Discussiones Mathematicae
Graph Theory 38 (2018) 811–828
doi:10.7151/dmgt.2045

TREES WITH UNIQUE LEAST CENTRAL SUBTREES

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Abstract

A subtree S of a tree T is a central subtree of T if S has the minimum eccentricity in the join-semilattice of all subtrees of T. Among all subtrees lying in the join-semilattice center, the subtree with minimal size is called the least central subtree. Hamina and Peltola asked what is the characterization of trees with unique least central subtree? In general, it is difficult to characterize completely the trees with unique least central subtree. Nieminen and Peltola [*The subtree center of a tree*, Networks 34 (1999) 272–278] characterized the trees with the least central subtree consisting just of a single vertex. This paper characterizes the trees having two adjacent vertices as a unique least central subtree.

Keywords: tree, central subtree, least central subtree.

2010 Mathematics Subject Classification: 05C05, 05C22.

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1. Introduction

The "central part" of a graph has many important applications in the facility location, and it has been well studied in the literature (see, for example, [2, 4, 5, 7, 11–15]). Applications of the center problem include the location of industrial plants, warehouses, distribution centers, and public service facilities in transportation networks, as well as the location of various facilities in telecommunication networks.

The concepts of central subtrees and least central subtrees were introduced in [10]. For every tree T, a *join-semilattice* $L(T)$ of subtrees of T is defined in [10] as follows. The *meet* $S_1 \wedge S_2$ of subtrees S_1 and S_2 equals the subtree induced by the intersection of the vertex sets of S_1 and S_2 whenever the intersection is nonempty, while the *join* $S_1 \vee S_2$ is the least subtree of T containing the subtrees S_1 and S_2. In other words, $S_1 \vee S_2$ is the subtree induced by the union of vertices of S_1 and S_2 whenever the intersection of the vertex sets of S_1 and S_2 is nonempty. In the case of nonintersection subtrees, $S_1 \vee S_2$ is the subtree induced by the union of vertices of S_1 and S_2 together with the vertices of the path from S_1 to S_2. A subtree S of a tree T is a *central subtree* of T if S has the minimum eccentricity in the join-semilattice of all subtrees of T. It can be a vertex, or a path, or some other kind of subtrees such that the subtree is the most central when compared with all subtrees of the tree. The Hasse diagram graph G_L of $L(T)$ is a median graph [1, 8, 9]. The graph center of the median graph G_L is closely related to central subtrees. The set of all central subtrees is, in fact, the set of central vertices of the graph G_L. Among all subtrees lying in the join-semilattice center, the best is the one with minimal size. That is the *least central subtree*. For paths and stars, the least central subtree is unique and coincides with the center of the tree.

Nieminen and Peltola [10] described the general properties of a least central subtree of a tree, they gave some connections between the least central subtree and the center/centroid of a tree, and proved that the intersection of two least central subtrees is nonempty. Hamina and Peltola [6] proved that every least central subtree of a tree contains the center and at least one vertex of the centroid of the tree.

Hamina and Peltola posed the following problem in [6].

Problem. What is the characterization of trees with unique least central subtree?

Motivated by this problem, Nieminen and Peltola [10] describes the structure of the trees with the least central subtree consisting just of a single vertex.

Theorem 1 [10]**.** *The least central subtree of a tree T is a single vertex if and only if T is either a path of odd order or a star $K_{1,p}$ $(p \geq 2)$.*

In general, characterizing completely the trees with unique least central subtrees seems to be difficult. In this paper we characterize the trees having two adjacent vertices as a unique least central subtree. Our main result is the following.

Theorem 2. *The unique least central subtree of a tree T is P_2 if and only if T is one of trees T_4, T_6, T_8, T_9, a double star and a path of even order.*

The trees T_4, T_6, T_8, T_9 in Theorem 2 are defined in Section 3. In the next section, we give some basic notation and terminology. In Section 3, we give the proof of Theorem 2.

2. Notation and Preliminaries

The vertex set of a graph G is referred to as $V(G)$, its edge set as $E(G)$. The number of vertices of G is its *order*, written as $|G|$. If $U \subseteq V(G)$, $G[U]$ is the subgraph of G induced by U and we write $G - U$ for $G[V(G) - U]$. In other words, $G - U$ is obtained from G by deleting all the vertices in $U \cap V(G)$ and their incident edges. If $U = \{v\}$ is a singleton, we write $G - v$ rather than $G - \{v\}$. As usual, P_n denotes the path of order n and the complete bipartite graph $K_{1,p}$ $(p \geq 1)$ is called a *star*. A *double star* is the tree obtained from two vertex disjoint stars by connecting their centers. The *subdivision* of a star $K_{1,p}$ is the tree obtained from $K_{1,p}$ by subdividing each edge of $K_{1,p}$ exactly once. The *distance* $d_G(x, y)$ in G of two vertices x, y is the length of a shortest $x - y$ path in G; if no such path exists, we set $d_G(x, y) = \infty$. The *eccentricity* $e(v)$ of a vertex v in a connected graph G is the distance to a vertex farthest from v, i.e., $e(v) = \max\{d_G(u, v) \,|\, u \in V(G)\}$. The number of components of a graph G is denoted by $\omega(G)$. The *center* $C(G)$ of G consists of vertices with minimum eccentricity, i.e., $C(G) = \{v \,|\, e(v) = \min\{e(u) \,|\, u \in V(G)\}\}$ and the *radius* $\text{Rad}(G)$ of G is its minimum eccentricity.

In a tree T, a vertex of degree one is referred to as a *leaf* and a vertex which is adjacent to a leaf is a *support vertex*. An edge incident to a leaf is a *pendant edge*. For subtrees S_1 and S_2 of a tree T, the *distance* $d_T(S_1, S_2)$ between S_1 and S_2 in T is the length of the shortest path joining two vertices of S_1 and S_2 in T. The *median graph* G_L of T is the graph on the set of all subtrees of T in which two subtrees S_1 and S_2 are adjacent as vertices of G_L if and only if $V(S_1) \supseteq V(S_2)$ and $|V(S_1) - V(S_2)| = 1$ or $V(S_2) \supseteq V(S_1)$ and $|V(S_2) - V(S_1)| = 1$. In G_L, we simply write $d_L(S_1, S_2)$ for the distance between two vertices S_1 and S_2 instead of $d_{G_L}(S_1, S_2)$. For a subtree S of T, the *L-eccentricity* $e_L(S)$ of S is the distance from S to a vertex most remote from it in G_L, that is, $e_L(S) = \max\{d_L(S, S') \,|\, S'$ is a subtree of $T\}$. A subtree S is a *central subtree* of T if it has the minimum eccentricity $e_L(S)$ in the graph G_L. A tree may contain several central subtrees [10]. A central subtree of T is called a *least central subtree* if it has the minimum number of vertices.

