Discussiones Mathematicae
Graph Theory 38 (2018) 683–701
doi:10.7151/dmgt.2033

ON THE BETA-NUMBER OF FORESTS WITH ISOMORPHIC COMPONENTS

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Abstract

The beta-number, $\beta(G)$, of a graph G is defined to be either the smallest positive integer n for which there exists an injective function $f : V(G) \to \{0, 1, \ldots, n\}$ such that each $uv \in E(G)$ is labeled $|f(u) - f(v)|$ and the resulting set of edge labels is $\{c, c+1, \ldots, c+|E(G)|-1\}$ for some positive integer c or $+\infty$ if there exists no such integer n. If $c = 1$, then the resulting beta-number is called the strong beta-number of G and is denoted by $\beta_s(G)$. In this paper, we show that if G is a bipartite graph and m is odd, then $\beta(mG) \leq m\beta(G) + m - 1$. This leads us to conclude that $\beta(mG) = m|V(G)| - 1$ if G has the additional property that G is a graceful nontrivial tree. In addition to these, we examine the (strong) beta-number of forests whose components are isomorphic to either paths or stars.

Keywords: beta-number, strong beta-number, graceful labeling, Skolem sequence, hooked Skolem sequence.

2010 Mathematics Subject Classification: Primary: 05C78; Secondary: 05C05.

1. Introduction

Terms and notation not defined below follow that used in the book by Chartrand and Lesniak [3]. All graphs considered in this paper are finite and undirected without loops or multiple edges. The vertex set of a graph G is denoted by $V(G)$, while the edge set is denoted by $E(G)$. Let G_1 and G_2 be vertex-disjoint graphs. Then the *union* of G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. If G_1, $G_2, \ldots, G_m$ are pairwise vertex-disjoint graphs that are isomorphic to G, then we write mG for $G_1 \cup G_2 \cup \cdots \cup G_m$. As usual, we denote a path with n vertices and a star with $n+1$ vertices by P_n and S_n, respectively.

For integers a and b with $a \leq b$, we write $[a, b]$ for the set $\{x \in \mathbb{Z} : a \leq x \leq b\}$, where $\mathbb{Z}$ denotes the set of integers. On the other hand, if $a > b$, then we treat $[a, b]$ as the empty set. If such situations occur in particular formulas for a given vertex labeling, then we ignore the corresponding portions of formulas.

The type of graph labelings that have received the most attention over the years was introduced by Rosa [13] in 1967 who called them β-valuations. For a graph G of size q, an injective function $f : V(G) \to [0, q]$ is called a *β-valuation* if each $uv \in E(G)$ is labeled $|f(u) - f(v)|$ and the resulting edge labels are distinct. Such a valuation is now commonly known as a *graceful labeling* (the term was coined by Golomb [8] in 1972) and a graph with a graceful labeling is called *graceful*. The concept of α-valuations (a particular type of graceful labeling) was also introduced by Rosa [13] as a tool for decomposing the complete graph into isomorphic subgraphs. A graceful labeling f is called an *α-valuation* if there exists an integer λ so that $\min\{f(u), f(v)\} \leq \lambda < \max\{f(u), f(v)\}$ for each $uv \in E(G)$.

The *gracefulness*, $\text{grac}(G)$, of a graph G is the smallest positive integer n for which there exists an injective function $f : V(G) \to [0, n]$ such that each $uv \in E(G)$ is labeled $|f(u) - f(v)|$ and the resulting set of edge labels consists of distinct integers. If G is a graph of size q with $\text{grac}(G) = q$, then G is graceful. Thus, the gracefulness of a graph G is a measure of how close G is to being graceful. This definition first appeared in a paper by Golomb [8].

Motivated by the concept of the gracefulness of a graph, Ichishima *et al.* [9] introduced the (strong) beta-number of a graph. The *beta-number*, $\beta(G)$, of a graph G with q edges is either the smallest positive integer n for which there exists an injective function $f : V(G) \to [0,n]$ such that each $uv \in E(G)$ is labeled $|f(u) - f(v)|$ and the resulting set of edge labels is $[c, c+q-1]$ for some positive integer c or $+\infty$ if there exists no such integer n. If $c = 1$, then the resulting beta-number is called the *strong beta-number* of G and is denoted by $\beta_s(G)$. These parameters can be regarded as measures of how close a graph is to being graceful. In literature, there is another kind of parameter that mesures how a graph is close to being graceful, namely, the 'gracesize' (see [15] for the definition).

To present the new results contained in this paper, the following lemmas taken from [9] will prove to be useful.

