Discussiones Mathematicae
Graph Theory 38 (2018) 889–898
doi:10.7151/dmgt.2043

GRAPHIC AND COGRAPHIC Γ-EXTENSIONS OF BINARY MATROIDS

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Abstract

Slater introduced the point-addition operation on graphs to characterize 4-connected graphs. The Γ-extension operation on binary matroids is a generalization of the point-addition operation. In general, under the Γ-extension operation the properties like graphicness and cographicness of matroids are not preserved. In this paper, we obtain forbidden minor characterizations for binary matroids whose Γ-extension matroids are graphic (respectively, cographic).

Keywords: splitting, Γ-extension, graphic, cographic, minor.

2010 Mathematics Subject Classification: 05B35, 05C50, 05C83.

1. Introduction

We refer to [5] for standard terminology in graphs and matroids. The matroids considered here are loopless and coloopless. Slater [9] introduced the point-addition operation on graphs and used it to classify 4-connected graphs. Azanchiler [1] extended this operation to binary matroids as follows.

Definition 1 [1]**.** Let M be a binary matroid with ground set S and standard matrix representation A over the field $GF(2)$. Let $X = \{x_1, x_2, \ldots, x_m\} \subset S$ be an independent set in M and let $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ be a set such that $S \cap \Gamma = \emptyset$. Suppose A' is the matrix obtained from the matrix A by adjoining m columns labeled by $\gamma_1, \gamma_2, \ldots, \gamma_m$ such that the column labeled by γ_i is same as the column labeled by x_i for $i = 1, 2, \ldots, m$. Let A^X be the matrix obtained by adjoining one extra row to A' which has entry 1 in the column labeled by γ_i for $i = 1, 2, \ldots, m$ and zero elsewhere. The vector matroid of the matrix A^X, denoted by M^X, is called as the Γ-extension of M with respect to X and the transition from M to M^X is called as the Γ-extension operation on M.

Note that the ground set of the matroid M^X is $S \cup \Gamma$ and $M^X \setminus \Gamma = M$. Therefore M^X is an extension of M. Some basic properties of M^X are studied in [1] and [2].

The Γ-extension operation is related to the *splitting operation* on binary matroids which is defined by Shikare *et al.* [8] as follows.

Definition 2 [8]**.** Let M be a binary matroid with standard matrix representation A over the field $GF(2)$ and let X be a set of elements of M. Let A_X be the matrix obtained by adjoining one extra row to the matrix A whose entries are 1 in the columns labeled by the elements of the set X and zero otherwise. The vector matroid of the matrix A_X, denoted by M_X, is called as the splitting matroid of M with respect to X, and the transition from M to M_X is called as the splitting operation.

Let M be a binary matroid with ground set S and let $X = \{x_1, x_2, \ldots, x_m\}$ be an independent set in M. Obtain the extension M' of M with ground set $S \cup \Gamma$, where $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ is disjoint from S, such that $\{x_i, \gamma_i\}$ is a 2-circuit in M' for each i. The matroid M'_Γ obtained from M' by splitting with respect to the set Γ is the Γ-extension matroid M^X.

Earlier, the splitting with respect to a pair of elements, which is a special case of Definition 2, was defined by Raghunathan *et al.* [6] for binary matroids as an extension of the corresponding graph operation due to Fleischner [4].

In general, under the splitting operation the properties like graphicness and cographicness of matroids are not preserved. Shikare and Waphare [7] obtained the following characterization for the class of graphic matroids M whose splitting matroids M_X, with $|X| = 2$, are again graphic.

Theorem 3 [7]**.** *Let M be a graphic matroid. For any $X \subset S$ with $|X| = 2$, the splitting matroid M_X is graphic if and only if M has no minor isomorphic to any of the circuit matroids $M(G_1), M(G_2), M(G_3)$ and $M(G_4)$, where G_1, G_2, G_3 and G_4 are the graphs as shown in Figure* 1.

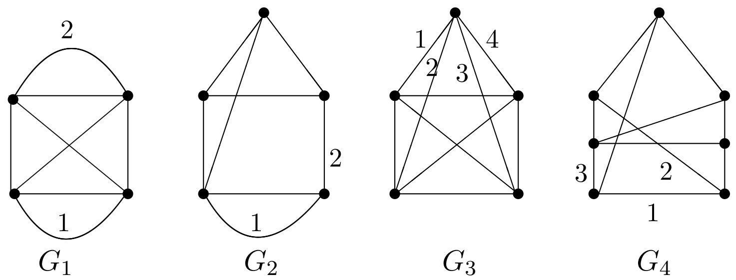

Figure 1

Borse *et al.* [3] obtained a similar characterization for the cographic matroids M whose splitting matroids M_X, with $|X| = 2$, are cographic.