The following are some basic results on the least central subtrees of a tree in [6, 10] that will be useful in the next section.

Lemma 3 [10]. *Let G_L be the semilattice graph of all subtrees of a tree T, and S_1 and S_2 be two subtrees of T. Then, the distance $d_L(S_1, S_2)$ between S_1 and S_2 in G_L is $|S_1| + |S_2| + 2(d_T(S_1, S_2) - 1)$ if $d_T(S_1, S_2) \geq 1$, and $|S_1 \vee S_2| - |S_1 \wedge S_2|$ if $d_T(S_1, S_2) = 0$.*

Let C_L be a least central subtree of a tree T and v a vertex adjacent to C_L. Let S_L be the component of $T - v$ containing C_L and C_v the subtree of T induced by $V(C_L) \cup \{v\}$. Furthermore, we set $S_v = T \setminus V(S_L)$ and let S_v^* be the subtree of T satisfying $e_L(C_v) = d_L(C_v, S_v^*)$.

Lemma 4 [6]. *Let C_L be a least central subtree of a tree T and v be a vertex adjacent to C_L. If $e_L(C_v) = d_L(C_v, S_v^*)$, then $S_v^* \neq T$ and*

(1) *if $V(C_L) \cap V(S_v^*) \neq \emptyset$, then $v \notin V(S_v^*)$,*

(2) *if $V(C_L) \cap V(S_v^*) = \emptyset$, then v does not lie in the $C_L - S_v^*$ path in T.*

Lemma 5 [6]. *$|S_v| \leq d_T(S_v, S_v^*)$.*

Theorem 6 [6]. *The center of a tree is a subtree of every least central subtree.*

The following theorem reveals a close connection between least central subtrees and leaves of a tree.

Theorem 7 [10]. *If C_L is a least central subtree of a tree T with at least three vertices, then the subtree C_L contains no leaf of T.*

3. Proof of Theorem 2

In this section we give the proof of our main result. For this purpose, we first give some special trees as follows.

T_1: the subdivision of $K_{1,3}$.

T_2: the tree obtained from $K_{1,3}$ by subdividing exactly two edges of $K_{1,3}$ once.

T_3: the tree obtained from $K_{1,3}$ by subdividing exactly one edge of $K_{1,3}$ twice.

T_4: the tree obtained from $K_{1,p}$ ($p \geq 4$) by subdividing exactly one edge of $K_{1,p}$ twice.

T_5: the tree obtained from $K_{1,3}$ by subdividing each edge of $K_{1,3}$ twice.

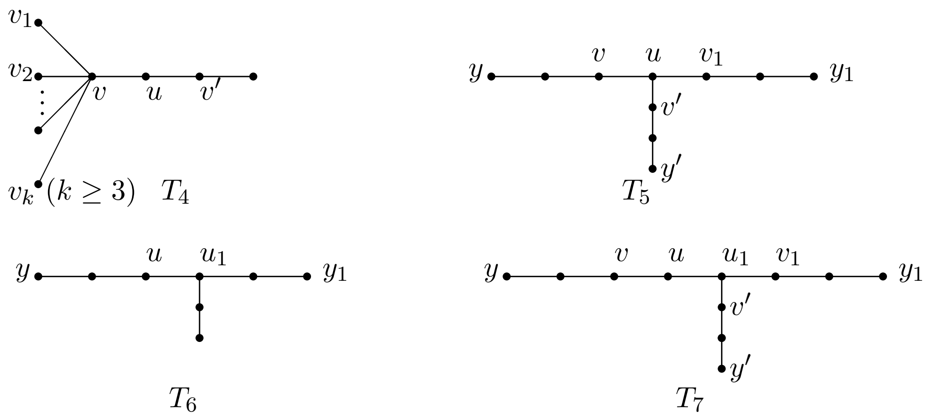

Figure 1. The trees $T_i, i = 4, 5, 6, 7$.

T_6: the tree obtained from $K_{1,3}$ by subdividing two edges of $K_{1,3}$ once and one edge of $K_{1,3}$ twice.

T_7: the tree obtained from $K_{1,3}$ by subdividing two edges of $K_{1,3}$ twice and one edge of $K_{1,3}$ three times.

T_8: the tree obtained from a path P with $\mathrm{Rad}(P) \geq 3$ by attaching two pendant edges to a support of P.

T_8': the tree obtained from a path P_{2k} of even order with $\mathrm{Rad}(P_{2k}) \geq 3$ by attaching two pendant edges to a support of P_{2k}.

T_8'': the tree obtained from a path P_{2k+1} of odd order with $\mathrm{Rad}(P_{2k+1}) \geq 3$ by attaching two pendant edges to a support of P_{2k+1}.

T_9: the tree obtained from a path P_{2k} of even order with $\mathrm{Rad}(P_{2k}) \geq 3$ by attaching one pendant edge to any vertex of degree two of P_{2k}.

For convenience, we identify the notation $C(T)$ with the subtree induced by the center $C(T)$ in a tree T. We obtain our main result by showing Lemmas 8, 9 and 10.

