Abstract

In this paper, we consider \( T \)-colorings of directed graphs. In particular, we consider as a \( T \)-set the set \( T_r = \{0, 1, 2, \ldots, r-1, r+1, \ldots\} \). Exact values and bounds of the \( T_r \)-span of directed graphs whose underlying graph is a wheel graph are presented.

Keywords: \( T \)-coloring, digraph, wheel graph, span.

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1. Introduction

\( T \)-colorings are a generalization of proper vertex colorings of graphs. They were introduced by Hale [2] to model the frequency assignment problem. The problem in assigning frequencies to requesters comes from the need to assign them in a manner that minimizes the use of the frequency spectrum (e.g., AM radio or UHF television) while avoiding the interferences and separation constraints that can occur amongst transmitters. \( T \)-colorings have been widely studied for over three decades since Hale’s paper appeared in 1980. Much of the research has optimized \( T \)-colorings for various classes of graphs, as well as classes of the separation constraints. For example, see Sivagami and Rajasingh [6], Juan et al. [4], and Janczewski [3]. Variants of \( T \)-colorings, in particular list \( T \)-colorings, have been studied by Junosza-Szaniawski and Rzążewski [5], Tesman [8], and Fiala et al. [1]. In this paper we are concerned with \( T \)-colorings of digraphs which were introduced by Tesman [7] to model the special case of unidirectional transmitters.
A T-set is a set of nonnegative integers. Given a digraph \( D = (V,A) \) and a T-set \( T \), a T-coloring of \( D \) is a function \( c : V(D) \to \mathbb{Z}^+ \) such that if \( (x, y) \in A(D) \), then \( c(x) - c(y) \notin T \). The span of a T-coloring \( c \) of \( D \) is defined as:

\[
\text{span of } c = \max_{x \in V(D)} c(x) - \min_{x \in V(D)} c(x).
\]

The minimum span over all T-colorings of a digraph \( D \) for a fixed T-set \( T \) is called the T-span of \( D \) and denoted \( sp_T(D) \). We will assume that \( 0 \in T \) because a coloring must be proper and, without loss of generality, 1 will be the minimum color in any T-coloring. We will find bounds and the exact T-span for some special classes of directed wheel graphs.

The underlying graph of the digraphs studied in this paper is a wheel graph, \( W_n \), i.e., a circuit for which every vertex on the circuit, \( b_1, b_2, \ldots, b_n \), is connected to a single “hub” vertex, \( y \). (See Figure 1.) The chromatic number of a wheel graph, \( \chi(W_n) \), will be used in Sections 3 and 4. Recall that \( \chi(W_n) = 3 \) or 4 depending on whether or not the wheel graph’s circuit is even or odd, respectively. These values provide lower bounds for the T-span of their respective directed wheel graphs since any T-coloring is also a proper vertex coloring because 0 is in every T-set.

The underlying wheel graph’s circuit edges \( \{b_i, b_{i+1}\} \), for \( i = 1, 2, \ldots, n - 1 \), and \( \{b_1, b_n\} \) will be directed clockwise, i.e., \( (b_i, b_{i+1}) \), for \( i = 1, 2, \ldots, n - 1 \), and \( (b_n, b_1) \) will be the circuit arcs of the digraph. (Note that a counterclockwise orientation of the circuit edges will yield the same results that we prove in this paper; the two digraphs are merely mirror images of one another.) Arcs directed from a circuit vertex to the hub vertex will be called introverted. Circuit vertices incident with introverted arcs will be called introverts. Similarly, arcs directed from the hub vertex to a circuit vertex will be called extroverted and circuit vertices incident with extroverted arcs will be called extroverts.

We will denote our directed wheel graphs by \( CW_{a_1,a_2,a_3,\ldots,a_2j} \) where, going clockwise and starting with “spoke” \( \{b_1, y\} \), there are \( a_1 \) introverted arcs, followed by \( a_2 \) extroverted arcs, \( \ldots \), and ending with \( a_2j \) extroverted arcs. The notation refers to a digraph with \( n + 1 \) vertices where \( n = a_1 + a_2 + \cdots + a_2j \) vertices are on the circuit and there is one hub vertex. We will assume that \( n > 1 \), otherwise, the digraph has a loop and no T-coloring is possible. See Figure 2 for a specific example, \( CW_{2,5,1,3} \). Note that for \( CW_{a_1,a_2,a_3,\ldots,a_2j} \), each \( a_i > 0 \) except in the special case of a directed wheel graph with only extroverted arcs, i.e., \( CW_{0,n} \).

The T-sets that we will be considering in this paper will be denoted by \( T_r \) and are of the form

\[
T_r = \{0, 1, 2, 3, \ldots, r - 1, r + 1, r + 2, \ldots\}.
\]

\(^1\)Our definition of \( W_n \) is slightly nonstandard to simplify the notation on the circuit vertices.

\(^2\)Note that \( CW_{a_1,a_2,a_3,\ldots,a_2j} \cong CW_{a_3,a_4,\ldots,a_2j,a_1,a_2} \cong CW_{a_5,a_6,\ldots,a_2j,a_1,a_2,a_3,a_4} \cong \cdots \) by rotational symmetry.
For $T$-sets of this type, we will refer to their $T$-coloring as a $T_r$-coloring. Again, we will only consider $T_r$ for $r \geq 1$. Note that
c is a $T_r$-coloring of $CW_{a_1,a_2,a_3,...,a_{2j}}$
(1) if and only if
$(x, y) \in A\left(CW_{a_1,a_2,a_3,...,a_{2j}}\right) \Rightarrow c(x) - c(y) = r$ or $c(x) - c(y) < 0$.

