EQUITABLE COLORING AND EQUITABLE CHOOSABILITY OF GRAPHS WITH SMALL MAXIMUM AVERAGE DEGREE

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Abstract

A graph is said to be equitably $k$-colorable if the vertex set $V(G)$ can be partitioned into $k$ independent subsets $V_1, V_2, \ldots, V_k$ such that $||V_i|-|V_j|| \leq 1$ ($1 \leq i, j \leq k$). A graph $G$ is equitably $k$-choosable if, for any given $k$-uniform list assignment $L$, $G$ is $L$-colorable and each color appears on at most $\left\lceil \frac{|V(G)|}{k} \right\rceil$ vertices. In this paper, we prove that if $G$ is a graph such that $mad(G) < 3$, then $G$ is equitably $k$-colorable and equitably $k$-choosable where $k \geq \max\{\Delta(G), 4\}$. Moreover, if $G$ is a graph such that $mad(G) < \frac{12}{5}$, then $G$ is equitably $k$-colorable and equitably $k$-choosable where $k \geq \max\{\Delta(G), 3\}$.

Keywords: graph coloring, equitable choosability, maximum average degree.

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1. Introduction

The terminology and notation used but undefined in this paper can be found in [1]. Let $G = (V(G), E(G))$ be a graph. Let $d_G(x)$, or simply $d(x)$, denote the number of edges incident with the vertex (face) $x$ in $G$. If $d(x) = k$, $d(x) \geq k$ and $d(x) \leq k$, then the vertex $x$ is called a $k$-vertex, $k^+$-vertex and $k^-$-vertex, respectively. We use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ to denote the vertex set, edge set, maximum degree, and minimum degree of $G$, respectively. The average degree of a graph $G$ is $\frac{\sum_{v \in V(G)} d(v)}{|V(G)|}$, and denote it by $ad(G)$. The maximum average degree $mad(G)$ of $G$ is the maximum of the average degree of its subgraphs. The girth of a planar graph is the length of a smallest cycle in the graph, and denote the girth of a graph $G$ by $g(G)$. We use $[x]$ to denote a minimum integer which is no less than $x$.

A proper $k$-coloring of a graph $G$ is a mapping $\pi$ from the vertex set $V(G)$ to the set of colors $\{1, 2, \ldots, k\}$ such that $\pi(x) \neq \pi(y)$ for every edge $xy \in E(G)$. A graph $G$ is equitable $k$-colorable if $G$ has a proper $k$-coloring such that the size of the color classes differ by at most 1. The equitable chromatic number of $G$, denoted by $\chi_e(G)$, is the smallest integer $k$ such that $G$ is equitably $k$-colorable. The equitable chromatic threshold of $G$, denoted by $\chi_e^*(G)$, is the smallest integer $k$ such that $G$ is equitably $l$-colorable (for any $l \geq k$).

In 1970, Hajnal and Szemerédi proved that $\chi_e^*(G) \leq \Delta(G) + 1$ for any graph $G$ [9]. This bound is sharp as shown in the example of $K_{2n+1,2n+1}$. In 1973, Meyer introduced the notion of equitable coloring and made the following conjecture.

Conjecture 1.1 (Meyer [18]). If $G$ is a connected graph which is neither a complete graph nor odd cycle, then $\chi_e(G) \leq \Delta(G)$.

In 1994, Chen, Lih and Wu put forth the following conjecture.

Conjecture 1.2 (Chen, Lih and Wu [2]). For any connected graph $G$, if it is different from a complete graph, a complete bipartite graph and an odd cycle, then $\chi_e^*(G) \leq \Delta(G)$.

