

## THE CROSSING NUMBERS OF JOIN OF SOME GRAPHS WITH $n$ ISOLATED VERTICES

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### Abstract

There are only few results concerning crossing numbers of join of some graphs. In this paper, for some graphs on five vertices, we give the crossing numbers of its join with  $n$  isolated vertices.

**Keywords:** disconnected graph, crossing number, join product.

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### 1. INTRODUCTION

For graph theory terminology not defined here, we direct the reader to [2]. A drawing of a graph  $G = (V, E)$  is a mapping  $\phi$  that assigns to each vertex in  $V$  a distinct point in the plane and to each edge  $uv$  in  $E$  a continuous arc (i.e., a homeomorphic image of a closed interval) connecting  $\phi(u)$  and  $\phi(v)$ , not passing through the image of any other vertex. For simplicity, we impose the following conditions on a drawing: (a) no three edges have an interior point in common, (b) if two edges share an interior point  $p$ , then they cross at  $p$ , and (c) any two edges of a drawing have only a finite number of crossings (common interior points). The crossing number,  $cr(G)$ , of a graph  $G$  is the minimum number of edge crossings in any drawing of  $G$ . Let  $D$  be a drawing of the graph  $G$ , we denote the number of crossings in  $D$  by  $cr_D(G)$ .

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For a graph  $G$ , let  $A, B \subseteq E(G)$ , then, for a drawing  $D$  of  $G$ , let

$$cr_D(A, B) = \sum_{a \in A, b \in B} |D(a) \cap D(b)|.$$

Additionally, let  $cr_D(A, A) = cr_D(A)$ . Informally,  $cr_D(A, B)$  denotes the number of crossings between every pair of edges where one edge is in  $A$  and the other in  $B$ . For three mutually disjoint subsets  $A, B, C \subseteq E(G)$ , the following equations hold:

$$\begin{aligned} cr_D(A \cup B) &= cr_D(A) + cr_D(B) + cr_D(A, B), \\ cr_D(A, B \cup C) &= cr_D(A, B) + cr_D(A, C). \end{aligned}$$

For more about crossing number, we refer the reader to [3]. The investigation on the crossing number of graphs is a classical but very difficult problem. It is well known that there are only few results concerning crossing numbers of join of some graphs. The join product of two graphs  $G_1$  and  $G_2$ , denoted by  $G_1 + G_2$ , is obtained from vertex-disjoint copies of  $G_1$  and  $G_2$  by adding all edges between  $V(G_1)$  and  $V(G_2)$ . For  $|V(G_1)| = m$  and  $|V(G_2)| = n$ , the edge set of  $G_1 + G_2$  is the union of disjoint edge sets of the graphs  $G_1, G_2$ , and the complete bipartite graph  $K_{m,n}$ . Let  $nK_1$  denote the graph on  $n$  isolated vertices (i.e., the complement of the complete graph  $K_n$ ).

It has been long conjectured in [8] that the crossing number of the complete bipartite graph  $K_{m,n}$  equals  $\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ . This conjecture was verified by Kleitman for  $\min\{m, n\} \leq 6$  in [6]. Similarly, most results about  $G_1 + G_2$  deal with the case  $|V(G_1)| \leq 6$ . Moreover, usually  $G_1$  is connected and  $G_2$  is some special graph, such as the graph  $nK_1$ , the path  $P_n$  on  $n$  vertices or the cycle  $C_n$ . The  $k$ -spoke wheel, denoted by  $W_k$ , has vertices  $v_0, v_1, \dots, v_k$ , where  $v_1, v_2, \dots, v_k$  form a cycle, and  $v_0$  is adjacent to all of  $v_1, v_2, \dots, v_k$ .

Using Kleitman's result [6], the crossing numbers for join of two paths, join of two cycles, or for join of a path and a cycle were studied in [7]. Moreover, the exact values for crossing numbers of  $G_1 + nK_1$  for all graphs  $G_1$  of order at most four are given. In 1986, Asano started to study crossing numbers of multipartite complete graphs. In [1], he established the crossing numbers of the tripartite graphs  $K_{1,3,n}$  and  $K_{2,3,n}$  (namely  $K_{1,3} + nK_1$  and  $K_{2,3} + nK_1$ ). For the graph  $K_{1,4,n}$  (namely  $K_{1,4} + nK_1$ ), the crossing number was given independently in [4] and [5].

