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EQUITABLE COLORING AND EQUITABLE CHOOSABILITY OF GRAPHS WITH SMALL MAXIMUM AVERAGE DEGREE [1]

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Abstract

A graph is said to be equitably k-colorable if the vertex set $V(G)$ can be partitioned into k independent subsets $V_1, V_2, \ldots, V_k$ such that $||V_i|-|V_j|| \leq 1$ $(1 \leq i, j \leq k)$. A graph G is equitably k-choosable if, for any given k-uniform list assignment L, G is L-colorable and each color appears on at most $\left\lceil \frac{|V(G)|}{k} \right\rceil$ vertices. In this paper, we prove that if G is a graph such that $mad(G) < 3$, then G is equitably k-colorable and equitably k-choosable where $k \geq \max\{\Delta(G), 4\}$. Moreover, if G is a graph such that $mad(G) < \frac{12}{5}$, then G is equitably k-colorable and equitably k-choosable where $k \geq \max\{\Delta(G), 3\}$.

Keywords: graph coloring, equitable choosability, maximum average degree.

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1. Introduction

The terminology and notation used but undefined in this paper can be found in [1]. Let $G = (V(G), E(G))$ be a graph. Let $d_G(x)$, or simply $d(x)$, denote the number of edges incident with the vertex (face) x in G. If $d(x) = k$, $d(x) \geq k$ and $d(x) \leq k$, then the vertex x is called a *k-vertex*, *k^+-vertex* and k^--vertex, respectively. We use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ to denote the vertex set, edge set, maximum degree, and minimum degree of G, respectively. The *average degree* of a graph G is $\frac{\sum_{v\in V(G)} d(v)}{|V(G)|}$, and denote it by $ad(G)$. The *maximum average degree* $mad(G)$ of G is the maximum of the average degree of its subgraphs. The *girth* of a planar graph is the length of a smallest cycle in the graph, and denote the girth of a graph G by $g(G)$. We use $\lceil x \rceil$ to denote a minimum integer which is no less than x.

A *proper k-coloring* of a graph G is a mapping π from the vertex set $V(G)$ to the set of colors $\{1, 2, \ldots, k\}$ such that $\pi(x) \neq \pi(y)$ for every edge $xy \in E(G)$. A graph G is *equitable k-colorable* if G has a proper k-coloring such that the size of the color classes differ by at most 1. The *equitable chromatic number* of G, denoted by $\chi_e(G)$, is the smallest integer k such that G is equitably k-colorable. The *equitable chromatic threshold* of G, denoted by $\chi_e^*(G)$, is the smallest integer k such that G is equitably l-colorable (for any $l \geq k$).

In 1970, Hajnál and Szemerédi proved that $\chi_e^*(G) \leq \Delta(G) + 1$ for any graph G [9]. This bound is sharp as shown in the example of $K_{2n+1,2n+1}$. In 1973, Meyer introduced the notion of equitable coloring and made the following conjecture.

Conjecture 1.1 (Meyer [18]). *If G is a connected graph which is neither a complete graph nor odd cycle, then $\chi_e(G) \leq \Delta(G)$.*

In 1994, Chen, Lih and Wu put forth the following conjecture.

Conjecture 1.2 (Chen, Lih and Wu [2]). *For any connected graph G, if it is different from a complete graph, a complete bipartite graph and an odd cycle, then $\chi_e^*(G) \leq \Delta(G)$.*

