{3}}\log n\right) \leq \mathbf{P}\left(\tau \geq C_3 n^{\frac{2}{3}}\log n\right) = O\left(n^{-\frac{C_3\lambda^2}{2}}\right).$$

Recall that $Z_{\geq k} = \sum_{v\in V} 1_{\{|\mathcal{C}(v)|\geq k\}}$. Therefore,

$$\begin{aligned}&\mathbf{P}(|\mathcal{C}_1| \geq C_3 n^{\frac{2}{3}}\log n) = \mathbf{P}\left(Z_{\geq C_3 n^{\frac{2}{3}}\log n} \geq C_3 n^{\frac{2}{3}}\log n\right)\\ &\leq \frac{\mathbf{E}\left(Z_{\geq C_3 n^{\frac{2}{3}}\log n}\right)}{C_3 n^{\frac{2}{3}}\log n} \leq \frac{\sum_{v\in V}\mathbf{E}\left(1_{\{|\mathcal{C}(v)|\geq C_3 n^{\frac{2}{3}}\log n\}}\right)}{C_3 n^{\frac{2}{3}}\log n}\\ &\leq \frac{n\mathbf{P}\left(|\mathcal{C}(v_0)| \geq C_3 n^{\frac{2}{3}}\log n\right)}{C_3 n^{\frac{2}{3}}\log n} = O\left(n^{-\left(C_3\lambda^2/2-1/3\right)}/\log n\right) = o(1).\end{aligned}$$

■

3.3. Above the critical window

In this subsection, the proof is almost the same as in Subsection 3.2.2. In fact, set

$$S_t = 1 + \sum_{i=1}^{t}(\xi_i - 1) \text{ and } \tau = \min\{t\colon S_t = 0,\ t \leq n\}.$$

For any $\theta \in \mathbb{R}$, define

$$\phi(\theta) = \mathbf{E}\left[e^{\theta(\xi_i-1)}\right] = e^{-\theta}\mathbf{E}\left[e^{\theta\xi_i}\right] \text{ and } \psi(\theta) = \log\phi(\theta).$$

Let $X_t := X_t(\theta) = \exp\left(-\theta S_t - t\psi(-\theta)\right)$. Then, it is easy to check that X_t is a martingale with $X_0 = e^{-\theta}$. By the optimal stopping theorem, $\mathbf{E}[X_\tau] = \mathbf{E}[X_0] = e^{-\theta}$. Similar to Subsection 3.2.2, we can show

$$\phi(-\epsilon(n)) \leq 1 - \frac{\epsilon^2(n)}{2} + o\left(\epsilon^2(n)\right),$$

and $\psi(-\epsilon(n)) < 0$ for a large enough n. So for any positive constant C_5, we have that

$$\begin{aligned}\mathbf{P}(\tau \geq C_5 n\epsilon(n)\log n) &\leq \mathbf{P}\left(e^{-\psi(-\epsilon(n))\tau} \geq e^{-C_5\psi(-\epsilon(n))n\epsilon(n)\log n}\right)\\ &\leq \frac{\mathbf{E}[X_\tau]}{e^{-C_5\psi(-\epsilon(n))n\epsilon(n)\log n}} = e^{-\epsilon(n)+C_5\psi(-\epsilon(n))n\epsilon(n)\log n}\\ &\leq e^{-\epsilon(n)}\left(1-\frac{\epsilon^2(n)}{2}+o\left(\epsilon^2(n)\right)\right)^{C_5 n\epsilon(n)\log n}\\ &\leq O\left(e^{-\frac{C_5}{2}n\epsilon^3(n)\log n}\right) = O\left(n^{-\frac{C_5}{2}n\epsilon^3(n)}\right).\end{aligned}$$

As in the proof of (16), we can determine that

$$\mathbf{P}\left(|\mathcal{C}(v_0)| \geq C_5 n\epsilon(n)\log n\right) \leq \mathbf{P}(\tau \geq C_5 n\epsilon(n)\log n) = O\left(n^{-\frac{C_5}{2}n\epsilon^3(n)}\right).$$