Lemma 1.1. *For every graph G of order p and size q,*

$$\max\{p-1, q\} \leq \operatorname{grac}(G) \leq \beta(G) \leq \beta_s(G).$$

Lemma 1.2. *For every two positive integers m and n,*

$$\beta(S_m \cup S_n) = \beta_s(S_m \cup S_n) = \begin{cases} m+n+1 & \text{if } mn \text{ is even,} \\ m+n+2 & \text{if } mn \text{ is odd.} \end{cases}$$

In Section 2, we prove that if G is a bipartite graph and m is odd, then $\beta(mG) \leq m\beta(G) + m - 1$. This leads us to conclude that if T is a graceful nontrivial tree of order p and m is odd, then $\beta(mT) = mp - 1$. In Section 3, we compute the (strong) beta-number of forests whose all components are paths of order 2. In Section 4, we examine the (strong) beta-number of forests that all components are isomorphic to either paths or stars. In Section 5, we propose new conjectures on the (strong) beta-number of forests.

2. Main Results

According to the survey on graph labelings by Gallian [7], various classes of bipartite graphs have been proved to admit graceful labelings. Furthermore, a number of techniques to construct trees from smaller ones with graceful labelings have been shown to yield graceful labelings in the resulting trees. In this section, we thus compute the beta-number for an odd number of copies of an isomorphic tree with a graceful labeling. To do this, we start with the following result.

Theorem 2.1. *If G is a bipartite graph and m is odd, then*

$$\beta(mG) \leq m\beta(G) + m - 1.$$

Proof. Let G be a bipartite graph of size q. First, notice that the result is trivial when $\beta(G) = +\infty$. Thus, assume that $\beta(G) = n$ for some positive integer n. Then there exists an injective function $f : V(G) \to [0, n]$ such that

$$\{|f(u) - f(v)| : uv \in E(G)\} = [c, c + q - 1]$$

for some positive integer c. If G has the partite sets U and V, then let $E(G) = UV$, where the juxtaposition of two partite sets denotes the edges between those two sets. Now, define $H \cong mG$ to be the graph with

$$V(H) = \bigcup_{i=1}^{m} (U_i \cup V_i) \text{ and } E(H) = \bigcup_{i=1}^{m} U_i V_i,$$

where $x_i \in X_i$ $(i \in [1, m])$ if and only if $x \in X$ (X is one of the sets U or V).

Next, consider the vertex labeling $g : V(H) \to [0, mn + m - 1]$ such that

$$g(x_i) = \begin{cases} mf(x) + i - 1 & \text{if } x \in U \text{ and } i \in [1, m], \\ mf(x) + (i-1)/2 & \text{if } x \in V \text{ and } i \text{ is odd}, \\ mf(x) + (m - 1 + i)/2 & \text{if } x \in V \text{ and } i \text{ is even}. \end{cases}$$

This leads us to conclude that $\beta(mG) \leq mn+m-1$ when m is odd. To verify this, notice that for each $uv \in E(G)$, where $u \in U$ and $v \in V$ such that $f(u) > f(v)$, we have

$$|g(u) - g(v)| = \begin{cases} m(f(u) - f(v)) + (i-1)/2 & \text{if } i \text{ is odd}, \\ m(f(u) - f(v)) + (i - 1 - m)/2 & \text{if } i \text{ is even}. \end{cases}$$

Notice also that for each $uv \in E(G)$, where $u \in U$ and $v \in V$ such that $f(u) < f(v)$, we have

$$|g(u) - g(v)| = \begin{cases} m(f(v) - f(u)) - (i-1)/2 & \text{if } i \text{ is odd}, \\ m(f(v) - f(u)) - (i - 1 - m)/2 & \text{if } i \text{ is even}. \end{cases}$$

Consequently, we have

$$\{|g(u) - g(v)| : uv \in E(H)\} = \left[c', c' + mq - 1\right],$$

where $c' = (2mc - m + 1)/2$. Finally, to see that

$$\{g(x) : x \in V(H)\} \subseteq [0, mn + m - 1],$$

notice that for each $x \in V(G)$, we have

$$\{f(x) : x \in V(G)\} \subseteq [0, n] \text{ and } \bigcup_{i=1}^{m} \{g(x_i)\} \subseteq \bigcup_{i=1}^{m} \{mf(x) + i - 1\}.$$

■

Figueroa-Centeno *et al.* [6] introduced the $\otimes_h$-product of digraphs as follows. Let D be a digraph and let $\Gamma = \{F_1, F_2, \ldots, F_n\}$ be a family of digraphs such that $V(F_i) = V$ for each $i \in [1, n]$. Consider a function $h : E(D) \to \Gamma$. Then the *product* $D \otimes_h \Gamma$ is defined to be the digraph with vertex set $V(D) \times V$ and $((a, b), (c, d)) \in E(D \otimes_h \Gamma)$ if and only if $(a, c) \in E(D)$ and $(b, d) \in E(h(a, c))$. Using this concept together with the techniques described in [12], it is possible to provide an alternative proof of Theorem 2.1. However, this approach requires to introduce a considerable amount of machinery. Since the proof presented in this paper is clear and easy to understand, we omit the alternative approach mentioned above.