Chen, Lih and Wu [2,3] proved Conjecture 1.2 for graphs with $\Delta(G) \leq 3$ or $\Delta(G) \geq \frac{|V(G)|}{2}$. In 2012, Chen *et al.* [4] improved the former result and confirmed the Conjecture 1.2 for graphs with $\Delta(G) \geq \frac{|V(G)|}{3} + 1$. Yap and Zhang [26,27] showed that Conjecture 1.2 holds for planar graphs with $\Delta(G) \geq 13$. In 2012, Nakprasit [19] confirmed the Conjecture 1.2 for planar graphs with $\Delta(G) \geq 9$. Lih and Wu [14] verified $\chi_e^*(G) \leq \Delta(G)$ for bipartite graphs other than complete bipartite graphs. Wang and Zhang [23] proved Conjecture 1.2 for line graphs, and Kostochka and Nakprasit [12,13] proved it for graphs with low average degree, and d-degenerate graphs with $\Delta(G) \geq 14d + 1$. Yan and Wang [25] showed that Conjecture 1.2 holds for Kronecker products of complete multipartite graphs and complete graphs. Wu and Wang [24], Luo *et al.* [17] confirmed Conjecture 1.2 for some planar graphs with large girth, respectively. Li *et al.* [16], Zhu *et al.* [29], Dong *et al.* [5–8], Nakprasit [20] confirmed Conjecture 1.2 for some planar graphs with some forbidden cycles. Zhang and Wu [28], Zhu and Bu [30] verified the Conjecture 1.2 for some series-parallel graphs and outerplanar graphs, respectively.

For a graph G and a list assignment L assigning to each vertex $v \in V(G)$ a set $L(v)$ of acceptable colors, an *L-coloring* of G is a proper vertex coloring such that for every $v \in V(G)$ the color on v belongs to $L(v)$. A list assignment L for G is *k-uniform* if $|L(v)| = k$ for all $v \in V(G)$. A graph G is *list equitably k-colorable* (also called *equitably k-choosable*) if, for any k-uniform list assignment L, G is L-colorable and each color appears on at most $\left\lceil \frac{|V(G)|}{k} \right\rceil$ vertices.

In 2003, Kostochka, Pelsmajer and West investigated the list equitable coloring of graphs. They proposed the following conjectures.

Conjecture 1.3 (Kostochka, Pelsmajer and West [11]). *Every graph G is equitably k-choosable whenever $k > \Delta(G)$.*

Conjecture 1.4 (Kostochka, Pelsmajer and West [11]). *If G is a connected graph with maximum degree at least 3, then G is equitably $\Delta(G)$-choosable, unless G is a complete graph or is $K_{k,k}$ for some odd k.*

It has been proved that Conjecture 1.3 holds for graphs with $\Delta(G) \leq 3$ in [21, 22] and then the result was strengthened by Kierstead and Kostochka. They confirmed the Conjecture 1.3 for graphs with $\Delta(G) \leq 7$ in [10]. Kostochka, Pelsmajer and West proved that a graph G is equitably k-choosable if either $G \neq K_{k+1}, K_{k,k}$ (with k odd in $K_{k,k}$) and $k \geq \max\left\{\Delta, \frac{|V(G)|}{2}\right\}$, or G is a connected interval graph and $k \geq \Delta(G)$ or G is a 2-degenerate graph and $k \geq \max\{\Delta(G), 5\}$ in [11]. Pelsmajer proved that every graph is equitably k-choosable for any $k \geq \frac{\Delta(G)(\Delta(G)-1)}{2} + 2$ in [21]. In 2009, Conjecture 1.4 were proved for planar graphs G without 4- and 6-cycles and with $\Delta(G) \geq 6$ by Li *et al.* in [16]. Zhu *et al.* confirmed Conjecture 1.4 for planar graph G without 3-cycles and with $\Delta(G) \geq 8$, planar graph G without 4- and 5-cycles and with $\Delta(G) \geq 7$ in [29], C_5-free planar graph G without adjacent triangles and with $\Delta(G) \geq 8$ in [30], outerplanar graphs in [31]. Zhang and Wu proved Conjecture 1.4 for series-parallel graphs in [28]. More results can be seen in [5–8] and [15].