Recall that $Z_{\geq k} = \sum_{v\in V} 1_{\{|\mathcal{C}(v)|\geq k\}}$. Therefore,

$$\begin{aligned}&\mathbf{P}(|\mathcal{C}_1| \geq C_5 n\epsilon(n)\log n) = \mathbf{P}\left(Z_{\geq C_5 n\epsilon(n)\log n} \geq C_5 n\epsilon(n)\log n\right)\\ &\leq \frac{\mathbf{E}\left(Z_{\geq C_5 n\epsilon(n)\log n}\right)}{C_5 n\epsilon(n)\log n} \leq \frac{\sum_{v\in V}\mathbf{E}\left(1_{\{|\mathcal{C}(v)|\geq C_5 n\epsilon(n)\log n\}}\right)}{C_5 n\epsilon(n)\log n}\\ &\leq \frac{n\mathbf{P}\left(|\mathcal{C}(v_0)| \geq C_5 n\epsilon(n)\log n\right)}{C_5 n\epsilon(n)\log n} = o(1).\end{aligned}$$

4. Questions

Recall the results for the critical Erdős-Rényi random graph from [5, Theorem 5.23] and [9, Theorem 1.1]. The following questions in $G(n,m,p)$ will be interesting.

- **$\mathbf{Q}_1$**: What is the size of the largest component within the critical window of $G(n,m,p)$ when $m = n^\alpha$ and $1 < \alpha < \frac{5}{3}$? With high probability, it may be of order $n^{\frac{3-\alpha}{2}}$.
- **$\mathbf{Q}_2$**: For the case $m = n^\alpha$ and $\alpha > \frac{5}{3}$ $\left(\text{or } 1 < \alpha < \frac{5}{3}\right)$, it is interesting to remove the factor $\log n$ in the upper bound for the supercritical phase and critical case when $\lambda > 0$.
- **$\mathbf{Q}_3$**: What is the size of the largest component in the critical $G(n,m,p)$ when $m = n^\alpha$ and $\alpha = 1$? Also, what is the width of the scaling window around the critical probability in this case?
- **$\mathbf{Q}_4$**: To study the critical $G(n,m,p)$ when $m = n^\alpha$ and $\alpha = \frac{5}{3}$ is of interest and a challenge.

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Appendix: Proof of Theorem 2

A.1. Below the critical window

When $nmp^2 = 1 - \epsilon(n)$ and $1 < \alpha < \frac{5}{3}$, let $\omega(n) := \epsilon(n) n^{\frac{\alpha-1}{2}}$ which means $\omega(n) \to \infty$ as $n \to \infty$, and define $G(n, p')$, where

$$p' = mp^2 \left(1 - (n-2)p - \frac{mp^2}{2}\right) = \frac{1-\epsilon(n)}{n} + o\left(\frac{\epsilon(n)}{n}\right).$$

Note that $\frac{\omega(n)}{n^{\frac{4}{3}}} < \frac{\epsilon(n)}{n} < \frac{\omega(n)}{n}$. Then, by Theorems 4 and 5, we can determine that w.h.p.

$$|\mathcal{C}_1| \geq (2 + o(1))\epsilon^{-2}(n) \log\left\{n\epsilon^3(n)\right\}.$$

