Abstract

Given a graph $H$, the Turán function $ex(n, H)$ is the maximum number of edges in a graph on $n$ vertices not containing $H$ as a subgraph. For two graphs $G$ and $H$, an $H$-decomposition of $G$ is a partition of the edge set of $G$ such that each part is either a single edge or forms a graph isomorphic to $H$. Let $\phi(n, H)$ be the smallest number $\phi$ such that any graph $G$ of order $n$ admits an $H$-decomposition with at most $\phi$ parts. Pikhurko and Sousa conjectured that $\phi(n, H) = ex(n, H)$ for $\chi(H) \geq 3$ and all sufficiently large $n$. Their conjecture has been verified by Özkahya and Person for all edge-critical graphs $H$. In this article, we consider the gem graphs $\text{gem}_4$ and $\text{gem}_5$. The graph $\text{gem}_4$ consists of the path $P_4$ with four vertices $a, b, c, d$ and edges $ab, bc, cd$ plus a universal vertex $u$ adjacent to $a, b, c, d$, and the graph $\text{gem}_5$ is similarly defined with the path $P_5$ on five vertices. We determine
the Turán functions \( \text{ex}(n, \text{gem}_4) \) and \( \text{ex}(n, \text{gem}_5) \), and verify the conjecture of Pikhurko and Sousa when \( H \) is the graph \( \text{gem}_4 \) and \( \text{gem}_5 \).

**Keywords:** gem graph, Turán function, extremal graph, graph decomposition.

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1. **Introduction**

Given a graph \( H \), the Turán function \( \text{ex}(n, H) \) is the maximum number of edges in a graph on \( n \) vertices, and not containing a copy of \( H \) as a subgraph. The important result of Turán [13] states that when \( H = K_r \) is the complete graph on \( r \geq 3 \) vertices, we have \( \text{ex}(n, K_r) = \frac{t_r - 1}{2} \binom{n}{2} \), where \( t_r - 1 \) is the number of edges in the Turán graph of order \( n \), \( T_r - 1(n) \), which is the unique complete \((r - 1)\)-partite graph on \( n \) vertices where every partition class has either \( \left\lfloor \frac{n}{r-1} \right\rfloor \) or \( \left\lceil \frac{n}{r-1} \right\rceil \) vertices. Moreover, \( T_r - 1(n) \) is the unique extremal graph on \( n \) vertices that has the maximum number of edges not containing \( K_r \) as a subgraph. For general graphs \( H \), the Turán function \( \text{ex}(n, H) \) has been well studied by numerous researchers, which led to many important results and open problems in extremal graph theory. For example, when \( H = C_{2k} \) is the even cycle of length \( 2k \), where \( k \geq 2 \), the exact determination of the function \( \text{ex}(n, C_{2k}) \) is still a wide open problem. It has been conjectured that \( \text{ex}(n, C_{2k}) = (c_k + o(1))n^1+1/k \) for some constant \( c_k > 0 \), and this conjecture is only known to be true for \( k = 2, 3, 5 \). See for example [8] and the references therein. When \( H = P_k \) is the path of order \( k \geq 3 \), Faudree and Schelp [5] have determined the function \( \text{ex}(n, P_k) \) exactly. In order to obtain \( \text{ex}(n, P_k) \), we can take the graph on \( n \) vertices containing as many disjoint copies of \( K_{k-1} \) as possible, and a smaller complete graph on the remaining vertices. For odd \( k \), this graph is the unique \( P_k \)-free extremal graph attaining \( \text{ex}(n, P_k) \), and for even \( k \) and certain values of \( n \), there are other such extremal graphs. Here we state the result of Faudree and Schelp as follows, which will be useful in this paper.

**Theorem 1.1** [5]. Let \( k \geq 3 \) and \( n = a(k-1) + b \), where \( a \geq 0 \) and \( 0 \leq b < k-1 \). Then \( \text{ex}(n, P_k) = a\binom{k-1}{2} + \binom{b}{2} \). Moreover, a \( P_k \)-free graph on \( n \) vertices attaining \( \text{ex}(n, P_k) \) is \( aK_{k-1} \cup K_b \), the disjoint union of \( a \) copies of \( K_{k-1} \) and one copy of \( K_b \).

For two graphs \( G \) and \( H \), an \( H \)-decomposition of \( G \) is a partition of the edge set of \( G \) such that each part is either a single edge or forms a graph isomorphic to \( H \). Let \( \phi(G, H) \) be the smallest possible number of parts in an \( H \)-decomposition of \( G \). It is easy to see that, for non-empty \( H \), we have \( \phi(G, H) = e(G) -
$p_H(G)(e(H) - 1)$, where $p_H(G)$ is the maximum number of pairwise edge-disjoint copies of $H$ that can be packed into $G$ and $e(G)$ denotes the number of edges in $G$. Dor and Tarsi [3] showed that if $H$ has a component with at least three edges, then the problem of checking whether a graph $G$ admits a partition into $H$-subgraphs is NP-complete. Thus, it is NP-hard to compute the function $\phi(G, H)$ for such $H$. Here we study the function

$$\phi(n, H) = \max\{\phi(G, H) \mid v(G) = n\},$$

which is the smallest number $\phi$ such that any graph $G$ of order $n$ admits an $H$-decomposition with at most $\phi$ parts.

This function was first studied, in 1966, by Erdős, Goodman and Pósa [4], who were motivated by the problem of representing graphs by set intersections. They proved that $\phi(n, K_3) = t_2(n)$. A decade later, this result was extended by Bollobás [2], who proved that $\phi(n, K_r) = t_{r-1}(n)$, for all $n \geq r \geq 3$.

General graphs $H$ were only considered recently by Pikhurko and Sousa [9]. They proved the following result.

**Theorem 1.2** (See Theorem 1.1 from [9]). Let $H$ be any fixed graph of chromatic number $r \geq 3$. Then,

$$\phi(n, H) = \text{ex}(n, H) + o(n^2).$$

Pikhurko and Sousa also made the following conjecture.

**Conjecture 1.3** [9]. For any graph $H$ of chromatic number $r \geq 3$, there exists $n_0 = n_0(H)$ such that $\phi(n, H) = \text{ex}(n, H)$ for all $n \geq n_0$.

A graph $H$ is edge-critical if there exists an edge $e \in E(H)$ such that $\chi(H) > \chi(H - e)$, where $\chi(H)$ denotes the chromatic number of $H$. For $r \geq 4$ a clique-extension of order $r$ is a connected graph that consists of a $K_{r-1}$ plus another vertex, say $v$, adjacent to at most $r - 2$ vertices of $K_{r-1}$. Conjecture 1.3 has been verified by Sousa for some edge-critical graphs, namely, clique-extensions of order $r \geq 4$ ($n \geq r$) [11] and the cycles of length $5$ ($n \geq 6$) and $7$ ($n \geq 10$) [10, 12]. Later, Özkahya and Person [7] verified the conjecture for all edge-critical graphs with chromatic number $r \geq 3$. Their result is the following.

**Theorem 1.4** (See Theorem 3 from [7]). For any edge-critical graph $H$ with chromatic number $r \geq 3$, there exists $n_0 = n_0(H)$ such that $\phi(n, H) = \text{ex}(n, H)$, for all $n \geq n_0$. Moreover, the only graph attaining $\text{ex}(n, H)$ is the Turán graph $T_{r-1}(n)$.

Recently, as an extension of Özkahya and Person’s work, Allen, Böttcher, and Person [1] improved the error term obtained by Pikhurko and Sousa in Theorem 1.2. In fact, they proved that the error term $o(n^2)$ can be replaced by $O(n^{2-\alpha})$.
for some $\alpha > 0$. Furthermore, they also showed that this error term has the correct order of magnitude. Their result is indeed an extension of Theorem 1.4 since the error term $O(n^{2-\alpha})$ that they obtained vanishes for every edge-critical graph $H$.

Conjecture 1.3 has also been verified by Liu and Sousa [6] for the $k$-fan graph $F_k$, which is the graph on $2k + 1$ vertices consisting of $k$ triangles intersecting in exactly one common vertex. Observe that $\chi(F_k) = 3$ and for $k \geq 2$ the graph $F_k$ is not edge-critical. Thus, the result of Liu and Sousa is not a particular case of Theorem 1.4 by Özkahya and Person.

In this article, we consider the gem graphs gem$_4$ and gem$_5$, defined as follows. For the graph gem$_4$, we take the path $P_4$ with vertices $a, b, c, d$ and edges $ab, bc, cd$ and add a universal vertex $u$ adjacent to $a, b, c, d$. Similarly for the graph gem$_5$, we take the path $P_5$ with vertices $a, b, c, d, e$ and edges $ab, bc, cd, de$ and add a universal vertex $u$ adjacent to $a, b, c, d, e$. See Figure 1 below. For convenience, we write $abcd + u$ and $abcde + u$ for these two graphs.

\begin{figure}[h]
\begin{center}
\includegraphics[width=0.5\textwidth]{gem4_gem5.png}
\end{center}
\caption{The graphs gem$_4$ and gem$_5$.}
\end{figure}

In Section 2, we will determine the Turán functions $\text{ex}(n, \text{gem}_4)$ for $n \geq 6$, and $\text{ex}(n, \text{gem}_5)$ for $n \geq 8$. Then, in Section 3, we will prove Pikhurko and Sousa conjecture for these two gem graphs. That is, we will show that $\phi(n, \text{gem}_4) = \text{ex}(n, \text{gem}_4)$ for $n \geq 6$, and $\phi(n, \text{gem}_5) = \text{ex}(n, \text{gem}_5)$ for $n \geq 8$. Note that $\chi(\text{gem}_4) = \chi(\text{gem}_5) = 3$, and that gem$_4$ and gem$_5$ are not edge-critical graphs. Thus, our results are again not implied by Theorem 1.4.

Our notations throughout the paper are fairly standard. For a vertex $v$ in a graph $G$, the \textit{neighbourhood} of $v$, denoted by $N(v)$, is the set of vertices in $G$ that are adjacent to $v$. The \textit{degree} of $v$ is $\text{deg}(v) = |N(v)|$, and the \textit{minimum degree} and \textit{maximum degree} of $G$ are $\delta(G)$ and $\Delta(G)$, respectively. For a set $U \subset V(G)$, let $\text{deg}(v, U)$ denote the number of vertices in $U$ that are adjacent to $v$, and let $G[U]$ denote the subgraph of $G$ induced by $U$.

\section{Turán Function for the Gem Graphs}

In this section, we will determine the Turán functions $\text{ex}(n, \text{gem}_4)$ for $n \geq 6$, and $\text{ex}(n, \text{gem}_5)$ for $n \geq 8$. Furthermore, we will determine the extremal graphs in
each case. That is, we will determine all $gem_4$-free graphs on $n \geq 6$ vertices with $ex(n, gem_4)$ edges, and all $gem_5$-free graphs on $n \geq 8$ vertices with $ex(n, gem_5)$ edges.