2. $CW_{a_1,a_2,a_3,...,a_{2j}}$

Our first two lemmas prove a relationship between the colors of most of the vertices on the circuit of the directed wheel graph $CW_{a_1,a_2,a_3,...,a_{2j}}$ and the color of the hub vertex for any $T_r$-coloring. Let the vertices of $CW_{a_1,a_2,a_3,...,a_{2j}}$ be labeled as in Figure 1.

**Lemma 1.** For any $T_r$-coloring $c$ of digraph $CW_{a_1,a_2,a_3,...,a_{2j}}$, $c(b_i) < c(y)$ for all introverts $b_i$ such that $b_{i+1}$ is also an introvert where $y$ is the hub vertex.

**Proof.** Note that $c(b_i) \neq c(y)$ since $(b_i, y) \in A\left(CW_{a_1,a_2,a_3,...,a_{2j}}\right)$. So, suppose that

(2) $c(b_i) > c(y)$ for some introvert $b_i$ such that $b_{i+1}$ is also an introvert.

Since $(b_i, y) \in A\left(CW_{a_1,a_2,a_3,...,a_{2j}}\right)$ and applying (1),

(3) $c(b_i) - c(y) = r$.

Now consider $c(b_{i+1})$. Since $(b_{i+1}, y) \in A\left(CW_{a_1,a_2,a_3,...,a_{2j}}\right)$, $c(b_{i+1}) \neq c(y)$.
Case 1. \( c(b_{i+1}) < c(y) \). By (2), we have \( c(b_{i+1}) < c(b_i) \). Since \((b_i, b_{i+1}) \in A\left(CW_{a_1, a_2, a_3, \ldots, a_{2j}}\right)\),

\[
(4) \quad c(b_i) - c(b_{i+1}) = r.
\]

By equations (3) and (4), \( c(y) = c(b_{i-1}) \), which is a contradiction since \((y, b_{i-1}) \in CW_{a_1, a_2, a_3, \ldots, a_{2j}}\).

Case 2. \( c(b_{i+1}) > c(y) \). Since \((b_{i+1}, y) \in A\left(CW_{a_1, a_2, a_3, \ldots, a_{2j}}\right)\) and applying (1),

\[
(5) \quad c(b_{i+1}) - c(y) = r.
\]

By equations (3) and (5), \( c(b_i) = c(b_{i+1}) \) which is a contradiction since \((b_i, b_{i+1}) \in A\left(CW_{a_1, a_2, a_3, \ldots, a_{2j}}\right)\).

Lemma 1 proved that for any \( T_r \)-coloring of \( CW_{a_1, a_2, a_3, \ldots, a_{2j}} \) and any maximal set of consecutive introverts, \( b_i, b_{i+1}, \ldots, b_{i+h} \), the color assigned to each introvert, except possibly \( b_{i+h} \), is less than the color assigned to the hub vertex \( y \).

Similarly, Lemma 2 proves that for any \( T_r \)-coloring of \( CW_{a_1, a_2, a_3, \ldots, a_{2j}} \) and any maximal set of consecutive extroverts, \( b_i, b_{i+1}, \ldots, b_{i+h} \), the color assigned to each extrovert, except possibly \( b_i \), is greater than the color assigned to the hub vertex \( y \).

Lemma 2. For any \( T_r \)-coloring \( c \) of digraph \( CW_{a_1, a_2, a_3, \ldots, a_{2j}} \), \( c(b_i) > c(y) \) for all extroverts \( b_i \) such that \( b_{i-1} \) is also an extrovert where \( y \) is the hub vertex.

Proof. Note that \( c(b_i) \neq c(y) \) since \((y, b_i) \in A\left(CW_{a_1, a_2, a_3, \ldots, a_{2j}}\right)\). So, suppose that

\[
(6) \quad c(b_i) < c(y) \quad \text{for some extrovert } b_i \text{ such that } b_{i-1} \text{ is also an extrovert}.
\]

Since \((y, b_i) \in A\left(CW_{a_1, a_2, a_3, \ldots, a_{2j}}\right)\) and applying (1),

\[
(7) \quad c(y) - c(b_i) = r.
\]

Now consider \( c(b_{i-1}) \). Since \((y, b_{i-1}) \in A\left(CW_{a_1, a_2, a_3, \ldots, a_{2j}}\right)\), \( c(y) \neq c(b_{i-1}) \).

Case 1. \( c(b_{i-1}) > c(y) \). By (6), we have \( c(b_{i-1}) > c(b_i) \). Since \((b_{i-1}, b_i) \in A\left(CW_{a_1, a_2, a_3, \ldots, a_{2j}}\right)\),

\[
(8) \quad c(b_{i-1}) - c(b_i) = r.
\]

By (7) and (8), we have \( c(y) = c(b_{i-1}) \), which is a contradiction since \((y, b_{i-1}) \in CW_{a_1, a_2, a_3, \ldots, a_{2j}}\).
Case 2. $c(b_{i-1}) < c(y)$. Since $(y, b_{i-1}) \in A(CW_{a_1,a_2,a_3,...,a_{2j}})$,

(9) \[ c(y) - c(b_{i-1}) = r. \]

By (7) and (9), we have $c(b_i) = c(b_{i-1})$ which is a contradiction since $(b_{i-1}, b_i) \in A(CW_{a_1,a_2,a_3,...,a_{2j}})$.