As it is difficult to determine the crossing number of join of the disconnected graph with  $nK_1$ , there are only few results concerning that. In this paper, we determine the crossing number for the join of  $nK_1$  with a disconnected graph  $G$  on five vertices, as shown in Figure 1. The approach is seemingly new. As the disconnected graph may be a subgraph of many connected graphs, we can gain

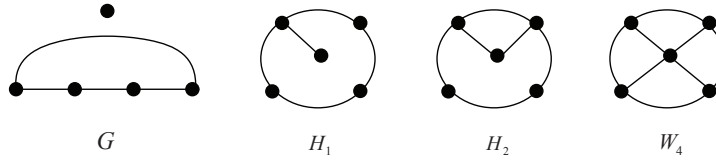


Figure 1. Some graphs on five vertices.

crossing numbers of join of some graphs  $H_1$  and  $H_2$  (see Figure 1) on five vertices with  $nK_1$  directly. The crossing number for the graph  $G + nK_1$  enables us, in Section 3, to give the crossing number of  $W_4 + nK_1$  cleverly and simply.

In the paper, some proofs are based on Kleitman’s result on crossing numbers of complete bipartite graphs. More precisely, he [6] proved that

$$cr(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \quad m \leq 6.$$

For convenience, the number  $\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$  is often denoted by  $Z(m, n)$  in our paper. In the proofs of the paper, we will often use the term “region” also in nonplanar drawings. In this case, crossings are considered to be vertices of the “map”.

## 2. THE CROSSING NUMBER OF $G + nK_1$

The graph  $G$  in Figure 1 is isomorphic to the graph  $C_4 \cup K_1$ . The graph  $G + nK_1$  consists of one copy of the graph  $G$  and  $n$  vertices  $t_1, t_2, \dots, t_n$ , where every vertex  $t_i, i = 1, 2, \dots, n$ , is adjacent to every vertex of  $G$ . For  $i = 1, 2, \dots, n$ , let  $T_i$  denote the subgraph induced by five edges incident with the vertex  $t_i$  and let  $F_i = G \cup T_i$ . For the simpler labelling, let  $G_n$  denote the graph  $G + nK_1$  in this paper. In Figure 2, we have  $G + nK_1 = G_n = G \cup (\bigcup_{i=1}^n T_i)$ , and we also have  $\bigcup_{i=1}^n T_i = K_{5,n}$ .

**Lemma 1.**  $cr(G + nK_1) \leq Z(5, n) + \left\lfloor \frac{n}{2} \right\rfloor$  for  $n \geq 1$ .

**Proof.** We will display a drawing  $\phi$  of  $G_n$  in the plane such that  $cr_\phi(G_n) = Z(5, n) + \left\lfloor \frac{n}{2} \right\rfloor$ . The desired drawing  $\phi$  is constructed as follows (see Figure 2).

- (i) Set all vertices of  $G$  on  $y$ -axis,  $\left\lceil \frac{n}{2} \right\rceil$  isolated vertices on the negative  $x$ -axis and  $\left\lfloor \frac{n}{2} \right\rfloor$  isolated vertices on the positive  $x$ -axis.
- (ii) The image of each edge for  $G$  is a thick line segment.

Then it is not difficult to see that  $cr_\phi(G_n) = Z(5, n) + \left\lfloor \frac{n}{2} \right\rfloor$ , and so  $cr(G + nK_1) \leq Z(5, n) + \left\lfloor \frac{n}{2} \right\rfloor$ . ■

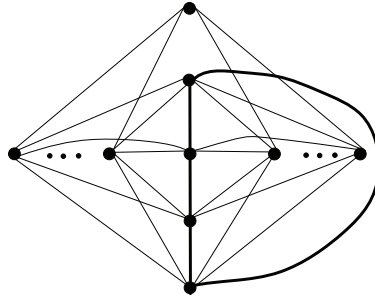


Figure 2. A drawing  $\phi$  of  $G_n$ .