2.1. Turán function for $gem_4$

We will now determine the function $ex(n, gem_4)$. In order to state our result, we first define the family of graphs $F_{n,4}$, which will consist of all the extremal graphs. Let $n \geq 6$ and $F_{n,4}$ be the family of graphs on $n$ vertices as follows. For $n \equiv 0 \pmod 4$, let $G^0_n$ be the graph obtained by taking the Turán graph $T_2(n)$ and embedding a maximum matching into a class of $T_2(n)$. For $n \equiv 1 \pmod 4$, let $G^{11}_n$ and $G^{12}_n$ be the graphs obtained by embedding a maximum matching into the smaller class and the larger class of $T_2(n)$, respectively. For $n \equiv 2 \pmod 4$, let $G^{21}_n$ and $G^{22}_n$ be the graphs obtained by embedding a maximum matching into a class of $T_2(n)$, and into the larger class of the complete bipartite graph $K_{n/2-1,n/2+1}$, respectively. For $n \equiv 3 \pmod 4$, let $G^3_n$ be the graph obtained by embedding a maximum matching into the larger class of $T_2(n)$. Let the vertex classes of $G^0_n$ be $A^0_n$ and $B^0_n$, with similar notations for the other graphs. Let $F_{n,4} = \{G^0_n\}$, $F_{n,4} = \{G^{11}_n, G^{12}_n\}$, $F_{n,4} = \{G^{21}_n, G^{22}_n\}$ and $F_{n,4} = \{G^3_n\}$ for $n \equiv 0, 1, 2, 3 \pmod 4$, respectively. Figure 2 below shows the graphs of $F_{n,4}$. Note that in $G^{12}_n$, we have an unmatched vertex in the class $B^{12}_n$, and similarly for $G^{21}_n$ with the class $B^{21}_n$.

![Graphs of F_{n,4}](image-url)
It is easy to see that every graph of $F_{n,4}$ is gem$_4$-free. Let $G \in F_{n,4}$, and suppose that there exists a copy of gem$_4$ in $G$, say $abcd + u$. We may consider in turn whether $u$ is in the independent class of $G$, or in the class containing the maximum matching. In each case, we can easily verify that no four neighbours of $u$ form a path $P_4$ in $G$, which is a contradiction. Also, for any graph of $F_{n,4}$, by adding an edge, we obtain a graph that contains a copy of gem$_4$. Indeed, let $G \in F_{n,4}$. Since $n \geq 6$, if an edge $cu$ is added to the independent class of $G$, then we may find an edge $ab$ and another vertex $d$ in the other class. If an edge $bu$ is added to the class of $G$ containing the maximum matching, then we may assume that $du$ is an edge in the matching, and choose vertices $a, c$ in the other class. In both cases, we have $abcd + u$ is a copy of gem$_4$.

We can easily check that for $n \geq 6$, all graphs of $F_{n,4}$ have the same number of edges. Thus for $G \in F_{n,4}$, we let $e_n$ denote the number of edges in the graph $G$. Then, we can easily check that the number of edges of $G$ is

\[(1) \quad e(G) = e_n = \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \begin{cases} 
0 & \text{if } n \equiv 0, 1, 2 \pmod{4}, \\
1 & \text{if } n \equiv 3 \pmod{4}.
\end{cases}\]

Moreover, for $n \geq 7$, $G \in F_{n,4}$ and $G' \in F_{n-1,4}$, we have

\[(2) \quad e(G) - e(G') = e_n - e_{n-1} = \left\lfloor \frac{n}{2} \right\rfloor + \begin{cases} 
0 & \text{if } n \equiv 0, 1, 2 \pmod{4}, \\
1 & \text{if } n \equiv 3 \pmod{4}.
\end{cases}\]

We have the following result for the Turán function $ex(n, \text{gem}_4)$.

**Theorem 2.1.** For $n \geq 6$, we have

\[ex(n, \text{gem}_4) = e_n = \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \begin{cases} 
0 & \text{if } n \equiv 0, 1, 2 \pmod{4}, \\
1 & \text{if } n \equiv 3 \pmod{4}.
\end{cases}\]

Moreover, the only gem$_4$-free graphs with $n$ vertices and $ex(n, \text{gem}_4)$ edges are the members of $F_{n,4}$.

We will prove Theorem 2.1 by induction on $n$. We first prove the base case as follows.

**Lemma 2.2.** $ex(6, \text{gem}_4) = e_6 = 10$ and the only gem$_4$-free graphs with six vertices and 10 edges are $G_{61}^{21}$ and $G_{62}^{22}$.

**Proof.** It suffices to prove that, for any graph $G$ with six vertices and $e_6 = 10$ edges, either $G$ contains a copy of the graph gem$_4$, or $G \in F_{6,4} = \{G_{61}^{21}, G_{62}^{22}\}$. Then for any graph $G'$ with six vertices and $e(G') \geq 11$, we can take a spanning subgraph $G \subset G'$ with $e(G) = e_6 = 10$, so that either $G$ contains a copy of gem$_4$, or $G \in F_{6,4}$. In either case, $G'$ contains a copy of gem$_4$ and we are done.
Let $G$ be a graph with six vertices and $e_6 = 10$ edges. Note that $G$ has either a vertex of degree 5, or two vertices of degree 4. Otherwise, we have $e(G) \leq \left\lfloor \frac{1}{2} (4 + 5 \cdot 3) \right\rfloor = 9 < 10 = e_6$, a contradiction.

Suppose first that $G$ has a vertex $u$ with $\deg(u) = 5$. By Theorem 1.1, we have $\text{ex}(5, P_4) = \left(\frac{5}{2}\right) + \left(\frac{5}{2}\right) = 4$. We have $e(G - u) = 10 - 5 = 5 > 4 = \text{ex}(5, P_4)$, and thus $G - u$ contains a copy of the path $P_4$, which together with $u$, form a copy of gem$_4$ in $G$.

Now, suppose that $G$ has two vertices of degree 4, say $u$ and $v$. Let $x_1, x_2, x_3, x_4$ be the remaining four vertices, and assume that $G$ does not contain a copy of gem$_4$. Suppose first that $uv \in E(G)$. If $u$ and $v$ have three common neighbours, say $x_1, x_2, x_3$, then we must have $x_i x_4 \in E(G)$ for $i = 1, 2, 3$, so that $G = G_{6}^{31}$. If $u$ and $v$ have two common neighbours, say $x_1, x_2$, then let $ux_3, vx_4 \in E(G)$ and $ux_4, vx_3 \notin E(G)$. We see that only the edges $x_1 x_2, x_3 x_4$ can be added to avoid creating a copy of gem$_4$, so that $G$ can only have at most nine edges, a contradiction. Now, suppose that $uv \notin E(G)$. Then $G$ contains all edges between $\{u, v\}$ and $\{x_1, x_2, x_3, x_4\}$. If $G$ does not contain a copy of gem$_4$, then the remaining two edges must be independent within $\{x_1, x_2, x_3, x_4\}$, so that $G = G_{6}^{32}$.

We conclude that either $G$ contains a copy of gem$_4$, or $G \in \mathcal{F}_{6,4}$, as required.

We are now able to prove Theorem 2.1.

**Proof of Theorem 2.1.** Let $n \geq 6$. The lower bound $\text{ex}(n, \text{gem}_4) \geq e_n$ follows instantly by considering any graph of $\mathcal{F}_{n,4}$. We prove the upper bound $\text{ex}(n, \text{gem}_4) \leq e_n$ by induction on $n$. Lemma 2.2 proves the result for $n = 6$. Now suppose that $n \geq 7$, and the theorem holds for $n - 1$. We will prove that if $G$ is a graph on $n$ vertices and $e(G) = e_n$, then either $G$ contains a copy of gem$_4$, or $G$ is one of the graphs of $\mathcal{F}_{n,4}$. This clearly implies the upper bound $\text{ex}(n, \text{gem}_4) \leq e_n$, and thus the theorem for $n$. Indeed, if we have a graph $G'$ with $n$ vertices and $e(G') > e_n$, then by taking a spanning subgraph $G \subset G'$ with $e(G) = e_n$, we see that either $G$ contains a copy of gem$_4$, or $G \in \mathcal{F}_{n,4}$. In either case, $G'$ contains a copy of gem$_4$.

First, suppose that $\delta(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$ and let $v \in V(G)$ be a vertex of minimum degree. Then by (2), we have

\begin{equation}
\tag{3}
e(G - v) = e(G) - \deg(v) \geq e_n - \left\lfloor \frac{n}{2} \right\rfloor \geq e_{n-1}.
\end{equation}

If $e(G - v) > e_{n-1}$, then by induction, $G - v$, and thus $G$, contains a copy of gem$_4$. Next, $e(G - v) = e_{n-1}$ holds if and only if $\deg(v) = \left\lfloor \frac{n}{2} \right\rfloor$ and $e_n - e_{n-1} = \left\lfloor \frac{n}{2} \right\rfloor$. The latter condition holds for $n \not\equiv 3 \pmod{4}$. By induction, either $G - v$, and thus $G$, contains a copy of gem$_4$ and we are done, or $G - v \in \mathcal{F}_{n-1,4}$, and we must consider the following cases.
Case 1. $n \equiv 0 \pmod{4}$. We have $G - v = G^3_{n-1}$ with classes $A^3_{n-1}$ and $B^3_{n-1}$, where $|A^3_{n-1}| = \frac{n}{2} - 1$ and $|B^3_{n-1}| = \frac{n}{2}$, and $B^3_{n-1}$ containing a perfect matching. Since $\deg(v) = \frac{n}{2}$, if $N(v) = B^3_{n-1}$, then $G = G^0_n$. Otherwise, if $v$ has neighbours $c \in A^3_{n-1}$ and $u \in B^3_{n-1}$, then $abcv + u$ is a copy of gem$_4$ in $G$, where $a \in A^3_{n-1} \setminus \{c\}$ and $b \in B^3_{n-1}$ is the vertex adjacent to $u$.

Case 2. $n \equiv 1 \pmod{4}$. We have $G - v = G^0_{n-1} = n-1$ with classes $A^0_{n-1}$ and $B^0_{n-1}$, where $|A^0_{n-1}| = |B^0_{n-1}| = \frac{n-1}{2}$, with $B^0_{n-1}$ containing a perfect matching. Since $\deg(v) = \frac{n-1}{2}$, it follows that if $N(v) = B^0_{n-1}$ then $G = G^{11}_n$, and if $N(v) = A^0_{n-1}$ then $G = G^{12}_n$. Otherwise, $v$ has a neighbour in both $A^0_{n-1}$ and $B^0_{n-1}$, so that as in Case 1, $G$ contains a copy of gem$_4$.

Case 3. $n \equiv 2 \pmod{4}$. We have $G - v \in \{G^{11}_{n-1}, G^{12}_{n-1}\}$. Suppose first that $G - v = G^{11}_{n-1}$. Then the classes of $G - v$ are $A^{11}_{n-1}$ and $B^{11}_{n-1}$, where $|A^{11}_{n-1}| = \frac{n-1}{2} - 1$ and $|B^{11}_{n-1}| = \frac{n}{2} + 1$, with $A^{11}_{n-1}$ containing a perfect matching. Since $\deg(v) = \frac{n}{2}$, it follows that if $N(v) = B^{11}_{n-1}$ then $G = G^{21}_n$. Otherwise, $v$ has a neighbour in both $A^{11}_{n-1}$ and $B^{11}_{n-1}$, and $G$ contains a copy of gem$_4$ as in Case 1. Now suppose that $G - v = G^{12}_{n-1}$. Then the classes are $A^{12}_{n-1}$ and $B^{12}_{n-1}$, where $|A^{12}_{n-1}| = \frac{n}{2} - 1$ and $|B^{12}_{n-1}| = \frac{n}{2}$, with $B^{12}_{n-1}$ containing a maximum matching with one unmatched vertex, say $w$. Since $\deg(v) = \frac{n}{2}$, it follows that if $N(v) = B^{12}_{n-1}$ then again $G = G^{21}_n$ and if $N(v) = A^{12}_{n-1} \cup \{w\}$ then $G = G^{22}_n$. Otherwise, $v$ has a neighbour in both $A^{12}_{n-1}$ and $B^{12}_{n-1} \setminus \{w\}$, and again as in Case 1, $G$ contains a copy of gem$_4$.