Chen, Lih and Wu [2, 3] proved Conjecture 1.2 for graphs with $\Delta(G) \leq 3$ or $\Delta(G) \geq \frac{|V(G)|}{2}$. In 2012, Chen et al. [4] improved the former result and confirmed the Conjecture 1.2 for graphs with $\Delta(G) \geq \frac{|V(G)|}{3} + 1$. Yap and Zhang [26, 27] showed that Conjecture 1.2 holds for planar graphs with $\Delta(G) \geq 13$. In 2012, Nakprasit [19] confirmed the Conjecture 1.2 for planar graphs with $\Delta(G) \geq 9$. Lih and Wu [14] verified $\chi_e^*(G) \leq \Delta(G)$ for bipartite graphs other than complete bipartite graphs. Wang and Zhang [23] proved Conjecture 1.2 for line graphs, and Kostochka and Nakprasit [12, 13] proved it for graphs with low average degree, and $d$-degenerate graphs with $\Delta(G) \geq 14d + 1$. Yan and Wang [25] showed that Conjecture 1.2 holds for Kronecker products of complete multipartite graphs and
complete graphs. Wu and Wang [24], Luo et al. [17] confirmed Conjecture 1.2 for some planar graphs with large girth, respectively. Li et al. [16], Zhu et al. [29], Dong et al. [5–8], Nakprasit [20] confirmed Conjecture 1.2 for some planar graphs with some forbidden cycles. Zhang and Wu [28], Zhu and Bu [30] verified the Conjecture 1.4 for planar graphs. They confirmed the Conjecture 1.3 for graphs with ∆(G) ≤ 3, then G is equitably k-colorable and equitably k-choosable where k ≥ max{∆(G), 4}. Moreover, if G is a complete graph or is a complete graph and show that if mad(G) < 3, then G is equitably k-colorable and each color appears on at most \( \frac{|V(G)|}{k} \) vertices.

In 2003, Kostochka, Pelsmajer and West investigated the list equitable coloring of graphs. They proposed the following conjectures.

**Conjecture 1.3** (Kostochka, Pelsmajer and West [11]). Every graph G is equitably k-choosable whenever k > ∆(G).

**Conjecture 1.4** (Kostochka, Pelsmajer and West [11]). If G is a connected graph with maximum degree at least 3, then G is equitably (∆(G))-choosable, unless G is a complete graph or is \( K_{k,k} \) for some odd k.

It has been proved that Conjecture 1.3 holds for graphs with ∆(G) ≤ 3 in [21, 22] and then the result was strengthened by Kierstead and Kostochka. They confirmed the Conjecture 1.3 for graphs with ∆(G) ≤ 7 in [10]. Kostochka, Pelsmajer and West proved that a graph G is equitably k-choosable if either G \( \not= \) \( K_{k+1}, K_{k,k} \) (with k odd in \( K_{k,k} \)) and \( k ≥ \max \left\{ \Delta, \frac{|V(G)|}{2} \right\} \), or G is a connected interval graph and \( k ≥ \Delta(G) \) or G is a 2-degenerate graph and \( k ≥ \max \{ \Delta(G), 5 \} \) in [11]. Pelsmajer proved that every graph is equitably k-choosable for any \( k ≥ \frac{\Delta(G)(\Delta(G)-1)}{2} + 2 \) in [21]. In 2009, Conjecture 1.4 were proved for planar graphs G without 4- and 6-cycles and with ∆(G) ≥ 6 by Li et al. in [16]. Zhu et al. confirmed Conjecture 1.4 for planar graph G without 3-cycles and with ∆(G) ≥ 8, planar graph G without 4- and 5-cycles and with ∆(G) ≥ 7 in [29], \( C_5 \)-free planar graph G without adjacent triangles and with ∆(G) ≥ 8 in [30], outerplanar graphs in [31]. Zhang and Wu proved Conjecture 1.4 for series-parallel graphs in [28]. More results can be seen in [5–8] and [15].

As for the sparse graph G with ∆(G) = 2, it is clear that G is equitably k-colorable and equitably k-choosable where \( k ≥ \max \{ \Delta(G), 3 \} \), if G is an odd cycle. Otherwise, G is equitably k-colorable and equitably k-choosable where \( k ≥ \max \{ \Delta(G), 2 \} \). In this paper, we consider the sparse graph G with ∆(G) ≥ 3 and show that if G is a graph such that mad(G) < 3, then G is equitably k-colorable and equitably k-choosable where \( k ≥ \max \{ \Delta(G), 4 \} \). Moreover, if G is
a graph such that \( \text{mad}(G) < \frac{12}{5} \), then \( G \) is equitably \( k \)-colorable and equitably \( k \)-choosable where \( k \geq \max\{\Delta(G), 3\} \).