**Lemma 2.**  $cr(G + K_1) = 0, cr(G + 2K_1) = 1$ .

*Proof.* The graph  $G + K_1$  is planar, so  $cr(G + K_1) = 0$ . As the graph  $G + 2K_1$  contains a subgraph which is homeomorphic to  $K_{3,3}$ , then  $cr(G + 2K_1) \geq 1$ . On the other hand, by Lemma 1, we have  $cr(G + 2K_1) \leq 1$ . This completes the proof. ■

**Lemma 3.** Let  $D$  be an optimal drawing of  $G_n$  and let  $C_4$  be the 4-cycle of  $G$ . Then we have  $cr_D(C_4) = 0$ .

*Proof.* We assume that there exists an optimal drawing  $D$  of  $G_n$  such that  $cr_D(C_4) \neq 0$ . There exist two crossed edges  $e, f \in E(C_4)$ . We assume that  $e = y_i y_j, f = y_k y_l$ , where  $i, j, k, l$  are distinct and the 4-cycle of  $G$  is  $y_i e y_j y_l f y_k y_i$ . For convenience, we denote the crossing between  $e$  and  $f$  by  $v$ . In the following, we shall produce a new good drawing  $D^*$  of  $G_n$  as shown in Figure 3.

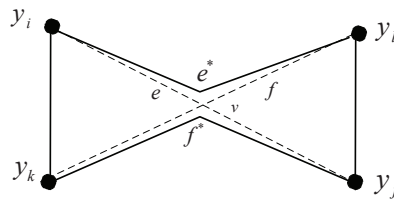


Figure 3. A 4-cycle of  $G$ .

First, we connect  $y_i$  to  $y_l$  sufficiently close to the section between  $y_i$  and  $v$  of  $e$  and the section between  $y_l$  and  $v$  of  $f$ , then we get a new edge  $e^* = y_i y_l$ . Analogously, we can get another new edge  $f^* = y_j y_k$ . Second, we delete two original edges  $e$  and  $f$ . In this way, we produce a new 4-cycle  $(y_i e^* y_l y_j f^* y_k y_i)$  and a new good drawing  $D^*$  of  $G_n$ , such that the crossing  $v$  in  $D$  is deleted in  $D^*$ , the other crossings are unchanged from  $D$  to  $D^*$ , and there is no new crossing occurring in  $D^*$ . Now  $cr_{D^*}(G_n) = cr_D(G_n) - 1$  contradicts to  $D$  being an optimal drawing. ■

**Theorem 4.**  $cr(G + nK_1) = Z(5, n) + \lfloor \frac{n}{2} \rfloor$  for  $n \geq 1$ .

*Proof.* Lemma 1 shows that  $cr(G + nK_1) \leq Z(5, n) + \lfloor \frac{n}{2} \rfloor$  and that the theorem is true if the equality holds. We prove the reverse inequality by induction on  $n$ . Lemma 2 implies that the result is true for the case  $n = 1, 2$ .

Suppose now that for  $3 \leq k \leq n - 1$

$$cr(G + kK_1) \geq Z(5, k) + \lfloor \frac{k}{2} \rfloor,$$

and assume there exists such an optimal drawing  $D$  of  $G_n$  that

$$(1) \quad cr_D(G + nK_1) < Z(5, n) + \lfloor \frac{n}{2} \rfloor.$$

Since

$$\begin{aligned} cr_D(G + nK_1) &= cr_D\left(G \cup \bigcup_{i=1}^n T_i\right) = cr_D\left(\bigcup_{i=1}^n T_i\right) + cr_D(G) + cr_D\left(G, \bigcup_{i=1}^n T_i\right) \\ &= cr_D(K_{5,n}) + cr_D(G) + cr_D\left(G, \bigcup_{i=1}^n T_i\right) \\ &\geq Z(5, n) + cr_D(G) + cr_D\left(G, \bigcup_{i=1}^n T_i\right), \end{aligned}$$

this implies that

$$(2) \quad cr_D(G) + cr_D\left(G, \bigcup_{i=1}^n T_i\right) < \lfloor \frac{n}{2} \rfloor.$$

The following claims hold for the drawing  $D$ .