Next, suppose that $\delta(G) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$. In view of (1), if $n$ is even, then we have $e(G) \geq \frac{n}{2} (\frac{n}{2} + 1) > e_n$. If $n \equiv 1 \pmod{4}$, then $e(G) \geq \left\lfloor \frac{n}{2} \right\rfloor \left( \left\lfloor \frac{n}{2} \right\rfloor + 1 \right) = \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + 1 > e_n$. We have a contradiction in these cases. Now let $n \equiv 3 \pmod{4}$. We have $e(G) \geq \frac{n}{2} \left( \left\lfloor \frac{n}{2} \right\rfloor + 1 \right) = \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + 1 = e_n$. We must have equality, and thus $G$ is a $\left( \left\lfloor \frac{n}{2} \right\rfloor + 1 \right)$-regular graph. Let $v \in V(G)$, so that by (2)

\begin{equation}
    e(G - v) = e(G) - \deg(v) = e_n - \left( \left\lfloor \frac{n}{2} \right\rfloor + 1 \right) = e_n - 1.
\end{equation}

By induction, either $G - v$, and thus $G$, contains a copy of gem$_4$, or $G - v \in \mathcal{F}_{n-1,4}$. If the latter holds, then $G - v \in \{G^{21}_{n-1}, G^{22}_{n-1}\}$. Suppose first that $G - v = G^{21}_{n-1}$. The classes are $A^{21}_{n-1}$ and $B^{21}_{n-1}$, where $|A^{21}_{n-1}| = |B^{21}_{n-1}| = \frac{n-1}{2}$, with $B^{21}_{n-1}$ containing a maximum matching with one unmatched vertex, say $w$. Since $\deg(v) = \frac{n-1}{2} + 1$, in order for $G$ to be $\left( \left\lfloor \frac{n}{2} \right\rfloor + 1 \right)$-regular, we must have $N(v) = A^{21}_{n-1} \cup \{w\}$. This gives $G = G^{0}_{n}$. Now, suppose that $G - v = G^{22}_{n-1}$. The classes are $A^{22}_{n-1}$ and $B^{22}_{n-1}$, where $|A^{22}_{n-1}| = \frac{n-1}{2} - 1$ and $|B^{22}_{n-1}| = \frac{n-1}{2} + 1$, with $B^{22}_{n-1}$ containing a perfect matching. Again since $G$ is $\left( \left\lfloor \frac{n}{2} \right\rfloor + 1 \right)$-regular, we must have $N(v) = B^{22}_{n-1}$, and this also implies $G = G^{0}_{n}$.

This completes the proof of Theorem 2.1.
2.2. Turán function for gem

We will next determine the function $ex(n, \text{gem}_5)$. Analogously, we first define the family of graphs $\mathcal{F}_{n,5}$, which will consist of all the extremal graphs. Let $n \geq 8$ and $\mathcal{F}_{n,5}$ be the family of graphs on $n$ vertices as follows. For $n \geq 11$, we let $\mathcal{F}_{n,5} = \mathcal{F}_{n,4}$. For $n = 8, 9, 10$, the family $\mathcal{F}_{n,5}$ will consist of all graphs of $\mathcal{F}_{n,4}$ and some additional graphs. Let $G'_n$ be the graph obtained by adding one edge into each class of $T_2(n)$. Also for $n = 8$, let $G''_8$ be the graph obtained by embedding two vertex-disjoint triangles into the larger class of the complete bipartite graph $K_{2,6}$. For $n = 9$, let $G''_9$ be the graph obtained by taking $G'_8$ and joining another vertex to the four unmatched vertices within the classes of $G'_8$.

As before, let $A'_8$ and $B'_8$ be the classes of $G'_8$, with similar notations for the other graphs. Figure 3 below shows these additional graphs. Let $\mathcal{F}_{8,5} = \{G'^0_8, G'_8, G''_8\}$, $\mathcal{F}_{9,5} = \{G'^0_9, G'_9, G''_9, G'_9\}$, and $\mathcal{F}_{10,5} = \{G'^0_{10}, G''_{10}, G'_9\}$.

![Figure 3](image)

Note that every graph of $\mathcal{F}_{n,5}$ is gem$_5$-free. Indeed, let $G \in \mathcal{F}_{n,5}$. If $G \notin \{G'^0_8, G''_8, G'_9, G''_9, G'_9\}$, then $G$ is gem$_4$-free as before, so that $G$ is gem$_5$-free. Suppose that $G \in \{G'^0_8, G''_8, G'_9, G''_9, G'_9\}$ and $G$ contains a copy of gem$_5$, say $abcde + u$. It is easy to check that in each choice for $G$, whichever vertex of $G$ is chosen for $u$, we have that $u$ does not have five neighbours that form a path $P_5$ in $G$. This is a contradiction.

Also, by adding an edge to any graph of $\mathcal{F}_{n,5}$, we obtain a graph that contains a copy of gem$_5$. To see this, let $G \in \mathcal{F}_{n,5}$. Suppose first that $G \notin \{G'^0_8, G''_8, G'_9, G''_9, G'_9\}$. Then similar to before, since $n \geq 8$, it follows that if an edge $cu$ is added to the independent class of $G$, then we can find two independent edges $ab, de$ in the other class. If an edge $bu$ is added to the class of $G$ containing the
maximum matching, then we may assume that \( du \) is an edge in the matching, and choose vertices \( a, c, e \) in the other class. In both cases, we have \( abcd + u \) is a copy of \( \text{gem}_5 \). Next, the case \( G \in \{ G', G'_0, G'_10 \} \) can be considered similarly, according to whether or not the added edge is incident with an edge within a class of \( G \). Now, consider \( G = G''_8 \). If the edge \( bu \) is added into \( A''_8 \), then let \( cde \) be a triangle and \( a \) be another vertex in \( B''_8 \), and we let \( u \in A''_8 \). In both cases, \( abcd + u \) is a copy of \( \text{gem}_5 \). Finally, consider \( G = G''_9 \). Since \( G''_9 \) contains \( G''_8 \) as a subgraph on \( A''_9 \cup B''_9 \), it follows that if an edge is added into \( A''_9 \) or \( B''_9 \), then we have a copy of \( \text{gem}_5 \). Thus, we may assume that the edge \( au \) is added to \( G''_9 \), where \( a \) is an end-vertex of the edge in \( A''_9 \), and \( u \) is the vertex outside of \( A''_9 \cup B''_9 \). Then if \( c, e \in A''_9 \) and \( b, d \in B''_9 \) are the neighbours of \( u \) in \( G''_9 \), we have \( abcd + u \) is a copy of \( \text{gem}_5 \). We can easily check that for \( n \geq 8 \), all graphs of \( F_{n,5} \) have the same number of edges, which is also the same as the number of edges in any graph of \( F_{n,4} \). Thus, we may also let \( e_n \) denote the number of edges in any graph of \( F_{n,5} \). Then, equations (1) and (2) remain true. That is, for \( G \in F_{n,5} \), we have

\[
e(G) = e_n = \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \left\{ \begin{array}{ll} 0 & \text{if } n \equiv 0, 1, 2 \pmod{4}, \\ 1 & \text{if } n \equiv 3 \pmod{4}. \end{array} \right.
\]

and for \( n \geq 9 \), \( G \in F_{n,5} \) and \( G' \in F_{n-1,5} \), we have

\[
e(G) - e(G') = e_n - e_{n-1} = \left\lfloor \frac{n}{2} \right\rfloor + \left\{ \begin{array}{ll} 0 & \text{if } n \equiv 0, 1, 2 \pmod{4}, \\ 1 & \text{if } n \equiv 3 \pmod{4}. \end{array} \right.
\]

We have the following result for the Turán function \( \text{ex}(n, \text{gem}_5) \).

**Theorem 2.3.** For \( n \geq 8 \), we have

\[
\text{ex}(n, \text{gem}_5) = e_n = \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \left\{ \begin{array}{ll} 0 & \text{if } n \equiv 0, 1, 2 \pmod{4}, \\ 1 & \text{if } n \equiv 3 \pmod{4}. \end{array} \right.
\]

Moreover, the only \( \text{gem}_5 \)-free graphs with \( n \) vertices and \( \text{ex}(n, \text{gem}_5) \) edges are the members of \( F_{n,5} \).

As before, Theorem 2.3 will be proved by induction on \( n \). We first prove the base case, which will involve a bit more of case analysis than in Lemma 2.2.

**Lemma 2.4.** \( \text{ex}(8, \text{gem}_5) = e_8 = 18 \) and the only \( \text{gem}_5 \)-free graphs with eight vertices and 18 edges are \( G'_8, G'_9 \) and \( G''_8 \).

To prove Lemma 2.4, the following lemma will be useful.
Lemma 2.5. Let $H$ be a graph with vertex set $A \cup B$, where $A = \{x, y\}$ and $B = \{z_1, z_2, z_3, z_4\}$. Suppose that $xy, xz_4 \in E(H)$, and $H$ also contains all edges between $\{x, y\}$ and $\{z_1, z_2, z_3\}$. Suppose that $H[B]$ contains two edges $f_1, f_2$, and either $z_4$ belongs to at least one of $f_1, f_2$, or $yz_4 \in E(H)$. Then $H$ contains a copy of $gem_5$.

Proof. First, if $z_4$ belongs to one of $f_1, f_2$, then we may assume that either $f_1 = z_1z_2, f_2 = z_3z_4$ or $f_1 = z_1z_2, f_2 = z_2z_4$ or $f_1 = z_1z_4, f_2 = z_2z_4$. Then $z_1z_2y, z_1z_4 + x$ or $z_3y_1z_2z_4 + x$ or $z_3y_1z_4z_2 + x$ is a copy of $gem_5$ in $H$, respectively.

Next, if $yz_4 \in E(H)$ and $z_4$ does not belong to $f_1$ and $f_2$, then we may assume that $f_1 = z_1z_2$ and $f_2 = z_2z_3$. Then $z_1z_2z_3y + x$ is a copy of $gem_5$ in $H$. ■

Proof of Lemma 2.4. Let $G$ be a graph with eight vertices and $e_8 = 18$ edges. As in Lemma 2.2, it suffices to prove that either $G$ contains a copy of $gem_5$, or $G \in \mathcal{F}_{8,5} = \{G_0^8, G_3^8, G_8^8\}$. Let $\Delta = \Delta(G)$ be the maximum degree of $G$. Note that $5 \leq \Delta \leq 7$, otherwise if $\Delta \leq 4$, then $e(G) \leq \left\lfloor \frac{1}{2} \cdot 8 \cdot 4 \right\rfloor = 16 < 18 = e_8$, a contradiction. Let $d_1 \geq d_2 \geq \cdots \geq d_8$ be the degree sequence of $G$. Let $u \in V(G)$ be a vertex of maximum degree, so that $deg(u) = \Delta = d_1$. We consider three cases according to the value of $\Delta$.

Case 1. $\Delta = 7$. By Theorem 1.1, we have $ex(7, P_5) = \left\lfloor \frac{3}{2} \right\rfloor + \left\lfloor \frac{3}{2} \right\rfloor = 9$. Thus $e(G-u) = 18 - 7 = 11 > 9 = ex(7, P_5)$, and there exists a copy of the path $P_5$ in $G - u$, which together with $u$, form a copy of $gem_5$ in $G$.

Case 2. $\Delta = 6$. Let $v \in V(G) \setminus \{u\}$ be a vertex with $deg(v) = d_2$. Note that $deg(v) = 6$ or $deg(v) = 5$, otherwise $e(G) \leq \left\lfloor \frac{1}{2} \cdot (6 + 7 \cdot 4) \right\rfloor = 17 < 18 = e_8$, a contradiction.

