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EXTREMAL IRREGULAR DIGRAPHS

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Abstract

A digraph is called irregular if its distinct vertices have distinct degree pairs. An irregular digraph is called minimal (maximal) if the removal of any arc (addition of any new arc) results in a non-irregular digraph. It is easily seen that the minimum sizes among irregular n-vertex whether digraphs or oriented graphs are the same and are asymptotic to $(\sqrt{2}/3)\, n^{3/2}$; maximum sizes, however, are asymptotic to n^2 and $n^2/2$, respectively. Let s stand for the sum of initial positive integers, $s = 1, 3, 6, \ldots$. An oriented graph H_s and a digraph F_s, both large (in terms of the size), minimal irregular, and on any such s vertices, $s \geq 21$, are constructed in [*Large minimal irregular digraphs*, Opuscula Math. 23 (2003) 21–24], co-authored by Z. D-H. and three more of the present co-authors (Z.M., J.M., Z.S.). In the present paper we nearly complete these constructions. Namely, a large minimal irregular digraph F_n, respectively oriented graph H_n, are constructed for any of remaining orders n, $n > 21$, and of size asymptotic to n^2, respectively to $n^2/2$. Also a digraph Φ_n and an oriented graph G_n, both small maximal irregular of any order $n \geq 6$, are constructed. The asymptotic value of the size of G_n is at least

$(\sqrt{2}/3)\, n^{3/2}$ and is just the least if $n = s \to \infty$, but otherwise the value is at most four times larger and is just the largest if $n = s - 1 \to \infty$. On the other hand, the size of Φ_n is of the asymptotic order $\Theta(n^{3/2})$.

Keywords: irregular digraph, oriented graph, minimal subdigraph, maximal subdigraph, asymptotic size.

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1. Introduction

For terminology and notation we refer to Chartrand and Lesniak [1]. Let $D = (V, A)$ be a digraph with vertex set $V = V(D)$ and arc set $A = A(D)$. A digraph without loops or 2-dicycles is called an *oriented graph.* Numbers of vertices and arcs of D are denoted by $|D|$ and $\|D\|$ and are called the *order* and the *size* of D, respectively. The ordered pair (a, b) comprising the two semi-degrees of a vertex, namely the outdegree a followed by the indegree b, is called the *degree pair* of the vertex. The sum of both semi-degrees is called the *degree* of a vertex. Moreover, $\delta(D)$ and $\Delta(D)$ denote respectively the minimum and the maximum degree over vertices in D.

A digraph D is called *diregular* (*ρ-diregular*) if all outdegrees and all indegrees are mutually equal (and equal to ρ). At the other extreme, D is said to be *irregular* if distinct vertices have distinct degree pairs, see Gargano, Kennedy and Quintas [4]. These digraphs were rediscovered and independently studied under the name fully irregular digraphs in [6, 7] and next in [2, 3, 5] with due credit to predecessors. The n-vertex transitive tournament, denoted by T_n (and coming from game theory), plays a prominent role in our study.

The following statements are well known and easily seen.

Theorem 1. *The transitive tournament T_n is the unique largest irregular oriented n-graph.*

$$\|T_n\| = \frac{1}{2}n(n-1) \sim \frac{1}{2}n^2. \tag{1}$$

An n-set of n degree pairs is called *minimum and symmetric* if the set is symmetric and with smallest possible sum of semi-degrees. In the paper [6], for each positive integer n, a minimum and symmetric n-set of degree pairs is presented. An oriented graph with those degree pairs is found. Consequently, a minimum n-vertex irregular digraph (i.e., with smallest size) is constructed. In the construction a few cases are considered, and the simplest of them is if n is the sum of the first t positive integers.

Notation. Throughout the paper the symbol s_t (or s as abbreviation) stands for the sum of the first t positive integers for some integer t,

$$s_t \ (= s) := 1 + 2 + \cdots + t = \frac{1}{2}t(t+1). \tag{2}$$

Proposition 2. *For each s, a minimum s-set of degree pairs is unique. Namely, the following union*

$$U_s := T_1 \cup T_2 \cup \cdots \cup T_t \tag{3}$$

of pairwise vertex disjoint tournaments $T_1, T_2, \ldots, T_t$ is an oriented graph which is a digraphic realization of the minimum s-set. That realization is not unique if $t \geq 3$.