As for the sparse graph G with $\Delta(G) = 2$, it is clear that G is equitably k-colorable and equitably k-choosable where $k \geq \max\{\Delta(G), 3\}$, if G is an odd cycle. Otherwise, G is equitably k-colorable and equitably k-choosable where $k \geq \max\{\Delta(G), 2\}$. In this paper, we consider the sparse graph G with $\Delta(G) \geq 3$ and show that if G is a graph such that $mad(G) < 3$, then G is equitably k-colorable and equitably k-choosable where $k \geq \max\{\Delta(G), 4\}$. Moreover, if G is a graph such that $mad(G) < \frac{12}{5}$, then G is equitably k-colorable and equitably k-choosable where $k \geq \max\{\Delta(G), 3\}$.

2. Some Important Lemmas

Lemma 2.1 (Kostochka, Pelsmajer and West [11]). *Let G be a graph with a k-uniform list assignment L. Let $S = \{v_1, v_2, \ldots, v_k\}$, where $\{v_1, v_2, \ldots, v_k\}$ are distinct vertices in G. If $G-S$ has an equitable L-coloring and $|N_G(v_i)-S| \leq k-i$ for $1 \leq i \leq k$, then G has an equitable L-coloring.*

Lemma 2.2 (Zhu and Bu [29]). *Let $S = \{v_1, v_2, \ldots, v_k\}$ be a set of k different vertices in G such that $G-S$ has an equitable k-coloring. If $|N_G(v_i)-S| \leq k-i$ for $1 \leq i \leq k$, then G has an equitable k-coloring.*

Lemma 2.3 (Hajnal and Szemerédi [9]). *Every graph has an equitable k-coloring whenever $k \geq \Delta(G)+1$.*

Lemma 2.4 (Pelsmajer, Wang and Lih [21, 22]). *Every graph G with maximum degree $\Delta(G) \leq 3$ is equitably k-choosable whenever $k \geq \Delta(G)+1$.*

Lemma 2.5. *Let G be a graph with $mad(G) < 3$. Then G is 2-degenerate.*

Proof. By contradiction, there is subgraph G' of G such that $\delta(G') \geq 3$. It is clear that $mad(G') \geq 3$, a contradiction. ■

Lemma 2.6 (Dong, Zou and Li [8]). *If G is a graph such that $mad(G) \leq 3$, then G is equitably k-colorable and equitably k-choosable where $k \geq \max\{\Delta(G), 5\}$.*

3. Graphs with $mad(G) < 3$

Lemma 3.1. *Let G be a connected graph with order at least 4 and $\delta(G) \geq 1$. If $\Delta(G) \leq 4$ and $mad(G) < 3$, then G has at least one of the structures in Figure 1.*

Proof. Let G be a counterexample. Then G does not contain any configuration $H_1 \sim H_6$ presented in Figure 1.

For each $v \in V(G)$, if $d(v) = 2$, then v is adjacent to at least one 4-vertex for the reason that G contains no structure H_1. If $d(v) = 4$, then v is adjacent to at most one 2-vertex for the reason that G contains no structure H_2. For convenience, let r denote the number of 4-vertices which are not adjacent to any 2-vertex. Obviously, G has the following property.

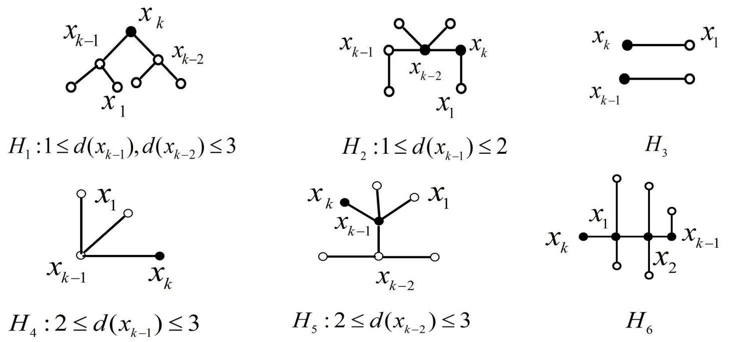

Figure 1

Each configuration depicted in Figure 1 is such that: (1) hollow vertices may be not distinct while solid vertices are distinct, (2) the degree of the solid vertices is fixed, and (3) except for specially pointed, the degree of a hollow vertices may be any integer from $[d, \Delta(G)]$, where d is the number of edges incident with the hollow vertex in the configuration.