Subcase 2.1. $deg(v) = 6$. Suppose first that $uv \notin E(G)$. We have $e(G - \{u, v\}) = 18 - 2 \cdot 6 = 6$. If there exists $x \in V(G) \setminus \{u, v\}$ with at least three neighbours in $V(G) \setminus \{u, v, x\}$, say $x_1, x_2, x_3$, then $x_1ux_2ux_3 + x$ is a copy of $gem_5$ in $G$. Otherwise, since $e(G - \{u, v\}) = 6$, we see that every vertex of $V(G) \setminus \{u, v\}$ must have exactly two neighbours in $V(G) \setminus \{u, v\}$, and thus, the subgraph $G - \{u, v\}$ must be either $C_6$ or two vertex-disjoint copies of $C_3$. If the former, then there is a copy of $P_5$ in $G - \{u, v\}$, which together with $u$, form a copy of $gem_5$. If the latter, then $G = G_0^8$.

Now, suppose that $uv \in E(G)$. Observe first that $u$ and $v$ have at least four common neighbours in $V(G) \setminus \{u, v\}$. If $G[N(u) \setminus \{v\}]$ contains two edges, then Lemma 2.5 implies that $G$ contains a copy of $gem_5$. Otherwise, we may assume that $G[N(u) \setminus \{v\}]$ contains at most one edge. If $y$ is the vertex not adjacent to $u$ in $G$, then $y$ has at most five neighbours in $N(u) \setminus \{v\}$. Therefore, we have $e(G - \{u, v\}) \leq 1 + 5 = 6$. This is a contradiction, since we have $e(G - \{u, v\}) = 18 - 1 - 2 \cdot 5 = 7$. 

Subcase 2.2. \( \deg(v) = 5 \). Let \( w \in V(G) \setminus \{u, v\} \) be a vertex with \( \deg(w) = d_3 \).

Note that \( \deg(w) = 5 \), otherwise, \( e(G) \leq \left\lfloor \frac{5}{2}(6 + 5 + 6 \cdot 4) \right\rfloor = 17 < 18 = e_g \).

Thus, without loss of generality, we may assume \( uw \in E(G) \), so that \( e(G - \{u, v\}) = 18 - 1 - 5 - 4 = 8 \). Let \( y \) be the vertex not adjacent to \( u \). Suppose that \( G \) does not contain a copy of \( \text{gem}_5 \).

Let \( vy \not\in E(G) \). Then \( v \) has exactly four neighbours in \( N(u) \setminus \{v\} \), and by Lemma 2.5, \( G[N(u) \setminus \{v\}] \) contains at most one edge, so that \( e(G - \{u, v\}) \leq 6 \), a contradiction.

Now let \( vy \in E(G) \). Let \( x_1, x_2, x_3 \) be the common neighbours of \( u \) and \( v \), and \( z_1, z_2 \) be the remaining two vertices, so that \( uz_1, uz_2 \in E(G) \) and \( vz_1, vz_2 \not\in E(G) \).

Again by Lemma 2.5, each of \( y, z_1, z_2 \) has at most one neighbour in \( \{x_1, x_2, x_3\} \).

If there are no edges between \( \{y, z_1, z_2\} \) and \( \{x_1, x_2, x_3\} \), then \( e(G - \{u, v\}) \leq 6 \), a contradiction. Otherwise, if there exists an edge between \( \{y, z_1, z_2\} \) and \( \{x_1, x_2, x_3\} \), then by Lemma 2.5, there are no edges in \( G[\{x_1, x_2, x_3\}] \). Since there are at most three edges in \( G[\{y, z_1, z_2\}] \) and at most three edges between \( \{y, z_1, z_2\} \) and \( \{x_1, x_2, x_3\} \), we have \( e(G - \{u, v\}) \leq 6 \), another contradiction.

Case 3. \( \Delta = 5 \). We have \( d_1 = d_2 = d_3 = d_4 = \Delta = 5 \), otherwise, \( e(G) \leq \left\lfloor \frac{5}{2}(3 \cdot 5 + 5 \cdot 4) \right\rfloor = 17 < 18 = e_g \). This means that, we may assume there exists \( v \in V(G) \setminus \{u\} \) with \( \deg(v) = 5 \) and \( uv \in E(G) \), so that \( e(G - \{u, v\}) = 18 - 1 - 2 \cdot 4 = 9 \). If \( G \) contains a copy of \( \text{gem}_5 \), then we are done, so assume otherwise.

Suppose first that \( u \) and \( v \) have four common neighbours, say \( x_1, x_2, x_3, x_4 \). Let \( y_1, y_2 \) be the remaining two vertices. By Lemma 2.5, \( G[\{x_1, x_2, x_3, x_4\}] \) contains at most one edge. If there is exactly one edge, say \( x_1 x_2 \in E(G) \), then there are 10 edges already in \( G \). The edges between \( \{y_1, y_2\} \) and \( \{x_1, x_2, x_3, x_4\} \), as well as \( y_1 y_2 \), may possibly be present, and since \( e(G) = 18 \), exactly one of these nine edges is not present. Suppose first that \( y_1 y_2 \in E(G) \). We may assume that \( y_1 x_1, y_1 x_2, y_2 x_1 \in E(G) \), but then \( uv x_2 y_1 y_2 + x_1 \) is a copy of \( \text{gem}_5 \). Otherwise, if \( y_1 y_2 \not\in E(G) \), then we have \( G = G_6' \). Finally, if there does not exist an edge in \( G[\{x_1, x_2, x_3, x_4\}] \), then a similar edge count shows that \( G \) contains all edges between \( \{y_1, y_2\} \) and \( \{x_1, x_2, x_3, x_4\} \), as well as \( y_1 y_2 \). This gives \( G = G_8' \).

Next, suppose that \( u \) and \( v \) have three common neighbours, say \( x_1, x_2, x_3 \). Let \( y, z_1, z_2 \) be the remaining vertices, where \( uz_1, vz_2 \in E(G) \) and \( vy, uv, uz_2, vz_1 \not\in E(G) \). By Lemma 2.5, each of \( z_1, z_2 \) has at most one neighbour in \( \{x_1, x_2, x_3\} \).

If there exists an edge between \( \{z_1, z_2\} \) and \( \{x_1, x_2, x_3\} \), then again by Lemma 2.5, there are no edges in \( G[\{x_1, x_2, x_3\}] \). Since there are at most three edges in \( G[\{y, z_1, z_2\}] \) and at most five edges between \( \{y, z_1, z_2\} \) and \( \{x_1, x_2, x_3\} \), we have \( e(G - \{u, v\}) \leq 8 \), a contradiction. Otherwise, suppose that there are no edges between \( \{z_1, z_2\} \) and \( \{x_1, x_2, x_3\} \). Then we have \( \deg(z_i) \leq 3 \) for \( i = 1, 2 \). This implies that the remaining six vertices must each have degree 5, otherwise \( e(G) \leq \left\lfloor \frac{5}{2}(5 \cdot 5 + 4 + 2 \cdot 3) \right\rfloor = 17 < 18 = e_g \). In particular, we have \( x_i x_j \in E(G) \).
for $1 \leq i \neq j \leq 3$ and $yx_i \in E(G)$ for $i = 1, 2, 3$. But then $uvx_2x_3y + x_1$ is a copy of $g_{e_5}$.

Finally, suppose that $u$ and $v$ have two common neighbours, say $x_1, x_2$. Let $y_1, y_2, z_1, z_2$ be the remaining vertices, where $uy_1, uy_2, vz_1, vz_2 \in E(G)$ and $uz_1, uz_2, vy_1, vy_2 \notin E(G)$. Suppose first that there are at most two edges in $G[(x_1, x_2, y_1, y_2)]$, and at most two edges in $G[(x_1, x_2, z_1, z_2)]$. Since there are at most four edges between $\{y_1, y_2\}$ and $\{z_1, z_2\}$, we have $e(G - \{u, v\}) \leq 2 \cdot 2 + 4 = 8$, a contradiction. Now, suppose that there are at least three edges in $G[(x_1, x_2, y_1, y_2)]$. If $x_1y_1, y_1y_2 \in E(G)$ or $x_1y_1, x_2y_2 \in E(G)$, then $x_2vy_1y_2 + u$ or $y_1x_1v_2y_2 + u$ is a copy of $g_{e_5}$. Thus, we may assume that $x_1x_2, x_1y_1, x_2y_1 \in E(G)$ and $x_1y_2, x_2y_2, y_1y_2 \notin E(G)$. If there are at most two edges in $G[(x_1, x_2, z_1, z_2)]$, including $x_1x_2$, then since there are at most four edges between $\{y_1, y_2\}$ and $\{z_1, z_2\}$, we have $e(G - \{u, v\}) \leq 3 + 4 = 8$, a contradiction. Thus, there are at least three edges in $G[(x_1, x_2, z_1, z_2)]$, and by similarly considering the edges in $G[(x_1, x_2, z_1, z_2)]$, we may assume that $x_1z_1, x_2z_1 \in E(G)$ and $x_1z_2, x_2z_2, z_1z_2 \notin E(G)$. But now, $y_1ux_2vz_1 + x_1$ is a copy of $g_{e_5}$.

Therefore, we conclude that either $G$ contains a copy of $g_{e_5}$, or $G \in F_{e_{n,5}}$. This completes the proof of Lemma 2.4.

We are now able to prove Theorem 2.3. The proof is generally similar to that of Theorem 2.1 but with a little more case analysis.

**Proof of Theorem 2.3.** Let $n \geq 8$. Again, the lower bound $ex(n, g_{e_5}) \geq c_n$ follows by considering any graph of $F_{n,5}$. We prove the upper bound $ex(n, g_{e_5}) \leq c_n$ by induction on $n$. Lemma 2.4 proves the result for $n = 8$. Now suppose that $n \geq 9$, and the theorem holds for $n - 1$. As before, it suffices to prove that if $G$ is a graph on $n$ vertices and $e(G) = c_n$, then either $G$ contains a copy of $g_{e_5}$, or $G \in F_{e_{n,5}}$.

First, suppose that $\delta(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$ and let $v \in V(G)$ be a vertex of minimum degree. Then exactly as in (3), we have $\varepsilon(G - v) \geq c_{n-1}$. Again we are done unless $\varepsilon(G - v) = c_{n-1}$, whence $\varepsilon(v) = \left\lfloor \frac{n}{2} \right\rfloor$ and $c_n - c_{n-1} = \left\lfloor \frac{n}{2} \right\rfloor$, and $n \neq 3$ (mod 4). By induction, either $G - v$, and thus $G$, contains a copy of $g_{e_5}$, and we are done, or $G - v \in F_{n-1,5}$, and we must consider the following cases.

**Case 1.** $n \equiv 0$ (mod 4). We have $G - v = G_{n-1}^3$ with classes $A_{n-1}^3$ and $B_{n-1}^3$, where $|A_{n-1}^3| = \frac{n}{2} - 1$ and $|B_{n-1}^3| = \frac{n}{2}$, and $B_{n-1}^3$ containing a perfect matching. We have $\varepsilon(v) = \frac{n}{2}$. If $N(v) = B_{n-1}^3$, then $G = G_{n-1}^0$. Otherwise, if $v$ has neighbours $c, d \in A_{n-1}^3$ and $u \in B_{n-1}^3$, then $acvd + u$ is a copy of $g_{e_5}$ in $G$, where $a \in A_{n-1}^3 \setminus \{c, d\}$ and $b \in B_{n-1}^3$ is the vertex adjacent to $u$. If $v$ has exactly one neighbour $u \in A_{n-1}^3$, then since $|B_{n-1}^3| = \frac{n}{2} > 4$, we can find $a, b, c, d \in B_{n-1}^3$ such that $ab, cd, bv, cv \in E(G)$. We have $acvd + u$ is a copy of $g_{e_5}$ in $G$. 