Related examples. 1. The set of degree pairs of the union $T_2 \cup T_3$ can be realized by orienting the edges of the path P_5 (in three ways), see Figure 1.

Figure 1. Three more realizations of the set of degree pairs of $T_2 \cup T_3$.

2. Let v_1, v_2 and w_1, w_2, w_3, w_4 be consecutive vertices of tournaments T_2 and T_4, respectively. Let D_6 be the digraph induced by the following set of arcs $A(D_6) = \{(v_1, w_4), (w_1, v_2), (w_1, w_3), (w_1, w_4), (w_2, w_3), (w_2, w_4), (w_3, w_2)\}$. Then D_6 includes a 2-dicycle $\vec{C_2}$, $\vec{C_2} = (w_2, w_3, w_2)$, and is a realization of the degree pairs in the union $T_2 \cup T_4$, see Figure 2.

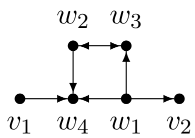

Figure 2. The digraph D_6 which is a realization of the set of degree pairs of $T_2 \cup T_4$.

Theorem 3. *Since U_s is among the smallest irregular digraphs, the complement $\overline{U_s}$ (in the complete symmetric digraph K_s^*) is one of the largest digraphs among irregular digraphs of order s.*

One can see the following consequence of (2):

$$t = \frac{1}{2}\left(\sqrt{8s+1} - 1\right) \sim \sqrt{2s}, \tag{4}$$

which implies the equality $t^2 = 2s - t$. Therefore, the asymptotic values of the sizes of U_s and $\overline{U_s}$ (which will be very useful) are as follows:

$$\|U_s\| = \frac{1}{2}\sum_{i=1}^{t} i(i-1) = \frac{1}{6}t(t^2-1) \;=\; \frac{1}{6}(t^3-t) \sim \frac{\sqrt{2}}{3}\,s^{3/2}, \tag{5}$$

$$= \frac{1}{6}t(2s-t-1) = \frac{1}{3}ts + O(t^2), \tag{6}$$

$$\|\overline{U_s}\| = \|K_s^*\| - \|U_s\| = s(s-1) - \|U_s\| \sim s^2. \tag{7}$$

An irregular digraph is called *minimal* if the removal of any arc spoils irregularity. Obviously, U_s is an example of minimal irregular oriented graph (and digraph). In the paper [2] a large minimal irregular oriented graph H_s and analogous digraph F_s, where $s \geq 21$ is the sum of six or more initial positive integers, are constructed. The sizes of H_s and F_s are asymptotically the largest possible since they are asymptotic to $s^2/2$ and s^2, respectively. We are going to construct large minimal irregular structures (an oriented graph H_n and a digraph F_n) of any remaining order $n > 21$ and with the same corresponding asymptotic sizes.

An irregular digraph (irregular oriented graph) is called *maximal* if the addition of any new arc spoils irregularity (or spoils being an oriented graph). Small maximal irregular structures (an oriented graph G_n and a digraph Φ_n) of arbitrary order $n \geq 6$ and with sizes of asymptotic order $\Theta(n^{3/2})$ will be constructed. In the special case when $n = s = s_t$, we construct a maximal oriented graph, G_s, with size asymptotic to $(\sqrt{2}/3)s^{3/2}$.

In the constructions which follow we assume that t_0, t, s, m and n are positive integers such that

$$t \geq t_0,\ s = s_t,\ 0 \leq m \leq t,\ n = s + m.$$

Consequently, due to (4), $m = O(t) = O(\sqrt{s})$. Moreover,

$$s \sim n,\ t \sim \sqrt{2n} \ \text{ and } \ m = O(\sqrt{n}). \tag{8}$$

2. Large Minimal Irregular Digraphs H_n and F_n

In this section we assume that

$$t \geq t_0 := 6 \quad \text{whence} \quad n \geq s = s_t \geq 21. \tag{9}$$

We first recall the construction (see [2]) of the large minimal irregular oriented graph H_s.