If G is a graceful bipartite graph of size q, then $\beta(G) = q$. This together with Theorem 2.1 gives us the following result.

Corollary 2.2. *If G is a graceful bipartite graph of size q, then*

$$\beta(mG) \leq mq + m - 1,$$

where m is odd.

For a graceful nontrivial tree, we have the following result, which shows that in this case, the bound provided in Corollary 2.2 is sharp.

Corollary 2.3. *If T is a graceful nontrivial tree of order p, then*

$$\beta(mT) = mp - 1,$$

where m is odd.

Proof. Let T be a graceful nontrivial tree of order p, and assume that m is odd. Since T has size $p - 1$, it follows from Corollary 2.2 that $\beta(mT) \leq mp - 1$. Further, the reverse inequality is easily obtained from Lemma 1.1. ■

The preceding result is the best possible in the sense that m cannot be even, since every star S_n is graceful (see [13]) and, by Lemma 1.2, we have $\beta(S_m \cup S_n) = m + n + 2$ when mn is odd. Thus, for every positive integer n, we have $\beta(2S_{2n-1}) = 4n$, but $2|V(S_{2n-1})| - 1 = 4n - 1$.

We end this section with the following result, which is a direct consequence of Lemma 1.1 and Theorem 2.1.

Corollary 2.4. *If F is a forest of order p such that $\beta(F) = p - 1$, then*

$$\beta(mF) = mp - 1,$$

where m is odd.

3. Connections with Skolem Type of Sequences

To present our results in this section, we utilize the notion of a Skolem sequence and its generalization, which has been widely used in the construction of combinatorial designs. The motivation for these sequences came from the area of balanced incomplete block designs, in particular, Steiner Triple Systems (see [16]). These systems are used for (among other things) interference resistant message code for missile guidance systems (see [4]).

A *Skolem sequence* of *order* m is a sequence $S = (s_1, s_2, \ldots, s_{2m})$ of $2m$ integers satisfying the conditions:

(S1) for every $k \in [1, m]$, there exist exactly two subscripts $i(k)$ and $j(k)$ such that $s_{i(k)} = s_{j(k)} = k$, and

(S2) if $s_{i(k)} = s_{j(k)} = k$ with $i(k) < j(k)$, then $j(k) - i(k) = k$.

The existence of Skolem sequences was settled in 1958 by Skolem [17] as we state in the following theorem.

Theorem 3.1. *A Skolem sequence of order m exists if and only if $m \equiv 0$ or $1 \pmod 4$.*

For $m \equiv 2$ or $3 \pmod 4$, the natural alternative is a hooked Skolem sequence. A *hooked Skolem sequence* of *order* m is a sequence $S = (s_1, s_2, \ldots, s_{2m+1})$ of $2m + 1$ integers satisfying conditions (S1) and (S2) of the definition of a Skolem sequence and

(S3) $s_{2m} = 0$.

The existence of hooked Skolem sequences was settled in 1961 by O'Keefe [10] as we state in the following theorem.

Theorem 3.2. *A hooked Skolem sequence of order m exists if and only if $m \equiv 2$ or $3 \pmod 4$.*

It is worth to mention that López and Muntaner-Batle [11] recently have found strong relations between super edge-magic labelings of graphs introduced independently in [1] and [5], and Skolem type of sequences. Their results can be used to improve the previously known bounds for the number of Skolem type of sequences. Hence, graph labelings and Skolem type of sequences are tightly related.

With the aid of Theorems 3.1 and 3.2, we are now able to provide the following result.

Theorem 3.3. *For every positive integer m,*

$$\beta_s(mP_2) = \begin{cases} 2m - 1 & \text{if } m \equiv 0 \text{ or } 1 \pmod 4, \\ 2m & \text{if } m \equiv 2 \text{ or } 3 \pmod 4. \end{cases}$$

Proof. Let $F \cong mP_2$ be the forest with

$$V(F) = \{x_k : k \in [1,m]\} \cup \{y_k : k \in [1,m]\} \text{ and } E(F) = \{x_k y_k : k \in [1,m]\}.$$

Since $|V(F)| = 2m$ and $|E(F)| = m$, it follows from Lemma 1.1 that $\beta_s(F) \geq 2m-1$ for every positive integer m.