Consider the special case when our directed wheel graph only has one set of introverts and one set of extroverts. We will use the notation $CW_{k,l}$ for such a digraph and use the vertex labeling as in Figure 3. The next corollary and lemma consider the relationship between $c(v_l)$ and $c(u_1)$, as well as $c(u_k)$ and $c(v_1)$, i.e., the colors of the last/first introvert and first/last extrovert, for any $T_r$-coloring $c$ of $CW_{k,l}$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{$CW_{k,l}$.}
\end{figure}

**Corollary 3.** In any $T_r$-coloring $c$ of $CW_{k,l}$ where $k, l \geq 2$,

\[ c(v_l) > c(u_1). \]

**Proof.** Since $k, l \geq 2$, $c(u_1) < c(y) < c(v_1)$ by Lemmas 1 and 2.

**Lemma 4.** In any $T_r$-coloring $c$ of $CW_{k,l}$, $c(v_1) > c(u_k)$.

**Proof.** Since $(u_k, v_1) \in A(CW_{k,l})$, $c(u_k) \neq c(v_1)$. So, suppose that $c(u_k) > c(v_1)$. Since $(u_k, v_1) \in A(CW_{k,l})$, $c(v_1) = c(u_k) - r$. Since $(u_k, y) \in A(CW_{k,l})$,

(a) $c(y) = c(u_k) - r$ or (b) $c(y) > c(u_k)$.

Also, since $(y, v_1) \in A(CW_{k,l})$,

(c) $c(y) = (c(u_k) - r) + r = c(u_k)$ or (d) $c(y) < c(u_k) - r$.

We reach a contradiction since no pair of these conditions, (a)–(c), (a)–(d), (b)–(c), or (b)–(d), can occur simultaneously.
Corollary 5. In any $T_r$-coloring $c$ of $CW_{k,l}$, it is never the case that both $c(u_k) > c(y)$ and $c(v_1) < c(y)$.

Our next lemma considers those vertices of $CW_{a_1, a_2, \ldots, a_2j}$ which cannot be colored 1, i.e., the minimum color of any $T$-coloring.

Lemma 6. For any $T_r$-coloring $c$ of digraph $CW_{a_1, a_2, \ldots, a_2j}$, $c(x) > 1$ for any extrovert $x \in V(CW_{a_1, a_2, \ldots, a_2j})$.

Proof. We prove this lemma for the digraph $CW_{a_1, a_2}$, i.e., $CW_{k,l}$. The general case for $CW_{a_1, a_2, \ldots, a_2j}$ follows immediately. Let the vertices of $CW_{k,l}$ be labeled as in Figure 3.

Case 1. consider $v_i \in V(CW_{k,l})$ for $i = 2, 3, \ldots, l$. By Lemma 2, $c(v_i) > c(y)$ Thus, $c(v_i) > 1$, for $i = 2, 3, \ldots, l$.

Case 2. consider $v_1 \in V(CW_{k,l})$ and suppose that $c(v_1) = 1$. Then, $c(y) = r + 1$ since $(y, v_1) \in A(CW_{k,l})$. Also, $c(u_k) = r + 1$ since $(u_k, v_1) \in A(CW_{k,l})$. Therefore, $c(y) = c(u_k)$ which is a contradiction since $(u_k, y) \in A(CW_{k,l})$. Thus, $c(v_1) > 1$.

We end this section by showing that the $T$-span of $CW_{a_1, a_2, \ldots, a_2j}$ is not entirely dependent on the order of the $a_i$s. Note that we will use the notation $\overrightarrow{D}$ to denote the digraph obtained by reversing all of the arcs of digraph $D$, i.e., the reverse digraph of $D$.

Lemma 7. $sp_{T_r}(CW_{k,l}) = sp_{T_r}(\overrightarrow{CW_{k,l}})$.

Proof. Let $c$ be a $T_r$-coloring of $CW_{k,l}$ and let $M_c = \max c(x)$ for $x \in V(CW_{k,l})$. Then define a new coloring $d$ of $CW_{k,l}$ by

$$d(x) = M_c - c(x) + 1, \text{ for } x \in V(CW_{k,l}).$$

Suppose that $(x, z) \in \overrightarrow{CW_{k,l}}$, i.e., $(z, x) \in CW_{k,l}$. Then $d(x) - d(z) = (M_c - c(x) + 1) - (M_c - c(z) + 1) = c(z) - c(x) < 0$ or equals $r$ by (1). Therefore, $d$ is a $T_r$-coloring of $\overrightarrow{CW_{k,l}}$ and not a $T_r$-coloring of $CW_{k,l}$.

The maximum and minimum values of $c$ are $M$ and 1, respectively. Then, $\max d(x) = M - 1 + 1 = M$ and $\min d(x) = M - M + 1 = 1$ since $d(x)$ is a strictly decreasing linear function of $c(x)$. Therefore, the span of $d$ is less than or equal to the span of $c$ which implies $sp_{T_r}(\overrightarrow{CW_{k,l}}) \leq sp_{T_r}(CW_{k,l})$.

A similar proof shows that $sp_{T_r}(CW_{k,l}) \leq sp_{T_r}(\overrightarrow{CW_{k,l}})$ using the fact that $\overrightarrow{CW_{k,l}} = CW_{k,l}$.

Therefore, $sp_{T_r}(CW_{k,l}) = sp_{T_r}(\overrightarrow{CW_{k,l}})$. 

Theorem 8. $sp_{Tr}(CW_{k,l}) = sp_{Tr}(CW_{l,k})$.

**Proof.** Note that $CW_{k,l}$ has a consecutive set of $k$ extroverted arcs and a consecutive set of $l$ introverted arcs since $CW_{k,l}$ has a consecutive set of $k$ introverted arcs and a consecutive set of $l$ extroverted arcs which are being reversed. Then, using a mirror image of $CW_{k,l}$, it is easy to see that $ CW_{k,l}$ is isomorphic to $ CW_{l,k}$. Therefore, by Lemma 7, the theorem follows.