Proof of the upper bound. Let $\{\xi_t\}_{t=1}^{\infty}$ be i.i.d. random variables distributed as $F(n, m, p)$. Set

$$S_t = 1 + \sum_{i=1}^{t} (\xi_i - 1).$$

Take $\beta = \xi_1 - 1$. By (6)–(7) and the following inequality

$$\begin{aligned} &\frac{1-\epsilon(n)}{n} - \frac{(1-\epsilon(n))^2}{2n^2} + \frac{(1-\epsilon(n))^3}{7n^3} \\ &< 1 - \left(1-p^2\right)^m < \frac{1-\epsilon(n)}{n} - \frac{(1-\epsilon(n))^2}{2n^2} + \frac{(1-\epsilon(n))^3}{5n^3}, \end{aligned}$$

we have that

$$\begin{aligned} \mathbf{E}\beta \;&=\; \mathbf{E}(\xi_i - 1) = (n-1)\left(1 - \left(1-p^2\right)^m\right) - 1 \\ &=\; -\epsilon(n) - \frac{3(1-\epsilon(n))}{2n} + O\left(\frac{\epsilon(n)}{n}\right); \\ \mathbf{E}\left[\beta^2\right] \;&=\; \mathbf{E}\left[(\xi_i - 1)^2\right] = \mathbf{E}\left[\xi_i(\xi_i - 1)\right] - \mathbf{E}(\xi_i - 1) \\ &=\; (n-1)(n-2)\left\{1 - 2\left(1-p^2\right)^m + \left[1 - p^2(2-p)\right]^m\right\} \\ &-\; (n-1)\left[1 - \left(1-p^2\right)^m\right] \\ &=\; \epsilon(n) + O\left(n^{-1}\right) + (n-1)(n-2)\left[1 - 2\left(1 - mp^2 + \frac{m(m-1)p^4}{2}\right)\right. \\ &+\; \left. 1 - mp^2(2-p) + \frac{m(m-1)p^4(2-p)^2}{2} + O\left(m^3p^6\right)\right] \\ &=\; 1 - \epsilon(n) + \epsilon^2(n) + \frac{(1-\epsilon(n))^{3/2}}{n^{\frac{\alpha-1}{2}}} + o\left(\frac{1}{n^{\frac{\alpha-1}{2}}}\right) + O\left(\frac{1}{n}\right); \end{aligned}$$

$$\begin{aligned}\mathbf{E}\left[\beta^3\right] &= \mathbf{E}\left[(\xi_i-1)^3\right] = \mathbf{E}[\xi_i(\xi_i-1)(\xi_i-2)] + \mathbf{E}(\xi_i-1)\\ &= -\epsilon(n) - \frac{3(1-\epsilon(n))}{2n} + O\left(\frac{\epsilon(n)}{n}\right) + (n-1)(n-2)(n-3)\\ &\cdot \left\{1-3\left(1-p^2\right)^m + 3\left[1-p^2(2-p)\right]^m - \left[1-p^2\left(3-3p+p^2\right)\right]^m\right\}\\ &= (1-\epsilon(n))^3 + \frac{3\left(1-\epsilon(n)\right)^{5/2}}{n^{\frac{\alpha-1}{2}}} + o\left(\frac{1}{n^{\frac{\alpha-1}{2}}}\right) - \epsilon(n) + O\left(\frac{1}{n}\right).\end{aligned}$$

Similar to (12) we can prove that there is a θ_0 satisfying

$$\frac{\epsilon(n)}{2} < \theta_0 < \epsilon(n) + \frac{\epsilon(n)}{2\log n} \quad \text{and} \quad \mathbf{E}\left(\beta e^{\theta_0\beta}\right) = 0.$$

Hence, we can deduce that

$$\mathbf{E}\left[e^{\theta_0\beta}\right] \le 1 - \frac{\epsilon^2(n)}{3} + o\left(\epsilon^2(n)\right).$$

Due to the same reason as in the subcritical case $\alpha > \frac{5}{3}$, $\sum_{i=1}^k Z_i$ is dominated above by $\sum_{i=1}^k \xi_i$ for all $k \le T$, where $T = \min\{t\colon Y_t = 0\}$ in the exploration process. Let $\mathcal{C}_t(v_0)$ denote the total inactive (explored) vertices which are explored from time 0 to t by the exploration process on $G(n,m,p)$ starting from vertex v_0 which is chosen uniformly at random from the vertex set V. Let $\tau = \min\{t\colon S_t = 0 \text{ and } t \le n\}$. Notice that

$$|\mathcal{C}_t(v_0)| = 1 + \sum_{i=1}^t (Z_i - 1) \quad \text{and} \quad S_t = 1 + \sum_{i=1}^t (\xi_i - 1).$$