Construction 1.

- Let D'_s be a ρ-diregular digraph on s vertices such that $\rho = \lfloor (s-1)/2) \rfloor$, $V(D'_s) = \mathbb{Z}_s$, and $A(D'_s) = \{(i, i+j)\colon\ i, j \in \mathbb{Z}_s,\ 1 \leq j \leq \rho\}$;
- split the vertex sequence $(0, 1, \ldots, s-1)$ into strings (initial sections) of decreasing lengths: $2t-1$ and next $t-2, t-3, \ldots, 1$;
- split the first string $(0, 1, \ldots, 2t-2)$ into two disjoint subsequences which make up sequences: $V_t := (0, 2, \ldots, 2t-2)$ and $V_{t-1} := (1, 3, \ldots, 2t-3)$. Denote the remaining strings by $V_{t-2}, V_{t-3}, \ldots, V_1$;
- let U'_s be the union of t subgraphs of D'_s induced by all t sequences V_j;
- $H_s := D'_s - A(U'_s)$.

Theorem 4. *Under the assumptions* (2) *and* (9)*, the digraph U'_s is isomorphic to U_s, which is defined in* (3) *within Proposition* 2*.*

Proof. Due to the above definition of the digraph D'_s, it is enough to prove, for the longest sequence V_t which is $V_t = (0, 2, \ldots, 2t-2)$, the following two properties:

(i) an arc exists which joins the initial vertex 0 to the terminal vertex $2t-2$,

(ii) no arc of the digraph joins the terminal vertex $2t-2$ to another vertex of the sequence.

It is so because then V_t and each shorter sequence induce transitive tournaments.

The properties (i) and (ii) are clearly implied by the respective inequalities:

(i') $2t-2 \leq \rho$, and (ii') $2t-2+\rho \leq s-1$. It remains to prove (i') for $t \geq 6$ because then, for $\rho = \lfloor (s-1)/2 \rfloor$, we have $2\rho \leq s-1$. To this end, we introduce the following notation.

$L_t = 2t-2$, $\rho = \rho_t$ where $s = s_t$ is involved, and next we use induction on $t \geq 6$. Note that equality holds in (i') if the initial $t = 6$. Assume that the inequality (i'), that is, $\rho_t \geq L_t$, holds for some $t \geq 6$. Then, for the next value $t+1$, we have $L_{t+1} = L_t + 2$, $s_{t+1} = s_t + t + 1$ whence $\rho_{t+1} = \lfloor (s_{t+1}-1)/2 \rfloor = \lfloor ((s_t - 1) + t + 1)/2 \rfloor \geq \rho_t + t/2 \geq L_{t+1}$, which ends the induction. ■

Theorem 5 (Theorem 1 in [2])**.** *Under the assumptions* (2) *and* (9)*, the digraph H_s obtained by Construction* 1 *is a minimal irregular oriented graph of size $\|H_s\| \sim s^2/2$, $s = 21, 28, \ldots$.*

Construction 2 (extension of Construction 1 under the assumption $n = s + m$, $s = s_t$ and $0 < m \leq t$)**.**

- Let S_m be an oriented graph on m new vertices $v_1, v_2, \ldots, v_m$ and with the following arc set:

$$A(S_m) = \left\{(v_i, v_j)\colon 1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor,\ m - i + 1 \leq j \leq m\right\};$$

- $H'_n := H_s \cup S_m$ where H_s is the digraph obtained by Construction 1 (and $0 < m \leq t$).

