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T_r-SPAN OF DIRECTED WHEEL GRAPHS

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Abstract

In this paper, we consider T-colorings of directed graphs. In particular, we consider as a T-set the set $T_r = \{0, 1, 2, \ldots, r-1, r+1, \ldots\}$. Exact values and bounds of the T_r-span of directed graphs whose underlying graph is a wheel graph are presented.

Keywords: T-coloring, digraph, wheel graph, span.

2010 Mathematics Subject Classification: 05C15.

1. Introduction

T-colorings are a generalization of proper vertex colorings of graphs. They were introduced by Hale [2] to model the frequency assignment problem. The problem in assigning frequencies to requesters comes from the need to assign them in a manner that minimizes the use of the frequency spectrum (e.g., AM radio or UHF television) while avoiding the interferences and separation constraints that can occur amongst transmitters. T-colorings have been widely studied for over three decades since Hale's paper appeared in 1980. Much of the research has optimized T-colorings for various classes of graphs, as well as classes of the separation constraints. For example, see Sivagami and Rajasingh [6], Juan *et al.* [4], and Janczewski [3]. Variants of T-colorings, in particular list T-colorings, have been studied by Junosza-Szaniawski and Rzążewski [5], Tesman [8], and Fiala *et al.* [1]. In this paper we are concerned with T-colorings of digraphs which were introduced by Tesman [7] to model the special case of unidirectional transmitters.

A T-*set* is a set of nonnegative integers. Given a digraph $D = (V, A)$ and a T-set T, a *T-coloring* of D is a function $c : V(D) \to \mathbb{Z}^+$ such that if $(x, y) \in A(D)$, then $c(x) - c(y) \notin T$. The *span* of a T-coloring c of D is defined as:

$$\text{span of } c = \max_{x \in V(D)} c(x) - \min_{x \in V(D)} c(x).$$

The minimum span over all T-colorings of a digraph D for a fixed T-set T is called the *T-span* of D and denoted $sp_T(D)$. We will assume that $0 \in T$ because a coloring must be proper and, without loss of generality, 1 will be the minimum color in any T-coloring. We will find bounds and the exact T-span for some special classes of directed wheel graphs.

The underlying graph of the digraphs studied in this paper is a wheel graph, W_n, i.e., a circuit for which every vertex on the circuit, $b_1, b_2, \ldots, b_n$, is connected to a single "hub" vertex, y.[1] (See Figure 1.) The chromatic number of a wheel graph, $\chi(W_n)$, will be used in Sections 3 and 4. Recall that $\chi(W_n) = 3$ or 4 depending on whether or not the wheel graph's circuit is even or odd, respectively. These values provide lower bounds for the T-span of their respective directed wheel graphs since any T-coloring is also a proper vertex coloring because 0 is in every T-set.

The underlying wheel graph's circuit edges $\{b_i, b_{i+1}\}$, for $i = 1, 2, \ldots, n-1$, and $\{b_1, b_n\}$ will be directed clockwise, i.e., (b_i, b_{i+1}), for $i = 1, 2, \ldots, n-1$, and (b_n, b_1) will be the circuit arcs of the digraph. (Note that a counterclockwise orientation of the circuit edges will yield the same results that we prove in this paper; the two digraphs are merely mirror images of one another.) Arcs directed from a circuit vertex to the hub vertex will be called *introverted*. Circuit vertices incident with introverted arcs will be called *introverts*. Similarly, arcs directed from the hub vertex to a circuit vertex will be called *extroverted* and circuit vertices incident with extroverted arcs will be called *extroverts*.

We will denote our directed wheel graphs by $CW_{a_1,a_2,a_3,\ldots,a_{2j}}$ where, going clockwise and starting with "spoke" $\{b_1, y\}$, there are a_1 introverted arcs, followed by a_2 extroverted arcs, ..., and ending with a_{2j} extroverted arcs.[2] The notation refers to a digraph with $n+1$ vertices where $n = a_1 + a_2 + \cdots + a_{2j}$ vertices are on the circuit and there is one hub vertex. We will assume that $n > 1$, otherwise, the digraph has a loop and no T-coloring is possible. See Figure 2 for a specific example, $CW_{2,5,1,3}$. Note that for $CW_{a_1,a_2,a_3,\ldots,a_{2j}}$, each $a_i > 0$ except in the special case of a directed wheel graph with only extroverted arcs, i.e., $CW_{0,n}$.

The T-sets that we will be considering in this paper will be denoted by T_r and are of the form

$$T_r = \{0, 1, 2, 3, \ldots, r-1, r+1, r+2, \ldots\}.$$

[1]Our definition of W_n is slightly nonstandard to simplify the notation on the circuit vertices.

[2]Note that $CW_{a_1,a_2,a_3,\ldots,a_{2j}} \cong CW_{a_3,a_4,\ldots,a_{2j},a_1,a_2} \cong CW_{a_5,a_6,\ldots,a_{2j},a_1,a_2,a_3,a_4} \cong \cdots$ by rotational symmetry.

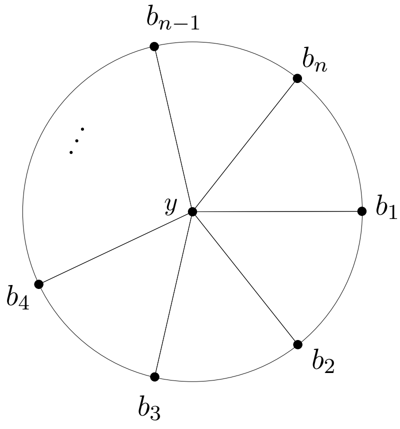


Figure 1. W_n.

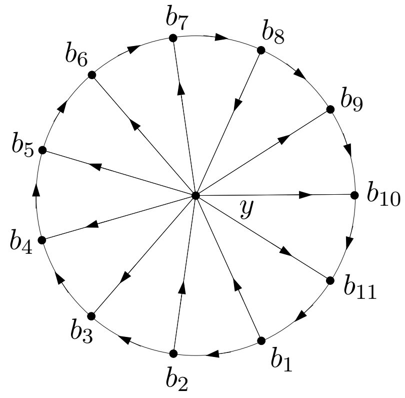


Figure 2. $CW_{2,5,1,3}$.

For T-sets of this type, we will refer to their T-coloring as a T_r-coloring. Again, we will only consider T_r for $r \geq 1$. Note that

$$\begin{gathered} c \text{ is a } T_r\text{-coloring of } CW_{a_1,a_2,a_3,\ldots,a_{2j}} \\ \text{if and only if} \\ (x,y) \in A\left(CW_{a_1,a_2,a_3,\ldots,a_{2j}}\right) \Rightarrow c(x) - c(y) = r \text{ or } c(x) - c(y) < 0. \end{gathered} \tag{1}$$

2. $CW_{a_1,a_2,a_3,\ldots,a_{2j}}$

Our first two lemmas prove a relationship between the colors of most of the vertices on the circuit of the directed wheel graph $CW_{a_1,a_2,a_3,\ldots,a_{2j}}$ and the color of the hub vertex for any T_r-coloring. Let the vertices of $CW_{a_1,a_2,a_3,\ldots,a_{2j}}$ be labeled as in Figure 1.