2. Some Important Lemmas

**Lemma 2.1** (Kostochka, Pelsmajer and West [11]). Let \( G \) be a graph with a \( k \)-uniform list assignment \( L \). Let \( S = \{v_1, v_2, \ldots, v_k\} \), where \( \{v_1, v_2, \ldots, v_k\} \) are distinct vertices in \( G \). If \( G - S \) has an equitable \( L \)-coloring and \( |N_G(v_i) - S| \leq k - i \) for \( 1 \leq i \leq k \), then \( G \) has an equitable \( L \)-coloring.

**Lemma 2.2** (Zhu and Bu [29]). Let \( S = \{v_1, v_2, \ldots, v_k\} \) be a set of \( k \) different vertices in \( G \) such that \( G - S \) has an equitable \( k \)-coloring. If \( |N_G(v_i) - S| \leq k - i \) for \( 1 \leq i \leq k \), then \( G \) has an equitable \( k \)-coloring.

**Lemma 2.3** (Hajnal and Szemerédi [9]). Every graph has an equitable \( k \)-coloring whenever \( k \geq \Delta(G) + 1 \).

**Lemma 2.4** (Pelsmajer, Wang and Lih [21, 22]). Every graph \( G \) with maximum degree \( \Delta(G) \leq 3 \) is equitably \( k \)-choosable whenever \( k \geq \Delta(G) + 1 \).

**Lemma 2.5**. Let \( G \) be a graph with \( \text{mad}(G) < 3 \). Then \( G \) is \( 2 \)-degenerate.

**Proof.** By contradiction, there is subgraph \( G' \) of \( G \) such that \( \delta(G') \geq 3 \). It is clear that \( \text{mad}(G') \geq 3 \), a contradiction.

**Lemma 2.6** (Dong, Zou and Li [8]). If \( G \) is a graph such that \( \text{mad}(G) \leq 3 \), then \( G \) is equitably \( k \)-colorable and equitably \( k \)-choosable where \( k \geq \max\{\Delta(G), 5\} \).

3. Graphs with \( \text{mad}(G) < 3 \)

**Lemma 3.1.** Let \( G \) be a connected graph with order at least 4 and \( \delta(G) \geq 1 \). If \( \Delta(G) \leq 4 \) and \( \text{mad}(G) < 3 \), then \( G \) has at least one of the structures in Figure 1.

**Proof.** Let \( G \) be a counterexample. Then \( G \) does not contain any configuration \( H_1 \sim H_6 \) presented in Figure 1.

For each \( v \in V(G) \), if \( d(v) = 2 \), then \( v \) is adjacent to at least one 4-vertex for the reason that \( G \) contains no structure \( H_1 \). If \( d(v) = 4 \), then \( v \) is adjacent to at most one 2-vertex for the reason that \( G \) contains no structure \( H_2 \). For convenience, let \( r \) denote the number of 4-vertices which are not adjacent to any 2-vertex. Obviously, \( G \) has the following property.
Each configuration depicted in Figure 1 is such that: (1) hollow vertices may be not distinct while solid vertices are distinct, (2) the degree of the solid vertices is fixed, and (3) except for specially pointed, the degree of a hollow vertex may be any integer from \([d, \Delta(G)]\), where \(d\) is the number of edges incident with the hollow vertex in the configuration.

**Observation 3.2.** \(n_4(G) \geq n_2(G) + r\).

By Lemma 2.5, we have \(\delta(G) \leq 2\).