Lemma 6. *The digraph S_m, see Construction* 2*, is a minimal irregular oriented graph of size $O(n)$.*

Proof. The structure is clearly correct if $m = 1, 2, 3$. Let $m \geq 4$. The outdegrees of initial vertices $1, 2, \ldots, \lfloor m/2 \rfloor$ grow by one from 1 to $\lfloor m/2 \rfloor$, the indegrees of next vertices grow analogously, either from 0 if m is odd or from 1 otherwise, up to $\lfloor m/2 \rfloor$, and all remaining semi-degrees of vertices in S_m equal 0. Hence S_m is irregular. Moreover, each arc of S_m is incident with a vertex which has a semi-degree larger than 1. This proves minimality. Furthermore, $\|S_m\| = \sum_{i=1}^{\lfloor m/2 \rfloor} i \leq 1 + 2 + \cdots + \lfloor t/2 \rfloor \leq t(t+2)/8 = O(n)$ due to (8). ■

Let

$$H_n = \begin{cases} H_s & \text{if } n = s = s_t \ (m = 0), \\ H'_n & \text{if } n = s + m, \ s = s_t \text{ and } 0 < m \leq t. \end{cases}$$

Theorem 7. *The digraph H_n is a large minimal irregular oriented graph of order n and size $\|H_n\| \sim n^2/2$.*

Proof. Due to Theorem 5, it remains to consider the case $m > 0$ and $H_n = H'_n$. By Lemma 6, S_m is a minimal irregular oriented graph. Moreover, digraphs H_s and S_m are vertex-disjoint, and the maximum degree in S_m is not greater than $t/2$ while the minimum degree in H_s (see Construction 1) is $\delta(H_s) = \delta(D'_s) - \Delta(U_s) = 2\rho - (t-1) \geq (s-2) - t + 1 = s - (t+1) = (t+1)(t-2)/2$ due to (2) and then $\delta(H_s) > t/2$ due to (9). Therefore, the digraph H'_n is irregular, too. It is clear that the removal of any arc from H_s or from S_m spoils irregularity of H'_n. On the other hand, by Theorem 5 and Lemma 6, since $s \sim n$, $\|H'_n\| = \|H_s\| + \|S_m\| \sim n^2/2$. ■

A large minimal irregular digraph F_n of order n, $n = s + m$ with $0 \leq m \leq t$, is obtained in a similar way. Namely, $F'_n = F_s \cup S_m$ with $m > 0$ is the disjoint union, where S_m is the digraph defined above, and digraphs F_s, F'_n and next F_n are constructed in the following way.

Construction 3 (under the assumption $n = s + m$, $s = s_t$ and $0 < m \leq t$)**.**

- Let D''_s be the digraph with vertex set $V(D''_s) = \mathbb{Z}_s$ obtained from the complete digraph K^*_s by deleting arcs $(i, i-1)$ and $(i, i-2j)$, for $i, j \in \mathbb{Z}_s$ and $j = 2, 3, \ldots, t-1$;
- let U_s be the union of transitive tournaments on sequences $V_1, V_2, \ldots, V_t$, cf. Construction 1;
- let $F_s = D''_s - A(U_s)$;
- let $F'_n = F_s \cup S_m$, with $m > 0$ and S_m as in Construction 2.

Hence $\|D_s''\| = \|K_s^*\| - s(t-1)$ and

$$\|F_s\| = \|D_s''\| - \|U_s\| = \|K_s^*\| - s(t-1) - \|U_s\|. \tag{10}$$

Let

$$F_n = \begin{cases} F_s & \text{if } n = s = s_t \ (m = 0), \\ F_n' = F_s \cup S_m & \text{if } n = s + m,\ s = s_t \text{ and } 0 < m \le t. \end{cases} \tag{11}$$

Theorem 8 (Theorem 2 in [2]). *Under the assumptions* (2) *and* (9)*, the digraph* F_s *obtained by Construction* 3 *is a minimal irregular digraph of size* $\|F_s\| = s(s-t) - \|U_s\| \sim s^2$, $s = 21, 28, \ldots$.

Theorem 9. *The digraph* F_n*, see* (11)*, is a large minimal irregular digraph of order* n *and size* $\|F_n\| \sim n^2$.

The proof of Theorem 9 is analogous to that of Theorem 7 because $\delta(F_s) > t/2 \ge \Delta(S_m)$.