Although we have no need for a stronger result, it can also be shown that

$$sp_{Tr} (CW_{a_1,a_2,...,a_{2j}}) = sp_{Tr} (CW_{a_{i+1},a_{i+2},...,a_{2j-1},a_{2j},a_1,a_2,...,a_i})$$

using a similar proof as above and the second footnote on page 2.

An example of Lemma 7 and Theorem 8 for a $T_5$-coloring of $CW_{5,3}$ is shown in Figure 4.

![Figure 4](image)

Figure 4. Equal span $T_5$-colorings of $CW_{5,3}$, $CW_{3,5}$.

3. $sp_{Tr}(CW_{k,l})$, $r > 1$

In this section and the next, we consider the special case of $CW_{k,l}$, i.e., where the directed wheel graph’s spokes only consist of a consecutive set of introverted arcs and a consecutive set of extroverted arcs. We assume that $r > 1$; the special case of $r = 1$ is addressed in Section 4. To prove our main result, i.e., the calculation of $sp_{Tr}(CW_{k,l})$, we consider different conditions on $k$ and $l$.

First, we consider $CW_k$ and $CW_{0,l}$, i.e., the directed wheel graphs with only introverted arcs and only extroverted arcs. Recall that we only consider $CW_k$ for $k \geq 3$, and $CW_{0,l}$, for $l \geq 3$.

**Theorem 9.** For $r > 1$,

$$sp_{Tr}(CW_k) = sp_{Tr}(CW_{0,l}) = r + 1.$$
Figure 5. Optimal $T_r$-colorings of $CW_k$.

Proof. Applying Theorem 8, it is sufficient to show that $sp_{T_r}(CW_k) = r + 1$. Figure 5 provides $T_r$-colorings of $CW_k$, depending on the parity of $k$, whose span is $r + 1$. Next we show that the span of any $T_r$-coloring of $CW_k$ is greater than or equal to $r + 1$.

Let the vertices of $CW_k$ be labeled as in Figure 1 and $c$ be a $T_r$-coloring of $CW_k$. By Lemma 1, $c(b_i) < c(y)$ for $i = 1, 2, \ldots, k$. Without loss of generality, let $c(b_1) = 1$. Then $c(b_k) = r + 1$ since $(b_k, b_1) \in A(CW_k)$. Therefore, $c(y) \geq r + 2$ which implies that the span of the $T_r$-coloring $c$ of $CW_k$ is greater than or equal to $r + 1$. Therefore, $sp_{T_r}(CW_k) = r + 1$.

Next, we prove our main result for $CW_{k,l}$. Note that the exceptions to the theorem are treated in Section 6.

**Theorem 10.** For $r > 1$,

$$sp_{T_r}(CW_{k,l}) = \begin{cases} r & k + l \leq r \text{ and } \min(k, l) > 0, \\ r + 1 & k + l > r \text{ and } \min(k, l) = 1, \\ \min(k, l) + r & k + l > r \text{ and } 1 < \min(k, l) \leq r, \\ 2r + 2 & k + l > r \text{ and } \min(k, l) > r, \end{cases}$$

except for:

$$\begin{array}{cccccccccccc}
 k & 5 & 4 & 4 & 3 & 3 & 3 & 2 & 2 & 2 & 2 & 2 \\
 l & 4 & 4 & 3 & 4 & 2 & 4 & 3 & 2 & 2 & 3 & 1 & 3
\end{array}$$

Proof. Case 1. $k + l \leq r$ and $\min(k, l) > 0$. Let $CW_{k,l}$ be labeled as in Figure 3 and $c$ be the following coloring:

$$c(u_s) = s, \text{ when } s = 1, 2, \ldots, k;$$
\[ c(y) = k + 1; \]
\[ c(v_t) = r - (l - 1) + t, \text{ when } t = 1, 2, \ldots, l. \]

See Figure 6. Note that the minimum color of the extroverts, \( c(v_1) = r - l + 2 \), is greater than \( c(y) = k + 1 \) since \( k + l \leq r \). Thus, the sequence of colors \( c(u_1), c(u_2), \ldots, c(u_k), c(v_1), c(v_2), \ldots, c(v_l) \) is strictly increasing and \( c(v_l) > c(u_i) \Rightarrow c(v_1) - c(v_l) = r. \) Also, \( c(y) > c(u_i) \) for \( i = 1, 2, \ldots, k \) and \( c(y) < c(v_j) \) for \( j = 1, 2, \ldots, l \). Thus, by (1), \( c \) is a \( T_r \)-coloring of \( CW_{k,l} \) and the span of \( c \) is \( r \).

![Figure 6. Optimal \( T_r \)-coloring of \( CW_{k,l} \), \( k + l \leq r \).](image)

Next, we show that the span of any \( T_r \)-coloring of \( CW_{k,l} \) is greater than or equal to \( r \) which implies \( sp_{T_r}(CW_{k,l}) = r \). Let \( d \) be a \( T_r \)-coloring of \( CW_{k,l} \) and let \( CW_{k,l} \) be labeled as in Figure 1. The sequence of colors on the non-hub vertices, \( d(b_1), d(b_2), \ldots, d(b_n), d(b_1) \), cannot be strictly increasing since the vertices form a cycle. When the sequence decreases, it must do so by exactly \( r \). Therefore, the span of \( T_r \)-coloring \( d \) of \( CW_{k,l} \) is greater than or equal to \( r \).