Lemma 8. *If T is a tree of order n with Rad$(T) \leq 2$, then the least central subtree of T is P_2 if and only if T is one of trees P_2, T_1, T_4 and a double star.*

Proof. Let T be a tree with $\mathrm{Rad}(T) \leq 2$. First, suppose that T is one of the trees P_2, a double star, T_1 and T_4. We show that the least central subtree of T is P_2. Clearly, the least central subtree of P_2 is itself. If T is a double star, then $V(T) - C(T)$ are leaves of T since $\mathrm{Rad}(T) = 2$. Theorems 6 and 7 imply that the least central subtree of T is P_2. If $T = T_1$, then clearly $|C(T_1)| = 1$. By Theorem 1, the least central subtree of T_1 is not a single vertex. Hence the least central subtree of T_1 has at least two vertices and it contains the center $C(T_1)$ by Theorem 6. A direct calculation shows that the least central subtree of T_1 is

P_2. If $T = T_4$, then $|C(T_4)| = 1$. Let $C(T_4) = \{u\}$ and v, v' be two neighbors of u (see Figure 1). Let S be the component of $T_4 - u$ containing v. By Theorem 7, the possible least central subtrees of T_4 are $T_4[\{u, v, v'\}]$ or $T_4[\{u, v\}] = P_2$. It is easy to verify that $e_L(T_4[\{u, v\}]) = d_L(T_4[\{u, v\}], T_4) = n - 2$. Note that $e_L(T_4[\{u, v, v'\}]) \geq d_L(S, T_4[\{u, v, v'\}]) = n - 2$, so the least subtree of T_4 is a path P_2.

Conversely, suppose that the least central subtree of T is P_2. Let $C_L = uu_1$ be a least central subtree of T. If $\mathrm{Rad}(T) = 1$, we see that T is P_2 or a star $K_{1,p}$ ($p \geq 2$). By Theorem 1, we see that the central subtree of a star $K_{1,p}$ ($p \geq 2$) consists of a single vertex, so $T = P_2$. Next we may assume that $\mathrm{Rad}(T) = 2$.

Suppose $|C(T)| = 2$. Then Theorem 6 and $\mathrm{Rad}(T) = 2$ imply that $V(C_L) = C(T) = \{u, u_1\}$, and $V(T) - \{u, u_1\}$ are leaves of T. Thus T is a double star.

Suppose $|C(T)| = 1$. Then $\mathrm{Rad}(T) = 2$ implies that $|T| \geq 5$. Let L be the set of leaves of T. If $|L| = 2$ then $T = P_5$. By Theorem 1, the least central subtree of P_5 is a single vertex, a contradiction. If $|L| = 3$, T is one of the trees T_1, T_2, T_3. It is easy to see that the least central subtrees of T_2 and T_3 contain three vertices, respectively. So T is the tree T_1. If $|L| \geq 4$, we claim that T is the tree T_4. If not, then either $\omega(T-u) \geq 3$ or $\omega(T-u) = 2$ and each component of $T-u$ contains at least two leaves. First note that $e_L(C_L) \geq d_L(C_L, T) = n-2$. On the other hand, we consider $e_L(T - L)$. Let S be any subtree of T. If $V(S) \cap (V(T) - L) = \emptyset$, then $d_L(S, T - L) = 1 + n - |L| \leq n - 3$ by Lemma 3. If $V(S) \cap (V(T) - L) \neq \emptyset$, then, by Lemma 3, we have

$$\begin{aligned} d_L(S, T - L) &\leq |S| + n - |L| - 2|S \wedge (T - L)| \\ &= |S| - 2|S \wedge (T - L)| + n - |L| \\ &\leq (|L| - 3) + n - |L| \leq n - 3. \end{aligned}$$

Hence $e_L(T - L) = \max\{d_L(S, T - L)|\ S \text{ is any subtree of } T\} \leq n - 3$. This implies that $e_L(T - L) < e_L(C_L)$, contradicting the fact that C_L is the least central subtree of T. So T is the tree T_4. The assertion follows. ■

Lemma 9. *Let T be a tree of order n with $|C(T)| = 2$ and $\mathrm{Rad}(T) \geq 3$. The least central subtree of T is P_2 if and only if T is one of trees T_6, T_8', T_9 and a path of even order with $\mathrm{Rad}(T) \geq 3$.*

Proof. Suppose that T is one of trees T_6, T_8', T_9 and a path of even order with $\mathrm{Rad}(T) \geq 3$. Clearly, $|C(T)| = 2$ and $\mathrm{Rad}(T) \geq 3$. By Theorem 6, the least central subtree of T contains the center $C(T)$. By a direct calculation, one can verify that the least central subtree of T is P_2.

Conversely, let T be a tree with $|C(T)| = 2$ and $\mathrm{Rad}(T) \geq 3$. Suppose that the least central subtree of T is P_2. Let $C_L = uu_1$ be a least central subtree of T. By Theorem 6, $C(T) = \{u, u_1\}$. Then there exist two vertices y, y_1 of T such that $d_T(u, y) = d_T(u_1, y_1) = \text{Rad}(T) - 1$. Let v be the vertex adjacent to u on the $u - y$ path and v_1 be the vertex adjacent to u_1 on the $u_1 - y_1$ path, respectively. Define

$$K = \{x \in V(T) |\ d_T(x, C_L) = \text{Rad}(T) - 1\}.$$

For any $x \in K$, let w be the vertex adjacent to C_L on the $x - C_L$ path. As defined in Section 2, let S_L be the component of $T - w$ containing C_L, and let $S_w = T \setminus V(S_L)$ and $e_L(C_w) = d_L(C_w, S_w^*)$, where C_w is the subtree of T induced by $V(C_L) \cup \{w\}$.

Claim 1. *S_w contains at most two vertices besides the vertices on $w - x$ path.*

Proof. Since $x \in K$ and the $w-x$ path is contained in S_w, $|S_w| \geq \text{Rad}(T)-1$. By Lemmas 4 and 5, we have $\text{Rad}(T) + 1 \geq d_T(S_w, S_w^*) \geq |S_w|$. Then $\text{Rad}(T) + 1 \geq d_T(S_w, S_w^*) \geq |S_w| \geq \text{Rad}(T)-1$. Hence S_w contains at most two vertices besides the vertices on the $w - x$ path, as claimed. □

Let r be the number of all vertices of T that do not lie in the path $y - y_1$. Then

$$\text{Rad}(T) \leq \frac{n-r}{2}. \tag{1}$$

Claim 2. $r \leq 3$.

Proof. We establish the claim by contradiction. Suppose $r \geq 4$. Let C_v be the subtree of T induced by $V(C_L) \cup \{v\}$. We shall show that $e_L(C_v) < e_L(C_L)$. For this purpose, we next show that $d_L(S, C_v) \leq n - 3$ for any subtree S of tree T.