**Claim 5.**  $cr_D(T_i, T_j) \geq 1$  for all  $i, j = 1, 2, \dots, n, i \neq j$ .

*Proof.* Assume that for some  $i \neq j, cr_D(T_i, T_j) = 0$ , implying that  $cr_D(T_i \cup T_j) = 0$  as  $cr_D(T_i) = cr_D(T_j) = 0$  due to  $D$  being a good drawing. The subgraph  $G \cup T_i \cup T_j$  is isomorphic to the graph  $G_2$ . Since  $cr(G_2) \leq cr_D(G \cup T_i \cup T_j) = cr_D(G, T_i \cup T_j) + cr_D(T_i \cup T_j) + cr_D(G)$  and  $cr_D(G) = cr_D(C_4) = 0$  (by Lemma 3), we have  $cr_D(G, T_i \cup T_j) \geq cr(G_2) - cr_D(T_i \cup T_j) - cr_D(G) = 1$ . For every subgraph  $T_k, k = 1, 2, \dots, n, k \neq i, j, T_k \cup T_i \cup T_j$  is isomorphic to the graph  $K_{3,5}$ . As  $cr(K_{3,5}) \leq cr_D(T_i \cup T_j \cup T_k) = cr_D(T_k, T_i \cup T_j) + cr_D(T_i \cup T_j) + cr_D(T_k)$ , we have  $cr_D(T_k, T_i \cup T_j) \geq cr(K_{3,5}) - cr_D(T_i \cup T_j) - cr_D(T_k) = 4$ . Thus,

$$\begin{aligned}
 cr_D(G_n) &= cr_D(G_{n-2} \cup T_i \cup T_j) \\
 &= cr_D(G_{n-2}) + cr_D(G_{n-2}, T_i \cup T_j) + cr_D(T_i \cup T_j) \\
 &= cr_D(G_{n-2}) + cr_D(G, T_i \cup T_j) + cr_D(K_{5, n-2}, T_i \cup T_j) + 0 \\
 &\geq Z(5, n-2) + \left\lfloor \frac{n-2}{2} \right\rfloor + 1 + 4(n-2) = Z(5, n) + \left\lfloor \frac{n}{2} \right\rfloor.
 \end{aligned}$$

This contradicts (1), and therefore  $cr_D(T_i, T_j) \geq 1$  for all  $i, j = 1, 2, \dots, n, i \neq j$ .  $\square$

The inequality (2) immediately implies Claim 6.

**Claim 6.** *In  $D$ , there are at least  $(\lfloor \frac{n}{2} \rfloor + 1)$  subgraphs  $T_i$ , neither of which crosses  $G$ .*

Assume, without loss of generality, that  $cr_D(G, T_n) = 0$ . According to Lemma 3 ( $cr_D(G) = cr_D(C_4) = 0$ ), the subgraph  $F_n = G \cup T_n$  of the graph  $G_n$  has a unique drawing as shown in Figure 4(1).

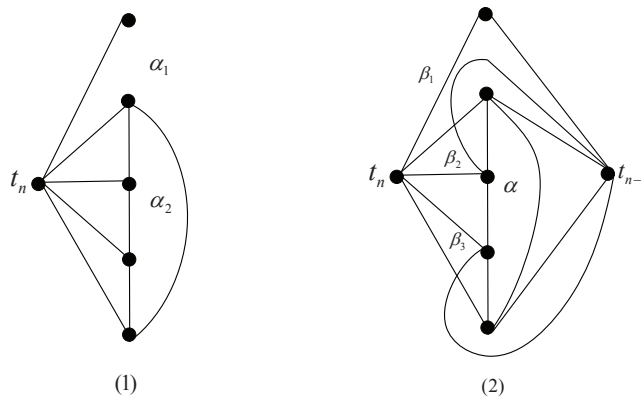


Figure 4.  $F_n$  and  $G^*$ .