Observation 3.2. $n_4(G) \geq n_2(G) + r$.

By Lemma 2.5, we have $\delta(G) \leq 2$.

Suppose $\delta(G) = 2$. By Observation 3.2, we have $ad(G) = \frac{2n_2(G)+3n_3(G)+4n_4(G)}{n_2(G)+n_3(G)+n_4(G)}$ $\geq \frac{2n_2(G)+3n_3(G)+4(n_2(G)+r)}{n_2(G)+n_3(G)+n_2(G)+r} = \frac{6n_2(G)+3n_3(G)+4r}{2n_2(G)+n_3(G)+r} = \frac{3[2n_2(G)+n_3(G)+r]+r}{2n_2(G)+n_3(G)+r} \geq 3$, a contradiction to $mad(G) < 3$.

Suppose $\delta(G) = 1$. Since G contains no structure H_3, there is only one 1-vertex v in G. Furthermore, the vertex v must be adjacent to a 4-vertex u for the reason that G contains no structure H_4. Since G contains no structure H_5, the other adjacent vertices of u must be 4-vertices. For convenience, we use u_i $(1 \leq i \leq 3)$ to denote the 4-vertices which are adjacent to u. Since G contains no structure H_6, u_i $(1 \leq i \leq 3)$ is not adjacent to any 2-vertex. From the above discussion, we have $r \geq 4$. Obviously, we have $ad(G) = \frac{n_1(G)+2n_2(G)+3n_3(G)+4n_4(G)}{n_1(G)+n_2(G)+n_3(G)+n_4(G)} = \frac{1+2n_2(G)+3n_3(G)+4(n_2(G)+r)}{1+n_2(G)+n_3(G)+n_2(G)+r} = \frac{1+6n_2(G)+3n_3(G)+4r}{1+2n_2(G)+n_3(G)+r} = \frac{1+6n_2(G)+3n_3(G)+3r+4}{1+2n_2(G)+n_3(G)+r} = \frac{3[1+2n_2(G)+n_3(G)+r]+2}{1+2n_2(G)+n_3(G)+r} \geq 3$, a contradiction to $mad(G) < 3$. ■

In the following, let us give the proof of the main theorems.

Theorem 3.3. *If G is a graph such that $mad(G) < 3$, then G is equitably k-colorable where $k \geq \max\{\Delta(G), 4\}$.*

Proof. By Lemma 2.6, we only need to focus on the situation where $\Delta(G) \leq 4$. Let G be a counterexample with the smallest number of vertices. Clearly, $\delta(G) \geq 1$. If each component of G has at most four vertices, then $\Delta(G) \leq 3$. So G is equitably k-colorable by Lemma 2.3. Otherwise, there is at least one component with at least four vertices. By Lemma 3.1, G has one of the structures $H_1 \sim H_6$, taking it and the vertices are labelled as they are in Figure 1. If there are vertices labelled repeatedly, then we take the larger (x_i is larger than x_{i-1}). In the following, we show how to find S in Lemma 2.2. If G has H_1, H_2 or H_5, then let $S' = \{x_k, x_{k-1}, x_{k-2}, x_1\}$. If G has H_3 or H_4, then let $S' = \{x_k, x_{k-1}, x_1\}$. If G has H_6, then let $S' = \{x_k, x_{k-1}, x_2, x_1\}$. By Lemma 2.5, G is 2-degenerate, thus we can find the remaining unspecified positions in S from highest to lowest indices by choosing a vertex with minimum degree in the graph obtained from G by deleting the vertices already being chosen for S at each step. By the minimality of $|V(G)|$ and since $k \geq \Delta(G) \geq \Delta(G - S)$, $G - S$ is equitably k-colorable. So G is also equitably k-colorable by Lemma 2.2. ■

Corollary 3.4. *Let G be a graph such that $mad(G) < 3$. If $\Delta(G) \geq 4$, then $\chi_e(G) \leq \Delta(G)$.*