Theorem 4 [3]**.** *Let M be a cographic matroid. For any $X \subset S$ with $|X| = 2$, the splitting matroid M_X is cographic if and only if M has no minor isomorphic to any of the circuit matroids $M(G_1)$ and $M(G_2)$, where G_1 and G_2 are the graphs as shown in Figure* 1*.*

It remains to find the effect of the splitting operation with respect to X where $|X| \geq 3$, on the properties like graphicness and cographicness of a matroid.

Like splitting operation, the Γ-extension operation also does not preserve graphicness and cographicness properties of a given matroid, in general. Azanchiler [2] obtained few results in this direction.

In this paper, we characterize binary matroids M whose Γ-extension matroids M^X with $|X| \geq 2$ are graphic (respectively, cographic).

The following are the main results of the paper.

Theorem 5. *Let M be a binary matroid. Then M^X is graphic* (*respectively, cographic*) *for every independent set X in M with $|X| = 2$ if and only if M does not contain a minor that is isomorphic to $M(K_4)$.*

Corollary 6. *Let M be a graphic* (*respectively, cographic*) *matroid. Then M^X is graphic* (*respectively, cographic*) *for every independent set X in M with $|X| = 2$ if and only if M does not contain a minor that is isomorphic to $M(K_4)$.*

Theorem 7. *Let M be a binary matroid. Then M^X is graphic* (*respectively, cographic*) *for every independent set X in M with $|X| \geq 3$ if and only if M does not contain a minor that is isomorphic to a 4-circuit.*

Corollary 8. *Let M be a graphic* (*respectively, cographic*) *matroid. Then M^X is graphic* (*respectively, cographic*) *for every independent set X in M with $|X| \geq 3$ if and only if M does not contain a minor that is isomorphic to a 4-circuit.*

2. Case $|X| = 2$

In this section, we prove Theorem 5. First, observe that there should be only three forbidden minors in Theorem 3. For the graphs G_2 and G_4 in Figure 1, $M(G_2) \cong M(G_4)\backslash\{3\}/\{1,2\}$. Therefore $M(G_2)$ is a minor of $M(G_4)$ and hence Theorem 3 can be restated as follows.

Theorem 9. *Let M be a graphic matroid. For any $X \subset S$ with $|X| = 2$ the splitting matroid M_X is graphic if and only if M has no minor isomorphic to any of the circuit matroids $M(G_1), M(G_2)$ and $M(G_3)$, where G_1, G_2 and G_3 are the graphs as shown in Figure* 1.

We need the following well-known characterizations.

Theorem 10 (Oxley [5])**.** *A binary matroid M is graphic if and only if no minor of M is isomorphic to any of the matroids $F_7, F_7^*, M^*(K_{3,3})$ and $M^*(K_5)$.*

Theorem 11 (Oxley [5])**.** *A binary matroid M is cographic if and only if no minor of M is isomorphic to any of the matroids $F_7, F_7^*, M(K_{3,3})$ and $M(K_5)$.*

Theorem 12 (Oxley [5])**.** *A binary matroid M is regular if and only if no minor of M is isomorphic to any of the matroids F_7, F_7^*.*

The proof of the following lemma is trivial.

Lemma 13. *If $\{x, y\}$ is a circuit in a matroid M, then $M\backslash\{x\} \cong M\backslash\{y\}$ and $M/\{x\} \cong M/\{y\}$.*

Lemma 14. *Let M be a binary matroid containing a minor isomorphic to $M(K_4)$. Then there is an independent set X in M with $|X| = 2$ such that the matroid M^X is not regular.*

Proof. Suppose M contains a minor N which is isomorphic to $M(K_4)$. Then there are subsets T_1 and T_2 of the ground set of M such that $N = M\backslash T_1/T_2$. Label the edges of the graph K_4 by the set $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ so that x_1, x_2, x_3, x_4, in order, form a 4-cycle and the edges x_5, x_6 are the chords of this cycle.