Case 2. $n \equiv 1 \pmod{4}$. If $n \geq 13$, we have $G - v = G_{n-1}^0$. If $n = 9$, we have $G - v \in \{G_8^0, G_8', G_8''\}$.

Subcase 2.1. $n \geq 9$ and $G - v = G_{n-1}^0$. The classes of $G - v$ are $A_{n-1}^0$ and $B_{n-1}^0$. Since $|B_{n-1}^0| = \frac{n-1}{2} \geq 4$, this subcase can be considered by combining the arguments used in Case 2 of Theorem 2.1 and in Case 1 above. We find that either $G$ contains a copy of gem$_5$, or $G \in \{G_{11}^1, G_{12}^1\}$.

Subcase 2.2. $n = 9$ and $G - v \in \{G_8', G_8''\}$. Suppose first that $G - v = G_8''$. so that the classes of $G - v$ are $A_8''$ and $B_8''$ with $|A_8''| = |B_8''| = 4$, and each class containing one edge, say $cu$ and $ab$ are the edges in $A_8''$ and $B_8''$. We have $\text{deg}(v) = 4$.

Suppose $N(v) = A_8''$ or $N(v) = B_8''$, then $G = G_9''$, and if $N(v) = (A_8'' \cup B_8'') \setminus \{a, b, c, u\}$, then $G = G_9''$. Otherwise, let $d \in B_8'' \setminus \{a, b\}$. We may assume that $uv \in E(G)$, and either $av \in E(G)$ or $dv \in E(G)$. Then $uabcd + u + abcdv + u$ is a copy of gem$_5$.

Now, suppose that $G - v = G_8'$. The classes of $G - v$ are $A_8'$ and $B_8'$ with $|A_8'| = |B_8'| = 6$, and there are two vertex-disjoint triangles embedded into $B_8'$. Let $A_8' = \{b, d\}$ and $acu$ be one of the triangles in $B_8'$. We have $\text{deg}(v) = 4$. If $bv, dv \in E(G)$, then we may assume that $uv \in E(G)$. We have $uabcd + u$ is a copy of gem$_5$. Otherwise, $v$ has at least three neighbours in $B_8'$, and we may assume that $av, wv \in E(G)$. Then $uabcd + u$ is a copy of gem$_5$.

Case 3. $n \equiv 2 \pmod{4}$. If $n \geq 14$, then we have $G - v \in \{G_{n-1}^{11}, G_{n-1}^{12}\}$. If $n = 10$, then we have $G - v \in \{G_9^{11}, G_9^{12}, G_9', G_9''\}$.

Subcase 3.1. $n \geq 10$ and $G - v \in \{G_{n-1}^{11}, G_{n-1}^{12}\}$. If $G - v = G_{10}^{11}$, then $|A_{n-1}^{11}| = \frac{n}{2} - 1 \geq 4$. If $G - v = G_{12}^{11}$, then $G - v$ has the class $B_{n-2}^{12}$ which contains a maximum matching with an unmatched vertex, say $w$. We have $|B_{n-2}^{12} \setminus \{w\}| = \frac{n}{2} - 1 \geq 4$. Since $\text{deg}(v) = \frac{n}{2}$, this subcase can be considered by combining the arguments used in Case 3 of Theorem 2.1 and in Case 1 above. We find that either $G$ contains a copy of gem$_5$, or $G \in \{G_{21}^{21}, G_{22}^{22}\}$.

Subcase 3.2. $n = 10$ and $G - v \in \{G_9', G_9''\}$. Suppose first that $G - v = G_9'$, so that the classes of $G - v$ are $A_9'$ and $B_9'$ with $|A_9'| = |B_9'| = 5$, and each class containing one edge. We have $\text{deg}(v) = 5$. If $N(v) = B_9'$, then $G = G_{10}'$. If $v$ has a neighbour which is incident with the edge in $A_9'$ or the edge in $B_9'$, then as in the argument in the first part of Subcase 2.2, $G$ contains a copy of gem$_5$. Otherwise, $N(v)$ consists of the five vertices not incident with the two edges within $A_9'$ and $B_9'$. Therefore, if $b, d \in A_9'$ and $a, c, e \in B_9'$ are these five neighbours of $v$, then $abedc + v$ is a copy of gem$_5$.

Now, suppose that $G - v = G_9''$. The graph $G - v$ consists of two sets $A_9''$ and $B_9''$ where $|A_9''| = |B_9''| = 4$, with one edge in each set, say $f_1$ in $A_9''$ and $f_2$ in $B_9''$, and another vertex, say $z$, joined to the four vertices not incident with $f_1, f_2$. Let $b, d \in A_9''$ and $a, c \in B_9''$ be the neighbours of $z$ in $G - v$. We have $\text{deg}(v) = 5$. 

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Again, if $v$ has a neighbour in each of $A'_n$ and $B''_n$ where at least one is incident with $f_1$ or $f_2$, then by the argument in Subcase 2.2, $G$ contains a copy of $\text{gem}_5$. Otherwise, we may assume that $N(v) = A''_n \cup \{z\}$ or $N(v) = \{a, b, c, d, z\}$, and $abcdv + z$ is a copy of $\text{gem}_5$.

This concludes the case when $\delta(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$.

Next, suppose that $\delta(G) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$. Then exactly as in the proof of Theorem 2.1, we must have $n \equiv 3 \pmod{4}$, and that $G$ is a $\left\lfloor \frac{n}{2} \right\rfloor + 1$-regular graph. Again for $v \in V(G)$, we have $e(G - v) = e_{n-1}$, using exactly the same argument as in (4). By induction, either $G - v$, and thus $G$, contains a copy of $\text{gem}_5$, or $G - v \in \mathcal{F}_{n-1,5}$. If the latter holds, then for $n \geq 15$ we have $G - v \in \{G_{n-1}^{21}, G_{n-1}^{22}\}$, and for $n = 11$ we have $G - v \in \{G_{10}^{11}, G_{10}^{22}, G_{10}^{10}\}$. If $n \geq 11$ and $G - v \in \{G_{n-1}^{21}, G_{n-1}^{22}\}$, then as in the proof of Theorem 2.1, the fact that $G$ is a $\left\lfloor \frac{n}{2} \right\rfloor + 1$-regular graph implies that $G = G_3^n$. Otherwise, we have $n = 11$ and $G - v = G_{10}^{10}$. Then $G$ is a 6-regular graph, which means that $N(v)$ consists of the six vertices not incident with the two edges within $A'_{10}$ and $B'_{10}$. Therefore, if $a, c, e \in A'_{10}$ and $b, d \in B'_{10}$ are neighbours of $v$, then $abcdv + z$ is a copy of $\text{gem}_5$.

This completes the proof Theorem 2.3.

3. Decompositions of Graphs into Gem Graphs and Single Edges

Recall that for a fixed graph $H$, $\phi(n, H)$ denotes the smallest integer $\phi$ such that any graph on $n$ vertices admits an $H$-decomposition with at most $\phi$ parts. In this section we will verify Pikhurko and Sousa conjecture (Conjecture 1.3) for the gem graphs $\text{gem}_4$ and $\text{gem}_5$. That is, we will show that $\phi(n, \text{gem}_4) = \text{ex}(n, \text{gem}_4)$ for $n \geq 6$, and $\phi(n, \text{gem}_5) = \text{ex}(n, \text{gem}_5)$ for $n \geq 8$.

3.1. $\text{gem}_4$-decompositions

We begin by considering $\text{gem}_4$-decompositions, and prove the following result.

**Theorem 3.1.** For $n \geq 6$ we have

$$\phi(n, \text{gem}_4) = \text{ex}(n, \text{gem}_4).$$

Moreover, the only graphs attaining $\text{ex}(n, \text{gem}_4)$ are the members of $\mathcal{F}_{n,4}$.

**Proof.** Let $n \geq 6$. The lower bound $\phi(n, \text{gem}_4) \geq \text{ex}(n, \text{gem}_4)$ holds by considering any graph of $\mathcal{F}_{n,4}$. We prove the matching upper bound. By Theorem 2.1, we know that $\text{ex}(n, \text{gem}_4) = e_n$ for $n \geq 6$. Let $G$ be a graph on $n \geq 6$ vertices. We must prove that $\phi(G, \text{gem}_4) \leq \text{ex}(n, \text{gem}_4) = e_n$, with equality if and only if $G \in \mathcal{F}_{n,4}$.

We proceed by induction on $n$. For $n = 6$, if $e(G) < e_6 = 10$, then we can simply decompose $G$ into single edges to obtain $\phi(G, \text{gem}_4) < e_6$. Otherwise, let
10 = e_6 \leq e(G) \leq 15. By Theorem 2.1, we either have \( G \in \mathcal{F}_{6,4} \), or \( G \) contains a copy of \( \text{gem}_4 \). If \( G \in \mathcal{F}_{6,4} \), then \( e(G) = e_6 = 10 \) and we must decompose \( G \) into single edges, thus, \( \phi(G, \text{gem}_4) = e_6 \) as required. If \( G \) contains a copy of \( \text{gem}_4 \), then \( \phi(G, \text{gem}_4) \leq 1 + e(G) - e(\text{gem}_4) \leq 9 < 10 = e_6 \). Thus, the theorem holds for \( n = 6 \).

Now, let \( n \geq 7 \), and suppose that the theorem holds for \( n - 1 \). Let \( G \) be a graph on \( n \) vertices. As before, if \( e(G) < e_n \), then \( \phi(G, \text{gem}_4) < e_n \), simply by decomposing \( G \) into single edges. If \( e(G) = e_n \), then by Theorem 2.1, either \( G \) contains a copy of \( \text{gem}_4 \), in which case \( \phi(G, \text{gem}_4) = 1 + e(G) - e(\text{gem}_4) = e_n - 6 < e_n \), or \( G \in \mathcal{F}_{n,4} \), in which case we can only decompose \( G \) into \( e_n \) single edges for a \( \text{gem}_4 \)-decomposition, and \( \phi(G, \text{gem}_4) = e_n \) as required.

Now, suppose that \( e(G) > e_n \), and let \( v \in V(G) \) be a vertex of minimum degree. If \( \deg(v) \leq \left\lfloor \frac{n}{2} \right\rfloor \), then by equation (2) we have \( e(G - v) = e(G) - \deg(v) > e_n - \left\lfloor \frac{n}{2} \right\rfloor \geq e_{n-1} \), that is, \( G - v \notin \mathcal{F}_{n-1,4} \) and by the induction hypothesis we have

\[
\phi(G - v, \text{gem}_4) < e(n-1, \text{gem}_4) = e_{n-1}.
\]

Therefore, when going from \( G - v \) to \( G \) we only need to use the edges joining \( v \) to the other vertices of \( G \), and there are at most \( \left\lfloor \frac{n}{2} \right\rfloor \) of these edges at \( v \). We have

\[
\phi(G, \text{gem}_4) \leq \phi(G - v, \text{gem}_4) + \deg(v) < e_{n-1} + \left\lfloor \frac{n}{2} \right\rfloor \leq e_n,
\]

as required.

Therefore, we may assume that \( \deg(v) \geq \left\lceil \frac{n}{2} \right\rceil + 1 \) and let \( \deg(v) = \left\lceil \frac{n}{2} \right\rceil + m \) for some integer \( m \geq 1 \). For every \( x \in N(v) \), we have

\[
\deg(x, N(v)) \geq \left\lceil \frac{n}{2} \right\rceil + m - \left( n - \left\lfloor \frac{n}{2} \right\rfloor - m \right) \]

\[
= 2 \left\lceil \frac{n}{2} \right\rceil + 2m - n \geq 2m - 1.
\]

This means that \( G[N(v)] \) must contain a path \( P_{2m} \) on \( 2m \) vertices. Otherwise, if the longest path in \( G[N(v)] \) has at most \( 2m - 1 \) vertices, say with an end-vertex \( y \), then all neighbours of \( y \) in \( N(v) \) must lie in the path, so that \( \deg(y, N(v)) \leq 2m - 2 \), contradicting (7).