Corollary 3.5. *Let G be a graph such that $mad(G) < 3$. If $\Delta(G) \geq 4$, then $\chi_e^*(G) \leq \Delta(G)$.*

Theorem 3.6. *If G is a graph such that $mad(G) < 3$ and $k \geq \max\{4, \Delta(G)\}$, then G is equitably k-choosable.*

Proof. Let G be a counterexample with the smallest number of vertices. If each component of G has at most 4 vertices, then $\Delta(G) \leq 3$. So G is equitably k-choosable by Lemma 2.4. Otherwise, the statement is similar to that in the corresponding cases of Theorem 3.3. By Lemma 2.1 and Lemma 2.4, we have this theorem. ■

Corollary 3.7. *Let G be a graph such that $mad(G) < 3$. If $\Delta(G) \geq 4$, then G is equitably $\Delta(G)$-choosable.*

For a planar graph with girth g, by $mad(G) < \frac{2g}{g-2}$, we have the following corollary.

Corollary 3.8. *Let G be a planar graph with girth $g \geq 6$. If $\Delta(G) \geq 4$, then G is equitably $\Delta(G)$-colorable and equitably $\Delta(G)$-choosable.*

4. Graphs with $mad(G) < \frac{12}{5}$

Lemma 4.1. *Let G be a connected graph with order at least 4 and $mad(G) < \frac{12}{5}$. Then G has at least one of the structures in Figure 2.*

Proof. Let G be a counterexample. Then G does not contain any configuration $F_1 \sim F_4$ presented in Figure 2.

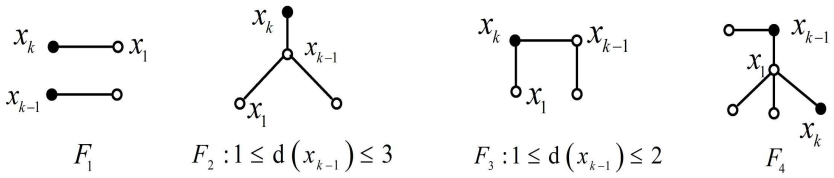

Figure 2

Each configuration depicted in Figure 2 is such that: (1) hollow vertices may be not distinct while solid vertices are distinct, (2) the degree of the solid vertices is fixed, and (3) except for specially pointed, the degree of a hollow vertices may be any integer from $[d, \Delta(G)]$, where d is the number of edges incident with the hollow vertex in the configuration.

In the following, we use the discharging method to get a contradiction. For every $v \in V(G)$, we define the original charge of v to be $w(v) = d(v) - \frac{12}{5}$. The total charge of the vertices of G is equal to

$$\sum_{v \in V(G)} \left(d(v) - \frac{12}{5}\right) = |V(G)| \times \left(ad(G) - \frac{12}{5}\right) \le |V(G)| \times \left(mad(G) - \frac{12}{5}\right) < 0.$$

In the following, we redistribute the charge according to the given discharging rules and let $w'(v)$ be the new charge of a vertex $v \in V(G)$, for convenience. If $\sum_{v \in V(G)} w'(v) > 0$ can be deduced, we can show that the assumption is wrong.

Define discharging rules as the following statements.

D1 Transfer charge $\frac{7}{5}$ from each 4^+-vertex to every adjacent 1-vertex.

D2 Transfer charge $\frac{1}{5}$ from each 3^+-vertex to every adjacent 2-vertex.

In the following, let us check the charge of each element v for $v \in V(G)$. For each $v \in V(G)$, if $d(v) = 1$, then $w(v) = -\frac{7}{5}$. Since G contains no structure F_1, there is at most one 1-vertex in G. Furthermore, the 1-vertex must be adjacent to a 4^+-vertex for the reason that G contains no structure F_2. So $w'(v) \ge -\frac{7}{5} + \frac{7}{5} = 0$ by $D1$.