Let $X = \{x_1, x_3\}$. Then X is disjoint from $T_1 \cup T_2$ and is independent in N as well as in M. Further, $N^X = M^X \setminus T_1/T_2$. Moreover, the edges x_1 and x_3 are not adjacent in K_4. Let A be the standard matrix representation of $M(K_4)$ over the field $GF(2)$. Then

$$\begin{array}{cc} & \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ A = & \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

and

$$A^X = \begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \gamma_1 & \gamma_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}.$$

Therefore

$$A^X/\{\gamma_1\} = \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \gamma_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}.$$

Since $A^X/\{\gamma_1\}$ is a matrix representation of the matroid $M(K_4)^X/\{\gamma_1\} \cong N^X/\{\gamma_1\}$, it follows from the standard matrix representation of the matroid F_7 that $N^X/\{\gamma_1\} \cong F_7$. Therefore $M^X \setminus T_1/T_2/\{\gamma_1\} \cong F_7$. This shows that F_7 is a minor of M^X. Hence, by Theorem 12, M^X is not regular. ■

Proposition 15. *Let M be a binary matroid such that no minor of M is isomorphic to $M(K_4)$. Then M^X is graphic as well as cographic for any independent set X in M with $|X| = 2$.*

Proof. Clearly, $M(K_4)$ is a minor of each of the six matroids $F_7, F_7^*, M(K_5)$, $M^*(K_5), M(K_{3,3})$ and $M^*(K_{3,3})$. Since no minor of M is isomorphic to $M(K_4)$, none of these six matroids can be a minor of M. Hence, by Theorems 10 and 11, M is graphic as well as cographic. Thus $M = M(G)$ for some planar graph G. Assume that M^X is not graphic or not cographic for some independent set $X = \{x_1, x_2\}$ in M. We obtain a contradiction by proving that M contains a minor isomorphic to $M(K_4)$.

Let M' be the extension of M obtained by adding two elements $\{\gamma_1, \gamma_2\}$ to the ground set S of M such that $\{x_1, \gamma_1\}$ and $\{x_2, \gamma_2\}$ are circuits in M'. Then $M' \setminus \{\gamma_1, \gamma_2\} = M$. The ground set of M' is $S \cup \{\gamma_1, \gamma_2\}$. Since M is graphic and cographic, so is M'. Therefore M' does not contain a minor isomorphic to $M(K_5) = M(G_3)$. By definition of M^X, we have $M^X = M'_{\{\gamma_1,\gamma_2\}}$, where $M'_{\{\gamma_1,\gamma_2\}}$ is the matroid obtained from M' by splitting with respect to the pair $\{\gamma_1, \gamma_2\}$. Therefore $M'_{\{\gamma_1,\gamma_2\}}$ is not graphic or not cographic.

By Theorems 4 and 9, there is a minor N' of M' such that $N' \cong M(G_1)$ or $N' \cong M(G_2)$, where G_1 and G_2 are the graphs as shown in Figure 1. Clearly, $M(K_4) \cong M(G_1)\setminus\{1,2\} \cong M(G_2)\setminus\{1\}/\{2\}$. Hence $M(K_4)$ is isomorphic to a minor of N'. If N' is a minor of M, then M has a minor isomorphic to $M(K_4)$, a contradiction. Consequently, N' is not a minor of M. It implies that N' contains γ_1 or γ_2 or both. By Lemma 13, we may assume that N' contains x_i whenever it contains γ_i. Thus N' contains at least one of the two 2-circuit $\{x_1, \gamma_1\}$ and $\{x_2, \gamma_2\}$ of M'. Suppose N' contains both γ_1 and γ_2. Then N' contains both the 2-circuits

$\{x_1, \gamma_1\}$ and $\{x_2, \gamma_2\}$. Therefore N' is isomorphic to $M(G_1)$ and the two 2-cycles present in G_1 corresponds to $\{x_1, \gamma_1\}$ and $\{x_2, \gamma_2\}$. Thus $M(K_4) \cong N' \backslash \{\gamma_1, \gamma_2\}$ is minor of $M' \backslash \{\gamma_1, \gamma_2\} = M$, a contradiction. Hence N' contains exactly one of γ_1 and γ_2.