**Case 2.** \( k + l > r \) and \( \min(k, l) = 1 \). Applying Theorem 8, it is sufficient to only consider \( \min(k, l) = k = 1 \), i.e., \( CW_{1,l} \). Figure 7 provides \( T_r \)-colorings with span \( r + 1 \), i.e., whose maximum color is \( r + 2 \), for \( CW_{1,l} \) except when \( r = 2 \) and \( l = 3 \). This special case, as well as when \( r = 2, \ k = 3 \), and \( l = 1 \), are addressed in Section 6.

Next we show that if \( c \) is any \( T_r \)-coloring of \( CW_{1,l} \), \( k + l > r \) and \( \min(k, l) = 1 \), then \( \max c(x) \geq r + 2 \) for \( x \in V(CW_{1,l}) \) which implies \( sp_{T_r}(CW_{1,l}) = r + 1 \), for all but the special case. Let the vertices of \( CW_{1,l} \) be labeled as in Figure 3 and \( c \) be a \( T_r \)-coloring of \( CW_{1,l} \). By Lemma 6, the minimum color 1 must be assigned to an introvert or the hub vertex.

**Case 2.a.** \( c(y) = 1 \). Each \( c(b_i) \neq 1 \) since \( (b_i, y) \) or \( (y, b_i) \in A(CW_{1,l}) \), for \( i = 1, 2, \ldots, n \). Let \( m = \min c(b_i) \), for \( i = 1, 2, \ldots, n \). Then \( m \geq 2 \) and the color on the circuit vertex preceding it in the sequence \( b_1, b_2, \ldots, b_n, b_1 \) must be colored \( r + m \geq r + 2 \).
Case 2.b. \( c(u_1) = 1 \). By Lemma 6, \( c(v_i) > 1 \), for \( i = 1, 2, \ldots, l \) and \( c(y) > 1 \) since \((u_1, y) \in A(CW_{1,l})\). Note that \( k + l > r \Rightarrow 1 + l > r \Rightarrow l > r - 1 \Rightarrow l \geq r \). Assume \( \max c(x) \leq r + 1 \), for \( x \in V(CW_{1,l}) \). Thus there are \( r \) available colors, \( 2, 3, \ldots, r + 1 \), to color \( l + 1 \geq r + 1 \) vertices, \( y, v_1, v_2, \ldots, v_l \). By the pigeonhole principle, at least two of these vertices must be colored the same and neither of them can be \( y \) since \((y, v_i) \in A(CW_{1,l})\) for all \( i \). Thus, the sequence of colors \( c(v_1), c(v_2), \ldots, c(v_l) \) is not strictly increasing. Suppose that \( c(v_i) > c(v_{i+1}) \).

Then, \( c(v_i) - c(v_{i+1}) = r \Rightarrow c(v_i) = c(v_{i+1}) + r \geq 2 + r \) which is a contradiction. Therefore, \( sp_{Tr}(CW_{1,l}) = r + 1 \) (for all but the special cases).

Case 3. \( k + l > r \) and \( 1 < \min(k, l) = k \leq r \). Applying Theorem 8, it is sufficient to only consider \( \min(k, l) = k \). Figures 8, 9 and 10 provide \( T_r \)-colorings with span \( k + r \), i.e., whose maximum color is \( k + r + 1 \), for \( CW_{k,l} \) except for the following six cases: \( l = 4 \) and \( k = r - 1 \) for \( r = 3, 4, 5 \); \( l = 3 \) and \( k = r \) for \( r = 2, 3 \); and \( l = 2 \) and \( k = r - 1 \) for \( r = 3 \), as well as their corresponding reverse digraphs. All of these special cases are addressed in Section 6.
Next we show that if \( c \) is any \( T_r \)-coloring of \( CW_{k,l} \), \( k + l > r \) and \( 1 < \min(k, l) = k \), then \( \max c(x) \geq k + r + 1 \) for \( x \in V(CW_{k,l}) \), which implies \( sp_{T_r}(CW_{k,l}) = k + r \), for all of the cases addressed in Figures 8, 9 and 10. Let the vertices of \( CW_{k,l} \) be labeled as in Figure 3 and \( c \) be a \( T_r \)-coloring of \( CW_{k,l} \). By Lemma 6, the minimum color 1 must be assigned to an introvert or the hub vertex.

**Case 3.a.** Suppose \( c(u_i) = 1 \), for \( i \neq 1 \). Then \( c(u_{i-1}) = r + 1 \) since \( (u_{i-1}, u_i) \in A(CW_{k,l}) \). Also, by Lemma 1, \( c(y) \geq r + 2 \) and, by Lemma 2, \( c(v_i) > c(y) \geq r + 2 \), for \( i = 2, 3, \ldots, l \). Assume \( \max c(x) \leq k + r \), for \( x \in V(CW_{k,l}) \). Thus there are \( k - 1 \) available colors, \( r + 2, r + 3, \ldots, r + k \), to color \( l \geq k \) vertices, \( y, v_2, v_3, \ldots, v_l \). By the pigeonhole principle, at least two of these vertices must be colored the same and neither of these same-colored vertices can be \( y \) since \( (y, v_i) \in A(CW_{k,l}) \) for all \( i \). Thus, the sequence of colors \( c(v_1), c(v_2), \ldots, c(v_l) \) is not strictly increasing. Suppose that \( c(v_i) > c(v_{i+1}) \). Then, \( c(v_i) - c(v_{i+1}) = r \Rightarrow c(v_i) = c(v_{i+1}) + r \geq (r + 2) + r \geq k + r + 2 \), since \( k \leq r \), which is a contradiction.
Thus, if \( c(u_i) = 1 \), for \( i \neq 1 \), then \( \max c(x) \geq k + r + 1 \), for \( x \in V(CW_{k,l}) \).