If \( m \geq 2 \), then the path \( P_{2m} \) contains \( \left\lfloor \frac{2m}{4} \right\rfloor = \left\lfloor \frac{m}{2} \right\rfloor \) vertex-disjoint paths of order 4. Thus, we have \( \left\lfloor \frac{2m}{4} \right\rfloor \) edge-disjoint copies of \( \text{gem}_4 \), where each copy is formed by a path of order 4, together with \( v \). Let \( F \subset G - v \) be the subgraph of order \( n - 1 \) obtained by deleting the edges of the paths of order 4 from \( G - v \). By induction and (2), and since \( m \geq 2 \), we have

\[
\phi(G, \text{gem}_4) \leq \phi(F, \text{gem}_4) + \left\lfloor \frac{m}{2} \right\rfloor + \deg(v) - 4 \left\lfloor \frac{m}{2} \right\rfloor
\]

\[
\leq e_{n-1} + \left\lfloor \frac{n}{2} \right\rfloor + m - 3 \left\lfloor \frac{m}{2} \right\rfloor < e_{n-1} + \left\lfloor \frac{n}{2} \right\rfloor \leq e_n.
\]
To complete the proof it remains to consider the case $m = 1$. For this case, we will repeatedly use the following claim.

**Claim 3.2.** Suppose that there exists a vertex $z \in V(G)$ with $\deg(z) = \left\lceil \frac{n}{2} \right\rceil + 1$, and $G$ has a copy of gem$_4$ with at least three edges incident to $z$. Then $\phi(G, \text{gem}_4) < e_n$.

**Proof.** Let $F \subset G - z$ be the subgraph on $n - 1$ vertices obtained from $G - z$ by deleting the edges of the copy of gem$_4$. By induction and (2), we have

$$\phi(G, \text{gem}_4) \leq \phi(F, \text{gem}_4) + 1 + \deg(z) - 3 \leq e_{n-1} + \left\lfloor \frac{n}{2} \right\rfloor - 1 < e_n. \quad \square$$

We now consider three cases. Let $\overline{N}(v) = V(G) \setminus (N(v) \cup \{v\})$, and note that

$$|N(v)| = \left\lceil \frac{n}{2} \right\rceil + 1 \geq 4 \text{ and } |\overline{N}(v)| = \left\lfloor \frac{n}{2} \right\rfloor - 2 \geq 2.$$

**Case 1.** $G[N(v)]$ contains a path $P$ of order 4. Then $P$ and $v$ form a copy of gem$_4$, and we have $\phi(G, \text{gem}_4) < e_n$ by Claim 3.2.

**Case 2.** The order of the longest path in $G[N(v)]$ is 3. Let $x_1x_2$ be a path of order 3 in $G[N(v)]$.

**Subcase 2.1.** $x_1x_2 \in E(G)$. We have $\deg(x, N(v)) = 2$, for otherwise $G[N(v)]$ would contain a $P_4$. We must have $\deg(x, \overline{N}(v)) \geq \left\lceil \frac{n}{2} \right\rceil + 1 - 3 \geq |\overline{N}(v)| - 1$. Similarly for $x_1, x_2$. This implies that two of $x, x_1, x_2$ have a common neighbour in $\overline{N}(v)$, say $y \in \overline{N}(v)$ is a common neighbour of $x, x_1$. Then $x_2vx_1y + x$ is a copy of gem$_4$, and by Claim 3.2 with $z = v$, we have $\phi(G, \text{gem}_4) < e_n$.

**Subcase 2.2.** $x_1x_2 \notin E(G)$. Let $N(v) = \{x, x_1, x_2, \ldots, x_{n/2}\}$. For $i = 1, 2$, we have $\deg(x_i, N(v)) = 1$, and

$$(8) \quad \deg(x_i, \overline{N}(v)) \geq \left\lceil \frac{n}{2} \right\rceil + 1 - 2 \geq \left\lfloor \frac{n}{2} \right\rfloor - 2 = |\overline{N}(v)|.$$

We must have equality to hold throughout, whence $n$ is odd, $\deg(x_1) = \deg(x_2) = \left\lceil \frac{n}{2} \right\rceil + 1$, and both $x_1, x_2$ are adjacent to all vertices of $\overline{N}(v)$. If $x$ has a neighbour $y \in \overline{N}(v)$, then $x_1vx_2y + x$ is a copy of gem$_4$, and again $\phi(G, \text{gem}) < e_n$ by Claim 3.2 with $z = v$.

Otherwise, suppose that $x$ does not have a neighbour in $\overline{N}(v)$. Then $\deg(x) \leq |N(v) \cup \{v\}| - 1 = \left\lceil \frac{n}{2} \right\rceil + 1$, so that $\deg(x) = \left\lceil \frac{n}{2} \right\rceil + 1$ and $xx_i \in E(G)$ for all $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$. Moreover, we have $x_ix_j \notin E(G)$ for all $i \neq j$, otherwise there would exist a copy of $P_4$ in $G[N(v)]$. By a similar argument as in (8), we have $\deg(x_i) = \left\lceil \frac{n}{2} \right\rceil + 1$, and $x_i$ is adjacent to all vertices of $\overline{N}(v)$ for all $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$. In order to get a contradiction, suppose that there does not exist a path of order 3.
in $G[N(v)]$. Then the maximum number of edges in $G[N(v)]$ is $\lfloor \frac{1}{2} |N(v)| \rfloor$. Recall that $n$ is odd. We have

$$e(G) \leq 2|N(v)| - 1 + (|N(v)| - 1)|N(v)| + \left\lfloor \frac{1}{2} |N(v)| \right\rfloor$$

$$= 2 \left\lfloor \frac{n}{2} \right\rfloor + 1 + \frac{n}{2} \left( \left\lceil \frac{n}{2} \right\rceil - 2 \right) + \frac{1}{2} \left( \left\lceil \frac{n}{2} \right\rceil - 2 \right)$$

$$= \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lceil \frac{n+1}{4} \right\rceil = e_n,$$

by (1), which contradicts the assumption $e(G) > e_n$. Therefore, $G[N(v)]$ must have a path of order 3, say $y_1y_2y_3$. Note that $|N(v)| = \left\lceil \frac{n}{2} \right\rceil - 2 \geq 3$ and thus we must have $n$ odd and $n \geq 9$. Then, $x_1y_1x_2y_3 + y_2$ is a copy of gem$_4$, and by Claim 3.2 with $z = x_1$, we have $\phi(G, \text{gem}) < e_n$.

Case 3. The longest path in $G[N(v)]$ has order 2. Note that this is indeed the remaining case, since $\deg(x, N(v)) \geq 2m - 1 = 1$ for all $x \in N(v)$ by (7). Moreover, $N(v)$ induces a perfect matching in $G$. Now by a similar argument as in (8), we must have $n$ odd, and for every $x \in N(v)$, we have $\deg(x) = \left\lfloor \frac{n}{2} \right\rfloor + 1$ and $x$ is adjacent to all vertices of $N(v)$. Thus, we can find an edge $x_1x_2$ in $G[N(v)]$ and a common neighbour $y \in N(v)$ of $x_1, x_2$. Now, since $vx_2y$ is a path of order 3 in $G[N(x_1)]$, we are done by applying Case 1 or Case 2 with $x_1$ in place of $v$.

The induction step is complete, and this completes the proof of Theorem 3.1.

### 3.2. gem$_5$-decompositions

By using the same ideas as in the proof of Theorem 3.1, but with more case analysis, we will be able to prove a similar result for gem$_5$-decompositions. That is, we will prove the following theorem.

**Theorem 3.3.** For $n \geq 8$ we have

$$\phi(n, \text{gem}_5) = \text{ex}(n, \text{gem}_5).$$

Moreover, the only graphs attaining $\text{ex}(n, \text{gem}_5)$ are the members of $F_{n,5}$.

**Proof.** Let $n \geq 8$. As before, we have $\phi(n, \text{gem}_5) \geq \text{ex}(n, \text{gem}_5)$ by considering any graph of $F_{n,5}$. By Theorem 2.3, to prove the matching upper bound, we must prove that if $G$ is a graph on $n \geq 8$ vertices, then $\phi(G, \text{gem}_5) \leq \text{ex}(n, \text{gem}_5) = e_n$, with equality if and only if $G \in F_{n,5}$.

We proceed by induction on $n$. For $n = 8$, if $e(G) < e_8 = 18$, then we can simply decompose $G$ into single edges to obtain $\phi(G, \text{gem}_4) < e_8$. Next,
suppose that $18 = e_8 \leq e(G) \leq 25$. By Theorem 2.3, we either have $G \in \mathcal{F}_{8,5}$, or $G$ contains a copy of gem$_5$. If $G \in \mathcal{F}_{8,5}$, then $e(G) = e_8 = 18$ and we must decompose $G$ into single edges, and $\phi(G, \text{gem}_5) = e_8$. If $G$ contains a copy of gem$_5$, then $\phi(G, \text{gem}_5) \leq 1 + e(G) - e(\text{gem}_5) \leq 17 < 18 = e_8$. Finally, suppose that $26 \leq e(G) \leq 28$. Clearly, there exist two vertices $x, y \in V(G)$ of degree 7, so that $\phi(G - \{x, y\}) \geq 26 - 1 - 2 \cdot 6 = 13$. Since $\text{ex}(6, P_5) = \left\lfloor \frac{1}{2} \right\rfloor + \left\lfloor \frac{5}{2} \right\rfloor = 7$ by Theorem 1.1, this means that we can find two edge-disjoint copies of $P_5$ in $G - \{x, y\}$. These two copies of $P_5$, together with $x$ and $y$, form two edge-disjoint copies of gem$_5$ in $G$. Thus, $\phi(G, \text{gem}_5) \leq 2 + e(G) - 2e(\text{gem}_5) \leq 12 < 18 = e_8$. The theorem holds for $n = 8$.

Now, let $n \geq 9$, and suppose that the theorem holds for $n - 1$. Let $G$ be a graph on $n$ vertices. As before, if $e(G) < e_n$, then $\phi(G, \text{gem}_5) < e_n$, simply by decomposing $G$ into single edges. If $e(G) = e_n$, then by Theorem 2.3, either $G$ contains a copy of gem$_5$, in which case $\phi(G, \text{gem}_5) \leq 1 + e(G) - e(\text{gem}_5) = e_n - 8 < e_n$, or $G \in \mathcal{F}_{n,5}$, in which case we can only decompose $G$ into $e_n$ single edges for a gem$_5$-decomposition, and $\phi(G, \text{gem}_5) = e_n$ as required.

Now, suppose that $e(G) > e_n$, and let $v \in V(G)$ be a vertex of minimum degree. If $\deg(v) \leq \left\lfloor \frac{n}{2} \right\rfloor$, then by equation (6), we have $e(G - v) = e(G) - \deg(v) > e_n - \left\lfloor \frac{n}{2} \right\rfloor \geq e_{n-1}$, that is, $G - v \notin \mathcal{F}_{n-1,5}$. By induction, we have $\phi(G - v, \text{gem}_5) < \text{ex}(n - 1, \text{gem}_5) = e_{n-1}$. Thus, when going from $G - v$ to $G$ we only need to use the edges joining $v$ to the other vertices of $G$. We have

$$\phi(G, \text{gem}_5) \leq \phi(G - v, \text{gem}_5) + \deg(v) < e_{n-1} + \left\lfloor \frac{n}{2} \right\rfloor \leq e_n.$$ 

Therefore, we may assume that $\deg(v) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$ and let $\deg(v) = \left\lfloor \frac{n}{2} \right\rfloor + m$ for some integer $m \geq 1$. As in (7), for every $x \in N(v)$, we have $\deg(x, N(v)) \geq 2m - 1$, and that $G[N(v)]$ must contain a path $P_{2m}$ on $2m$ vertices.