We may assume that N' contains γ_1 but not γ_2. Then $N' \backslash \gamma_1$ is a minor of $M' \backslash \gamma_1$ and hence is a minor of M. Suppose N' is isomorphic to $M(G_2)$. Then the 2-cycle present in G_2 corresponds to the 2-circuit $\{x_1, \gamma_1\}$ in N'. Hence $M(K_4) \cong N' \backslash \{\gamma_1\}/\{2\}$. But $N' \backslash \{\gamma_1\}/\{2\}$ is minor of $N' \backslash \{\gamma_1\}$ and so is a minor of M. Consequently, $M(K_4)$ is isomorphic to a minor of M, a contradiction. Therefore $N' \cong M(G_1)$. We may assume that the 2-circuit $\{x_1, \gamma_1\}$ of N' corresponds to the 2-cycle of G_1 containing the edge labeled by 1. Clearly, $M(K_4) \cong N' \backslash \{\gamma_1, 2\}$. Thus $M(K_4)$ is isomorphic to a minor of $N' \backslash \{\gamma_1\}$ and so is isomorphic to a minor of M, a contradiction. ∎

***Proof of Theorem* 5.** Suppose M contains a minor isomorphic to $M(K_4)$. By Lemma 14, M^X is not regular for some independent set X in M with $|X| = 2$. Therefore, by Theorems 10, 11 and 12, M^X is neither graphic nor cographic. Conversely, if no minor of M is isomorphic to $M(K_4)$, then, by Proposition 15, M^X is graphic as well as cographic for any independent set X in M with $|X| = 2$. ∎

3. Case $|X| \geq 3$

In this section, we prove Theorem 7.

Lemma 16. *Let M be a binary matroid containing a minor isomorphic to a 4-circuit. Then there is an independent set X in M with $|X| \geq 3$ such that M^X is not regular.*

Proof. Suppose M contains a minor N which is isomorphic to a 4-circuit. Let the ground set of N be $\{x_1, x_2, x_3, x_4\}$. Let $X = \{x_1, x_2, x_3\}$. Then X is independent in N and so in M. The following matrix A represents N over the field $GF(2)$.

$$A = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array}. \text{ Therefore } A^X = \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & \gamma_1 & \gamma_2 & \gamma_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array}.$$

In A^X, by adding the fourth row to the first row and then interchanging the fourth and fifth columns, we get the following matrix which is the standard matrix representation of the matroid F_7^* over $GF(2)$:

$$\begin{array}{ccccccc} x_1 & x_2 & x_3 & \gamma_1 & x_4 & \gamma_2 & \gamma_3 \end{array}$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

The vector matroid of the matrix A^X is the Γ-extension N^X of N. Hence N^X is isomorphic to F_7^*. Since N is a minor of M, there are disjoint subsets T_1 and T_2 of the ground set of M such that $N = M \setminus T_1/T_2$. Since $X \cap T_1 = \emptyset$ and $X \cap T_2 = \emptyset$, it follows that $N^X = (M \setminus T_1/T_2)^X = M^X \setminus T_1/T_2$. Hence N^X is a minor of M^X. Therefore M^X has a minor isomorphic to F_7^*. By Theorem 12, M^X is not regular. ■

Proposition 17. *Let M be a binary matroid such that no minor of M is isomorphic to a 4-circuit. Then M^X is graphic as well as cographic for any independent set X in M with $|X| \geq 3$.*

Proof. Clearly, each of the six matroids $F_7, F_7^*, M(K_5), M(K_{3,3}), M^*(K_5)$ and $M^*(K_{3,3})$ contains a 4-circuit. Hence none of these six matroids can be a minor of M. Therefore, by Theorems 10 and 11, M is graphic as well as cographic. Hence $M = M(G)$ for some graph G without isolated vertices.

Let X be an independent set in M with $|X| \geq 3$. We prove that M^X is graphic as well as cographic. Let $D_1, D_2, \ldots, D_m$ be components of M. Since M is graphic and cographic, each component D_i is also graphic and cographic. Therefore $D_i = M(H_i)$ for some planar graph H_i for $i = 1, 2, \ldots, m$. If the set X does not intersect a component D_i of M, then D_i is a component of M^X, too. Therefore we may assume that X intersects each D_i. Let $X_i = X \cap D_i$ for $i = 1, 2, \ldots, m$. Then $X = X_1 \cup X_2 \cup \cdots \cup X_m$. Since X is independent in M, each X_i is independent in D_i and so it does not contain parallel edges. Since $M(H_i)$ is component of M for all $i = 1, 2, \ldots, m$, we may assume that graphs $H_i (i = 1, 2, \ldots, m)$ are vertex-disjoint.