Case 3.b. Suppose that \( c(u_1) = 1 \). Then \( c(u_i) \neq 1 \) for \( i = 2, 3, \ldots, k \) and \( c(v_1) = r + 1 \) since \( (v_1, u_1) \in A(CW_{k,l}) \).

Case 3.b.i. \( c(u_k) < c(y) \) and \( c(v_1) < c(y) \). Since \( c(v_1) = r + 1 \) then, by Lemma 2, \( c(y) \leq r \). Also since \( (y, v_1) \in A(CW_{k,l}) \) and \( c(y) > c(v_1) \) then \( c(y) - c(v_1) = r \Rightarrow c(v_1) = c(y) - r \leq 0 \) which is a contradiction.

Case 3.b.ii. \( c(u_k) < c(y) \) and \( c(v_1) > c(y) \). Since \( c(u_k) < c(y) \), then the color of each of the \( k \) introverts is less than \( c(y) \). If the introverts’ colors are distinct then \( c(y) \geq k + 1 \). If the introverts’ colors are not distinct then there exists a \( u_i \) such that \( c(u_i) > c(v_{i+1}) \Rightarrow c(u_i) = r + c(u_{i+1}) \geq r + 2 \) since \( c(u_{i+1}) > 1 \). Thus, \( c(y) > c(u_i) \geq r + 2 \geq k + 2 \Rightarrow c(y) > k + 2 \).

Thus, \( c(y) \geq k + 3 \).

Case 3.b.ii.a. \( c(v_i) > c(v_{i+1}) \) for some \( i \in \{1, 2, \ldots, l - 1\} \). Thus, \( c(v_i) = c(v_{i+1}) + r \). Since \( c(v_{i+1}) > c(y) \geq k + 3 \), then \( c(v_i) = c(v_{i+1}) + r > k + 3 + r \).

Therefore, \( c(x) \geq k + r + 1 \) for \( x \in V(CW_{k,l}) \).

Case 3.b.ii.b. \( c(v_i) < c(v_{i+1}) \) for all \( i \in \{1, 2, \ldots, l - 1\} \). Thus, the extroverts’ colors are all distinct and all greater than \( c(y) \). They are also strictly increasing from \( v_1 \) to \( v_l \) and they are all less than or equal to \( r + 1 \) since \( c(v_1) = r + 1 \). Since \( c(v_1) > c(y) \geq k + 3 \), then \( c(v_l) \geq k + 3 + l > r + 3 \) which contradicts \( c(v_l) = r + 1 \).

Case 3.b.iii. \( c(u_k) > c(y) \). Then \( c(u_k) = c(y) + r \) since \( (u_k, y) \in A(CW_{k,l}) \).

Case 3.b.iii.a. \( c(v_1) < c(u_k) \). Thus, we reach a contradiction by Lemma 4.

Case 3.b.iii.b. \( c(v_1) > c(u_k) \). Thus, \( c(v_1) > r + c(y) \). Since \( c(y) \geq k \) by Lemma 1, then \( c(v_1) \geq k + r + 1 \), i.e., \( \max c(x) \geq k + r + 1 \) for any \( x \in V(CW_{k,l}) \).

Case 3.c. \( c(y) = 1 \). By Lemma 1, it must be the case that \( k = 1 \) otherwise there will be an introvert’s color which is less than \( 1 \). This contradicts a condition of this case.

Therefore, for all but the special cases, \( \max c(x) \geq k + r + 1 \) for any \( x \in V(CW_{k,l}) \) and our \( T_r \)-colorings in Figures 8, 9 and 10 are optimal with respect to span.

Case 4. \( k + l > r \) and \( \min(k, l) > r \). Note that Figure 11 provides a \( T_r \)-coloring of \( CW_{k,l} \) whose span is \( 2r + 2 \). Next, we show that the span of any \( T_r \)-coloring, \( c \), of \( CW_{k,l} \) is greater than or equal to \( 2r + 2 \).

Again, let the vertices of \( CW_{k,l} \) be labeled as in Figure 3. By Lemmas 1 and 2, \( c(u_i) < c(y) \), for all \( 1 \leq i \leq k - 1 \), and \( c(v_j) > c(y) \), for all \( 2 \leq j \leq l \). We consider four cases:

- (a) \( c(u_k) > c(y) \) and \( c(v_1) < c(y) \);
- (b) \( c(u_k) < c(y) \) and \( c(v_1) < c(y) \);
- (c) \( c(u_k) < c(y) \) and \( c(v_1) > c(y) \);
- (d) \( c(u_k) > c(y) \) and \( c(v_1) > c(y) \).
Figure 11. Optimal $T_r$-coloring of $CW_{k,l}$, $k+l > r$ and $\min(k,l) > r$.

Case 4.a. $c(u_k) > c(y)$ and $c(v_1) < c(y)$. This case is not possible by Corollary 5.

Case 4.b. $c(u_k) < c(y)$ and $c(v_1) > c(y)$. Since $c(u_k) < c(y)$, then $c(u_i) < c(y)$ for $i = 1, 2, \ldots, k$. If the $c(u_i)$s are all distinct then $c(y) \geq r + 2$ since $k > r$. If the $c(u_i)$s are not all distinct then $c(u_j) < c(u_{j-1})$ for at least one $j$ since colors on consecutive $u_i$s can not be equal. Given that $c(u_j) \geq 1$ then $c(u_{j-1}) \geq r + 1$ since $(u_{j-1}, u_j) \in A(CW_{k,l})$. So, in general, $c(y) \geq r + 2$.