Suppose \(\delta(G) = 2\). By Observation 3.2, we have
\[
ad(G) = \frac{2n_2(G) + 3n_3(G) + 4n_4(G)}{n_2(G) + n_3(G) + n_4(G)} \geq \frac{6n_2(G) + 3n_3(G) + 4r}{2n_2(G) + n_3(G) + r} = \frac{3(2n_2(G) + n_3(G) + r) + r}{2n_2(G) + n_3(G) + r} \geq 3,
\]
contradiction to \(mad(G) < 3\).

Suppose \(\delta(G) = 1\). Since \(G\) contains no structure \(H_3\), there is only one 1-vertex \(v\) in \(G\). Furthermore, the vertex \(v\) must be adjacent to a 4-vertex \(u\) for the reason that \(G\) contains no structure \(H_4\). Since \(G\) contains no structure \(H_5\), the other adjacent vertices of \(u\) must be 4-vertices. For convenience, we use \(u_i\) \((1 \leq i \leq 3)\) to denote the 4-vertices which are adjacent to \(u\). Since \(G\) contains no structure \(H_6\), \(u_i\) \((1 \leq i \leq 3)\) is not adjacent to any 2-vertex. From the above discussion, we have \(r \geq 4\). Obviously, we have
\[
ad(G) = \frac{n_1(G) + 2n_2(G) + 3n_3(G) + 4n_4(G)}{n_1(G) + 2n_2(G) + 3n_3(G) + 4n_4(G)} = \frac{1 + 2n_2(G) + 3n_3(G) + 4n_4(G)}{1 + 2n_2(G) + 3n_3(G) + 4n_4(G)} = \frac{1 + 6n_2(G) + 3n_3(G) + 3r + 4}{1 + 2n_2(G) + 3n_3(G) + 4n_4(G)} \geq 3,
\]
contradiction to \(mad(G) < 3\).

In the following, let us give the proof of the main theorems.

**Theorem 3.3.** If \(G\) is a graph such that \(mad(G) < 3\), then \(G\) is equitably \(k\)-colorable where \(k \geq \max\{\Delta(G), 4\}\).
Proof. By Lemma 2.6, we only need to focus on the situation where $\Delta(G) \leq 4$. Let $G$ be a counterexample with the smallest number of vertices. Clearly, $\delta(G) \geq 1$. If each component of $G$ has at most four vertices, then $\Delta(G) \leq 3$. So $G$ is equitably $k$-colorable by Lemma 2.3. Otherwise, there is at least one component with at least four vertices. By Lemma 3.1, $G$ has one of the structures $H_1 \sim H_6$, taking it and the vertices are labelled as they are in Figure 1. If there are vertices labelled repeatedly, then we take the larger ($x_i$ is larger than $x_{i-1}$). In the following, we show how to find $S$ in Lemma 2.2. If $G$ has $H_1$, $H_2$ or $H_5$, then let $S' = \{x_k, x_{k-1}, x_{k-2}, x_1\}$. If $G$ has $H_3$ or $H_4$, then let $S' = \{x_k, x_{k-1}, x_1\}$. If $G$ has $H_6$, then let $S' = \{x_k, x_{k-1}, x_2, x_1\}$. By Lemma 2.5, $G$ is 2-degenerate, thus we can find the remaining unspecified positions in $S$ from highest to lowest indices by choosing a vertex with minimum degree in the graph obtained from $G$ by deleting the vertices already being chosen for $S$ at each step. By the minimality of $|V(G)|$ and since $k \geq \Delta(G) \geq \Delta(G - S)$, $G - S$ is equitably $k$-colorable. So $G$ is also equitably $k$-colorable by Lemma 2.2.

Corollary 3.4. Let $G$ be a graph such that $\text{mad}(G) < 3$. If $\Delta(G) \geq 4$, then $\chi_e(G) \leq \Delta(G)$.

Corollary 3.5. Let $G$ be a graph such that $\text{mad}(G) < 3$. If $\Delta(G) \geq 4$, then $\chi^*_e(G) \leq \Delta(G)$.