3. Small Maximal Irregular Digraphs G_s, G_n, and Φ_s, Φ_n

In this section we assume that

$$t \ge t_0 := 3 \quad \text{whence} \quad n \ge s = s_t \ge 6.$$

We now present a construction of a small maximal irregular oriented graph G_s of order s and with the spanning cycle $\vec{C}_s = (0, 1, 2, \ldots, s-1, 0)$.

Construction 4.

- Assume that the sequence $V_t, V_{t-1}, \ldots, V_2, V_1$ represents a partition of the vertex set $V(\vec{C}_s)$ into i-sets V_i, $i = t, t-1, \ldots, 2, 1$. Subsets V_i are chosen in such a way that in all $\lfloor t/2 \rfloor$ pairs (V_i, V_{i-1}) with $i = t, t-2, \ldots$ down to $i = 3$ or $i = 2$, depending on the parity of t, the sets V_i, V_{i-1} intertwine along the cycle $\vec{C}_s$. For instance, $V_t = \{0, 2, \ldots, 2t-2\}$ and $V_{t-1} = \{1, 3, \ldots, 2t-3\}$. Consequently, each V_i is an independent subset of $V(\vec{C}_s)$;
- for $k = 1, 2, \ldots, t$, let T_k be the transitive tournament such that $V(T_k) = V_k$ and for $i, j \in V_k$, $(i, j) \in A(T_k) \Leftrightarrow i < j$;
- let $U_s = \bigcup_{i=1}^{t} T_i$;
- let $G_s = \vec{C}_s + A(U_s)$.

Theorem 10. *The digraph* G_s *obtained by Construction* 4 *is a small maximal irregular oriented graph of order* s *and size* $\|G_s\| \sim (\sqrt{2}/3)s^{3/2}$.

Proof. To show maximality is to prove that, for any arc e which joins nonadjacent vertices of G_s, the digraph $G_s + e$ is not irregular. To this end, let an arc e join a vertex, say $x \in V_i$, with a vertex, say x', of some V_j such that $j > i$ and vertices $x,\ x'$ are nonadjacent in G_s. Because each subset V_k with $1 \le k \le t$ induces a transitive tournament in G_s, $|V_k|$ is the number of degree pairs in G_s of vertices of that subset. Consequently, if $k = j < t$ and $z = x' \in V_k$ or $k = i < j - 1$ and $z = x \in V_k$, then the degree pair $p_{G_s+e}(z)$ coincides with degree pair $p_{G_s}(y)$ of a vertex $y \in V_{k+1}$. The case which still remains is $i + 1 = j = t$. Then the degree pair $p_{G_s+e}(x)$ coincides with that of a neighbor of x on the cycle $\vec{C}_s$. This shows maximality of G_s. Note that $\|G_s\| = |A(\vec{C})| + |A(U_s)| = s + \|U_s\| \sim (\sqrt{2}/3)s^{3/2}$ due to (5). ∎

We now construct a small maximal irregular oriented graph G'_n of order $n = s + m$ with $0 < m \le t$.

Construction 5 (under the assumption $n = s + m$, $s = s_t$ and $0 < m \le t$)**.**

- Let G_s be the oriented graph as in Construction 4;
- let T_m be the transitive tournament on m new vertices $v_1, v_2, \ldots, v_m$;
- let $G'_n = G_s \cup T_m + A_m$ where $m > 0$ and A_m is the set of all arcs which go from vertices of T_m to those of G_s, $A_m = \{(u, v)\colon u \in V(T_m), v \in V(G_s)\}$.

  Let
$$G_n = \begin{cases} G_s & \text{if } n = s = s_t \ (m = 0), \\ G'_n & \text{if } n = s + m,\ s = s_t \text{ and } 0 < m \le t. \end{cases}$$

The following theorem complements Theorem 10.