Hence, there is a coupling $\left(\left(\left|\widehat{\mathcal{C}}_t(v_0)\right|\right)_t, \left(\widehat{S}_t\right)_t\right)$ of the processes $(|\mathcal{C}_t(v_0)|)_t$ and $(S_t)_t$ such that

$$\left|\widehat{\mathcal{C}}_t(v_0)\right| \le \widehat{S}_t,\ t \le \min\left\{s\colon \left|\widehat{\mathcal{C}}_s(v_0)\right| = 0\right\},$$

which means that τ is stochastically larger than T. Therefore, by Lemma 7, we determine

$$\begin{aligned}\mathbf{P}\left(|\mathcal{C}(v_0)| \ge \eta\right) &= \mathbf{P}\left(T \ge \eta\right) \le \mathbf{P}(\tau \ge \eta) = \sum_{\ell \ge \eta} \mathbf{P}(\tau = \ell)\\ &= \sum_{\ell\ge\eta} O\left(\ell^{-3/2}\left[1 - \frac{\epsilon^2(n)}{3} + o\left(\epsilon^2(n)\right)\right]^\ell\right).\end{aligned}$$

For a positive constant $C_6 > 3$, set $\eta = C_6(1-\epsilon(n))\epsilon(n)^{-2}\log\left(n\epsilon^3(n)\right)$. Then, by the inequality that $1-x \le e^{-x},\ x \ge 0$, we obtain

$$\begin{aligned}
\mathbf{P}(\tau \ge \eta) &= O\left(\eta^{-3/2}\right)\sum_{\ell \ge \eta}\left(1-\frac{\epsilon^2(n)}{3}+o\left(\epsilon^2(n)\right)\right)^{\ell} \\
&= O\left(\epsilon^{-2}(n)\eta^{-3/2}\right)\left(1-\frac{\epsilon^2(n)}{3}+o\left(\epsilon^2(n)\right)\right)^{\eta} \\
&= O\left(\epsilon^{-2}(n)\eta^{-3/2}\right)\exp\left\{-\frac{\eta\epsilon^2(n)}{3}+o\left(\eta\epsilon^2(n)\right)\right\} \\
&= O\left(\epsilon(n)\left(\log\left(n\epsilon^3(n)\right)\right)^{-3/2}\left(n\epsilon^3(n)\right)^{-C_6/3+o(1)}\right).
\end{aligned}$$

Let $Z_{\ge k} = \sum_{v\in V} 1_{\{|\mathcal{C}(v)|\ge k\}}$. Then, by the fact that $n\epsilon^3(n) = n^{\frac{5-3\alpha}{2}}\omega^3(n) \to \infty$ as $n \to \infty$,

$$\begin{aligned}
\mathbf{P}(|\mathcal{C}_1| \ge \eta) &= \mathbf{P}(Z_{\ge\eta} \ge \eta) \le \frac{\mathbf{E}\left(Z_{\ge\eta}\right)}{\eta} = \frac{\sum_{v\in V}\mathbf{E}\left(1_{\{|\mathcal{C}(v)|\ge\eta\}}\right)}{\eta} \\
&= \frac{n\mathbf{P}(|\mathcal{C}(v_0)| \ge \eta)}{\eta} \le \frac{n\mathbf{P}(\tau \ge \eta)}{\eta} \\
&= O\left(\left(n\epsilon^3(n)\right)^{-(C_6/3-1-o(1))}\left(\log\left(n\epsilon^3(n)\right)\right)^{-5/2}\right) = o(1).
\end{aligned}$$