Suppose $V(S) \cap V(C_v) \neq \emptyset$. Then $|V(S) \cap V(C_v)| \leq 3$. If $|V(S) \cap V(C_v)| = 1$, then $|S| \leq |T| - 4 = n - 4$ by $\text{Rad}(T) \geq 3$, thus $d_L(S, C_v) = 3 + |S| - 2 \leq n - 3$ by Lemma 3. If $|V(S) \cap V(C_v)| = 2$, then $|S| \leq n - 3$ by $\text{Rad}(T) \geq 3$, thus $d_L(S, C_v) = 3 + |S| - 4 \leq n - 3$ by Lemma 3. If $|V(S) \cap V(C_v)| = 3$, then $d_L(S, C_v) = |S| - 3 \leq n - 3$.

Suppose that $V(S) \cap V(C_v) = \emptyset$ and the subtree S contains at least one vertex on $y - y_1$ path. By Claim 1,

$$d_T(S, C_v) \leq \begin{cases} \text{Rad}(T) - |S| & \text{if } |S| \leq 2; \\ \text{Rad}(T) - |S| + 1 & \text{if } |S| = 3; \\ \text{Rad}(T) - |S| + 2 & \text{if } |S| \geq 4. \end{cases}$$

If $|S| \leq 2$, by Lemma 3, we have

$$\begin{aligned} d_L(S, C_v) &= |S| + 3 + 2(d_T(S, C_v) - 1) \\ &\leq |S| + 3 + 2(\text{Rad}(T) - |S| - 1) \\ &\leq 1 + 2\text{Rad}(T) - |S| \\ &\leq 1 + (n - r) - |S| \qquad \text{(by (1))} \\ &\leq n - 3. \end{aligned}$$

If $|S| = 3$, by Lemma 3, we have

$$\begin{aligned} d_L(S, C_v) &= |S| + 3 + 2(d_T(S, C_v) - 1) \\ &\leq |S| + 3 + 2(\mathrm{Rad}(T) - |S|) \\ &\leq n - r \qquad \text{(by (1))} \\ &\leq n - 3. \end{aligned}$$

If $|S| \geq 4$, by Lemma 3, we have

$$\begin{aligned} d_L(S, C_v) &= |S| + 3 + 2(d_T(S, C_v) - 1) \\ &\leq |S| + 3 + 2(\mathrm{Rad}(T) - |S| + 1) \\ &\leq n - r + 5 - |S| \qquad \text{(by (1))} \\ &\leq n - 3. \end{aligned}$$

Suppose that $V(S) \cap V(C_v) = \emptyset$ and the subtree S contains no vertex on $y - y_1$ path. Then $|S| \leq r$. If $|S| = 1$, then $d_T(S, C_v) \leq \mathrm{Rad}(T) - 1$. By Lemma 3, we have

$$\begin{aligned} d_L(S, C_v) &= 1 + 3 + 2(d_T(S, C_v) - 1) \\ &\leq 1 + 3 + 2(\mathrm{Rad}(T) - 2) \\ &\leq n - r \qquad \text{(by (1))} \\ &\leq n - 3. \end{aligned}$$

If $|S| \geq 2$, then $d_T(S, C_v) \leq \mathrm{Rad}(T) - 2$. By Lemma 3, we have

$$\begin{aligned} d_L(S, C_v) &= |S| + 3 + 2(d_T(S, C_v) - 1) \\ &\leq |S| + 3 + 2(\mathrm{Rad}(T) - 3) \\ &\leq |S| + (n - r) - 3 \qquad \text{(by (1))} \\ &\leq n - 3, \end{aligned}$$

where the last inequality follows from $|S| \leq r$. Thus

$$e_L(C_v) = \max\{d_L(S, C_v) \mid S \text{ is any subtree of } T\} \leq n - 3.$$

Note that $e_L(C_L) \geq d_L(T, C_L) = n - 2$. But then $e_L(C_L) > e_L(C_v)$, a contradiction. Consequently, $r \leq 3$. $\square$

Let $K_0 = \{x \in K \mid \text{the } u - x \text{ path or } u_1 - x \text{ path pass through neither } v \text{ nor } v_1\}$.

Claim 3. *If $K_0 \neq \emptyset$, then T is the tree T_6.*

Proof. If $K_0 \neq \emptyset$, Claim 2 implies $\mathrm{Rad}(T) = 3$ or $\mathrm{Rad}(T) = 4$. Then T is isomorphic to the tree T_6 or T_7. If T is isomorphic to the tree T_7, let $C(T_7) =$

$\{u, u_1\}$ and v, v_1, v' be neighbors of u or u_1, where v' is adjacent to the vertex u_1 on the $u - y'$ path (see Figure 1). Set $M = \{u, u_1, v, v_1, v'\}$. It is easy to check that $e_L(T_7[M]) = 8 = |T_7| - 3 \leq n - 3$. Note that $e_L(C_L) \geq d_L(T_7, C_L) = |T_7| - 2 = n - 2$. But then $e_L(C_L) > e_L(T_7[M])$, a contradiction. Thus T is isomorphic to the tree T_6. □

Claim 4. *If $K_0 = \emptyset$, then $r \leq 2$.*

Proof. Suppose not, then $r = 3$ by Claim 2. Let $C_{vv_1} = T[\{u, u_1, v, v_1\}]$. We show that $e_L(C_{vv_1}) \leq n - 3$. For this, it suffices to verify that $d_L(S, C_{vv_1}) \leq n - 3$ for any subtree S of T.

Suppose that $V(S) \cap V(C_{vv_1}) \neq \emptyset$. As indicated in the beginning of the proof for Claim 2, one can easily verify that $d_L(S, C_{vv_1}) \leq n - 3$.