Since $c(v_1) > c(y)$, then $c(v_j) > c(y)$ for $j = 1, 2, \ldots, l$. If the $c(v_j)$s are all distinct and since $l > r$ then $\max(c(v_j)) \geq c(y) + l \geq c(y) + (r + 1) \geq (r + 2) + (r + 1) = 2r + 3$. If the $c(v_j)$s are not all distinct then $c(v_i) < c(v_{i-1})$ for at least one $i$ since colors on consecutive $v_j$s can not be equal. Since $c(v_i) > c(y)$ and $c(y) \geq r + 2$ then $c(v_i) \geq r + 3$, for all $i$. Thus, $c(v_{i-1}) \geq (r + 3) + r = 2r + 3$ since $(v_{i-1}, v_i) \in A(CW_{k,l})$. So, in general, $c(v_j) \geq 2r + 3$. Therefore, in this case, the span of any $T_r$-coloring of $CW_{k,l}$ is greater than or equal to $2r + 2$.

Case 4.c. $c(u_k) < c(y)$ and $c(v_1) < c(y)$. As in Case 4.b, since $c(u_k) < c(y)$ and thus $c(u_i) < c(y)$ for all $i = 1, 2, \ldots, k$, we know that $c(y) \geq r + 2$.

Case 4.c.i. $c(v_i) > c(v_{i+1})$ for some $i = 2, 3, \ldots, l-1$. Thus, $c(v_i) - c(v_{i+1}) = r \Rightarrow c(v_i) = c(v_{i+1}) + r$ since $(v_i, v_{i+1}) \in A(CW_{k,l})$. Since $c(v_{i+1}) > c(y)$, then $c(v_{i+1}) \geq r + 3$. Therefore, $c(v_i) \geq (r + 3) + r = 2r + 3$.

Case 4.c.ii. $c(v_i) < c(v_{i+1})$, for all $i = 2, 3, \ldots, l-1$. Since $c(v_i) > c(y)$, for $i = 2, 3, \ldots, l$, and $c(y) \geq r + 2$, then $c(v_i) \geq c(y) + (l - 1) \geq c(y) + r$ since $l > r$. Thus, $c(v_l) \geq c(y) + l - 1 \geq c(y) + r$. However, $c(u_1) \leq c(y) - 1$. We reach a contradiction since $(v_l, u_1) \in A(CW_{k,l})$ but $c(v_l) - c(u_1) \geq c(y) + r - (c(y) - 1) = r + 1 \not\in T_r$. 

So, it must be the case that \( \max c(v_j) \geq 2r + 3 \). Therefore, in this case, the span of any \( T_r \)-coloring of \( CW_{k,l} \) is greater than or equal to \( 2r + 2 \).

**Case 4.d.** \( c(u_k) > c(y) \) and \( c(v_1) > c(y) \).

**Case 4.d.i.** \( c(u_1) = 1 \). Given that \( c(u_k) > c(y) \), then \( c(u_i) < c(y) \) for \( i = 1, 2, \ldots, k - 1 \). If these \( c(u_i) \)'s are all distinct then \( c(y) \geq r + 1 \) since \( k > r \). However, by Lemma 2, \( c(v_1) > c(y) \Rightarrow c(v_1) \geq r + 2 \). We reach a contradiction since \( (v_1, u_1) \in A(CW_{k,l}) \) but \( c(v_1) - c(u_1) \geq (r + 2) - 1 = r + 1 \notin T_r \). If \( c(u_i) \), for \( i = 1, 2, \ldots, k - 1 \), are not all distinct then \( c(u_j) < c(u_{j-1}) \) for at least one \( j \) since colors on consecutive \( u_i \)'s cannot be equal. Since \( c(u_j) \geq 1 \), then \( c(u_{j-1}) \geq r + 1 \) since \( (u_{j-1}, u_j) \in A(CW_{k,l}) \). So, in general, \( c(y) \geq r + 2 \). Now that we know \( c(y) \geq r + 2 \), then, as in Case 4.b, since \( c(v_j) > c(y) \) for all \( j = 1, 2, \ldots, l \), \( \max c(v_j) \geq 2r + 3 \).

**Case 4.d.ii.** \( c(u_1) \neq 1 \). By Lemma 1, Lemma 6, and given that \( c(u_k) > c(y) \), then \( c(y) \neq 1 \) and \( c(u_i) = 1 \) for some \( i = 2, 3, \ldots, k - 1 \). This implies \( c(u_{i-1}) = r + 1 \Rightarrow c(y) \geq r + 2 \). Then, as in Case 4.b, since \( c(v_j) > c(y) \) for all \( j = 1, 2, \ldots, l \), \( \max c(v_j) \geq 2r + 3 \).

Thus, in this final case, the span of any \( T_r \)-coloring of \( CW_{k,l} \) is again greater than or equal to \( 2r + 2 \).

Therefore, \( sp_{T_1}(CW_{k,l}) = 2r + 2 \) for \( k + l > r \) and \( \min(k, l) > r \).

4. \( CW_{k,l}, r = 1 \)

In this section, we consider the case of \( T_r \)-coloring \( CW_{k,l} \) where \( k, l \geq 0 \) and \( r = 1 \). First, we consider the “all-introvert” case, \( CW_n \), and the “all-extrovert” case, \( CW_{0,n} \).