Lemma 1. *For any T_r-coloring c of digraph $CW_{a_1,a_2,a_3,\ldots,a_{2j}}$, $c(b_i) < c(y)$ for all introverts b_i such that b_{i+1} is also an introvert where y is the hub vertex.*

Proof. Note that $c(b_i) \neq c(y)$ since $(b_i, y) \in A\left(CW_{a_1,a_2,a_3,\ldots,a_{2j}}\right)$. So, suppose that

$$c(b_i) > c(y) \text{ for some introvert } b_i \text{ such that } b_{i+1} \text{ is also an introvert.} \tag{2}$$

Since $(b_i, y) \in A\left(CW_{a_1,a_2,a_3,\ldots,a_{2j}}\right)$ and applying (1),

$$c(b_i) - c(y) = r. \tag{3}$$

Now consider $c(b_{i+1})$. Since $(b_{i+1}, y) \in A\left(CW_{a_1,a_2,a_3,\ldots,a_{2j}}\right)$, $c(b_{i+1}) \neq c(y)$.

Case 1. $c(b_{i+1}) < c(y)$. By (2), we have $c(b_{i+1}) < c(b_i)$. Since $(b_i, b_{i+1}) \in A\left(CW_{a_1,a_2,a_3,\ldots,a_{2j}}\right)$,

$$c(b_i) - c(b_{i+1}) = r. \tag{4}$$

By equations (3) and (4), $c(y) = c(b_{i+1})$, which is a contradiction since $(b_{i+1}, y) \in A\left(CW_{a_1,a_2,a_3,\ldots,a_{2j}}\right)$.

Case 2. $c(b_{i+1}) > c(y)$. Since $(b_{i+1}, y) \in A\left(CW_{a_1,a_2,a_3,\ldots,a_{2j}}\right)$ and applying (1),

$$c(b_{i+1}) - c(y) = r. \tag{5}$$

By equations (3) and (5), $c(b_i) = c(b_{i+1})$ which is a contradiction since $(b_i, b_{i+1}) \in A\left(CW_{a_1,a_2,a_3,\ldots,a_{2j}}\right)$. ■

Lemma 1 proved that for any T_r-coloring of $CW_{a_1,a_2,a_3,\ldots,a_{2j}}$ and any maximal set of consecutive introverts, $b_i, b_{i+1}, \ldots, b_{i+h}$, the color assigned to each introvert, except possibly b_{i+h}, is less than the color assigned to the hub vertex y.

Similarly, Lemma 2 proves that for any T_r-coloring of $CW_{a_1,a_2,a_3,\ldots,a_{2j}}$ and any maximal set of consecutive extroverts, $b_i, b_{i+1}, \ldots, b_{i+h}$, the color assigned to each extrovert, except possibly b_i, is greater than the color assigned to the hub vertex y.

Lemma 2. *For any T_r-coloring c of digragh $CW_{a_1,a_2,a_3,\ldots,a_{2j}}$, $c(b_i) > c(y)$ for all extroverts b_i such that b_{i-1} is also an extrovert where y is the hub vertex.*

Proof. Note that $c(b_i) \neq c(y)$ since $(y, b_i) \in A\left(CW_{a_1,a_2,a_3,\ldots,a_{2j}}\right)$. So, suppose that

(6) $c(b_i) < c(y)$ for some extrovert b_i such that b_{i-1} is also an extrovert.

Since $(y, b_i) \in A\left(CW_{a_1,a_2,a_3,\ldots,a_{2j}}\right)$ and applying (1),

$$c(y) - c(b_i) = r. \tag{7}$$

Now consider $c(b_{i-1})$. Since $(y, b_{i-1}) \in A\left(CW_{a_1,a_2,a_3,\ldots,a_{2j}}\right)$, $c(y) \neq c(b_{i-1})$.

Case 1. $c(b_{i-1}) > c(y)$. By (6), we have $c(b_{i-1}) > c(b_i)$. Since $(b_{i-1}, b_i) \in A\left(CW_{a_1,a_2,a_3,\ldots,a_{2j}}\right)$,

$$c(b_{i-1}) - c(b_i) = r. \tag{8}$$

By (7) and (8), we have $c(y) = c(b_{i-1})$, which is a contradiction since $(y, b_{i-1}) \in \left(CW_{a_1,a_2,a_3,\ldots,a_{2j}}\right)$.

Case 2. $c(b_{i-1}) < c(y)$. Since $(y, b_{i-1}) \in A\left(CW_{a_1,a_2,a_3,\ldots,a_{2j}}\right)$,

$$c(y) - c(b_{i-1}) = r. \tag{9}$$

By (7) and (9), we have $c(b_i) = c(b_{i-1})$ which is a contradiction since $(b_{i-1}, b_i) \in A\left(CW_{a_1,a_2,a_3,\ldots,a_{2j}}\right)$. ■

Consider the special case when our directed wheel graph only has one set of introverts and one set of extroverts. We will use the notation $CW_{k,l}$ for such a digraph and use the vertex labeling as in Figure 3. The next corollary and lemma consider the relationship between $c(v_l)$ and $c(u_1)$, as well as $c(u_k)$ and $c(v_1)$, i.e., the colors of the last/first introvert and first/last extrovert, for any T_r-coloring c of $CW_{k,l}$.

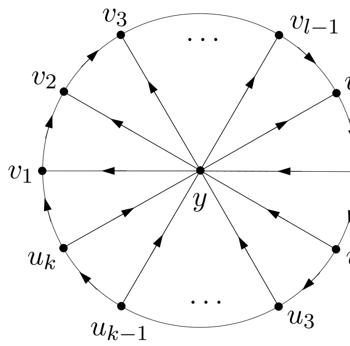


Figure 3. $CW_{k,l}$.

Corollary 3. *In any T_r-coloring c of $CW_{k,l}$ where $k, l \geq 2$,*

$$c(v_l) > c(u_1).$$

Proof. Since $k, l \geq 2$, $c(u_1) < c(y) < c(v_l)$ by Lemmas 1 and 2. ■

Lemma 4. *In any T_r-coloring c of $CW_{k,l}$, $c(v_1) > c(u_k)$.*

Proof. Since $(u_k, v_1) \in A(CW_{k,l})$, $c(u_k) \neq c(v_1)$. So, suppose that $c(u_k) > c(v_1)$. Since $(u_k, v_1) \in A(CW_{k,l})$, $c(v_1) = c(u_k) - r$. Since $(u_k, y) \in A(CW_{k,l})$,

$$(a)\ c(y) = c(u_k) - r \text{ or } (b)\ c(y) > c(u_k).$$

Also, since $(y, v_1) \in A(CW_{k,l})$,

$$(c)\ c(y) = (c(u_k) - r) + r = c(u_k) \text{ or } (d)\ c(y) < c(u_k) - r.$$

We reach a contradiction since no pair of these conditions, (a)–(c), (a)–(d), (b)–(c), or (b)–(d), can occur simultaneously. ■

Corollary 5. *In any T_r-coloring c of $CW_{k,l}$, it is never the case that both $c(u_k) > c(y)$ and $c(v_1) < c(y)$.*

Our next lemma considers those vertices of $CW_{a_1,a_2,a_3,\ldots,a_{2j}}$ which cannot be colored 1, i.e., the minimum color of any T-coloring.

Lemma 6. *For any T_r-coloring c of digragh $CW_{a_1,a_2,a_3,\ldots,a_{2j}}$, $c(x) > 1$ for any extrovert $x \in V\big(CW_{a_1,a_2,a_3,\ldots,a_{2j}}\big)$.*

Proof. We prove this lemma for the digraph CW_{a_1,a_2}, i.e., $CW_{k,l}$. The general case for $CW_{a_1,a_2,a_3,\ldots,a_{2j}}$ follows immediately. Let the vertices of $CW_{k,l}$ be labeled as in Figure 3.