Suppose that $V(S) \cap V(C_{vv_1}) = \emptyset$. Hence, by Claim 1,

$$d_T(S, C_{vv_1}) \leq \begin{cases} \mathrm{Rad}(T) - |S| - 1 & \text{if } |S| \leq 2; \\ \mathrm{Rad}(T) - |S| & \text{if } |S| = 3; \\ \mathrm{Rad}(T) - |S| + 1 & \text{if } |S| \geq 4. \end{cases}$$

If $|S| \leq 2$, by Lemma 3, we have

$$\begin{aligned} d_L(S, C_{vv_1}) &= |S| + 4 + 2(d_T(S, C_{vv_1}) - 1) \\ &\leq |S| + 4 + 2(\mathrm{Rad}(T) - |S| - 2) \\ &\leq 2\mathrm{Rad}(T) - |S| \\ &\leq n - r - |S| \qquad \text{(by (1))} \\ &\leq n - 3. \end{aligned}$$

If $|S| = 3$, by Lemma 3, we have

$$\begin{aligned} d_L(S, C_{vv_1}) &= |S| + 4 + 2(d_T(S, C_{vv_1}) - 1) \\ &\leq |S| + 4 + 2(\mathrm{Rad}(T) - |S| - 1) \\ &\leq n - r - 1 \qquad \text{(by (1))} \\ &\leq n - 3. \end{aligned}$$

If $|S| \geq 4$, by Lemma 3, we have

$$\begin{aligned} d_L(S, C_{vv_1}) &= |S| + 4 + 2(d_T(S, C_{vv_1}) - 1) \\ &\leq |S| + 4 + 2(\mathrm{Rad}(T) - |S|) \\ &\leq n - r + 4 - |S| \qquad \text{(by (1))} \\ &\leq n - 3. \end{aligned}$$

So $e_L(C_{vv_1}) \leq n - 3$. But then $e_L(C_L) \geq d_L(T, C_L) = n - 2 > e_L(C_{vv_1})$, a contradiction. Thus $r \leq 2$. □

We discuss the case $K_0 = \emptyset$. For $r = 0$, clearly T is a path of even order with $\mathrm{Rad}(T) \geq 3$. For $r = 1$, it is easy to see that T is the tree T_9. We next consider the case $r = 2$, i.e., T has precisely two vertices that do not lie in the $y-y_1$ path. We shall show that the two vertices of T must be adjacent to a support vertex of $y-y_1$ path, i.e., T is the tree T_8'.

Suppose not, that is, T is not the tree T_8'. We shall obtain a contradiction by showing $e_L(C_L) > e_L(C_{vv_1})$. Note that $e_L(C_L) \geq d_L(T, C_L) = n-2$. We next show that $e_L(C_{vv_1}) \leq n-3$. It suffices to show that $d_L(S, C_{vv_1}) \leq n-3$ for any subtree S of T. If $V(S) \cap V(C_{vv_1}) \neq \emptyset$, then, as before, one can easily verify that $d_L(S, C_{vv_1}) \leq n-3$. Suppose that $V(S) \cap V(C_{vv_1}) = \emptyset$. Then

$$d_T(S, C_{vv_1}) \leq \begin{cases} \mathrm{Rad}(T) - |S| - 1 & \text{if } |S| \leq 2; \\ \mathrm{Rad}(T) - |S| & \text{if } |S| \geq 3. \end{cases}$$

So if $|S| \leq 2$, by Lemma 3, we have

$$\begin{aligned} d_L(S, C_{vv_1}) &= |S| + 4 + 2(d_T(S, C_{vv_1}) - 1) \\ &\leq |S| + 4 + 2(\mathrm{Rad}(T) - 2 - |S|) \\ &\leq (n-r) - |S| \qquad \text{(by (1))} \\ &\leq n-3. \end{aligned}$$

If $|S| \geq 3$, by Lemma 3, we have

$$\begin{aligned} d_L(S, C_{vv_1}) &= |S| + 4 + 2(d_T(S, C_{vv_1}) - 1) \\ &\leq |S| + 4 + 2(\mathrm{Rad}(T) - 1 - |S|) \\ &\leq 2 + (n-r) - |S| \qquad \text{(by (1))} \\ &\leq n-3. \end{aligned}$$

So $e_L(C_{vv_1}) = \max\{d_L(S, C_{vv_1}) |\ S \text{ is any subtree of } T\} \leq n-3$. But then $e_L(C_L) > e_L(C_{vv_1})$, a contradiction. Consequently, T is the tree T_8'. ■

Lemma 10. *Let T be a tree of order n with $|C(T)| = 1$ and $Rad(T) \geq 3$. The least central subtree of T is P_2 if and only if T is the tree T_8''.*

Proof. Suppose that $T = T_8''$. By Theorem 1, the least central subtree of T_8'' is not a single vertex. Note that $|C(T_8'')| = 1$, so the least central subtree of T_8'' has at least two vertices and it contains the center $C(T_8'')$ by Theorem 6. By a direct calculation, one may easily verify that the least central subtree of T_8'' is P_2.

Conversely, let T be a tree of order n with $|C(T)| = 1$ and $\mathrm{Rad}(T) \geq 3$. Let $C_L = uu_1$ be a least central subtree of T. Since $|C(T)| = 1$, we may assume that $u \in C(T)$ and $u_1 \notin C(T)$ by Theorem 6. Then there exist vertices y, y_1 such that $d_T(u, y) = d_T(u, y_1) = \mathrm{Rad}(T)$. Let v, v_1 be the vertices adjacent to u on the

$u-y$ path and $u-y_1$ path, respectively. Let $K = \{x \in V(T) | \, d_T(u,x) = \mathrm{Rad}(T)\}$ and let r be the number of all vertices of T that do not lie in the $y - y_1$ path. Then

$$\mathrm{Rad}(T) \le \frac{n-r-1}{2}. \tag{2}$$

Claim 1. *There exists a vertex $y' \in K$ such that u_1 is on the $u - y'$ path.*

Proof. Suppose not, then $u_1 \neq v_1$. Let the subtrees S_v, S_{v_1} and S_v^* of T be defined as in Lemma 4. We have the following fact.

Fact 1. S_v and S_{v_1} contain at most one vertex besides the vertices on $v-y$ path and $v_1 - y_1$ path, respectively.