**Claim 7.** *If for any  $j$  with  $1 \leq j \leq n - 1$  we have  $cr_D(T_j, F_n) = 2$ , then  $cr_D(T_j, G) \geq 1$ .*

**Proof.** Without loss of generality, let  $j = n - 1$ . If  $cr_D(T_{n-1}, F_n) = 2$  and  $cr_D(T_{n-1}, G) = 0$ , then the subgraph  $G^* = T_{n-1} \cup F_n$  of the graph  $G_n$  has a unique drawing (see Figure 4(2)). We claim that no matter where  $t_i$  ( $1 \leq i \leq n - 2$ ) is placed, we have either  $cr_D(T_i, G^*) \geq 6$ , or both  $cr_D(T_i, G) \geq 1$  and  $cr_D(T_i, G^*) \geq 3$ . It is also worth mentioning that it is a straightforward case analysis and some details are left to the reader.

It is easy to verify that, for every subgraph  $T_i, 1 \leq i \leq n - 2$ , if  $t_i$  is placed in the region  $\alpha$ , we have  $cr_D(T_i, G) \geq 1$ . According to Claim 5 ( $cr_D(T_i, T_j) \geq 1, i \neq j$ ), we have  $cr_D(T_i, G^*) = cr_D(T_i, G) + cr_D(T_i, T_{n-1}) + cr_D(T_i, T_n) \geq 3$ .

If  $t_i$  is placed in the region  $\beta_1, \beta_2$  or  $\beta_3$ , then  $cr_D(T_i, G^*) \geq 5$ . Especially, if  $cr_D(T_i, G^*) = 5$ , we have  $cr_D(T_i, G) \geq 1$ .

If  $t_i$  is placed in other regions, we have  $cr_D(T_i, G^*) \geq 4$ . Especially, if  $cr_D(T_i, G^*) = 4$ , then  $cr_D(T_i, G) \geq 1$ . If  $cr_D(T_i, G^*) = 5$ , we have  $cr_D(T_i, G) \geq 1$ . Thus, for all  $T_i, 1 \leq i \leq n - 2$ , let

$$A_1 = \{T_i | 3 \leq cr_D(T_i, G^*) \leq 5, cr_D(T_i, G) \geq 1\},$$

$$A_2 = \{T_i | cr_D(T_i, G^*) \geq 6\}.$$

The inequality (2) implies  $|A_1| \leq \lfloor \frac{n}{2} \rfloor - 1$ . Therefore, using  $|A_1| + |A_2| = n - 2$ , we have

$$\begin{aligned} cr_D(G_n) &= cr_D\left(G \cup \bigcup_{i=1}^n T_i\right) = cr_D\left(\bigcup_{i=1}^{n-2} T_i\right) + cr_D\left(G^*, \bigcup_{i=1}^{n-2} T_i\right) + cr_D(G^*) \\ &= cr_D(K_{5, n-2}) + cr_D\left(G^*, \bigcup_{i=1}^{n-2} T_i\right) + cr_D(G^*) \\ &\geq Z(5, n - 2) + 3|A_1| + 6|A_2| + 2 \\ &= Z(5, n) - (4n - 8) + 3|A_1| + 6|A_2| + 2 \\ &= Z(5, n) - (4n - 8) + 6(n - 2) - 3|A_1| + 2 = Z(5, n) + 2n - 2 - 3|A_1| \\ &\geq Z(5, n) + 2n - 2 - 3\left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right) \geq Z(5, n) + \left\lfloor \frac{n}{2} \right\rfloor. \end{aligned}$$

This contradicts (1). □

Now, we consider Figure 4(1). It is easy to verify that, for every subgraph  $T_i, 1 \leq i \leq n - 1$ , if  $t_i$  is placed in the region  $\alpha_1$ , we have  $cr_D(T_i, F_n) \geq 2$ . Especially, according to Claim 7, if  $cr_D(T_i, F_n) = 2$ , then  $cr_D(T_i, G) \geq 1$ .