Case 1. consider $v_i \in V(CW_{k,l})$ for $i = 2, 3, \ldots, l$. By Lemma 2, $c(v_i) > c(y)$ Thus, $c(v_i) > 1$, for $i = 2, 3, \ldots, l$.

Case 2. consider $v_1 \in V(CW_{k,l})$ and suppose that $c(v_1) = 1$. Then, $c(y) = r + 1$ since $(y, v_1) \in A(CW_{k,l})$. Also, $c(u_k) = r + 1$ since $(u_k, v_1) \in A(CW_{k,l})$. Therefore, $c(y) = c(u_k)$ which is a contradiction since $(u_k, y) \in A(CW_{k,l})$. Thus, $c(v_1) > 1$. ■

We end this section by showing that the T-span of $CW_{a_1,a_2,\ldots,a_{2j}}$ is not entirely dependent on the order of the a_is. Note that we will use the notation $\overleftarrow{D}$ to denote the digraph obtained by reversing all of the arcs of digraph D, i.e., the *reverse digraph* of D.

Lemma 7. $sp_{T_r}(CW_{k,l}) = sp_{T_r}\left(\overleftarrow{CW}_{k,l}\right)$.

Proof. Let c be a T_r-coloring of $CW_{k,l}$ and let $M_c = \max c(x)$ for $x \in V(CW_{k,l})$. Then define a new coloring d of $CW_{k,l}$ by

$$d(x) = M_c - c(x) + 1, \text{ for } x \in V(CW_{k,l}).$$

Suppose that $(x, z) \in \overleftarrow{CW}_{k,l}$, i.e., $(z, x) \in CW_{k,l}$. Then $d(x) - d(z) = (M_c - c(x) + 1) - (M_c - c(z) + 1) = c(z) - c(x) < 0$ or equals r by (1). Therefore, d is a T_r-coloring of $\overleftarrow{CW}_{k,l}$ and *not* a T_r-coloring of $CW_{k,l}$.

The maximum and minimum values of c are M and 1, respectively. Then, $\max d(x) = M - 1 + 1 = M$ and $\min d(x) = M - M + 1 = 1$ since $d(x)$ is a strictly decreasing linear function of $c(x)$. Therefore, the span of d is less than or equal to the span of c which implies $sp_{T_r}\left(\overleftarrow{CW}_{k,l}\right) \leq sp_{T_r}(CW_{k,l})$.

A similar proof shows that $sp_{T_r}(CW_{k,l}) \leq sp_{T_r}\left(\overleftarrow{CW}_{k,l}\right)$ using the fact that $\overleftarrow{\overleftarrow{CW}}_{k,l} = CW_{k,l}$.

Therefore, $sp_{T_r}(CW_{k,l}) = sp_{T_r}\left(\overleftarrow{CW}_{k,l}\right)$. ■

Theorem 8. $sp_{T_r}(CW_{k,l}) = sp_{T_r}(CW_{l,k})$.

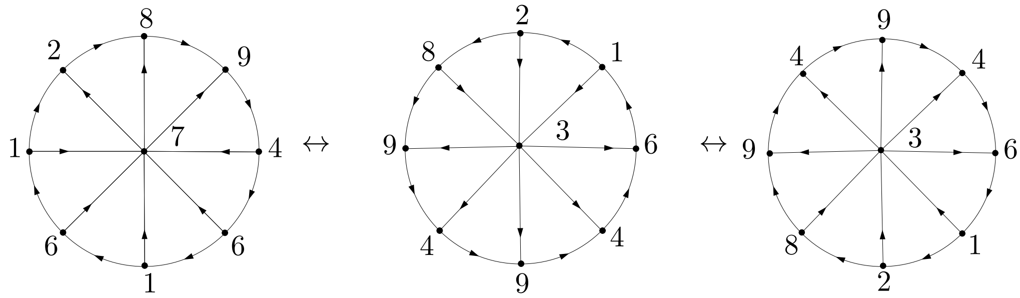
Proof. Note that $\overleftarrow{CW}_{k,l}$ has a consecutive set of k extroverted arcs and a consecutive set of l introverted arcs since $CW_{k,l}$ has a consecutive set of k introverted arcs and a consecutive set of l extroverted arcs which are being reversed. Then, using a mirror image of $CW_{k,l}$, it is easy to see that $\overleftarrow{CW}_{k,l}$ is isomorphic to $CW_{l,k}$. Therefore, by Lemma 7, the theorem follows. ■

Although we have no need for a stronger result, it can also be shown that

$$sp_{T_r}\left(CW_{a_1,a_2,\ldots,a_{2j}}\right) = sp_{T_r}\left(CW_{a_{i+1},a_{i+2},\ldots,a_{2j-1},a_{2j},a_1,a_2,\ldots,a_i}\right)$$

using a similar proof as above and the second footnote on page 2.

An example of Lemma 7 and Theorem 8 for a T_5-coloring of $CW_{5,3}$ is shown in Figure 4.



(a) $CW_{5,3}$ (b) $\overleftarrow{CW}_{5,3}$ (c) $CW_{3,5}$

Figure 4. Equal span T_5-colorings of $CW_{5,3}$, $\overleftarrow{CW}_{5,3}$, $CW_{3,5}$.

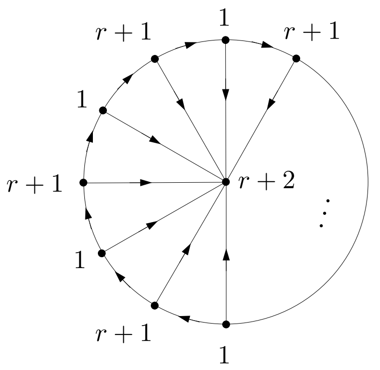
3. $sp_{T_r}(CW_{k,l})$, $r > 1$

In this section and the next, we consider the special case of $CW_{k,l}$, i.e., where the directed wheel graph's spokes only consist of a consecutive set of introverted arcs and a consecutive set of extroverted arcs. We assume that $r > 1$; the special case of $r = 1$ is addressed in Section 4. To prove our main result, i.e., the calculation of $sp_{T_r}(CW_{k,l})$, we consider different conditions on k and l.

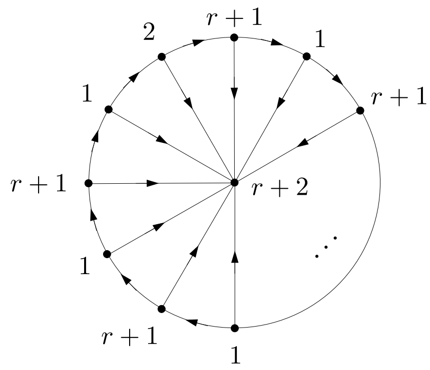
First, we consider CW_k and $CW_{0,l}$, i.e., the directed wheel graphs with only introverted arcs and only extroverted arcs. Recall that we only consider CW_k for $k \geq 3$, and $CW_{0,l}$, for $l \geq 3$.

Theorem 9. *For* $r > 1$,

$$sp_{T_r}(CW_k) = sp_{T_r}(CW_{0,l}) = r + 1.$$



(a) k even



(b) k odd

Figure 5. Optimal T_r-colorings of CW_k.