First, assume that $m \equiv 0$ or $1 \pmod 4$. By Theorem 3.1, there exists a Skolem sequence of order m. That is, for each $k \in [1,m]$, there exist exactly two subscripts $i(k)$ and $j(k)$ satisfying all the conditions of the definition of a Skolem sequence. Thus, the vertex labeling $f : V(F) \to [0, 2m-1]$ such that

$$f(x_k) = i(k) - 1 \text{ and } f(y_k) = j(k) - 1 \ (k \in [1,m])$$

is a bijective function and satisfies that

$$\begin{aligned}&\{|f(x_k) - f(y_k)| : k \in [1,m]\}\\ &= \left\{|i(k) - j(k)| : s_{i(k)} = s_{j(k)} = k \text{ and } k \in [1,m]\right\} = [1, |E(F)|],\end{aligned}$$

since $i(k)$, $j(k) \in [1,2m]$ and $i(k) < j(k)$ for all $k \in [1,m]$, and $|E(F)| = m$. Therefore, $\beta_s(F) \leq 2m-1$, which leads us to conclude that $\beta_s(F) = 2m-1$ when $m \equiv 0$ or $1 \pmod 4$.

For $m \equiv 2$ or $3 \pmod 4$, suppose, to the contrary, that $\beta_s(F) = 2m-1$. Since $|V(F)| = 2m$, it follows that there exists a bijective function $g : V(F) \to [0, 2m-1]$ such that $\{|g(x_k) - g(y_k)| : k \in [1,m]\} = [1,m]$. This produces a Skolem sequence of order m by letting

$$i(k) = g(x_k) + 1 \text{ and } j(k) = g(y_k) + 1 \ (k \in [1,m]),$$

which is impossible by Theorem 3.1. Thus, $\beta_s(F) \geq 2m$ when $m \equiv 2$ or $3 \pmod 4$. On the other hand, it follows from Theorem 3.2 that there exists a hooked Skolem sequence of order m. That is, for every $k \in [1,m]$, there exist exactly two subscripts $i(k)$ and $j(k)$ satisfying all the conditions of the definition of a hooked Skolem sequence. Thus, the vertex labeling $h : V(F) \to [0, 2m]$ such that

$$h(x_k) = i(k) - 1 \text{ and } h(y_k) = j(k) - 1 \ (k \in [1,m])$$

is an injective function and satisfies that

$$\begin{aligned}&\{|h(x_k) - h(y_k)| : k \in [1,m]\}\\ &= \left\{|i(k) - j(k)| : s_{i(k)} = s_{j(k)} = k \text{ and } k \in [1,m]\right\} = [1, |E(F)|],\end{aligned}$$

since $i(k)$, $j(k) \in [1, 2m+1]$ and $i(k) < j(k)$ for all $k \in [1,m]$, and $|E(F)| = m$. Therefore, $\beta_s(F) \leq 2m$, which leads us to conclude that $\beta_s(F) = 2m$ when $m \equiv 2$ or $3 \pmod 4$. ■

With the aid of Corollary 2.3 and Theorem 3.3, we are now able to present the following result.

Corollary 3.4. *For every positive integer* m*,*

$$\beta(mP_2) = \begin{cases} 2m-1 & \text{if } m \not\equiv 2 \pmod{4}, \\ 2m & \text{if } m \equiv 2 \pmod{4}. \end{cases}$$

Proof. Let $F \cong mP_2$ be the forest as in the proof of Theorem 3.3. Since P_2 is clearly graceful, it follows from Corollary 2.3 that $\beta(F) = 2m-1$ for $m \equiv 1$ or 3 (mod 4). It also follows from Lemma 1.1 and Theorem 3.3 that $\beta(F) = 2m-1$ for $m \equiv 0 \pmod{4}$ and $\beta(F) \leq 2m$ for $m \equiv 2 \pmod{4}$.

Next, assume that $m \equiv 2$ (mod 4), and suppose, to the contrary, that $\beta(F) = 2m-1$. Since $|V(F)| = 2m$, it certainly follows that there exists a bijective function $f : V(F) \to [0, 2m-1]$ such that

$$\{|f(x_i) - f(y_i)| : i \in [1, m]\} = [c, c+m-1]$$

for some positive integer c. Thus, the sum of induced edge labels of F is

$$\begin{aligned} \sum_{i=1}^{m} |f(x_i) - f(y_i)| &\equiv \sum_{i=1}^{m} (f(x_i) - f(y_i)) \pmod{2} \\ &\equiv \sum_{i=1}^{m} (f(x_i) + f(y_i)) \pmod{2} \\ &\equiv m(2m-1) \equiv 0 \pmod{2}. \end{aligned}$$

That is, the sum of induced edge labels of F is even. However, the sum of the edge labels is

$$\sum_{i=0}^{m-1} (i+c) = m(m+2c-1)/2,$$

which is odd for any positive integer c. This produces a contradiction and thus $\beta(F) \geq 2m$ for $m \equiv 2 \pmod{4}$, which leads us to conclude that $\beta(F) = 2m$ for $m \equiv 2 \pmod{4}$. ■

4. Further Results on Forests with Isomorphic Components

As we have already mentioned in Section 2, we cannot extend Corollary 2.3 to the case that the number of components is even. However, in this section, we are able to determine the (strong) beta-number for some classes of forests with even number of isomorphic components.