If  $t_i$  is placed in the region  $\alpha_2$ , then  $cr_D(T_i, G) \geq 1$  and  $cr_D(T_i, F_n) = cr_D(T_i, T_n) + cr_D(T_i, G) \geq 2$ .

If  $t_i$  is placed in other regions, we have  $cr_D(T_i, F_n) \geq 3$ . Thus, for all  $T_i, 1 \leq i \leq n - 1$ , let

$$B_1 = \{T_i | cr_D(T_i, F_n) = 2, cr_D(T_i, G) \geq 1\},$$

$$B_2 = \{T_i | cr_D(T_i, F_n) \geq 3\}.$$

The inequality (2) implies  $|B_1| \leq \lfloor \frac{n}{2} \rfloor - 1$ .

Hence, using  $|B_1| + |B_2| = n - 1$ , we have

$$\begin{aligned} cr_D(G_n) &= cr_D\left(G \cup \bigcup_{i=1}^n T_i\right) = cr_D\left(\bigcup_{i=1}^{n-1} T_i\right) + cr_D\left(F_n, \bigcup_{i=1}^{n-1} T_i\right) + cr_D(F_n) \\ &= cr_D(K_{5,n-1}) + cr_D\left(F_n, \bigcup_{i=1}^{n-1} T_i\right) + 0 \\ &\geq Z(5, n - 1) + 2|B_1| + 3|B_2| = Z(5, n) - 4 \left\lfloor \frac{n-1}{2} \right\rfloor + 3(n-1) - |B_1| \\ &\geq Z(5, n) - 4 \left\lfloor \frac{n-1}{2} \right\rfloor + 3(n-1) - \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right) \geq Z(5, n) + \left\lfloor \frac{n}{2} \right\rfloor. \end{aligned}$$

This contradicts our assumption (1) and completes the proof. ■

**Corollary 8.** For  $n \geq 1$ ,  $cr(H_1 + nK_1) = cr(H_2 + nK_1) = Z(5, n) + \lfloor \frac{n}{2} \rfloor$ .

*Proof.* The graphs  $H_1$  and  $H_2$  are shown in Figure 1. Figure 2 shows the drawing of the graph  $G + nK_1$  with  $Z(5, n) + \lfloor \frac{n}{2} \rfloor$  crossings. One can easily see that in this drawing it is possible to add one edge which form the graph  $H_1$  or add two edges which form the graph  $H_2$  on the vertices of  $G$  in such a way that there is no new crossing occurring. Hence, we have  $cr(H_1 + nK_1) \leq Z(5, n) + \lfloor \frac{n}{2} \rfloor$  and  $cr(H_2 + nK_1) \leq Z(5, n) + \lfloor \frac{n}{2} \rfloor$ . On the other hand, as  $G$  is a subgraph of  $H_1$  and is a subgraph of  $H_2$ , clearly, we have  $cr(H_2 + nK_1) \geq cr(H_1 + nK_1) \geq cr(G + nK_1)$ . Thus, this completes the proof. ■

### 3. THE CROSSING NUMBER OF $W_4 + nK_1$

**Lemma 9.**  $cr(W_4 + nK_1) \leq Z(5, n) + n + \lfloor \frac{n}{2} \rfloor$  for  $n \geq 1$ .

*Proof.* We will display a drawing  $\varphi$  of  $W_4 + nK_1$  in the plane such that  $cr_\varphi(W_4 + nK_1) = Z(5, n) + n + \lfloor \frac{n}{2} \rfloor$ . The desired drawing  $\varphi$  is constructed as follows (see Figure 5).

- (i) Set all vertices of  $W_4$  on  $y$ -axis,  $\lceil \frac{n}{2} \rceil$  isolated vertices on the negative  $x$ -axis and  $\lfloor \frac{n}{2} \rfloor$  isolated vertices on the positive  $x$ -axis.
- (ii) The image of each edge for  $W_4$  is a thick line segment.