If $m \geq 3$, then the path $P_{2m}$ contains $\left\lfloor \frac{2m}{5} \right\rfloor$ vertex-disjoint paths of order 5. Thus, we have $\left\lfloor \frac{2m}{5} \right\rfloor$ edge-disjoint copies of gem$_5$, where each copy is formed by a path of order 5, together with $v$. Let $F \subset G - v$ be the subgraph of order $n - 1$ obtained by deleting the edges of the paths of order 5 from $G - v$. By induction and (6), and since $m \geq 3$, we have

$$\phi(G, \text{gem}_5) \leq \phi(F, \text{gem}_5) + \left\lfloor \frac{2m}{5} \right\rfloor + \deg(v) - 5 \left\lfloor \frac{2m}{5} \right\rfloor$$

$$\leq e_{n-1} + \left\lfloor \frac{n}{2} \right\rfloor + m - 4 \left\lfloor \frac{2m}{5} \right\rfloor < e_{n-1} + \left\lfloor \frac{n}{2} \right\rfloor \leq e_n.$$ 

For the rest of the proof, let $\overline{N}(v) = V(G) \setminus (N(v) \cup \{v\})$. Next, suppose that $m = 2$, so that $|N(v)| = \left\lfloor \frac{n}{2} \right\rfloor + 2 \geq 6$ and $|\overline{N}(v)| = \left\lfloor \frac{n}{2} \right\rfloor - 3 \geq 2$. If $G[\overline{N}(v)]$ contains a path $P_5$ of order 5, then this path together with $v$ form a copy of gem$_5$. 
Let $F \subset G - v$ be the subgraph of order $n - 1$, obtained by deleting the edges of the $P_5$. Then,

$$\phi(G, \text{gem}_5) \leq \phi(F, \text{gem}_5) + 1 + \deg(v) - 5 \leq e_{n-1} + \left\lfloor \frac{n}{2} \right\rfloor + 2 - 4 < e_n.$$ 

Therefore, we may assume that the longest path in $G[N(v)]$ has order 4. Let $x_1x_2x_3x_4$ be such a path in $G[N(v)]$. Since $\deg(x_1, N(v)) \geq 2 \cdot 2 - 1 = 3$, we must have $x_1x_3, x_1x_4 \in E(G)$. Moreover, the only neighbours of $x_1$ in $N(v)$ are $x_2, x_3, x_4$, so that

$$\deg(x_1, N(v)) \geq \left\lfloor \frac{n}{2} \right\rfloor + 2 - 4 \geq \left\lfloor \frac{n}{2} \right\rfloor - 3 = |N(v)|.$$ 

We must have equality, so that $n$ is odd, $\deg(x_1) = \left\lfloor \frac{n}{2} \right\rfloor + 2$, and $x_1$ is adjacent to every vertex of $N(v)$. The same argument holds for $x_4$, so that $x_1, x_4$ have a common neighbour $y \in N(v)$. Now, since $vx_2x_3x_4y$ is a path of order 5 in $G[N(x_1)]$, we are done by applying the previous argument with $x_1$ in place of $v$.

To complete the proof it remains to consider the case $m = 1$. As before, we will repeatedly use the following claim which is analogous to Claim 3.2.

**Claim 3.4.** Suppose that there exists a vertex $z \in V(G)$ with $\deg(z) = \left\lfloor \frac{n}{2} \right\rfloor + 1$, and $G$ has a copy of $\text{gem}_5$ with at least three edges incident to $z$. Then $\phi(G, \text{gem}_5) < e_n$.

**Proof.** Exactly the same as the proof of Claim 3.2. □

We now consider four cases. Note that we have

$$|N(v)| = \left\lfloor \frac{n}{2} \right\rfloor + 1 \geq 5 \quad \text{and} \quad |N(v)| = \left\lfloor \frac{n}{2} \right\rfloor - 2 \geq 3.$$ 

**Case 1.** $G[N(v)]$ contains a path $P$ of order 5. Then $P$ and $v$ form a copy of $\text{gem}_5$, and we have $\phi(G, \text{gem}_5) < e_n$ by Claim 3.4.

**Case 2.** The order of the longest path in $G[N(v)]$ is 4. Let $x_1x_2x_3x_4$ be such a path in $G[N(v)]$. It suffices to consider the following subcases.

**Subcase 2.1.** $x_1x_3, x_1x_4 \in E(G)$. For $i = 1, 2, 3, 4$, $x_i$ does not have a neighbour in $N(v) \setminus \{x_1, x_2, x_3, x_4\}$, so that $\deg(x_i, N(v)) \leq 3$. Thus,

$$\deg(x_i, N(v)) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1 - 4 \geq \left\lfloor \frac{n}{2} \right\rfloor - 4 = |N(v)| - 2.$$ 

If $x_2x_4 \not\in E(G)$, then we have $\deg(x_j, N(v)) = 2$, and $\deg(x_j, N(v)) \geq |N(v)| - 1$ for $j = 2, 4$. With (9), this implies that either $x_1, x_2$ or $x_2, x_3$ or $x_1, x_3$, have a common neighbour $y \in N(v)$. Then, either $x_4vx_3x_2y + x_1$; or $x_4vx_1x_2y + x_3$; or $x_4vx_2x_3y + x_1$, is a copy of $\text{gem}_5$, respectively. By Claim 3.4 with $z = v$, we
have \( \phi(G, \text{gem}_5) < e_n \). Now, if \( x_2x_4 \in E(G) \), then by (9), two of \( x_1, x_2, x_3, x_4 \) have a common neighbour in \( \overline{N}(v) \). We may assume that \( x_1, x_2 \) have a common neighbour \( y \in \overline{N}(v) \). Then we have \( \phi(G, \text{gem}_5) < e_n \) by the same argument.

**Subcase 2.2.** \( x_1x_3 \in E(G) \) and \( x_1x_4, x_2x_4 \notin E(G) \). We see that \( x_3 \) is the only neighbour of \( x_4 \) in \( N(v) \), so that

\[
\deg(x_4, \overline{N}(v)) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1 - 2 \geq \left\lfloor \frac{n}{2} \right\rfloor - 2 = |\overline{N}(v)|.
\]

We must have equality throughout, so that \( \deg(x_4) = \left\lfloor \frac{n}{2} \right\rfloor + 1 \) and \( n \) is odd. Moreover, \( x_4 \) is adjacent to every vertex of \( \overline{N}(v) \). If \( x_4 \) has a neighbour \( y \in \overline{N}(v) \), then \( x_1x_2v, x_4y + x_4, x_3 \) is a copy of \( \text{gem}_5 \), and we have \( \phi(G, \text{gem}_5) < e_n \) by Claim 3.4 with \( z = v \). Now suppose that \( x_3 \) does not have a neighbour in \( \overline{N}(v) \). Let \( x_5, x_6, \ldots, x_{[n/2]} + 1 \) be the remaining vertices of \( N(v) \). Then \( \deg(x_3) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1 \) implies that \( x_3x_i \in E(G) \) for every \( i \geq 5 \). Moreover, we have \( x_1x_i, x_2x_i \notin E(G) \) for all \( i \geq 5 \), otherwise we are in Subcase 2.1. This means that \( \deg(x_i) = \left\lfloor \frac{n}{2} \right\rfloor + 1 \) and \( x_i \) is adjacent to every vertex of \( \overline{N}(v) \) for all \( i \geq 4 \). Also, note that for \( i = 1, 2 \),

\[
\deg(x_i, \overline{N}(v)) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1 - 3 = \left\lfloor \frac{n}{2} \right\rfloor - 3 = |\overline{N}(v)| - 1.
\]

Suppose first that \( G[\overline{N}(v)] \) contains a path of order \( 3 \), say \( y_1y_2y_3 \). If \( n \geq 11 \) so that \( |N(v)| = \left\lfloor \frac{n}{2} \right\rfloor + 1 \geq 6 \), then \( x_4y_1x_5y_3x_6 + y_2 \) is a copy of \( \text{gem}_5 \), and we have \( \phi(G, \text{gem}_5) < e_n \) by Claim 3.4 with \( z = x_5 \). Now let \( n = 9 \), and suppose that \( x_1y_1, x_1y_2 \in E(G) \). Then \( x_1y_1x_4y_3x_5 + y_2 \) is a copy of \( \text{gem}_5 \), and we have \( \phi(G, \text{gem}_5) < e_n \) by Claim 3.4 with \( z = x_4 \). Thus, we may assume that \( x_1y_1, x_1y_3, x_2y_1, x_2y_3 \in E(G) \) and \( x_1y_2, x_2y_2 \notin E(G) \). It is easy to check that \( G \) is the graph \( G'_9 \) with \( A'_9 = \{x_1, x_2, x_4, x_5\} \), \( B'_9 = \{v, x_3, y_1, y_3\} \), and \( y_2 \) is the remaining vertex, so that \( \phi(G, \text{gem}_5) = e_9 = \text{ex}(9, \text{gem}_5) \).

Now, suppose that \( G[\overline{N}(v)] \) contains an edge, say \( y_1y_2 \). If \( v \) is not adjacent to every vertex in \( \overline{N}(v) \), then we may assume that \( x_2y_1 \in E(G) \). Then \( x_4v, x_2y_2 + x_1 \) is a copy of \( \text{gem}_5 \), and we have \( \phi(G, \text{gem}_5) < e_n \) by Claim 3.4 with \( z = v \). Thus we may assume that \( x_1 \) and \( x_2 \) are not adjacent to exactly one vertex in \( \overline{N}(v) \). Since there are at most \( |N(v)| \) edges in \( G[\overline{N}(v)] \) and at most \( \left\lfloor \frac{1}{2}|\overline{N}(v)| \right\rfloor \) edges in \( G[\overline{N}(v)] \), we have

\[
e(G) \leq 2|N(v)| + 2(|\overline{N}(v)| - 1) + (|N(v)| - 3)|\overline{N}(v)| + \left\lfloor \frac{1}{2}|\overline{N}(v)| \right\rfloor
\]

\[
= 2n - 4 + \left( \left\lfloor \frac{n}{2} \right\rfloor \right)^2 + \left( \left\lfloor \frac{n}{2} \right\rfloor - 2 \right) + \frac{1}{2} \left( \left\lfloor \frac{n}{2} \right\rfloor - 2 \right)
\]

\[
= \left\lfloor \frac{n^2}{4} \right\rfloor + \frac{n + 1}{4} = e_n.
\]
by (5) and since $n$ is odd, which contradicts the assumption $e(G) > e_n$. Finally, if $G[\overline{N(v)}]$ does not contain an edge, then

$$e(G) \leq 2|N(v)| + (|N(v)| - 1)|\overline{N(v)}|$$

$$= 2\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) + \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor - 2\right) = \left\lfloor \frac{n^2}{4} \right\rfloor + 2 \leq e_n,$$

another contradiction.