Proof. Applying Theorem 8, it is sufficient to show that $sp_{T_r}(CW_k) = r + 1$. Figure 5 provides T_r-colorings of CW_k, depending on the parity of k, whose span is $r + 1$. Next we show that the span of any T_r-coloring of CW_k is greater than or equal to $r + 1$.

Let the vertices of CW_k be labeled as in Figure 1 and c be a T_r-coloring of CW_k. By Lemma 1, $c(b_i) < c(y)$ for $i = 1, 2, \ldots, k$. Without loss of generality, let $c(b_1) = 1$. Then $c(b_k) = r+1$ since $(b_k, b_1) \in A(CW_k)$. Therefore, $c(y) \geq r+2$ which implies that the span of the T_r-coloring c of CW_k is greater than or equal to $r + 1$. Therefore, $sp_{T_r}(CW_k) = r + 1$. ■

Next, we prove our main result for $CW_{k,l}$. Note that the exceptions to the theorem are treated in Section 6.

Theorem 10. *For $r > 1$,*

$$sp_{T_r}(CW_{k,l}) = \begin{cases} r & k+l \leq r \textit{ and } \min(k,l) > 0, \\ r+1 & k+l > r \textit{ and } \min(k,l) = 1, \\ \min(k,l)+r & k+l > r \textit{ and } 1 < \min(k,l) \leq r, \\ 2r+2 & k+l > r \textit{ and } \min(k,l) > r, \end{cases}$$

except for:

r	5	4	4	3	3	3	3	2	2	2	2
k	4	3	4	2	4	3	2	2	3	1	3
l	4	4	3	4	2	3	2	3	2	3	1

Proof. *Case* 1. $k + l \leq r$ and $\min(k, l) > 0$. Let $CW_{k,l}$ be labeled as in Figure 3 and c be the following coloring:

$$c(u_s) = s, \text{ when } s = 1, 2, \ldots, k;$$

$$c(y) = k + 1;$$

$$c(v_t) = r - (l - 1) + t, \text{ when } t = 1, 2, \ldots, l.$$

See Figure 6. Note that the minimum color of the extroverts, $c(v_1) = r - l + 2$, is greater than $c(y) = k + 1$ since $k + l \leq r$. Thus, the sequence of colors $c(u_1)$, $c(u_2), \ldots, c(u_k), c(v_1), c(v_2), \ldots, c(v_l)$ is strictly increasing and $c(v_l) > c(u_1) \Rightarrow c(v_1) - c(v_1) = r$. Also, $c(y) > c(u_i)$ for $i = 1, 2, \ldots, k$ and $c(y) < c(v_j)$ for $j = 1, 2, \ldots, l$. Thus, by (1), c is a T_r-coloring of $CW_{k,l}$ and the span of c is r.

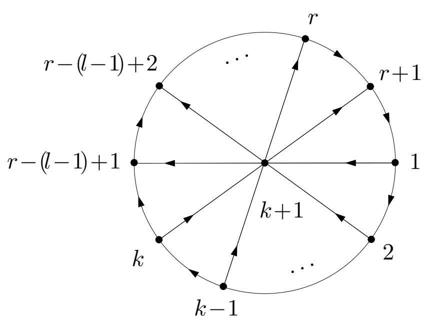


Figure 6. Optimal T_r-coloring of $CW_{k,l}$, $k + l \leq r$.

Next, we show that the span of any T_r-coloring of $CW_{k,l}$ is greater than or equal to r which implies $sp_{T_r}(CW_{k,l}) = r$. Let d be a T_r-coloring of $CW_{k,l}$ and let $CW_{k,l}$ be labeled as in Figure 1. The sequence of colors on the non-hub vertices, $d(b_1), d(b_2), \ldots, d(b_n), d(b_1)$, can not be strictly increasing since the vertices form a cycle. When the sequence decreases, it must do so by exactly r. Therefore, the span of T_r-coloring d of $CW_{k,l}$ is greater than or equal to r.

Case 2. $k + l > r$ and $\min(k, l) = 1$. Applying Theorem 8, it is sufficient to only consider $\min(k, l) = k = 1$, i.e., $CW_{1,l}$. Figure 7 provides T_r-colorings with span $r + 1$, i.e., whose maximum color is $r + 2$, for $CW_{1,l}$ except when $r = 2$ and $l = 3$. This special case, as well as when $r = 2$, $k = 3$, and $l = 1$, are addressed in Section 6.

Next we show that if c is any T_r-coloring of $CW_{1,l}$, $k + l > r$ and $\min(k, l) = k = 1$, then $\max c(x) \geq r + 2$ for $x \in V(CW_{1,l})$ which implies $sp_{T_r}(CW_{1,l}) = r + 1$, for all but the special case. Let the vertices of $CW_{1,l}$ be labeled as in Figure 3 and c be a T_r-coloring of $CW_{1,l}$. By Lemma 6, the minimum color 1 must be assigned to an introvert or the hub vertex.

Case 2.a. $c(y) = 1$. Each $c(b_i) \neq 1$ since (b_i, y) or $(y, b_i) \in A(CW_{1,l})$, for $i = 1, 2, \ldots, n$. Let $m = \min c(b_i)$, for $i = 1, 2, \ldots, n$. Then $m \geq 2$ and the color on the circuit vertex preceding it in the sequence $b_1, b_2, \ldots, b_n, b_1$ must be colored $r + m \geq r + 2$.

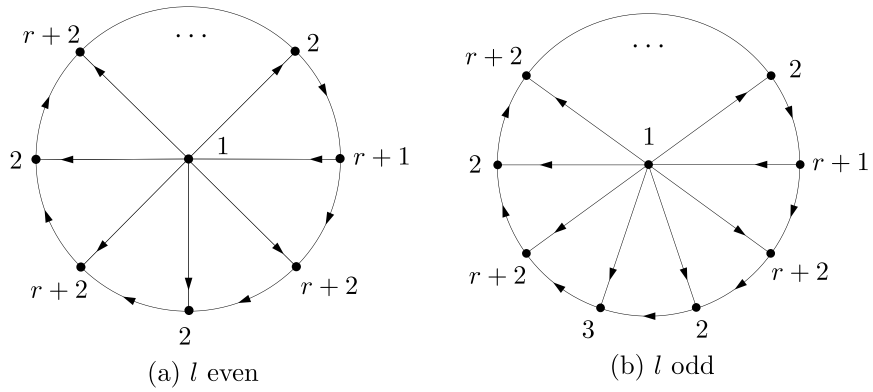


Figure 7. Optimal T_r-colorings of $CW_{1,l}$.

Case 2.b. $c(u_1) = 1$. By Lemma 6, $c(v_i) > 1$, for $i = 1, 2, \ldots, l$ and $c(y) > 1$ since $(u_1, y) \in A(CW_{1,l})$. Note that $k + l > r \Rightarrow 1 + l > r \Rightarrow l > r - 1 \Rightarrow l \geq r$. Assume $\max c(x) \leq r + 1$, for $x \in V(CW_{1,l})$. Thus there are r available colors, $2, 3, \ldots, r + 1$, to color $l + 1 \geq r + 1$ vertices, $y, v_1, v_2, \ldots, v_l$. By the pigeonhole principle, at least two of these vertices must be colored the same and neither of them can be y since $(y, v_i) \in A(CW_{1,l})$ for all i. Thus, the sequence of colors $c(v_1), c(v_2), \ldots, c(v_l)$ is not strictly increasing. Suppose that $c(v_i) > c(v_{i+1})$. Then, $c(v_i) - c(v_{i+1}) = r \Rightarrow c(v_i) = c(v_{i+1}) + r \geq 2 + r$ which is a contradiction.