Theorem 3.6. If $G$ is a graph such that $\text{mad}(G) < 3$ and $k \geq \max\{4, \Delta(G)\}$, then $G$ is equitably $k$-choosable.

Proof. Let $G$ be a counterexample with the smallest number of vertices. If each component of $G$ has at most 4 vertices, then $\Delta(G) \leq 3$. So $G$ is equitably $k$-choosable by Lemma 2.4. Otherwise, the statement is similar to that in the corresponding cases of Theorem 3.3. By Lemma 2.1 and Lemma 2.4, we have this theorem.

Corollary 3.7. Let $G$ be a graph such that $\text{mad}(G) < 3$. If $\Delta(G) \geq 4$, then $G$ is equitably $\Delta(G)$-choosable.

For a planar graph with girth $g$, by $\text{mad}(G) < \frac{2g}{g-2}$, we have the following corollary.

Corollary 3.8. Let $G$ be a planar graph with girth $g \geq 6$. If $\Delta(G) \geq 4$, then $G$ is equitably $\Delta(G)$-colorable and equitably $\Delta(G)$-choosable.

4. Graphs with $\text{mad}(G) < \frac{12}{5}$

Lemma 4.1. Let $G$ be a connected graph with order at least 4 and $\text{mad}(G) < \frac{12}{5}$. Then $G$ has at least one of the structures in Figure 2.
Proof. Let $G$ be a counterexample. Then $G$ does not contain any configuration $F_1 \sim F_4$ presented in Figure 2.

![Figure 2](image)

Each configuration depicted in Figure 2 is such that: (1) hollow vertices may be not distinct while solid vertices are distinct, (2) the degree of the solid vertices is fixed, and (3) except for specially pointed, the degree of a hollow vertices may be any integer from $[d, \Delta(G)]$, where $d$ is the number of edges incident with the hollow vertex in the configuration.

In the following, we use the discharging method to get a contradiction. For every $v \in V(G)$, we define the original charge of $v$ to be $w(v) = d(v) - \frac{12}{5}$. The total charge of the vertices of $G$ is equal to

$$\sum_{v \in V(G)} \left( d(v) - \frac{12}{5} \right) = |V(G)| \left( ad(G) - \frac{12}{5} \right) \leq |V(G)| \times \left( mad(G) - \frac{12}{5} \right) < 0.$$

In the following, we redistribute the charge according to the given discharging rules and let $w'(v)$ be the new charge of a vertex $v \in V(G)$, for convenience. If $\sum_{v \in V(G)} w'(v) > 0$ can be deduced, we can show that the assumption is wrong.

Define discharging rules as the following statements.

**D1** Transfer charge $\frac{7}{5}$ from each $4^+$-vertex to every adjacent $1$-vertex.

**D2** Transfer charge $\frac{1}{5}$ from each $3^+$-vertex to every adjacent $2$-vertex.

In the following, let us check the charge of each element $v$ for $v \in V(G)$. For each $v \in V(G)$, if $d(v) = 1$, then $w(v) = -\frac{7}{5}$. Since $G$ contains no structure $F_1$, there is at most one $1$-vertex in $G$. Furthermore, the $1$-vertex must be adjacent to a $4^+$-vertex for the reason that $G$ contains no structure $F_2$. So $w'(v) \geq -\frac{7}{5} + \frac{7}{5} = 0$ by $D1$.

If $d(v) = 2$, then $w(v) = -\frac{2}{5}$. Since $G$ contains no structure $F_3$, $v$ is not adjacent to any $2^-$-vertex. We have $w'(v) \geq -\frac{2}{5} + \frac{1}{5} \times 2 = 0$ by $D2$.

If $d(v) = 3$, then $w(v) = \frac{3}{5}$. Since $G$ contains no structure $F_2$, $v$ is not adjacent to any $1$-vertex. Then we have $w'(v) \geq \frac{3}{5} - \frac{1}{5} \times 3 = 0$ by $D2$.