Subcase 2.3. $x_1x_4 \in E(G)$ and $x_1x_3, x_2x_4 \notin E(G)$. For $i = 1, 2, 3, 4$, $x_i$ does not have a neighbour in $N(v) \setminus \{x_1, x_2, x_3, x_4\}$, so that $\deg(x_i, \overline{N(v)}) = 2$. Thus,

$$\deg(x_i, \overline{N(v)}) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1 - 3 \geq \left\lfloor \frac{n}{2} \right\rfloor - 3 = |\overline{N(v)}| - 1.$$  

If $\deg(x_1, \overline{N(v)}) = |\overline{N(v)}|$, then we can find $y_1, y_2 \in \overline{N(v)}$ such that $y_1$ is a common neighbour of $x_1, x_2$, and $y_2$ is a common neighbour of $x_2, x_3$. Then $y_1x_1yx_2y_2 + x_2$ is a copy of gem$_5$, and we have $\phi(G, \text{gem}_5) < e_n$ by Claim 3.4 with $z = v$. Otherwise, we must have equality in (10) for $i = 1, 2, 3, 4$, so that $n$ is odd, and for $i = 1, 2, 3, 4$, we have $\deg(x_i) = \left\lfloor \frac{n}{2} \right\rfloor + 1$, and $x_i$ is not adjacent to exactly one vertex in $\overline{N(v)}$. If $n \geq 11$ so that $|\overline{N(v)}| = \left\lfloor \frac{n}{2} \right\rfloor - 2 \geq 4$, then we can again find the vertices $y_1, y_2 \in \overline{N(v)}$ and we are done as before. Now let $n = 9$, so that $|\overline{N(v)}| = 5, |\overline{N(v)}| = 3$, and each $x_i$ has exactly two neighbours in $\overline{N(v)}$. If $x_1$ and $x_2$ have two common neighbours in $\overline{N(v)}$, then we can again find $y_1, y_2 \in \overline{N(v)}$ as before and we are done. Otherwise, we may assume that $\overline{N(v)} = \{z_1, z_2, z_3\}$ with $x_1z_1, x_1z_2, x_2z_1, x_2z_3 \in E(G)$. If $z_1z_2 \in E(G)$, then $x_4x_2z_2z_2 + x_1$ is a copy of gem$_5$, and again $\phi(G, \text{gem}_5) < e_n$ by Claim 3.4 with $z = v$. A similar argument holds if $z_1z_3 \in E(G)$. Otherwise, we have at most one edge in $G[\overline{N(v)}]$, and since there are exactly nine edges in $G[\overline{N(v)} \cup \{v\}]$ and at most $4 \cdot 2 + 3 = 11$ edges between $N(v)$ and $\overline{N(v)}$, we have $e(G) \leq 1 + 9 + 11 = 21 < 22 = e_9$, which is a contradiction.

Subcase 2.4. $x_1x_3, x_1x_4, x_2x_4 \notin E(G)$. We first note that $x_2$ is the only neighbour of $x_1$ in $N(v)$, so that

$$\deg(x_1, \overline{N(v)}) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1 - 2 \geq \left\lfloor \frac{n}{2} \right\rfloor - 2 = |\overline{N(v)}|.$$ 

We must have equality throughout, so that $n$ is odd, $\deg(x_1) = \left\lfloor \frac{n}{2} \right\rfloor + 1$, and $x_1$ is adjacent to all vertices of $\overline{N(v)}$. The exact same properties hold for $x_4$. Next, suppose that $x_2$ has $p$ neighbours in $N(v) \setminus \{x_1, x_2, x_3, x_4\}$, where $0 \leq p \leq \left\lfloor \frac{n}{2} \right\rfloor - 3$. Let $S_2$ be the set of these $p$ neighbours. We have

$$\deg(x_2, \overline{N(v)}) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1 - 3 - p = \left\lfloor \frac{n}{2} \right\rfloor - 3 - p.$$
Now, $x_3$ does not have a neighbour in $S_2$, otherwise there would exist a path of order 5 in $G[N(v)]$. Thus, $x_3$ has at most $|N(v)| - 4 - p = \left\lfloor \frac{n}{2} \right\rfloor - 3 - p$ neighbours in $N(v) \setminus \{x_1, x_2, x_3, x_4\}$. Let $S_3$ be these neighbours of $x_3$, so that $S_2 \cap S_3 = \emptyset$. We have

\begin{equation}
\deg(x_3, \overline{N}(v)) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1 - 3 - \left( \left\lfloor \frac{n}{2} \right\rfloor - 3 - p \right) = p + 1.
\end{equation}

Suppose that $x_2, x_3$ have a common neighbour $y_1 \in \overline{N}(v)$. Clearly, from (11) and (12), at least one of $x_2, x_3$ has at least two neighbours in $\overline{N}(v)$. If $x_2$ has this property, then $x_1, x_2$ have a common neighbour $y_2 \in \overline{N}(v) \setminus \{y_1\}$. Thus, $y_1x_1y_2 + x_2$ is a copy of gem$_5$, and by Claim 3.4 with $z = v$, we have $\phi(G, \text{gem}_5) < e_n$. A similar argument holds if $x_3$ has at least two neighbours in $\overline{N}(v)$, with $x_4$ in place of $x_1$.

Thus, if $T_2, T_3 \subset \overline{N}(v)$ are the sets of neighbours of $x_2, x_3$ in $\overline{N}(v)$, respectively, then we may assume that $T_2 \cap T_3 = \emptyset$. Note that from (11) and (12), we have

$$\deg(x_2, \overline{N}(v)) + \deg(x_3, \overline{N}(v)) \geq \left\lfloor \frac{n}{2} \right\rfloor - 2 = |\overline{N}(v)|.$$

Thus, we must have equality above, as well as in (11) and (12). This means that $\deg(x_2) = \deg(x_3) = \left\lfloor \frac{n}{2} \right\rfloor + 1$, and we have the partitions $N(v) \setminus \{x_1, x_2, x_3, x_4\} = S_2 \cup S_3$ and $\overline{N}(v) = T_2 \cup T_3$. Clearly, there are no edges in $G[S_2 \cup S_3]$, otherwise there would exist a path of order 5 in $G[N(v)]$. Next, suppose that there is a path of order 3 in $G[\overline{N}(v)]$, say $y_1y_2y_3$. Suppose that $y_2 \in T_2$. Then $x_2y_1y_2y_3 + y_2$ is a copy of gem$_5$, so that by Claim 3.4 with $z = x_1$, we have $\phi(G, \text{gem}_5) < e_n$. A similar argument holds if $y_2 \in T_3$. Otherwise, we have $|N(v)| - 1$ edges in $G[N(v)], |\overline{N}(v)|$ edges between $\{x_2, x_3\}$ and $\overline{N}(v)$, and at most $\left\lfloor \frac{1}{2} |\overline{N}(v)| \right\rfloor$ edges in $G[\overline{N}(v)]$. By (5) and since $n$ is odd,

$$e(G) \leq 2|N(v)| - 1 + |\overline{N}(v)| + (|N(v)| - 2)|\overline{N}(v)| + \left\lfloor \frac{1}{2} |\overline{N}(v)| \right\rfloor$$

$$= 2\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor - 1 + \left( \left\lfloor \frac{n}{2} \right\rfloor - 1 \right)\left( \left\lfloor \frac{n}{2} \right\rfloor - 2 \right) + \left\lfloor \frac{1}{2} \left( \left\lfloor \frac{n}{2} \right\rfloor - 2 \right) \right\rfloor$$

$$= \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n + 1}{4} \right\rfloor = e_n,$$

which contradicts the assumption $e(G) > e_n$.

**Case 3.** The order of the longest path in $G[N(v)]$ is 3. Let $x_1x_2$ be such a path in $G[N(v)]$. We consider the following subcases.

**Subcase 3.1.** $x_1x_2 \in E(G)$. We have $\deg(x, N(v)) = 2$, for otherwise $G[N(v)]$ would contain a $P_4$. Thus

$$\deg(x, \overline{N}(v)) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1 - 3 \geq \left\lfloor \frac{n}{2} \right\rfloor - 3 = |\overline{N}(v)| - 1.$$
Similar inequalities hold for $x_1, x_2$. If $\deg(x, \overline{N}(v)) = |\overline{N}(v)|$, then there exist $y_1, y_2 \in \overline{N}(v)$ such that $y_i$ is a common neighbor of $x, x_i$ for $i = 1, 2$. Then $y_1x_1x_2y_2 + x$ is a copy of gem$_5$, and by Claim 3.4 with $z = v$, we have $\phi(G, \text{gem}_5) < e_n$. Otherwise, we have $\deg(x, \overline{N}(v)) = |\overline{N}(v)| - 1$, whence $n$ is odd and $\deg(x) = \left\lfloor \frac{n}{2} \right\rfloor + 1$. We may assume that $x, x_1$ have a common neighbor $y \in \overline{N}(v)$. Now, $v_2x_1y$ is a path of order 4 in $G[N(x)]$, and we are done by applying Case 1 or Case 2 with $x$ in place of $v$.

**Subcase 3.2.** $x_1x_2 \notin E(G)$. Let $\mathcal{N}(v) = \{x, x_1, x_2, \ldots, x_{\lceil n/2 \rceil}\}$. For $i = 1, 2$, we have

\begin{equation}
\deg(x_i, \overline{N}(v)) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1 - 2 \geq \left\lceil \frac{n}{2} \right\rceil - 2 = |\overline{N}(v)|.
\end{equation}

We must have equality to hold throughout, whence $n$ is odd, $\deg(x_1) = \deg(x_2) = \left\lfloor \frac{n}{2} \right\rfloor + 1$, and both $x_1, x_2$ are adjacent to all vertices of $\overline{N}(v)$. If $x$ has neighbors $y_1, y_2 \in \overline{N}(v)$, then we are done as in Subcase 3.1. If $x$ has exactly one neighbor $y \in \overline{N}(v)$, then we have

$$\deg(x, \mathcal{N}(v) \setminus \{x, x_1, x_2\}) \geq \left\lceil \frac{n}{2} \right\rceil + 1 - 4 \geq 1,$$

and we may assume that $xx_3 \in E(G)$. Then $x_1yx_2yx_3 + x$ is a copy of gem$_5$, and we have $\phi(G, \text{gem}_5) < e_n$ by Claim 3.4 with $z = v$. Otherwise, suppose that $x$ does not have a neighbor in $\overline{N}(v)$. We may apply the exact same argument as in Subcase 2.2 of Theorem 3.1 to deduce that $x_i$ is adjacent to all vertices of $\overline{N}(v)$ for all $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$, and $G[\overline{N}(v)]$ must contain a path of order 3, say $y_1y_2y_3$. Then $x_1y_1x_2y_3y_3y_2$ is a copy of gem$_5$, and by Claim 3.4 with $z = x_2$, we have $\phi(G, \text{gem}_5) < e_n$.

**Case 4.** The longest path in $G[\mathcal{N}(v)]$ has order 2. Note that this is indeed the remaining case, since $\deg(x, \mathcal{N}(v)) \geq 2m - 1 = 1$ for all $x \in \mathcal{N}(v)$. Moreover, $\mathcal{N}(v)$ induces a perfect matching in $G$. By a similar argument as in (13), we must have $n$ odd, and for every $x \in \mathcal{N}(v)$, we have $\deg(x) = \left\lceil \frac{n}{2} \right\rceil + 1$ and $x$ is adjacent to all vertices of $\overline{N}(v)$. Thus, we can find an edge $x_1x_2$ in $G[\mathcal{N}(v)]$ and a common neighbor $y \in \overline{N}(v)$ of $x_1, x_2$. Now, since $v_2x_1y$ is a path of order 3 in $G[\mathcal{N}(x_1)]$, we are done by applying Case 1, Case 2 or Case 3 with $x_1$ in place of $v$.

The induction step is complete, and this completes the proof of Theorem 3.3. ■

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