■

A.2. Inside the critical window

When $\lambda < 0$, define $G(n, p^*)$, where

$$p^* = mp^2\left(1-(n-2)p-\frac{mp^2}{2}\right) = \frac{1-(\lambda+1)\,n^{-\frac{\alpha-1}{2}}}{n} + o\left(\frac{1}{n^{\frac{\alpha+1}{2}}}\right).$$

Notice that in this case $(|\lambda|+1)\,n^{-\frac{\alpha-1}{2}} > \widetilde{\omega}(n)n^{-\frac{1}{3}}$, where $\widetilde{\omega}(n) \to \infty$ in a sufficiently slow rate as $n \to \infty$. By Theorems 4 and 5, we can determine that w.h.p.

$$\begin{aligned}
|\mathcal{C}_1| &\ge (2+o(1))\left((|\lambda|+1)n^{-\frac{\alpha-1}{2}}\right)^{-2}\log\left\{n\left((|\lambda|+1)n^{-\frac{\alpha-1}{2}}\right)^3\right\} \\
&= 5-3\alpha+o(1)\,(|\lambda|+1)^{-2}\,n^{\alpha-1}\log n.
\end{aligned}$$

Alternatively, when $\lambda > 0$, define $G(n, p')$, where

$$p' = mp^2\left(1-(n-2)p-\frac{mp^2}{2}\right) = \frac{1+(\lambda+1)\,n^{-\frac{\alpha-1}{2}}}{n} + o\left(\frac{1}{n^{\frac{\alpha+1}{2}}}\right).$$

Notice $(\lambda+1)\, n^{-\frac{\alpha-1}{2}} > \widetilde{\omega}(n) n^{-\frac{1}{3}}$, where $\widetilde{\omega}(n) \to \infty$ in a sufficiently slow rate as $n \to \infty$. By Theorems 4 and 5, we can determine that w.h.p.

$$|\mathcal{C}_1| \;\; \geq \;\; (2(\lambda+1)+o(1))\, n^{\frac{3-\alpha}{2}}.$$

For the upper bound, we follow the line of the proof in Subsection 3.2.2. Similar to Lemma 8, we have the following lemma for the case $\lambda < 0$.

Lemma A. *Let $m = n^{\alpha}\left(1<\alpha<\frac{5}{3}\right)$ and p be such that $nmp^2 = 1+\epsilon(n)$ with $\epsilon(n) = \lambda n^{-\frac{\alpha-1}{2}}$ $(\lambda < 0)$ and $\{\xi_t\}_{t\geq 1}$ be i.i.d. random variables distributed as $F\,(n,m,p)$. Let $W_t = 1+\sum_{i=1}^{t}(\xi_i - 1)$ for $t \geq 0$. Define*

$$\gamma = \min\left\{t > 1 \colon W_t \geq n^{\frac{3-\alpha}{4}} \text{ or } W_t = 0\right\}.$$

Then, we have

$$\mathbf{E}(\gamma) = O\left(n^{\frac{\alpha-1}{2}}\right) \text{ and } \mathbf{P}\left(W_{\gamma} \geq n^{\frac{3-\alpha}{4}}\right) = O\left(n^{-\frac{\alpha-1}{2}}\right).$$

Now fix a vertex v. To analyze the component of v in $G\,(n,m,p)$, due to the same reason as the subcritical case, we can couple the sequence $\{Z_t\}_{t\geq 1}$ to a sequence of the i.i.d. random variable $\{\xi_t\}_{t\geq 1}$ with distribution $F\,(n,m,p)$ so that $\sum_{i=1}^{t}\xi_i \geq \sum_{i=1}^{t} Z_i$ for $t \leq \min\{\gamma, n\}$. This means that $W_t \geq Y_t$ for all t, where Y_t is defined in the exploration process. Define γ the same as in Lemma A. Let $\kappa = \gamma \wedge n^{\frac{3-\alpha}{2}}\omega(n)$, where $\omega(n)$ tends to infinity as $n \to \infty$ and is of order less than $\log n$. By Lemma A, we have