If $d(v) = 2$, then $w(v) = -\frac{2}{5}$. Since G contains no structure F_3, v is not adjacent to any 2^--vertex. We have $w'(v) \ge -\frac{2}{5} + \frac{1}{5} \times 2 = 0$ by $D2$.

If $d(v) = 3$, then $w(v) = \frac{3}{5}$. Since G contains no structure F_2, v is not adjacent to any 1-vertex. Then we have $w'(v) \ge \frac{3}{5} - \frac{1}{5} \times 3 = 0$ by $D2$.

Suppose $d(v) \ge 4$. Then $w(v) = d(v) - \frac{12}{5}$. Since G contains no structure F_4, the vertex v is adjacent to at most one 1-vertex. If v is adjacent to a 1-vertex, then v is not adjacent to any 2^--vertex for the reason that G contains no structure F_4. We have $w'(v) \geq d(v) - \frac{12}{5} - \frac{7}{5} \geq 4 - \frac{12}{5} - \frac{7}{5} = \frac{1}{5} > 0$ by $D1$. Otherwise, we have $w'(v) \geq d(v) - \frac{12}{5} - \frac{1}{5} \times d(v) = \frac{4}{5}d(v) - \frac{12}{5} \geq \frac{4}{5} \times 4 - \frac{12}{5} = \frac{4}{5} > 0$ by $D2$.

From the above discussion, we have $\sum_{v \in V(G)} w'(v) \geq 0$, a contradiction. ■

In the following, let us give the proof of the main theorem.

Theorem 4.2. *If G is a graph such that $mad(G) < \frac{12}{5}$, then G is equitably k-colorable where $k \geq \max\{\Delta(G), 3\}$.*

Proof. Let G be a counterexample with smallest number of vertices. If each component of G has at most 3 vertices, then $\Delta(G) \leq 2$. So G is equitably k-colorable by Lemma 2.3. Otherwise, there is at least one component with at least four vertices. By Lemma 4.1, G has one of the structures $F_1 \sim F_4$, taking it and the vertices are labelled as they are in Figure 1. If there are vertices labelled repeatedly, then we take the larger (x_i is larger than x_{i-1}). In the following, we show how to find S in Lemma 2.2. Let $S' = \{x_k, x_{k-1}, x_1\}$. By Lemma 2.5, G is 2-degenerate, hence we can find the remaining unspecified positions in S from highest to lowest indices by choosing a vertex with minimum degree in the graph obtained from G by deleting the vertices already being chosen for S at each step. By the minimality of $|V(G)|$ and since $k \geq \Delta(G) \geq \Delta(G-S)$, $G-S$ is equitably k-colorable. So G is also equitably k-colorable by Lemma 2.2. ■

Corollary 4.3. *Let G be a graph such that $mad(G) < \frac{12}{5}$. If $\Delta(G) \geq 3$, then $\chi_e(G) \leq \Delta(G)$.*

Corollary 4.4. *Let G be a graph such that $mad(G) < \frac{12}{5}$. If $\Delta(G) \geq 3$, then $\chi_e^*(G) \leq \Delta(G)$.*

Theorem 4.5. *If G is a graph such that $mad(G) < \frac{12}{5}$ and $k \geq \max\{3, \Delta(G)\}$, then G is equitably k-choosable.*

Proof. Let G be a counterexample with the smallest number of vertices. If each component of G has at most 3 vertices, then $\Delta(G) \leq 2$. So G is equitably k-choosable by Lemma 2.4. Otherwise, the statement is similar to that in the corresponding cases of Theorem 4.2. By Lemma 2.1 and Lemma 2.4, we have this theorem. ■

Corollary 4.6. *Let G be a graph such that $mad(G) < \frac{12}{5}$. If $\Delta(G) \geq 3$, then G is equitably $\Delta(G)$-choosable.*

For a planar graph with girth g, we have the following corollary.

Corollary 4.7. *Let G be a planar graph with girth $g \geq 12$. If $\Delta(G) \geq 3$, then G is equitably $\Delta(G)$-colorable and equitably $\Delta(G)$-choosable.*

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