Before presenting our next result, we briefly discuss some previously known results on α-valuations and graceful labelings of paths. We first mention the result obtained by Rosa [14].

Theorem 4.1. *For any positive integer n and any vertex v of the path P_n, there exists a graceful labeling f of P_n with $f(v) = 0$. Moreover, an α-valuation of P_n with $f(v) = 0$ exists if and only if v is not the central vertex of P_5.*

When concerning a graceful labeling of a path, Cattell [2] has shown that one has almost complete freedom to choose a particular label i for any given vertex v. In particular, he proved the following result.

Theorem 4.2. *Let v be any vertex of the nontrivial path P_n. Then there exists a graceful labeling f of P_n with $f(v) = i$ for any $i \in [1, n-1]$ unless $n \equiv 3 \pmod 4$ or $n \equiv 1 \pmod{12}$; v is in the smaller of the two partite sets of vertices, and $i = (n-1)/2$.*

As an immediate consequence of Theorem 4.2, we obtain the following result.

Corollary 4.3. *Let v be an end-vertex of the path P_n $(n \geq 3)$. Then there exists a graceful labeling f of P_n with $f(v) \leq \lfloor (n-1)/2 \rfloor - 1$.*

With the aid of Theorem 4.1 and Corollary 4.3, we are able to determine the (strong) beta-number of the forest whose components are two copies of nontrivial paths as our next result indicates. Our proof uses the concept of induced subgraph. Let S be a nonempty set of vertices of a graph G. The *subgraph induced by S* is the maximal subgraph of G with vertex set S, and is denoted by $\langle S \rangle$, that is, contains precisely those edges of G joining two vertices in S. A subgraph H of a graph G is an *induced subgraph* if $H \cong \langle S \rangle$ for some nonempty set S of vertices of G.

Theorem 4.4. *For every integer $n \geq 2$,*

$$\beta(2P_n) = \beta_s(2P_n) = \begin{cases} 2n & \text{if } n = 2, \\ 2n-1 & \text{if } n \geq 3. \end{cases}$$

Proof. For $n = 2$, the result easily follows from Theorem 3.3 and Corollary 3.4. It also follows from Lemma 1.2 that $\beta(S_m \cup S_n) = \beta_s(S_m \cup S_n) = m + n + 1$ when mn is even. Combining this with the fact that $2S_2 \cong 2P_3$, we have the result for $n = 3$. Let $F \cong 2P_n$ be the forest with

$$V(F) = \{x_i : i \in [1, n]\} \cup \{y_i : i \in [1, n]\}$$

and

$$E(F) = \{x_i x_{i+1} : i \in [1, n-1]\} \cup \{y_i y_{i+1} : i \in [1, n-1]\}.$$

In light of Lemma 1.1 and Table 2, it suffices to show that $\beta_s(F) \leq 2n-1$ when $n \geq 12$. Thereby, we first consider the vertex labels of the induced subgraphs of F. Let F_1 be the induced subgraph of F with $V(F_1) = \{x_i : i \in [1,n]\}$. Then $F_1 \cong P_n$, and Theorem 4.1 guarantees the existence of an α-valuation f_1 of F_1 with $f_1(x_1) = 0$. If we define the induced subgraph F_2 of F with $V(F_2) = \{y_i : i \in [1, \lceil n/2 \rceil + 1]\}$, then $F_2 \cong P_{\lceil n/2 \rceil + 1}$, and Corollary 4.3 guarantees the existence of a graceful labeling f_2 of F_2 with $f_2\left(y_{\lceil n/2 \rceil + 1}\right) = \lfloor n/4 \rfloor - 1$. Let F_3 be the induced subgraph of F with $V(F_3) = \{y_i : i \in [\lceil n/2 \rceil + 2, n]\}$. Then $F_3 \cong P_{\lfloor n/2 \rfloor - 1}$, and Theorem 4.1 guarantees the existence of an α-valuation f_3 of F_3 with $f_3\left(y_{\lceil n/2 \rceil + 2}\right) = 0$.