Suppose the rank of D_i is at least 3. Then H_i contains at least four vertices. Since D_i is connected, H_i is 2-connected. It follows that H_i contains an r-circuit and so M contains an r-circuit for some $r \geq 4$. This implies that M has a 4-circuit as a minor, a contradiction. Hence the rank of each D_i is one or two. If the rank of D_i is one, then H_i has exactly two vertices. Therefore H_i is K_2 or a graph in which any two edges are parallel. Thus X_i contains exactly one edge of H_i. Suppose the rank of D_i is two. Then H_i has exactly three vertices and further, H_i is 2-connected and so it contains a triangle, say T. Any edge of H_i which is not in T is parallel to one of the three edges of T. This implies that any two edges of H_i are adjacent. Since X_i is independent, it contains one edge or two non-parallel edges of H_i. Consequently, $|X_i| = 1$ or 2. Let $X_i = \{e_i\}$ if $|X_i| = 1$

and let $X_i = \{e_i, f_i\}$ if $|X_i| = 2$ for $i = 1, 2, \ldots, m$. Let $e_i = u_i v_i$. Then $f_i = u_i w_i$ for some $w_i \neq v_i$.

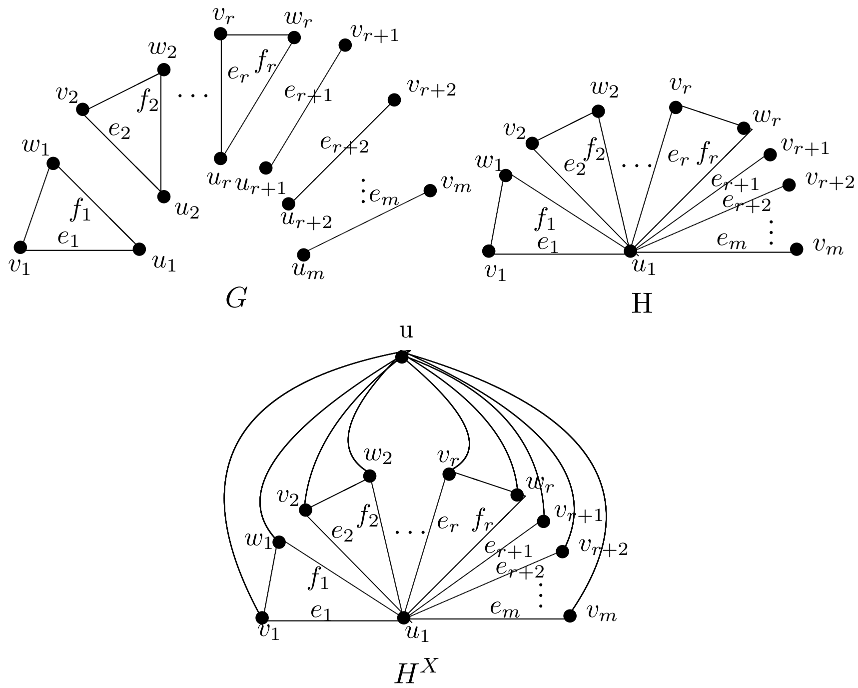

Figure 2

Let H be the graph obtained from $H_1, H_2, \ldots, H_m$ by identifying the vertices $u_2, u_3, \ldots, u_m$ to u_1 (see Figure 2). Then $M(G)$ is isomorphic to $M(H)$. Therefore $M(G)^X$ is isomorphic to $M(H)^X$. Let H^X be the graph obtained from H by adding an additional vertex u and edges uv_i for $i = 1, 2, \ldots, m$, and the edge uw_i if $f_i \in X$ for each i. By Definition 1, $M(H)^X$ is isomorphic to the matroid $M(H^X)$. Thus $M^X = M(G)^X$ is isomorphic to $M(H^X)$. Hence M^X is graphic.

Now, we prove that $M(H^X)$ is cographic, that is, H^X is planar. Assume that $M(H^X)$ is not cographic. Then, by Theorem 11, it has $M(K_5)$ or $M(K_{3,3})$ as a minor. Each of K_5 and $K_{3,3}$ are simple graphs. Also, addition or deletion of parallel edges to a graph does not change its planarity. Further, X does not contain parallel edges. Therefore we may assume that each graph H_i is simple. Hence each H_i is a K_2 or a triangle. Clearly, the graph H is planar. All vertices of H^X other than u_1 and u have degree two or three. However, K_5 has five vertices with degree four. Contractions and deletions in H^X does not increase degree of any vertex in H^X other than u and u_1. Hence $M(K_5)$ cannot be a minor of $M(H^X)$.