Suppose $d(v) \geq 4$. Then $w(v) = d(v) - \frac{12}{5}$. Since $G$ contains no structure $F_4$, the vertex $v$ is adjacent to at most one $1$-vertex. If $v$ is adjacent to a $1$-vertex,
then \( v \) is not adjacent to any \( 2^- \)-vertex for the reason that \( G \) contains no structure \( F_1 \). We have \( w'(v) \geq d(v) - \frac{12}{5} - \frac{7}{5} \geq 4 - \frac{12}{5} - \frac{7}{5} = \frac{1}{5} > 0 \) by \( D1 \). Otherwise, we have \( w'(v) \geq d(v) - \frac{12}{5} - \frac{1}{5} \times d(v) = \frac{4}{5}d(v) - \frac{12}{5} \geq \frac{4}{5} \times 4 - \frac{12}{5} = \frac{4}{5} > 0 \) by \( D2 \).

From the above discussion, we have \( \sum_{v \in V(G)} w'(v) \geq 0 \), a contradiction. 

In the following, let us give the proof of the main theorem.

**Theorem 4.2.** If \( G \) is a graph such that \( \text{mad}(G) < \frac{12}{5} \), then \( G \) is equitably \( k \)-colorable where \( k \geq \max\{\Delta(G), 3\} \).

**Proof.** Let \( G \) be a counterexample with smallest number of vertices. If each component of \( G \) has at most 3 vertices, then \( \Delta(G) \leq 2 \). So \( G \) is equitably \( k \)-colorable by Lemma 2.3. Otherwise, there is at least one component with at least four vertices. By Lemma 4.1, \( G \) has one of the structures \( F_1 \sim F_4 \), taking it and the vertices are labelled as they are in Figure 1. If there are vertices labelled repeatedly, then we take the larger \( (x_i \) is larger than \( x_{i-1} \)). In the following, we show how to find \( S \) in Lemma 2.2. Let \( S' = \{x_k, x_{k-1}, x_1\} \). By Lemma 2.5, \( G \) is \( 2 \)-degenerate, hence we can find the remaining unspecified positions in \( S \) from highest to lowest indices by choosing a vertex with minimum degree in the graph obtained from \( G \) by deleting the vertices already being chosen for \( S \) at each step. By the minimality of \( |V(G)| \) and since \( k \geq \Delta(G) \geq \Delta(G - S) \), \( G - S \) is equitably \( k \)-colorable. So \( G \) is also equitably \( k \)-colorable by Lemma 2.2.

**Corollary 4.3.** Let \( G \) be a graph such that \( \text{mad}(G) < \frac{12}{5} \). If \( \Delta(G) \geq 3 \), then \( \chi_e(G) \leq \Delta(G) \).

**Corollary 4.4.** Let \( G \) be a graph such that \( \text{mad}(G) < \frac{12}{5} \). If \( \Delta(G) \geq 3 \), then \( \chi_e^*(G) \leq \Delta(G) \).

**Theorem 4.5.** If \( G \) is a graph such that \( \text{mad}(G) < \frac{12}{5} \) and \( k \geq \max\{3, \Delta(G)\} \), then \( G \) is equitably \( k \)-choosable.

**Proof.** Let \( G \) be a counterexample with the smallest number of vertices. If each component of \( G \) has at most 3 vertices, then \( \Delta(G) \leq 2 \). So \( G \) is equitably \( k \)-choosable by Lemma 2.4. Otherwise, the statement is similar to that in the corresponding cases of Theorem 4.2. By Lemma 2.1 and Lemma 2.4, we have this theorem.

**Corollary 4.6.** Let \( G \) be a graph such that \( \text{mad}(G) < \frac{12}{5} \). If \( \Delta(G) \geq 3 \), then \( G \) is equitably \( \Delta(G) \)-choosable.

For a planar graph with girth \( g \), we have the following corollary.

**Corollary 4.7.** Let \( G \) be a planar graph with girth \( g \geq 12 \). If \( \Delta(G) \geq 3 \), then \( G \) is equitably \( \Delta(G) \)-colorable and equitably \( \Delta(G) \)-choosable.
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