We now consider the vertex labeling $g : V(F) \to [0, 2n-1]$ such that

$$g(x_j) = \begin{cases} 1 + f_1(x_j) & \text{if } j = 2i-1 \text{ and } i \in [1, \lceil n/2 \rceil], \\ n + f_1(x_j) & \text{if } j = 2i \text{ and } i \in [1, \lfloor n/2 \rfloor], \end{cases}$$

and

$$g(y_j) = \begin{cases} n - \lfloor n/4 \rfloor + f_2(y_j) & \text{if } j \in [1, \lceil n/2 \rceil + 1], \\ 0 & \text{if } j = \lceil n/2 \rceil + 2, \\ n - 1 - f_3(y_j) & \text{if } j = \lceil n/2 \rceil + 1 + 2i \text{ and } i \in [1, \lceil (n-1)/4 \rceil - 1], \\ \lceil 3n/2 \rceil - f_3(y_j) & \text{if } j = \lceil n/2 \rceil + 2 + 2i \text{ and } i \in [1, \lfloor n/4 \rfloor - 1]. \end{cases}$$

Then g is a bijective function. To show this, we first compute the vertex labels of F_1. Since f_1 is an α-valuation of F_1 with $f_1(x_1) = 0$, it follows that

$$\{f_1(x_{2i-1}) : i \in [1, \lceil n/2 \rceil]\} = [0, \lceil n/2 \rceil - 1]$$

and

$$\{f_1(x_{2i}) : i \in [1, \lfloor n/2 \rfloor]\} = [\lceil n/2 \rceil, n-1].$$

This in turn implies that

$$\{g(x_{2i-1}) : i \in [1, \lceil n/2 \rceil]\} = [1, \lceil n/2 \rceil]$$

and

$$\{g(x_{2i}) : i \in [1, \lfloor n/2 \rfloor]\} = [n + \lceil n/2 \rceil, 2n-1] = [\lceil 3n/2 \rceil, 2n-1],$$

which implies that

$$\{g(v) : v \in V(F_1)\} = [1, \lceil n/2 \rceil] \cup [\lceil 3n/2 \rceil, 2n-1].$$

We next compute the vertex labels of F_2. Since f_2 is a graceful labeling of F_2, it follows that

$$\{f_2(y_i) : i \in [1, \lceil n/2 \rceil + 1]\} = [0, \lceil n/2 \rceil],$$

which implies that

$$\{g(y_i) : i \in [1, \lceil n/2 \rceil + 1]\} = [n - \lfloor n/4 \rfloor, n - \lfloor n/4 \rfloor + \lceil n/2 \rceil],$$

that is,

$$\{g(v) : v \in V(F_2)\} = [n - \lfloor n/4 \rfloor, n - \lfloor n/4 \rfloor + \lceil n/2 \rceil].$$

It remains to compute the vertex labels of F_3. Since f_3 is an α-valuation of F_3 with $f_3 \left(y_{\lceil n/2 \rceil + 2}\right) = 0$, it follows that

$$\left\{f_3 \left(y_{\lceil n/2 \rceil + 1 + 2i}\right) : i \in [1, \lceil (n-1)/4 \rceil - 1]\right\} = [\lfloor n/4 \rfloor, n - \lceil n/2 \rceil - 2]$$

and

$$\left\{f_3 \left(y_{\lceil n/2 \rceil + 2 + 2i}\right) : i \in [1, \lfloor n/4 \rfloor - 1]\right\} = [1, \lfloor n/4 \rfloor - 1].$$

This in turn implies that

$$\begin{aligned}\left\{g \left(y_{\lceil n/2 \rceil + 1 + 2i}\right) : i \in [1, \lceil (n-1)/4 \rceil - 1]\right\} &= [n - \lfloor n/2 \rfloor + 1, n - \lfloor n/4 \rfloor - 1] \\ &= [\lceil n/2 \rceil + 1, n - \lfloor n/4 \rfloor - 1]\end{aligned}$$

and

$$\begin{aligned}\left\{g \left(y_{\lceil n/2 \rceil + 2 + 2i}\right) : i \in [1, \lfloor n/4 \rfloor - 1]\right\} &= [\lceil 3n/2 \rceil - \lfloor n/4 \rfloor + 1, \lceil 3n/2 \rceil - 1] \\ &= [n - \lfloor n/4 \rfloor + \lceil n/2 \rceil + 1, \lceil 3n/2 \rceil - 1].\end{aligned}$$

This together with $g \left(y_{\lceil n/2 \rceil + 2}\right) = 0$ implies that

$$\begin{aligned}\{g(v) : v \in V(F_3)\} &= \{0\} \cup [\lceil n/2 \rceil + 1, n - \lfloor n/4 \rfloor - 1] \\ &\quad \cup [n - \lfloor n/4 \rfloor + \lceil n/2 \rceil + 1, \lceil 3n/2 \rceil - 1].\end{aligned}$$

Thus, we have

$$\{g(v) : v \in V(F)\} = [0, 2n - 1],$$

which shows that g is a bijective function.