$$\begin{aligned}
\mathbf{P}\left(W_{\kappa} > 0\right) \;&=\; \mathbf{P}\left(W_{\gamma} > 0 \,\middle|\, \gamma < n^{\frac{3-\alpha}{2}}\omega(n)\right)\mathbf{P}\left(\gamma < n^{\frac{3-\alpha}{2}}\omega(n)\right)\\
&\quad + \mathbf{P}\left(W_{\gamma} > 0 \,\middle|\, \gamma \geq n^{\frac{3-\alpha}{2}}\omega(n)\right)\mathbf{P}\left(\gamma \geq n^{\frac{3-\alpha}{2}}\omega(n)\right)\\
&\leq \mathbf{P}\left(W_{\gamma} \geq n^{\frac{3-\alpha}{4}}\right) + \mathbf{P}\left(\gamma \geq n^{\frac{3-\alpha}{2}}\omega(n)\right)\\
&\leq \mathbf{P}\left(W_{\gamma} \geq n^{\frac{3-\alpha}{4}}\right) + \frac{\mathbf{E}\,(\gamma)}{n^{\frac{3-\alpha}{2}}\omega(n)}\\
&= O\left(n^{-\frac{\alpha-1}{2}}\right) + O\left(n^{\alpha-2}/\omega(n)\right).
\end{aligned}$$

Note that when $|\mathcal{C}(v)| \geq n^{\frac{3-\alpha}{2}}\omega(n)$, then $W_{\kappa} > 0$. Thus

$$\mathbf{P}\left(|\mathcal{C}(v)| \geq n^{\frac{3-\alpha}{2}}\omega(n)\right) \leq \mathbf{P}\left(W_{\kappa} > 0\right) = O\left(n^{-\frac{\alpha-1}{2}}\right) + O\left(n^{\alpha-2}/\omega(n)\right).$$

Denote $Z_{\geq k} = \sum_{v \in V} 1_{\{|\mathcal{C}(v)| \geq k\}}$. Then we have

$$\begin{aligned}\mathbf{P}\left(|\mathcal{C}_1| \geq n^{\frac{3-\alpha}{2}}\omega(n)\right) &= \mathbf{P}\left(Z_{\geq n^{\frac{1+\alpha}{4}}\omega(n)} \geq n^{\frac{3-\alpha}{2}}\omega(n)\right) \\ &\leq \frac{\mathbf{E}\left(Z_{\geq n^{\frac{3-\alpha}{2}}\omega(n)}\right)}{n^{\frac{3-\alpha}{2}}\omega(n)} \leq \frac{\sum_{v\in V}\mathbf{E}\left(1_{\left\{|\mathcal{C}(v)|\geq n^{\frac{3-\alpha}{2}}\omega(n)\right\}}\right)}{n^{\frac{3-\alpha}{2}}\omega(n)} \\ &= O\left(\frac{1}{\omega(n)}\right) + O\left(\frac{n^{\frac{3\alpha-5}{2}}}{\omega^2(n)}\right) \to 0.\end{aligned}$$

When $\lambda > 0$, the proof is almost the same as in the following proof of the supercritical phase. We omit this for simplicity.

A.3. Above the critical window

For the lower bound, we only notice that in this case

$$p' = mp^2\left(1 - (n-2)p - \frac{mp^2}{2}\right) = \frac{1+\epsilon(n)}{n} + o\left(\frac{\epsilon(n)}{n}\right).$$

By Theorems 4 and 5, we can determine that in $G(n,m,p)$ w.h.p.

$$|\mathcal{C}_1| \geq (2+o(1))n\epsilon(n).$$

For the upper bound, recall that $\{\xi_i\}_{i=1}^{\infty}$ is a sequence of the i.i.d. random variables distributed as $F(n,m,p)$, $S_t = 1 + \sum_{i=1}^{t}(\xi_i - 1)$ and $\tau = \min\{t\colon S_t = 0$ and $t \leq n\}$. For any $\theta \in \mathbb{R}$, let

$$\phi(\theta) = \mathbf{E}\left[e^{\theta(\xi_i - 1)}\right] = e^{-\theta}\mathbf{E}\left[e^{\theta\xi_i}\right],\ \psi(\theta) = \log\phi(\theta),\ X_t = \exp\left(-\theta S_t - t\psi(-\theta)\right).$$