Therefore, $sp_{T_r}(CW_{1,l}) = r + 1$ (for all but the special cases).

Case 3. $k + l > r$ and $1 < \min(k, l) = k \leq r$. Applying Theorem 8, it is sufficient to only consider $\min(k, l) = k$. Figures 8, 9 and 10 provide T_r-colorings with span $k + r$, i.e., whose maximum color is $k + r + 1$, for $CW_{k,l}$ except for the following six cases: $l = 4$ and $k = r - 1$ for $r = 3, 4, 5$; $l = 3$ and $k = r$ for $r = 2, 3$; and $l = 2$ and $k = r - 1$ for $r = 3$, as well as their corresponding reverse digraphs. All of these special cases are addressed in Section 6.

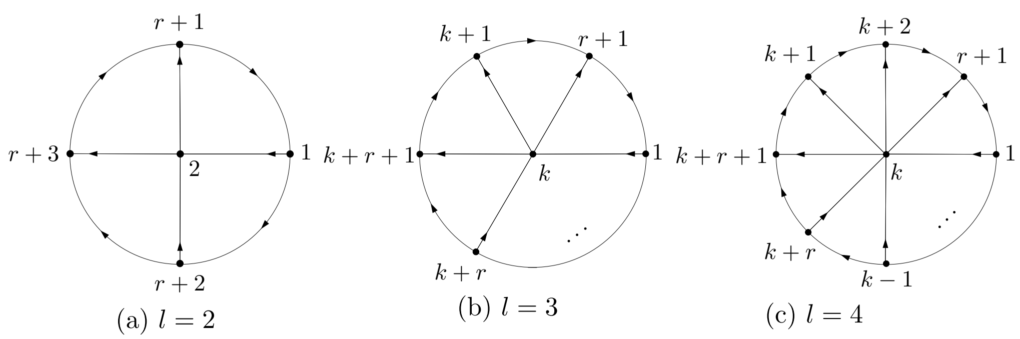
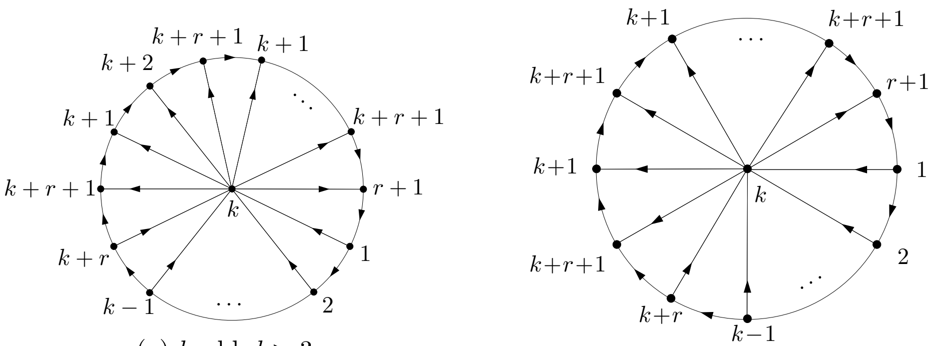


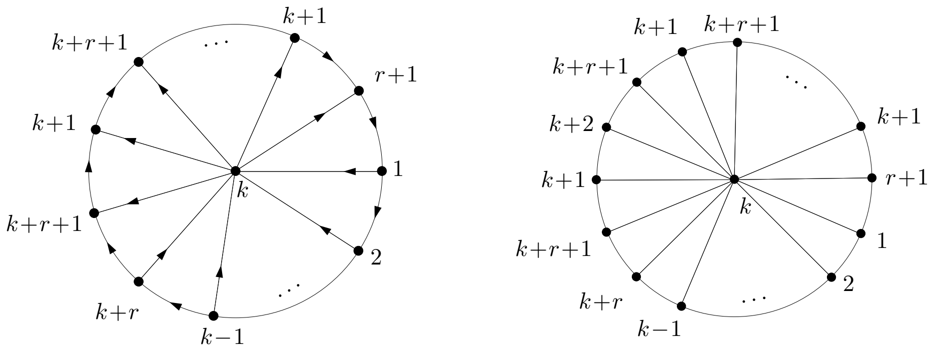
Figure 8. Optimal T_r-coloring of $CW_{k,l}$, $k + l > r$ and $1 < \min(k, l) = k \leq r$.



(a) l odd, $l > 3$

(b) l even, $l > 4$

Figure 9. Optimal T_r-colorings of $CW_{k,l}$, $k + l > r$ and $1 < \min(k, l) = k = r$.



(a) l odd, $l > 3$

(b) l even, $l > 4$

Figure 10. Optimal T_r-colorings of $CW_{k,l}$, $k + l > r$ and $1 < \min(k, l) = k < r$.

Next we show that if c is any T_r-coloring of $CW_{k,l}$, $k + l > r$ and $1 < \min(k, l) = k \le r$, then $\max c(x) \ge k + r + 1$ for $x \in V(CW_{k,l})$ which implies $sp_{T_r}(CW_{k,l}) = k + r$, for all of the cases addressed in Figures 8, 9 and 10. Let the vertices of $CW_{k,l}$ be labeled as in Figure 3 and c be a T_r-coloring of $CW_{k,l}$. By Lemma 6, the minimum color 1 must be assigned to an introvert or the hub vertex.

Case 3.a. Suppose $c(u_i) = 1$, for $i \ne 1$. Then $c(u_{i-1}) = r+1$ since $(u_{i-1}, u_i) \in A(CW_{k,l})$. Also, by Lemma 1, $c(y) \ge r+2$ and, by Lemma 2, $c(v_i) > c(y) \ge r+2$, for $i = 2, 3, \ldots, l$. Assume $\max c(x) \le k + r$, for $x \in V(CW_{k,l})$. Thus there are $k-1$ available colors, $r+2, r+3, \ldots, r+k$, to color $l \ge k$ vertices, $y, v_2, v_3, \ldots, v_l$. By the pigeonhole principle, at least two of these vertices must be colored the same and neither of these same-colored vertices can be y since $(y, v_i) \in A(CW_{k,l})$ for all i. Thus, the sequence of colors $c(v_1), c(v_2), \ldots, c(v_l)$ is not strictly increasing. Suppose that $c(v_i) > c(v_{i+1})$. Then, $c(v_i) - c(v_{i+1}) = r \Rightarrow c(v_i) = c(v_{i+1}) + r \ge (r + 2) + r \ge k + r + 2$, since $k \le r$, which is a contradiction.

Thus, if $c(u_i) = 1$, for $i \neq 1$, then $\max c(x) \geq k + r + 1$, for $x \in V(CW_{k,l})$.

Case 3.b. Suppose that $c(u_1) = 1$. Then $c(u_i) \neq 1$ for $i = 2, 3, \ldots, k$ and $c(v_l) = r + 1$ since $(v_l, u_1) \in A(CW_{k,l})$.

Case 3.b.i. $c(u_k) < c(y)$ and $c(v_1) < c(y)$. Since $c(v_l) = r + 1$ then, by Lemma 2, $c(y) \leq r$. Also since $(y, v_1) \in A(CW_{k,l})$ and $c(y) > c(v_1)$ then $c(y) - c(v_1) = r \Rightarrow c(v_1) = c(y) - r \leq 0$ which is a contradiction.