We finally compute the induced edge labels by ascending order. Since f_2 is a graceful labeling of F_2, it follows that

$$\{|f_2(u) - f_2(v)| : uv \in E(F_2)\} = [1, \lceil n/2 \rceil],$$

which implies that

$$\{|g(u) - g(v)| : uv \in E(F_2)\} = [1, \lceil n/2 \rceil].$$

Next, notice that $f_3 \left(y_{\lceil n/2 \rceil + 3}\right) = n - \lceil n/2 \rceil - 2$. This implies that

$$g \left(y_{\lceil n/2 \rceil + 3}\right) = n - 1 - f_3 \left(y_{\lceil n/2 \rceil + 3}\right) = \lceil n/2 \rceil + 1.$$

This together with $g\left(y_{\lceil n/2\rceil+2}\right)=0$ implies that

$$\left|g\left(y_{\lceil n/2\rceil+3}\right)-g\left(y_{\lceil n/2\rceil+2}\right)\right|=\lceil n/2\rceil+1.$$

For all $uv\in E\left(F_3\right)-\left\{y_{\lceil n/2\rceil+2}y_{\lceil n/2\rceil+3}\right\}$, where

$$u\in\left\{y_{\lceil n/2\rceil+2+2i}:i\in\left[1,\lfloor n/4\rfloor-1\right]\right\}$$

and

$$v\in\left\{y_{\lceil n/2\rceil+1+2i}:i\in\left[1,\lceil(n-1)/4\rceil-1\right]\right\},$$

we know from the vertex labels computed in the above that $f_3\left(u\right)<f_3\left(v\right)$. This implies that $g\left(u\right)>g\left(v\right)$ and

$$\begin{aligned}\left|g\left(u\right)-g\left(v\right)\right| &= g\left(u\right)-g\left(v\right)\\ &= \lceil 3n/2\rceil-n+1-\left(f_3\left(u\right)-f_3\left(v\right)\right)\end{aligned}$$

for all $uv\in E\left(F_3\right)-\left\{y_{\lceil n/2\rceil+2}y_{\lceil n/2\rceil+3}\right\}$, where

$$u\in\left\{y_{\lceil n/2\rceil+2+2i}:i\in\left[1,\lfloor n/4\rfloor-1\right]\right\}$$

and

$$v\in\left\{y_{\lceil n/2\rceil+1+2i}:i\in\left[1,\lceil(n-1)/4\rceil-1\right]\right\}.$$

It is now easy to see that

$$\left\{f_3\left(v\right)-f_3\left(u\right):uv\in E\left(F_3\right)-\left\{y_{\lceil n/2\rceil+2}y_{\lceil n/2\rceil+3}\right\}\right\}=\left[1,\lfloor n/2\rfloor-3\right],$$

since f_3 is an α-valuation of F_3 with $f_3\left(y_{\lceil n/2\rceil+2}\right)=0$. From this and the last equation on the induced edge labels $\left|g\left(u\right)-g\left(v\right)\right|$, where $uv\in E\left(F_3\right)-\left\{y_{\lceil n/2\rceil+2}y_{\lceil n/2\rceil+3}\right\}$, we obtain

$$\left\{\left|g\left(u\right)-g\left(v\right)\right|:uv\in E\left(F_3\right)-\left\{y_{\lceil n/2\rceil+2}y_{\lceil n/2\rceil+3}\right\}\right\}=\left[\lceil n/2\rceil+2,n-2\right].$$

Recall that $f_2\left(y_{\lceil n/2\rceil+1}\right)=\lfloor n/4\rfloor-1$, which implies that

$$g\left(y_{\lceil n/2\rceil+1}\right)=n-\lfloor n/4\rfloor+f_2\left(y_{\lceil n/2\rceil+1}\right)=n-1.$$

This together with $g\left(y_{\lceil n/2\rceil+2}\right)=0$ implies that

$$\left|g\left(y_{\lceil n/2\rceil+1}\right)-g\left(y_{\lceil n/2\rceil+2}\right)\right|=n-1.$$

It remains to compute the induced edge labels of F_1. As we can see from the vertex labels computed above, we have $f_1\left(u\right)>f_1\left(v\right)$ for all $uv\in E\left(F_1\right)$, where $u\in$