Thus $M(H^X)$ contains $M(K_{3,3})$ as a minor. If H does not contain a triangle, then it is the star $K_{1,m}$ and hence H^X is $K_{2,m}$. Therefore $M(H^X)$ does not have

$M(K_{3,3})$ as a minor, a contradiction. Suppose H contains a triangle. The vertices of a triangle in H are u_1 and v_i, w_i for some $i \geq 1$. Hence u is adjacent to v_i or w_i or both in H^X. The graph $M(K_{3,3})$ does not contain a triangle and also has all vertices of degree three. Suppose u is not adjacent to w_i in H^X. Then the degree of w_i in H^X is two. In order to get $K_{3,3}$ as a minor of H^X, we need to delete or contract one edge incident to w_i and then delete the other edge incident to w_i. This also can be done by just deleting both edges incident to w_i. But then the degree of v_i becomes two. Suppose u is adjacent to both v_i and w_i. Then u, v_i, w_i induces a triangle in H^X. Since $M(K_{3,3})$ does not contain a triangle, we need to delete or contract one of the edges in this triangle. The contraction creates a parallel edge which is to be deleted later on. Thus, at least one edge of the triangle with vertices u, v_i, w_i is deleted. Hence the degree of v_i or w_i or both becomes two. It follows that in order to remove triangles from H^X we are left with a subgraph isomorphic to $K_{2,r}$ for some $r \geq 1$. However $M(K_{2,r})$ does not contain $M(K_{3,3})$ as a minor and hence $M(H^X)$ does not contain $M(K_{3,3})$ as a minor, a contradiction. Thus $M(H^X)$ is cographic. ∎

***Proof of Theorem* 7.** If M contains a minor isomorphic to a 4-circuit, then, by Lemma 16, M^X is not regular and hence, by Theorems 10, 11 and 12, M^X is neither graphic nor cographic for every independent set X in M with $|X| \geq 3$. Conversely, if no minor of M is isomorphic to a 4-circuit, then, by Proposition 17, M^X is graphic as well as cographic for any independent set X with $|X| \geq 3$. ∎

Remark 18. As pointed out by one of the referees, Theorem 5 can be proved using graph-theoretic approach, as a binary matroid without $M(K_4)$-minor is the cycle matroid of some series-parallel graph. There is no change in the proof of the "only if" part of Theorem 5. The referee outlined the proof of the "if" part as follows.

Suppose $M = M(G)$ for some series-parallel graph G. To show that $M(G)^X$ is graphic and cographic, it suffices to show that $M(G)^X$ is graphic and planar. To show that $M(G)^X$ is graphic, it suffices to show that, for any pair of edges e and f of G, there exists a graph G' that is 2-isomorphic to G in which e and f are adjacent. (Showing e and f are adjacent in G' implies that every matroid splitting operation in $M(G)$ can be realized as a graphic splitting operation in G'.) Showing that such a G' exists is easily done by induction: first reduce to the 2-connected case, which is trivial, and then take a 2-sum $\{G_1, G_2\}$ of G. (Such a 2-sum always exists in series-parallel graphs having at least four edges.) If e and f are in G_1 (say), then just apply induction. If e is in G_1 and f in G_2, then apply induction to e and q in G_1, and f and q in G_2, where q is the edge common to G_1 and G_2. Now, given that e and f are adjacent in G', and G' is series-parallel, it is easy to verify that the graph splitting operation of e and f in G' produces a planar graph, which proves Theorem 5.

Theorem 7 can be handled in a similar fashion. In particular, binary matroids with no 4-circuit minor are graphic, and can be constructed from 1-sums of "fat" triangles (a triangle plus parallel edges) and "fat" edges (an edge plus parallel edges).

Acknowledgments

The authors are grateful to the referees for their fruitful suggestions to improve the quality of the paper. We thank the referee who suggested an alternative proof of Theorem 5. The first author is supported by DST-SERB, Government of India through the project SR/S4/MS:750/12.

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Received 12 October 2016
Revised 2 March 2017
Accepted 2 March 2017