Then it is easy to see that X_t is a martingale with $X_0 = e^{-\theta}$. By the optimal stopping theorem, we have

$$\mathbf{E}[X_\tau] = \mathbf{E}[X_0] = e^{-\theta}.$$

Similar to the proof in Subsection 3.2.2 we have

$$\begin{aligned}\phi(-\epsilon(n)) &\leq 1 - \epsilon(n)\left(e^{\epsilon(n)} - 1\right) + \frac{(1+\epsilon(n))^2\left(e^{\epsilon(n)}-1\right)\left(1-e^{-\epsilon(n)}\right)}{2} \\ &\quad + \frac{2\left(e^{\epsilon(n)}-1\right)\left(1-e^{-\epsilon(n)}\right)}{n^{(\alpha-1)/2}} + \frac{e^{\epsilon(n)}-1}{n} = 1 - \frac{\epsilon^2(n)}{2} + o\left(\epsilon^2(n)\right),\end{aligned}$$

and further $\psi(-\epsilon(n)) < 0$ for a large enough n. So for any positive constant C_{12}, we have that

$$\begin{aligned}\mathbf{P}(\tau \geq C_{12}n\epsilon(n)\log n) &\leq \mathbf{P}\left(e^{-\psi(-\epsilon(n))\tau} \geq e^{-C_{12}\psi(-\epsilon(n))n\epsilon(n)\log n}\right)\\ &\leq \frac{\mathbf{E}[X_\tau]}{e^{-C_{12}\psi(-\epsilon(n))n\epsilon(n)\log n}} = e^{-\epsilon(n)+C_{12}\psi(-\epsilon(n))n\epsilon(n)\log n}\\ &\leq e^{-\epsilon(n)}\left(1-\frac{\epsilon^2(n)}{2}+o\left(\epsilon^2(n)\right)\right)^{C_{12}n\epsilon(n)\log n}\\ &\leq O\left(e^{-\frac{C_{12}}{2}n\epsilon^3(n)\log n}\right) = O\left(n^{-\frac{C_{12}}{2}n\epsilon^3(n)}\right).\end{aligned}$$

As in the proof of (16), we can determine that

$$\mathbf{P}\left(|\mathcal{C}(v_0)| \geq C_{12}n\epsilon(n)\log n\right) \leq \mathbf{P}(\tau \geq C_{12}n\epsilon(n)\log n) = O\left(n^{-\frac{C_{12}}{2}n\epsilon^3(n)}\right).$$

Recall that $Z_{\geq k} = \sum_{v\in V} 1_{\{|\mathcal{C}(v)|\geq k\}}$. Therefore,

$$\begin{aligned}&\mathbf{P}\left(|\mathcal{C}_1| \geq C_{12}n\epsilon(n)\log n\right) = \mathbf{P}\left(Z_{\geq C_{12}n\epsilon(n)\log n} \geq C_{12}n\epsilon(n)\log n\right)\\ &\leq \frac{\mathbf{E}\left(Z_{\geq C_{12}n\epsilon(n)\log n}\right)}{C_{12}n\epsilon(n)\log n} \leq \frac{\sum_{v\in V}\mathbf{E}\left(1_{\{|\mathcal{C}(v)|\geq C_{12}n\epsilon(n)\log n\}}\right)}{C_{12}n\epsilon(n)\log n}\\ &\leq \frac{n\mathbf{P}\left(|\mathcal{C}(v_0)| \geq C_{12}n\epsilon(n)\log n\right)}{C_{12}n\epsilon(n)\log n} = o(1).\end{aligned}$$