Case 3.b.ii. $c(u_k) < c(y)$ and $c(v_1) > c(y)$. Since $c(u_k) < c(y)$, then the color of each of the k introverts is less than $c(y)$. If the introverts' colors are distinct then $c(y) \geq k + 1$. If the introverts' colors are not distinct then there exists a u_i such that $c(u_i) > c(u_{i+1}) \Rightarrow c(u_i) = r + c(u_{i+1}) \geq r + 2$ since $c(u_{i+1}) > 1$. Thus, $c(y) > c(u_i) \geq r + 2 \geq k + 2 \Rightarrow c(y) > k + 2$.
Thus, $c(y) \geq k + 3$.

Case 3.b.ii.α. $c(v_i) > c(v_{i+1})$ for some $i \in \{1, 2, \ldots, l - 1\}$. Thus, $c(v_i) = c(v_{i+1}) + r$. Since $c(v_{i+1}) > c(y) \geq k + 3$, then $c(v_i) = c(v_{i+1}) + r > k + 3 + r$. Thus, $\max c(x) \geq k + r + 1$ for $x \in V(CW_{k,l})$.

Case 3.b.ii.β. $c(v_i) < c(v_{i+1})$ for all $i \in \{1, 2, \ldots, l-1\}$. Thus, the extroverts' colors are all distinct and all greater than $c(y)$. They are also strictly increasing from v_1 to v_l and they are all less than or equal to $r+1$ since $c(v_l) = r+1$. Since $c(v_1) > c(y) \geq k+3$, then $c(v_l) \geq k+3+l > r+3$ which contradicts $c(v_l) = r+1$.

Case 3.b.iii. $c(u_k) > c(y)$. Then $c(u_k) = c(y) + r$ since $(u_k, y) \in A(CW_{k,l})$.

Case 3.b.iii.α. $c(v_1) < c(u_k)$. Thus, we reach a contradiction by Lemma 4.

Case 3.b.iii.β. $c(v_1) > c(u_k)$. Thus, $c(v_1) > r + c(y)$. Since $c(y) \geq k$ by Lemma 1, then $c(v_1) \geq k+r+1$, i.e., $\max c(x) \geq k+r+1$ for any $x \in V(CW_{k,l})$.

Case 3.c. $c(y) = 1$. By Lemma 1, it must be the case that $k = 1$ otherwise there will be an introvert's color which is less than 1. This contradicts a condition of this case.

Therefore, for all but the special cases, $\max c(x) \geq k + r + 1$ for any $x \in V(CW_{k,l})$ and our T_r-colorings in Figures 8, 9 and 10 are optimal with respect to span.

Case 4. $k + l > r$ and $\min(k, l) > r$. Note that Figure 11 provides a T_r-coloring of $CW_{k,l}$ whose span is $2r + 2$. Next, we show that the span of any T_r-coloring, c, of $CW_{k,l}$ is greater than or equal to $2r + 2$.

Again, let the vertices of $CW_{k,l}$ be labeled as in Figure 3. By Lemmas 1 and 2, $c(u_i) < c(y)$, for all $1 \leq i \leq k - 1$, and $c(v_j) > c(y)$, for all $2 \leq j \leq l$. We consider four cases:

(a) $c(u_k) > c(y)$ and $c(v_1) < c(y)$; (b) $c(u_k) < c(y)$ and $c(v_1) < c(y)$;
(c) $c(u_k) < c(y)$ and $c(v_1) > c(y)$; (d) $c(u_k) > c(y)$ and $c(v_1) > c(y)$.

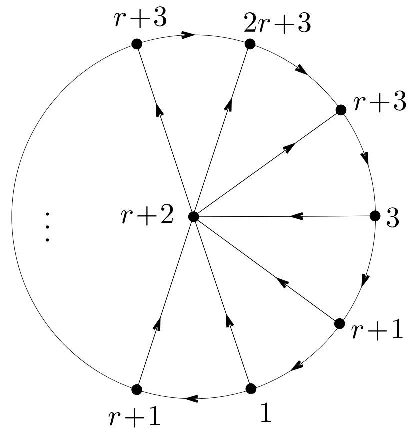


Figure 11. Optimal T_r-coloring of $CW_{k,l}$, $k+l > r$ and $\min(k,l) > r$.

Case 4.a. $c(u_k) > c(y)$ and $c(v_1) < c(y)$. This case is not possible by Corollary 5.

Case 4.b. $c(u_k) < c(y)$ and $c(v_1) > c(y)$. Since $c(u_k) < c(y)$, then $c(u_i) < c(y)$ for $i = 1, 2, \ldots, k$. If the $c(u_i)$s are all distinct then $c(y) \geq r+2$ since $k > r$. If the $c(u_i)$s are not all distinct then $c(u_j) < c(u_{j-1})$ for at least one j since colors on consecutive u_is can not be equal. Given that $c(u_j) \geq 1$ then $c(u_{j-1}) \geq r+1$ since $(u_{j-1}, u_j) \in A(CW_{k,l})$. So, in general, $c(y) \geq r+2$.

Since $c(v_1) > c(y)$, then $c(v_j) > c(y)$ for $j = 1, 2, \ldots, l$. If the $c(v_j)$s are all distinct and since $l > r$ then $\max c(v_j) \geq c(y) + l \geq c(y) + (r+1) \geq (r+2) + (r+1) = 2r+3$. If the $c(v_j)$s are not all distinct then $c(v_i) < c(v_{i-1})$ for at least one i since colors on consecutive v_is can not be equal. Since $c(v_i) > c(y)$ and $c(y) \geq r+2$ then $c(v_i) \geq r+3$, for all i. Thus, $c(v_{i-1}) \geq (r+3) + r = 2r+3$ since $(v_{i-1}, v_i) \in A(CW_{k,l})$. So, in general, $\max c(v_j) \geq 2r+3$. Therefore, in this case, the span of any T_r-coloring of $CW_{k,l}$ is greater than or equal to $2r+2$.

Case 4.c. $c(u_k) < c(y)$ and $c(v_1) < c(y)$. As in Case 4.b, since $c(u_k) < c(y)$ and thus $c(u_i) < c(y)$ for all $i = 1, 2, \ldots, k$, we know that $c(y) \geq r+2$.

Case 4.c.i. $c(v_i) > c(v_{i+1})$ for some $i = 2, 3, \ldots, l-1$. Thus, $c(v_i) - c(v_{i+1}) = r \Rightarrow c(v_i) = c(v_{i+1}) + r$ since $(v_i, v_{i+1}) \in A(CW_{k,l})$. Since $c(v_{i+1}) > c(y)$, then $c(v_{i+1}) \geq r+3$. Therefore, $c(v_i) \geq (r+3) + r = 2r+3$.

Case 4.c.ii. $c(v_i) < c(v_{i+1})$, for all $i = 2, 3, \ldots, l-1$. Since $c(v_i) > c(y)$, for $i = 2, 3, \ldots l$, and $c(y) \geq r+2$, then $c(v_l) \geq c(y) + (l-1) \geq c(y) + r$ since $l > r$. Thus, $c(v_l) \geq c(y) + l - 1 \geq c(y) + r$. However, $c(u_1) \leq c(y) - 1$. We reach a contradiction since $(v_l, u_1) \in A(CW_{k,l})$ but $c(v_l) - c(u_1) \geq c(y) + r - (c(y) - 1) = r + 1 \notin T_r$.