Then it is not difficult to see that  $cr_\varphi(W_4 + nK_1) = Z(5, n) + n + \lfloor \frac{n}{2} \rfloor$ , and so  $cr(W_4 + nK_1) \leq Z(5, n) + n + \lfloor \frac{n}{2} \rfloor$ . ■

**Theorem 10.**  $cr(W_4 + nK_1) = Z(5, n) + n + \lfloor \frac{n}{2} \rfloor$  for  $n \geq 1$ .



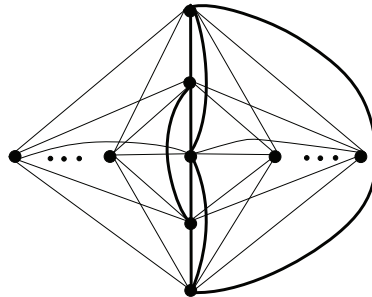


Figure 5. A drawing  $\varphi$  of  $W_4 + nK_1$ .

**Proof.** Lemma 9 shows that  $cr(W_4 + nK_1) \leq Z(5, n) + n + \lfloor \frac{n}{2} \rfloor$ . Thus, in order to prove the theorem, we need only to prove that  $cr_\phi(W_4 + nK_1) \geq Z(5, n) + n + \lfloor \frac{n}{2} \rfloor$  for any drawing  $\phi$  of  $W_4 + nK_1$ . Let  $e_i$  denote the edge  $xy_i$  of  $W_4, i = 1, 2, 3, 4$ , wherein  $x$  is a vertex of degree 4 and  $y_i$  are vertices of degree 3 in  $W_4$ . Without loss of generality, assume that under any drawing  $\phi$ , the clockwise order of these four images  $\phi(e_i)$  around  $\phi(x)$  is  $\phi(e_1) \rightarrow \phi(e_2) \rightarrow \phi(e_3) \rightarrow \phi(e_4)$ . The graph  $W_4 + nK_1$  has additional  $n$  edges  $f_j = t_jx$  incident with  $x$  ( $1 \leq j \leq n$ ). Let  $A_i$  denote the set of all those images  $f_j$ , each of which lies in the angle  $\alpha_i$  formed between  $\phi(e_i)$  and  $\phi(e_{i+1})$ , where the indices are read modulo 4 (see Figure 6(1)). We note that  $|A_1| + |A_2| + |A_3| + |A_4| = n$ . Again, we see that in the plane  $\mathbb{R}^2$ , there exists a circular neighborhood around  $\phi(x), N(\phi(x), \varepsilon) = \{s \in \mathbb{R}^2 : \|s - \phi(x)\| < \varepsilon\}$ , where  $\varepsilon$  is a sufficiently small positive number, such that for any other edge  $e$  of  $W_4 + nK_1$  not incident with  $x, \phi(e) \cap N(\phi(x), \varepsilon) = \emptyset$ . We now consider two cases.

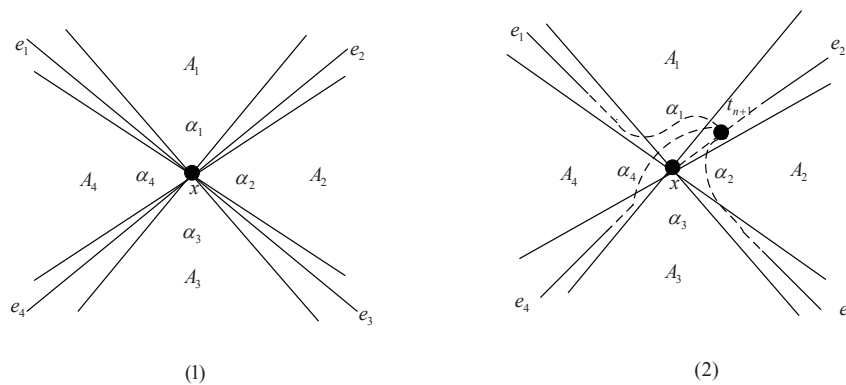


Figure 6. A drawing  $\phi^*$  of  $G_{n+1}$  obtained from  $\phi(W_4 + nK_1)$ .