Proof. Since $d_T(u,y) = \mathrm{Rad}(T)$, $|S_v| \ge \mathrm{Rad}(T)$. By Lemmas 4 and 5, we have

$$\mathrm{Rad}(T) + 1 \ge d_T(S_v, S_v^*) \ge |S_v| \ge \mathrm{Rad}(T).$$

So $|S_v| = \mathrm{Rad}(T)$ or $|S_v| = \mathrm{Rad}(T)+1$. Thus S_v contains at most one vertex besides the vertices of $v - y$ path. Similarly, S_{v_1} contains at most one vertex besides the vertices of $v_1 - y_1$ path. □

Let $C_{vv_1} = T[\{u, v, v_1\}]$. We shall show that $e_L(C_{vv_1}) \le n-3$. For this, it suffices to show that $d_L(S, C_{vv_1}) \le n-3$ for any subtree S of T. Suppose that $V(S) \cap V(C_{vv_1}) \neq \emptyset$. Then, as before, one can easily verify that $d_L(S, C_{vv_1}) \le n-3$. Now we may assume that $V(S) \cap V(C_{vv_1}) = \emptyset$.

Suppose that the subtree S contains at least one vertex on $y-y_1$ path. Then $V(S) \subseteq V(S_v)$ or $V(S) \subseteq V(S_{v_1})$. By Fact 1, we have

$$d_T(S, C_{vv_1}) \le \begin{cases} \mathrm{Rad}(T) - |S| & \text{if } |S| \le 2; \\ \mathrm{Rad}(T) - |S| + 1 & \text{if } |S| \ge 3. \end{cases} \tag{3}$$

Note that u_1 is not on $y - y_1$ path, so $r \ge 2$ by Theorem 7. Hence, if $|S| \le 2$, by Lemma 3, we have

$$\begin{aligned} d_L(S, C_{vv_1}) &= |S| + 3 + 2(d_T(S, C_{vv_1}) - 1) \\ &\le |S| + 3 + 2(\mathrm{Rad}(T) - |S| - 1) \\ &\le 1 + 2\mathrm{Rad}(T) - |S| \\ &\le n - r - 1 \qquad \text{(by (2))} \\ &\le n - 3. \end{aligned}$$

If $|S| \ge 3$, by Lemma 3, we have

$$d_L(S, C_{vv_1}) = |S| + 3 + 2(d_T(S, C_{vv_1}) - 1)$$

$$\begin{aligned}
&\leq |S| + 3 + 2(\text{Rad}(T) - |S|) \\
&\leq 3 + 2\text{Rad}(T) - |S| \\
&\leq n - r - |S| + 2 \qquad \text{(by (2))} \\
&\leq n - 3.
\end{aligned}$$

Suppose that the subtree S contains no vertex on $y - y_1$ path. Let

$$K_0 = \{x \in K \mid \text{the } x - u \text{ path pass through neither } v \text{ nor } v_1\}.$$

If $K_0 \neq \emptyset$, note that u_1 is not on $y - y_1$ path, so $r \geq 4$ by Theorem 7. Furthermore, $K_0 \neq \emptyset$ implies that $|S| \leq r - 1$. It is easily seen that

$$d_T(S, C_{vv_1}) \leq \begin{cases} \text{Rad}(T) & \text{if } |S| = 1; \\ \text{Rad}(T) - 1 & \text{if } |S| \geq 2. \end{cases}$$

So if $|S| = 1$, by Lemma 3, we have

$$\begin{aligned}
d_L(S, C_{vv_1}) &= 1 + 3 + 2(d_T(S, C_{vv_1}) - 1) \\
&\leq 1 + 3 + 2(\text{Rad}(T) - 1) \\
&\leq n - r + 1 \qquad \text{(by (2))} \\
&\leq n - 3.
\end{aligned}$$

If $|S| \geq 2$, by Lemma 3, we have

$$\begin{aligned}
d_L(S, C_{vv_1}) &= |S| + 3 + 2(d_T(S, C_{vv_1}) - 1) \\
&\leq |S| + 3 + 2(\text{Rad}(T) - 2) \\
&\leq |S| - 1 + (n - r - 1) \qquad \text{(by (2))} \\
&\leq n - 3,
\end{aligned}$$

where the last inequality follows from $|S| \leq r - 1$. If $K_0 = \emptyset$, we can see that

$$d_T(S, C_{vv_1}) \leq \begin{cases} \text{Rad}(T) - 1 & \text{if } |S| = 1; \\ \text{Rad}(T) - 2 & \text{if } |S| \geq 2. \end{cases}$$

So if $|S| = 1$, by Lemma 3, we have

$$\begin{aligned}
d_L(S, C_{vv_1}) &= 1 + 3 + 2(d_T(S, C_{vv_1}) - 1) \\
&\leq 1 + 3 + 2(\text{Rad}(T) - 2) \\
&\leq n - r - 1 \qquad \text{(by (2))} \\
&\leq n - 3.
\end{aligned}$$

If $|S| \geq 2$, by Lemma 3, we have

$$\begin{aligned} d_L(S, C_{vv_1}) &= |S| + 3 + 2(d_T(S, C_{vv_1}) - 1) \\ &\leq |S| + 3 + 2(\text{Rad}(T) - 3) \\ &\leq |S| - 3 + (n - r - 1) \qquad \text{(by (2))} \\ &\leq n - 3, \end{aligned}$$

where the last inequality follows from $|S| \leq r$.

Therefore, we obtain $e_L(C_{vv_1}) \leq n - 3$. But then $e_L(C_L) \geq d_L(T, C_L) = n - 2 > e_L(C_{vv_1})$, a contradiction. The claim follows. □

By Claim 1, we may assume that $u_1 = v_1$. Let $K_1 = \{x \in K \mid$ the $u - x$ path does not pass through vertex $u_1\}$, $K_2 = \{x \in K \mid u - x$ path passes through the vertex $u_1\}$. For any $x \in K_1$, let w be the vertex adjacent to u on the $u - x$ path. For any $x \in K_2$, let w_1 be the vertex adjacent to u_1 on the $u_1 - x$ path. Let S_w, S_w^*, S_{w_1} and $S_{w_1}^*$ be defined as in Lemma 4. Then we have the following claims.