**Theorem 11.** For \( n \geq 2 \), \( sp_{T_1}(CW_n) = sp_{T_1}(CW_{0,n}) = \begin{cases} 2 & \text{for } n \text{ even}, \\ 3 & \text{for } n \text{ odd}. \end{cases} \)

**Proof.** Applying Theorem 8, it is sufficient to prove this theorem for only \( CW_n \).

**Case 1.** \( n \) even. The same proof as in Theorem 9 holds here for \( CW_n \).

**Case 2.** \( n \) odd. Figure 12 provides a \( T_1 \)-coloring of \( CW_n \) whose span is 3. Next we show that the span of any \( T_1 \)-coloring of \( CW_n \) is greater than or equal to 3.

Let \( c \) be any \( T_1 \)-coloring of \( CW_n \). At least four distinct colors must be used to \( T_1 \)-color the vertices of \( CW_n \) since its underlying graph is a wheel graph with an odd cycle and \( 0 \in T_1 \). Therefore, the span of \( c \) is greater than or equal to \( 4 - 1 = 3 \).

The next theorem considers the case of \( CW_{k,l} \) where \( r = 1 \) and \( \min(k, l) = 1 \).
Theorem 12. \( sp_{T_1}(CW_{1,m}) = sp_{T_1}(CW_{m,1}) = \begin{cases} 2 & \text{for } m \text{ odd,} \\ 3 & \text{for } m \text{ even.} \end{cases} \)

Proof. Applying Theorem 8, it is sufficient to prove this theorem for only \( CW_{1,m} \).

Case 1. \( m \) odd. Figure 13 provides a \( T_1 \)-coloring of \( CW_{1,m} \) whose span is 2. Next, let \( c \) be any \( T_1 \)-coloring of \( CW_{1,m} \). At least three distinct colors must be used to \( T_1 \)-color the vertices of \( CW_{1,m} \) since its underlying graph is a wheel graph with an even cycle and \( 0 \in T_1 \). Therefore, the span of \( c \) is greater than or equal to \( 3 - 1 = 2 \).

Case 2. \( m \) even. Figure 13(b) provides a \( T_1 \)-coloring of \( CW_{1,m} \) whose span is 3. Next, let \( c \) be any \( T_1 \)-coloring of \( CW_{1,m} \). At least four distinct colors must
be used to $T_1$-color the vertices of $CW_{1,m}$ since its underlying graph is a wheel graph with an odd cycle and $0 \in T_1$. Therefore, the span of $c$ is greater than or equal to $4 - 1 = 3$.

The case $r = 1$ is unique in the sense that it is the only case where it is not possible to $T$-color some directed wheel graphs $CW_{k,l}$.

**Theorem 13.** If $k, l \geq 2$, then it is impossible to $T_1$-color $CW_{k,l}$.

**Proof.** Let $CW_{k,l}$ be labeled as in Figure 3. By Lemmas 1 and 2, $c(v_1) > c(y) > c(u_1)$. Since $(v_1, u_1) \in A(CW_{k,l})$, $c(v_1) - c(u_1) = r = 1 \Rightarrow c(y) = c(u_1) + 1$. We reach a contradiction because $c(y)$ must be strictly between two consecutive colors: $c(u_1)$ and $c(v_1)$.

We conclude this paper with bounds on $sp_{T_r}(CW_{a_1,a_2,a_3,...,a_{2j}})$.

**Theorem 14.** $r \leq sp_{T_r}(CW_{a_1,a_2,...,a_{2j}}) \leq 2r + 2$.

**Proof.** For $CW_{a_1,a_2,...,a_{2j}}$, let the vertices be labeled as in Figure 1 and $c$ be any $T_r$-coloring. Consider the sequence of colors of the non-hub vertices, $c(b_1), c(b_2), \ldots, c(b_n), c(b_1)$. This sequence of colors can not be strictly increasing since the vertices form a circuit. When the sequence decreases, by (1), it must do so by exactly $r$. Therefore, the span of $c$ is greater than or equal to $r$ which implies $sp_{T_r}(CW_{a_1,a_2,...,a_{2j}}) \geq r$.

Next, for each consecutive set of $a_1, a_3, a_5, \ldots, a_{2j-1}$ introverted arcs, we color their corresponding introverts, $x_1, x_2, \ldots, x_{a_1}$, as follows:

if $r > 2$: \quad $3, r + 1, 1, r + 1, 1, \ldots$; \quad or if $r = 2$: \quad $3, 1, 3, 1, \ldots$

For each consecutive set of $a_2, a_4, a_6, \ldots, a_{2j}$ extroverted arcs, we color their corresponding extroverts, $x_{a_2}, x_{a_4}, \ldots, x_{a_{2j}}$, as follows:

$r + 3, 2r + 3, r + 3, 2r + 3, \ldots$

Finally, for the hub vertex: $c(y) = r + 2$.

Using the labeling of Figure 1, note that $c(b_n) - c(b_1) = r$ and $c(b_i) - c(b_{i+1})$, for each $i = 1, 2, \ldots, n - 1$, equals $r$ or is negative. Also, $c(b_i) - c(y) < 0$ for every introvert $b_i$ and $c(y) - c(b_j) < 0$ for every extrovert $b_j$. Therefore, $c$ is a $T_r$-coloring of $CW_{a_1,a_2,...,a_{2j}}$ and $sp_{T_r}(CW_{a_1,a_2,...,a_{2j}}) \leq 2r + 2$.

### 6. Special Cases

This section considers the eleven special cases which cannot be $T_r$-colored using the optimal colorings from Theorem 10. In fact, each of their $T$-spans disagrees...
Figure 14. Optimal $T_r$-colorings of the special cases.
with the value that Theorem 10 would have given (see Table 1). Optimal $T_r$-colorings for seven of the special cases are given in Figure 14. The remaining four special cases follow from Theorem 8. Proof of their optimality is left to the reader.

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<th>Digraph</th>
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References


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