*Case 1.* Assume  $n$  is even. We consider arbitrarily a pair  $A_1$  and  $A_3$  or  $A_2$  and  $A_4$ , say  $A_1$  and  $A_3$ . Without loss of generality, assume  $|A_1| \leq |A_3|$ . In the following, we produce the graph  $G_{n+1}$  ( $G_{n+1} = G + (n + 1)K_1$  and  $G = C_4 \cup K_1$  in Figure 1) together with its drawing  $\phi^*$ .

**Step 1.** Add a new vertex  $t_{n+1}$  in some location of  $\phi(e_2) \cap N(\phi(x), \varepsilon)$ .

**Step 2.** For all  $1 \leq i \leq 4$ , remove the part of  $\phi(e_i)$  lying in  $N(\phi(x), \varepsilon)$  (do not remove the vertex  $t_{n+1}$ ).

**Step 3.** Connect  $t_{n+1}$  to each vertex in  $\{\phi(x), \phi(y_1), \phi(y_2), \phi(y_3), \phi(y_4)\}$  in such a way as described in Figure 6(2).

For example, connect  $t_{n+1}$  to  $\phi(x)$  along the section of  $\phi(e_2) \cap N(\phi(x), \varepsilon)$ , connect  $t_{n+1}$  to  $\phi(y_2)$  first along the section of  $\phi(e_2) \cap N(\phi(x), \varepsilon)$  and then along the original section  $\phi(e_2)$  outside  $N(\phi(x), \varepsilon)$ . Again, the way of connecting  $t_{n+1}$  to  $\phi(y_4)$  is first by successively traversing through the angles  $\alpha_1$  and  $\alpha_4$  (near to  $\phi(x)$ ), and then along the original section of  $\phi(e_4)$  outside  $N(\phi(x), \varepsilon)$ . Thus, we obtain a drawing  $\phi^*$  of the graph  $G_{n+1}$ . It is easy to see that

$$\begin{aligned} cr_{\phi^*}(G_{n+1}) &= cr_{\phi}(W_4 + nK_1) + 2|A_1| + |A_2| + |A_4| \\ &\leq cr_{\phi}(W_4 + nK_1) + n. \end{aligned}$$

Under any drawing  $\phi$  of  $W_4 + nK_1$ , using Theorem 4,

$$\begin{aligned} cr_{\phi}(W_4 + nK_1) &\geq cr_{\phi^*}(G_{n+1}) - n \geq Z(5, n + 1) + \left\lfloor \frac{n + 1}{2} \right\rfloor - n \\ &\geq Z(5, n) + n + \left\lfloor \frac{n}{2} \right\rfloor. \end{aligned}$$

*Case 2.* Assume  $n$  is odd. First, if  $|A_1| = |A_3|$  and  $|A_2| = |A_4|$ , then it follows that  $n = \sum_{i=1}^4 |A_i| = 2(|A_1| + |A_2|)$ . This contradicts that  $n$  is odd. Therefore, either  $|A_1| \neq |A_3|$  or  $|A_2| \neq |A_4|$ . Without loss of generality, let  $|A_1| \neq |A_3|$ , and moreover let  $|A_3| \geq |A_1| + 1$ . Completely analogously to Case 1 above,

$$\begin{aligned} cr_{\phi^*}(G_{n+1}) &= cr_{\phi}(W_4 + nK_1) + 2|A_1| + |A_2| + |A_4| \\ &\leq cr_{\phi}(W_4 + nK_1) + n - 1. \end{aligned}$$

Thus, we have

$$\begin{aligned} cr_{\phi}(W_4 + nK_1) &\geq cr_{\phi^*}(G_{n+1}) - n + 1 \geq Z(5, n + 1) + \left\lfloor \frac{n + 1}{2} \right\rfloor - n + 1 \\ &\geq Z(5, n) + n + \left\lfloor \frac{n}{2} \right\rfloor. \end{aligned}$$

Thereby, from the arguments above we finish the proof of the theorem. ■

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