Claim 2. *S_w contains at most one vertex besides the vertices on the $w - x$ path.*

Proof. By definition, w lies in the $x - u$ path. Since $d_T(x, u) = \text{Rad}(T)$ and the vertices on $w - x$ path are contained in S_w, $|S_w| \geq \text{Rad}(T)$. By Lemmas 4 and 5,

$$\text{Rad}(T) + 1 \geq d(S_w, S_w^*) \geq |S_w| \geq \text{Rad}(T)$$

So $|S_w| = \text{Rad}(T)$ or $|S_w| = \text{Rad}(T) + 1$. Thus S_w contains at most one vertex besides the vertices on the $w - x$ path. □

Claim 3. *S_{w_1} contains at most three vertices besides the vertices on the $w_1 - y_1$ path.*

Proof. By definition, w_1 lies in the $x - u_1$ path. Since $d_T(x, u_1) = \text{Rad}(T) - 1$ and the vertices on the $x - w_1$ path are contained in S_{w_1}, $|S_{w_1}| \geq \text{Rad}(T) - 1$. By Lemmas 4 and 5, $\text{Rad}(T) + 2 \geq d_T(S_{w_1}, S_{w_1}^*) \geq |S_{w_1}| \geq \text{Rad}(T) - 1$. Hence S_{w_1} contains at most three vertices besides the vertices on the $w_1 - y_1$ path. □

Claim 4. $2 \leq r \leq 3$.

Proof. If $r = 1$, we observe that $e_L(C_L) \geq d_L(y, C_L) = 1 + 2 + 2(\text{Rad}(T) - 1) = n - 1$ by Lemma 3. On the other hand, it is easy to verify that $e_L(C_{vv_1}) \leq n - 2$ by Lemma 3. Then $e_L(C_L) > e_L(C_{vv_1})$, a contradiction. Thus $r \geq 2$. We claim that $r \leq 3$. Suppose to the contrary that $r \geq 4$. As before, let C_v be the subtree of T induced by $V(C_L) \cup \{v\}$. We shall prove that $e_L(C_v) \leq n - 3$. It suffices to show that $d_L(S, C_v) \leq n - 3$ for any subtree S of T.

Suppose that $V(S) \cap V(C_v) \neq \emptyset$. By Lemma 3, one can easily verify that $d_L(S, C_v) \leq n - 3$ for any subtree S of T.

Suppose that $V(S) \cap V(C_v) = \emptyset$ and the subtree S contains at least one vertex on $y - y_1$ path. Then either $V(S) \subseteq V(S_v)$ or there exists a vertex $x \in K_2$ such that $V(S) \subseteq V(S_{w_1}) \cup \{u_1\}$. By Claims 2 and 3,

$$d_T(S, C_v) \leq \begin{cases} \text{Rad}(T) - |S| & \text{if } |S| \leq 2; \\ \text{Rad}(T) - |S| + 2 & \text{if } 3 \leq |S| \leq 4; \\ \text{Rad}(T) - |S| + 3 & \text{if } |S| \geq 5. \end{cases}$$

Then if $|S| \leq 2$, by Lemma 3, we have

$$\begin{aligned} d_L(S, C_v) &= |S| + 3 + 2(d_T(S, C_v) - 1) \\ &\leq |S| + 3 + 2(\text{Rad}(T) - |S| - 1) \\ &\leq 1 + 2\text{Rad}(T) - |S| \\ &\leq n - r - 1 \qquad (\text{by } (2)) \\ &\leq n - 3. \end{aligned}$$

If $3 \leq |S| \leq 4$, by Lemma 3, we have

$$\begin{aligned} d_L(S, C_v) &= |S| + 3 + 2(d_T(S, C_v) - 1) \\ &\leq |S| + 3 + 2(\text{Rad}(T) - |S| + 1) \\ &\leq 5 + 2\text{Rad}(T) - |S| \\ &\leq n - r - |S| + 4 \qquad (\text{by } (2)) \\ &\leq n - 3. \end{aligned}$$

If $|S| \geq 5$, by Lemma 3, we have

$$\begin{aligned} d_L(S, C_v) &= |S| + 3 + 2(d_T(S, C_v) - 1) \\ &\leq |S| + 3 + 2(\text{Rad}(T) - |S| + 2) \\ &\leq 7 + 2\text{Rad}(T) - |S| \\ &\leq n - r - |S| + 6 \qquad (\text{by } (2)) \\ &\leq n - 3. \end{aligned}$$

Suppose that $V(S) \cap V(C_v) = \emptyset$ and the subtree S contains no vertex on $y - y_1$ path. Then by Claim 2,

$$d_T(S, C_v) \leq \begin{cases} \text{Rad}(T) & \text{if } |S| = 1; \\ \text{Rad}(T) - 1 & \text{if } 2 \leq |S| \leq 3; \\ \text{Rad}(T) - 2 & \text{if } |S| \geq 4. \end{cases}$$

So if $|S| = 1$, by Lemma 3, we have

$$\begin{aligned} d_L(S, C_v) &= 1 + 3 + 2(d_T(S, C_v)) \\ &\leq 1 + 3 + 2(\text{Rad}(T) - 1) \\ &\leq n - r + 1 \qquad \text{(by (2))} \\ &\leq n - 3. \end{aligned}$$

If $2 \leq |S| \leq 3$, by Lemma 3, we have

$$\begin{aligned} d_L(S, C_v) &= |S| + 3 + 2(d_T(S, C_v) - 1) \\ &\leq |S| + 3 + 2(\text{Rad}(T) - 2) \\ &\leq |S| - 1 + (n - r - 1) \qquad \text{(by (2))} \\ &\leq n - 3, \end{aligned}$$

If $|S| \geq 4$, by Lemma 3, we have

$$\begin{aligned} d_L(S, C_v) &= |S| + 3 + 2(d_T(S, C_v) - 1) \\ &\leq |S| + 3 + 2(\text{Rad}(T) - 3) \\ &\leq |S| - 3 + (n - r - 1) \qquad \text{(by (2))} \\ &\leq n - 3, \end{aligned}$$

where the last inequality follows from $|S| \leq r$. Thus $e_L(C_v) \leq n - 3$. But then $e_L(C_L) \geq d_L(T, C_L) = n - 2 > e_L(C_v)$, a contradiction. Consequently, $2 \leq r \leq 3$. □

Let $K' = \{x \in K \mid x - u \text{ path pass through neither } v \text{ nor } v_1\}$.

Claim 5. $K' = \emptyset$.