So, it must be the case that $\max c(v_j) \geq 2r+3$. Therefore, in this case, the span of any T_r-coloring of $CW_{k,l}$ is greater than or equal to $2r+2$.

Case 4.d. $c(u_k) > c(y)$ and $c(v_1) > c(y)$.

Case 4.d.i. $c(u_1) = 1$. Given that $c(u_k) > c(y)$, then $c(u_i) < c(y)$ for $i = 1, 2, \ldots, k-1$. If these $c(u_i)$s are all distinct then $c(y) \geq r+1$ since $k > r$. However, by Lemma 2, $c(v_l) > c(y) \Rightarrow c(v_l) \geq r+2$. We reach a contradiction since $(v_l, u_1) \in A(CW_{k,l})$ but $c(v_l) - c(u_1) \geq (r+2) - 1 = r+1 \notin T_r$. If $c(u_i)$, for $i = 1, 2, \ldots, k-1$, are not all distinct then $c(u_j) < c(u_{j-1})$ for at least one j since colors on consecutive u_is can not be equal. Since $c(u_j) \geq 1$, then $c(u_{j-1}) \geq r+1$ since $(u_{j-1}, u_j) \in A(CW_{k,l})$. So, in general, $c(y) \geq r+2$. Now that we know $c(y) \geq r+2$, then, as in Case 4.b, since $c(v_j) > c(y)$ for all $j = 1, 2, \ldots, l$, $\max c(v_j) \geq 2r+3$.

Case 4.d.ii. $c(u_1) \neq 1$. By Lemma 1, Lemma 6, and given that $c(u_k) > c(y)$, then $c(y) \neq 1$ and $c(u_i) = 1$ for some $i = 2, 3, \ldots, k-1$. This implies $c(u_{i-1}) = r+1 \Rightarrow c(y) \geq r+2$. Then, as in Case 4.b, since $c(v_j) > c(y)$ for all $j = 1, 2, \ldots, l$, $\max c(v_j) \geq 2r+3$.

Thus, in this final case, the span of any T_r-coloring of $CW_{k,l}$ is again greater than or equal to $2r+2$.

Therefore, $sp_{T_r}(CW_{k,l}) = 2r+2$ for $k+l > r$ and $\min(k,l) > r$. ∎

4. $CW_{k,l}$, $r = 1$

In this section, we consider the case of T_r-coloring $CW_{k,l}$ where $k, l \geq 0$ and $r = 1$. First, we consider the "all-introvert" case, CW_n, and the "all-extrovert" case, $CW_{0,n}$.

Theorem 11. *For $n \geq 2$,* $sp_{T_1}(CW_n) = sp_{T_1}(CW_{0,n}) = \begin{cases} 2 & \text{for } n \text{ even,} \\ 3 & \text{for } n \text{ odd.} \end{cases}$

Proof. Applying Theorem 8, it is sufficient to prove this theorem for only CW_n.

Case 1. n even. The same proof as in Theorem 9 holds here for CW_n.

Case 2. n odd. Figure 12 provides a T_1-coloring of CW_n whose span is 3. Next we show that the span of any T_1-coloring of CW_n is greater than or equal to 3.

Let c be any T_1-coloring of CW_n. At least four distinct colors must be used to T_1-color the vertices of CW_n since its underlying graph is a wheel graph with an odd cycle and $0 \in T_1$. Therefore, the span of c is greater than or equal to $4 - 1 = 3$. ∎

The next theorem considers the case of $CW_{k,l}$ where $r = 1$ and $\min(k,l) = 1$.

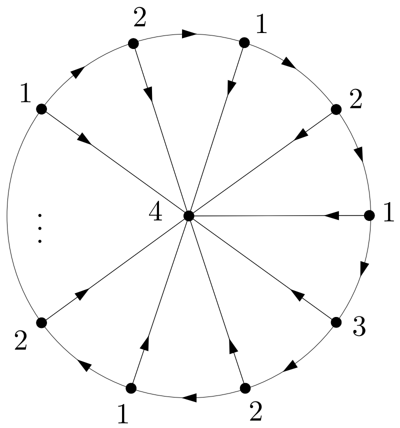
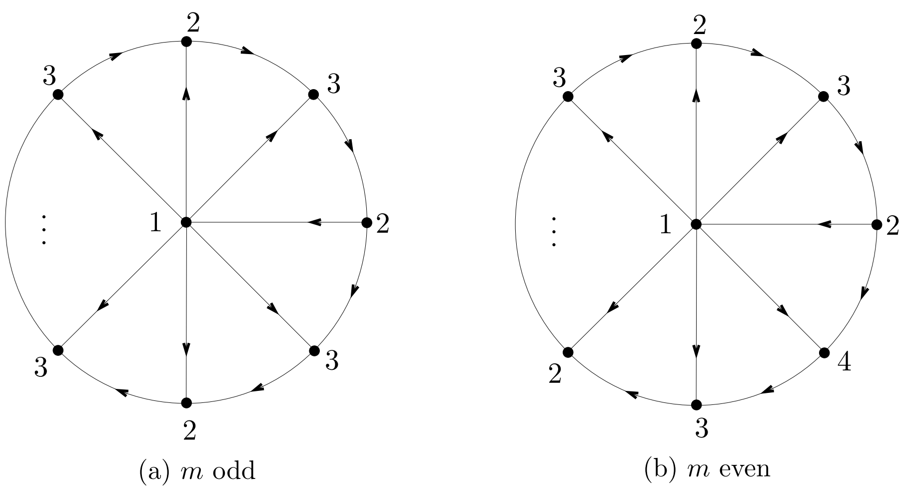


Figure 12. Optimal T_1-coloring of CW_n, n odd.

Theorem 12. $sp_{T_1}(CW_{1,m}) = sp_{T_1}(CW_{m,1}) = \begin{cases} 2 & \textit{for } m \textit{ odd,} \\ 3 & \textit{for } m \textit{ even.} \end{cases}$

Proof. Applying Theorem 8, it is sufficient to prove this theorem for only $CW_{1,m}$.

Case 1. m odd. Figure 13 provides a T_1-coloring of $CW_{1,m}$ whose span is 2. Next, let c be any T_1-coloring of $CW_{1,m}$. At least three distinct colors must be used to T_1-color the vertices of $CW_{1,m}$ since its underlying graph is a wheel graph with an even cycle and $0 \in T_1$. Therefore, the span of c is greater than or equal to $3 - 1 = 2$.



(a) m odd (b) m even

Figure 13. Optimal T_1-colorings of $CW_{1,m}$.

Case 2. m even. Figure 13(b) provides a T_1-coloring of $CW_{1,m}$ whose span is 3. Next, let c be any T_1-coloring of $CW_{1,m}$. At least four distinct colors must be used to T_1-color the vertices of $CW_{1,m}$ since its underlying graph is a wheel graph with an odd cycle and $0 \in T_1$. Therefore, the span of c is greater than or equal to $4 - 1 = 3$. ■

The case $r = 1$ is unique in the sense that it is the only case where it is not possible to T-color some directed wheel graphs $CW_{k,l}$.