Theorem 11. *The digraph G_n is a small maximal irregular oriented graph of order n and size $\|G_n\| = \Theta(n^{3/2})$. Moreover, if $m = t$ and $n = s_t + t$, then $\|G_n\| \sim (4\sqrt{2}/3)n^{3/2}$ wherein the asymptotic coefficient is the largest possible.*

Proof. Due to Theorem 10, it remains to consider the case $m > 0$ and $G_n = G'_n$. It is easy to see that G'_n is an irregular oriented graph. Note that only an arc with both endvertices in the subgraph G_s can be added to G'_n so that the property of being an oriented graph could be preserved. However, adding any such arc spoils irregularity, cf. the above proof of Theorem 10. This shows maximality.

Let $n = s + t$ where, due to (2), $s = t(t+1)/2 = O(t^2)$. Then $\|G_s\| = s + \|U_s\| = t^3/6 + O(t^2)$ due to (2) and (6), $\|T_t\| = O(t^2)$ due to (1), and $|A_m| \le |A_t| = ts = t^3/2 + O(t^2)$ due to (2) whence $\|G'_n\| \le \|G'_{s+t}\| = \|G_s\| + \|T_t\| + |A_t| = t^3/6 + t^3/2 + O(t^2) \sim 2t^3/3 \sim (4\sqrt{2}/3)n^{3/2}$ due to (8). ∎

Proposition 12. *The complement of a* (*large*) *minimal irregular digraph is a* (*small*) *maximal irregular digraph, and conversely.*

Using (11) we define

$$\Phi_n = \begin{cases} \overline{F_s} & \text{if } n = s = s_t \ (m = 0), \\ \overline{F'_n} & \text{if } n = s + m,\ s = s_t \text{ and } 0 < m \le t. \end{cases} \tag{12}$$

Theorem 13. *We refer to* (12), (11) *and Construction* 3 *under the assumptions* (2) *and* (9). *The digraph* Φ_n *is a small maximal irregular digraph of order* n *and with size* $\|\Phi_n\| = \Theta(n^{3/2})$ *where* $\|\Phi_s\| \sim (4\sqrt{2}/3)\, s^{3/2}$ *if* $n = s$ $(m = 0)$. *The largest asymptotic coefficient is if* $n = s + t$ $(m = t)$ *and then* $\|\Phi_n\| \sim (10\sqrt{2}/3)\, n^{3/2}$.

Proof. The digraph Φ_n is maximal irregular due to Proposition 12 and Theorems 8 and 9. Note that, due to (12) and (10), $\|\Phi_s\| = \|K_s^*\| - \|F_s\| = s(t-1) + \|U_s\|$.

Hence, due to (4) and (5),

$$\|\Phi_s\| \sim s\sqrt{2s} + \frac{\sqrt{2}}{3}\, s^{3/2} = 4\frac{\sqrt{2}}{3}\, s^{3/2}. \tag{13}$$

Next if $n = s + m$ and $0 < m \le t$ then, due to (11) and (10), $\|\Phi_{s+m}\| = \|\overline{F_s \cup S_m}\| = \|K_{s+m}^*\| - \|F_s\| - \|S_m\| = s(s-1) - \|F_s\| + 2sm + m(m-1) - \|S_m\| = \|\Phi_s\| + 2sm + \|\overline{S_m}\| = \|\Phi_s\| + 2sm + O(m^2)$. Using this equality for $m = t$ and $n = s + t$, and using asymptotic formulas (8) and (13), we get $O(m^2) = O(s)$, $2sm = 2st \sim (2s)^{3/2}$, and $\|\Phi_{s+t}\| \sim (4\sqrt{2}/3)\, s^{3/2} + 2\sqrt{2}\, s^{3/2} = (10\sqrt{2}/3)\, s^{3/2}$, which ends the proof because $s \sim n$. ∎

4. Concluding Remarks

Constructed in this paper are large minimal irregular oriented graphs H_n (respectively digraphs F_n) of order $n \ge 21$ on one hand, and small maximal irregular oriented graphs G_n/digraphs Φ_n, both of order $n \ge 6$, on the other hand. They have sizes which are asymptotically best possible as compared to the corresponding smallest/largest irregular structures, see Theorems 7, 9, 11, 13 versus Theorems 1, 3 and asymptotics in formulas (1), (5) and (7).

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