Proof. If $K' \neq \emptyset$, then there exists a vertex $x \in K'$ such that the $x - u$ path pass through neither v nor v_1. Claim 4 implies that $\text{Rad}(T) = 3$ and T is the tree T_5. Let $C(T_5) = \{u\}$ and v, v_1, v' be neighbors of u, where v' is adjacent to u on the $u - x$ path (see Figure 1). Set $M = \{u, v, u_1, v'\}$ (where $u_1 = v_1$). One may verify that $e_L(T_5[M]) \leq n - 3$ by Lemma 3. But then $e_L(C_L) \geq d_L(T_5, C_L) = n - 2 > e_L(T_5[M])$, a contradiction. □

Let $C_{vv'} = T[\{u, u_1, v, v'\}]$, where v' is the vertex adjacent to u_1 on the $u_1 - y_1$ path.

Claim 6. $r = 2$.

Proof. Suppose not, then Claim 4 implies that $r = 3$. Note the fact that $e_L(C_L) \geq d_L(T, C_L) = n - 2$. If we can show $e_L(C_{vv'}) \leq n - 3$, we will derive a contradiction. It is sufficient to show that $d_L(C_{vv'}) \leq n - 3$ for any subtree S of T.

Suppose that $V(S)\cap V(C_{vv'})\neq\emptyset$. Then one can easily check that $d_L(S,C_{vv'})\leq n-3$ by Lemma 3.

Suppose that $V(S)\cap V(C_{vv'})=\emptyset$. By Claims 2, 3, we have

$$d_T(S,C_{vv'})\leq\begin{cases}\text{Rad}(T)-|S| & \text{if } |S|\leq 2;\\ \text{Rad}(T)-|S|+1 & \text{if } 3\leq|S|\leq 4;\\ \text{Rad}(T)-|S|+2 & \text{if } |S|\geq 5.\end{cases}$$

So if $|S|\leq 2$, by Lemma 3, we have

$$\begin{aligned} d_L(S,C_{vv'}) &= |S|+4+2(d_T(S,C_{vv'})-1)\\ &\leq |S|+4+2(\text{Rad}(T)-|S|-1)\\ &\leq n-r \qquad \text{(by (2))}\\ &\leq n-3.\end{aligned}$$

If $3\leq|S|\leq 4$, by Lemma 3, we have

$$\begin{aligned} d_L(S,C_{vv'}) &= |S|+4+2(d_T(S,C_{vv'})-1)\\ &\leq |S|+4+2(\text{Rad}(T)-|S|)\\ &\leq (n-r)-|S|+3 \qquad \text{(by (2))}\\ &\leq n-3.\end{aligned}$$

If $|S|\geq 5$, by Lemma 3, we have

$$\begin{aligned} d_L(S,C_{vv'}) &= |S|+4+2(d_T(S,C_{vv'})-1)\\ &\leq |S|+4+2(\text{Rad}(T)-|S|+1)\\ &\leq (n-r)-|S|+5 \qquad \text{(by (2))}\\ &\leq n-3.\end{aligned}$$

So $e_L(C_{vv'})<e_L(C_L)$, a contradiction. Thus $r=2$. □

Claim 6 implies that T has precisely two vertices that do not lie in the $y-y_1$ path.

Claim 7. *The two vertices of T that do not lie in the $y-y_1$ path are adjacent to a support vertex on the $y-y_1$ path, i.e., T is the tree T''_8.*

Proof. Otherwise, we shall derive a contradiction by showing that $e_L(C_L)>e_L(C_{vu_1})$ where $C_{vu_1}=T[\{v,u,u_1\}]$. It is sufficient to show that $d_L(S,C_{vu_1})\leq n-3$ for any subtree S of T since $e_L(C_L)\geq d_L(T,C_L)=n-2$. Suppose that $V(S)\cap V(C_{vu_1})\neq\emptyset$. One may directly verify that $d_L(S,C_{vu_1})\leq n-3$ by Lemma 3. Suppose that $V(S)\cap V(C_{vu_1})=\emptyset$. Then by Claims 2, 3 and 5, we have

$$d_T(S,C_{vu_1})\leq\begin{cases}\text{Rad}(T)-|S| & \text{if } |S|\leq 2;\\ \text{Rad}(T)-|S|+1 & \text{if } |S|\geq 3.\end{cases}$$

So if $|S| \leq 2$, by Lemma 3, we have

$$\begin{aligned} d_L(S, C_{vu_1}) &= |S| + 3 + 2(d_T(S, C_{vu_1}) - 1) \\ &\leq |S| + 3 + 2(\text{Rad}(T) - |S| - 1) \\ &\leq (n - r) - 1 \qquad \text{(by (2))} \\ &\leq n - 3. \end{aligned}$$

If $|S| \geq 3$, by Lemma 3, we have

$$\begin{aligned} d_L(S, C_{vu_1}) &= |S| + 3 + 2(d_T(S, C_{vu_1}) - 1) \\ &\leq |S| + 3 + 2(\text{Rad}(T) - |S|) \\ &\leq (n - r) - |S| + 2 \qquad \text{(by (2))} \\ &\leq n - 3. \end{aligned}$$

So $e_L(C_{vu_1}) < e_L(C_L)$. This contradiction implies that T is the tree T_8'', as desired. □

■

***Proof of Theorem* 2.** Note that $T_8 = T_8'$ or T_8''. By Lemmas 8, 9 and 10, we conclude that the least central subtree of a tree T is P_2 if and only if T is one of the trees T_1, T_4, T_6, T_8, T_9, a double star and a path of even order. Furthermore, we observe that the tree T_1 has three least central subtrees, while each one of the trees T_4, T_6, T_8, T_9, a double star and a path of even order has a unique least central subtree P_2. This completes the proof of Theorem 2. ■

4. Conclusion

In this paper we give a complete characterization of trees with the unique least central subtree consisting of two adjacent vertices. In light of the idea in the proof of Theorem 2, it is possible to characterize the trees having the unique least central subtree with small order.

Acknowledgments

This work was partially supported by the National Nature Science Foundation of China (grant no. 11471210, 11571222). We are very thankful to the referees for their careful reading of this paper and all helpful comments.

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Received 29 July 2016
Revised 12 January 2017
Accepted 9 February 2017