Theorem 13. *If $k, l \geq 2$, then it is impossible to T_1-color $CW_{k,l}$.*

Proof. Let $CW_{k,l}$ be labeled as in Figure 3. By Lemmas 1 and 2, $c(v_l) > c(y) > c(u_1)$. Since $(v_l, u_1) \in A(CW_{k,l})$, $c(v_l) - c(u_1) = r = 1 \Rightarrow c(v_l) = c(u_1) + 1$. We reach a contradiction because $c(y)$ must be strictly between two consecutive colors: $c(u_1)$ and $c(v_l)$. ■

5. $sp_{T_r}\left(CW_{a_1,a_2,a_3,\ldots,a_{2j}}\right)$

We conclude this paper with bounds on $sp_{T_r}\left(CW_{a_1,a_2,a_3,\ldots,a_{2j}}\right)$.

Theorem 14. $r \leq sp_{T_r}\left(CW_{a_1,a_2,\ldots,a_{2j}}\right) \leq 2r + 2$.

Proof. For $CW_{a_1,a_2,\ldots,a_{2j}}$, let the vertices be labeled as in Figure 1 and c be any T_r-coloring. Consider the sequence of colors of the non-hub vertices, $c(b_1), c(b_2), \ldots, c(b_n), c(b_1)$. This sequence of colors can not be strictly increasing since the vertices form a circuit. When the sequence decreases, by (1), it must do so by exactly r. Therefore, the span of c is greater than or equal to r which implies $sp_{T_r}\left(CW_{a_1,a_2,\ldots,a_{2j}}\right) \geq r$.

Next, for each consecutive set of $a_1, a_3, a_5, \ldots, a_{2j-1}$ introverted arcs, we color their corresponding introverts, $x_1, x_2, \ldots, x_{a_i}$, as follows:

$$\text{if } r > 2: \quad 3, r+1, 1, r+1, 1, \ldots; \quad \text{or} \quad \text{if } r = 2: \quad 3, 1, 3, 1, \ldots$$

For each consecutive set of $a_2, a_4, a_6, \ldots, a_{2j}$ extroverted arcs, we color their corresponding extroverts, $x_{a_j}, x_{a_{j-1}}, \ldots, x_1$, as follows:

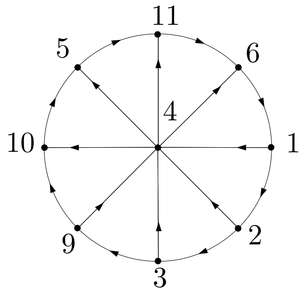
$$r+3, 2r+3, r+3, 2r+3, \ldots$$

Finally, for the hub vertex: $c(y) = r + 2$.

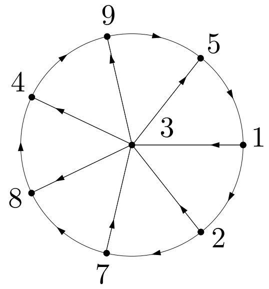
Using the labeling of Figure 1, note that $c(b_n) - c(b_1) = r$ and $c(b_i) - c(b_{i+1})$, for each $i = 1, 2, \ldots, n-1$, equals r or is negative. Also, $c(b_i) - c(y) < 0$ for every introvert b_i and $c(y) - c(b_j) < 0$ for every extrovert b_j. Therefore, c is a T_r-coloring of $CW_{a_1,a_2,\ldots,a_{2j}}$ and $sp_{T_r}\left(CW_{a_1,a_2,\ldots,a_{2j}}\right) \leq 2r + 2$. ■

6. Special Cases

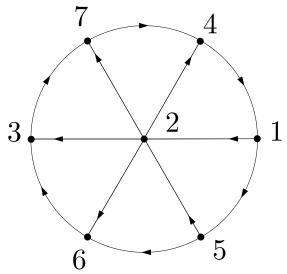
This section considers the eleven special cases which cannot be T_r-colored using the optimal colorings from Theorem 10. In fact, each of their T-spans disagrees



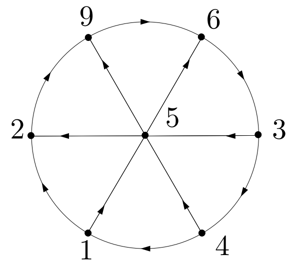
(a) $r = 5, k = 4, l = 4$



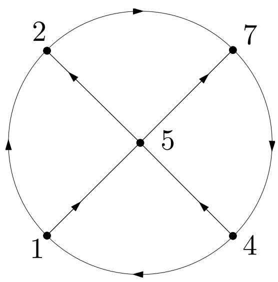
(b) $r = 4, k = 3, l = 4$



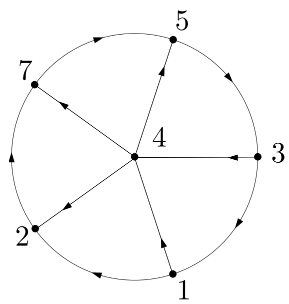
(c) $r = 3, k = 2, l = 4$



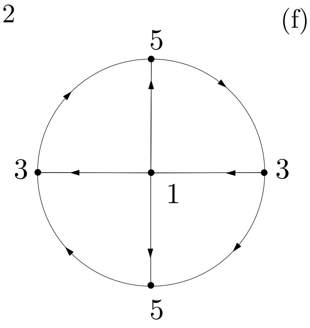
(d) $r = 3, k = 3, l = 3$



(e) $r = 3, k = 2, l = 2$



(f) $r = 2, k = 2, l = 3$



(g) $r = 2, k = 1, l = 3$

Figure 14. Optimal T_r-colorings of the special cases.

with the value that Theorem 10 would have given (see Table 1). Optimal T_r-colorings for seven of the special cases are given in Figure 14. The remaining four special cases follow from Theorem 8. Proof of their optimality is left to the reader.

Digraph	r	Theorem 10 predicted span	Actual T_r-span	Figure 14
$CW_{4,4}$	5	9	10	(a)
$CW_{3,4}$	4	7	8	(b)
$CW_{4,3}$	4	7	8	
$CW_{2,4}$	3	5	6	(c)
$CW_{4,2}$	3	5	6	
$CW_{3,3}$	3	6	8	(d)
$CW_{2,2}$	3	5	6	(e)
$CW_{2,3}$	2	4	6	(f)
$CW_{3,2}$	2	4	6	
$CW_{1,3}$	2	3	4	(g)
$CW_{3,1}$	2	3	4	

References

[1] J. Fiala, D. Král and R. Škrekovski, *A Brooks-type theorem for the generalized list T-coloring*, SIAM J. Discrete Math. **19** (2005) 588–609.
doi:10.1137/S0895480103437870

[2] W.K. Hale, *Frequency assignment: theory and applications*, Proc. IEEE **68** (1980) 1497–1514.
doi:10.1109/PROC.1980.11899

[3] R. Janczewski, *Greedy T-colorings of graphs*, Discrete Math. **309** (2009) 1685–1690.
doi:10.1016/j.disc.2008.01.049

[4] J.S.-T. Juan, I. Sun and P.-W. Wu, *T-coloring of folded hypercubes*, Taiwanese J. Math. **13** (2009) 1331–1341.
doi:10.11650/twjm/1500405511

[5] K. Junosza-Szaniawski and P. Rzążewski, *An exact algorithm for the generalized list T-coloring problem*, Discrete Math. Theor. Comput. Sci. **16** (2014) 77–94.

[6] P. Sivagami and I. Rajasingh, *T-coloring of certain networks*, Math. Comput. Sci. **10** (2016) 239–248.
doi:10.1007/s11786-016-0260-6

[7] B.A. Tesman, *TcColorings of digraphs*, Congr. Numer. **159** (2002) 167–176.

[8] B.A. Tesman, *Meaningfulness of list T-colorings of graphs*, Congr. Numer. **207** (2011) 105–109.

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