$\{x_{2i} : i \in [1, \lceil n/2 \rceil]\}$ and $v \in \{x_{2i-1} : i \in [1, \lfloor n/2 \rfloor]\}$. This implies that $g(u) > g(v)$ and

$$\begin{aligned} |g(u) - g(v)| &= g(u) - g(v) \\ &= n - 1 + (f_1(u) - f_1(v)) \end{aligned}$$

for all $uv \in E(F_1)$, where $u \in \{x_{2i} : i \in [1, \lceil n/2 \rceil]\}$ and $v \in \{x_{2i-1} : i \in [1, \lfloor n/2 \rfloor]\}$. By definition,

$$\{f_1(u) - f_1(v) : uv \in E(F_1)\} = [1, n-1],$$

since f_1 is an α-valuation of F_1 with $f_1(x_1) = 0$. From this and the last equation on the induced edge labels $|g(u) - g(v)|$, where $uv \in E(F_1)$, we obtain

$$\{|g(u) - g(v)| : uv \in E(F_1)\} = [n, 2n-2].$$

It is now immediate that

$$\{|g(u) - g(v)| : uv \in E(F)\} = [1, |E(F)|].$$

Therefore, we conclude that $\beta_s(F) \leq 2n - 1$ for all integers $n \geq 12$, completing the proof. ∎

For each integer n with $4 \leq n \leq 11$, we illustrate the construction described in Theorem 4.4 using Table 1, which contains the vertex labels of F_1, F_2 and F_3. The resulting vertex labeling of F appears in Table 2.

Table 1. The vertex labeling of F_1, F_2 and F_3 for small values of n.

n	F_1	F_2	F_3
4	$(0, 3, 1, 2)$	$(1, 2, 0)$	(0)
5	$(0, 4, 1, 3, 2)$	$(2, 1, 3, 0)$	(0)
6	$(0, 5, 1, 4, 2, 3)$	$(2, 1, 3, 0)$	$(0, 1)$
7	$(0, 6, 1, 5, 2, 4, 3)$	$(2, 3, 1, 4, 0)$	$(0, 1)$
8	$(0, 7, 1, 6, 2, 5, 3, 4)$	$(3, 0, 4, 2, 1)$	$(0, 2, 1)$
9	$(0, 8, 1, 7, 2, 6, 3, 5, 4)$	$(4, 2, 3, 0, 5, 1)$	$(0, 2, 1)$
10	$(0, 9, 1, 8, 2, 7, 3, 6, 4, 5)$	$(4, 2, 3, 0, 5, 1)$	$(0, 3, 1, 2)$
11	$(0, 10, 1, 9, 2, 8, 3, 7, 4, 6, 5)$	$(2, 5, 3, 4, 0, 6, 1)$	$(0, 3, 1, 2)$

It is well known that every nontrivial path P_n is graceful. This together with Corollary 2.3 implies that $\beta(mP_n) = mn - 1$ when m is odd and $n \geq 2$. It also follows from Corollary 2.4 and Theorem 4.4 that $\beta(mP_n) = mn - 1$ when $m \equiv 2 \pmod 4$ and $n \geq 3$. Thus, we have the following result.

Table 2. The vertex labeling of F for small values of n.

n	F
4	$(1,7,2,6),(4,5,3,0)$
5	$(1,9,2,8,3),(6,5,7,4,0)$
6	$(1,11,2,10,3,9),(7,6,8,5,0,4)$
7	$(1,13,2,12,3,11,4),(8,9,7,10,6,0,5)$
8	$(1,15,2,14,3,13,4,12),(9,6,10,8,7,0,5,11)$
9	$(1,17,2,16,3,15,4,14,5),(11,9,10,7,12,8,0,6,13)$
10	$(1,19,2,18,3,17,4,16,5,15),(12,10,11,8,13,9,0,6,14,7)$
11	$(1,21,2,20,3,19,4,18,5,17,6),(11,14,12,13,9,15,10,0,7,16,8)$

Corollary 4.5. *For every two positive integers m and n such that $m \not\equiv 0 \pmod 4$ and $n \geq 3$,*

$$\beta(mP_n) = mn - 1.$$

With the aid of Corollary 4.5, we have the following result.

Corollary 4.6. *For every positive integer m,*

$$\beta(mP_3) = 3m - 1.$$

Proof. In light of Lemma 1.1 and the preceding corollary, it suffices to verify that $\beta(mP_3) \leq 3m-1$ when $m \equiv 0 \pmod 4$. To do so, let $F \cong mP_3$ be the forest with

$$V(F) = \{x_i : i \in [1,m]\} \cup \{y_i : i \in [1,m]\} \cup \{z_i : i \in [1,m]\}$$

and

$$E(F) = \{x_i y_i : i \in [1,m]\} \cup \{y_i z_i : i \in [1,m]\},$$

and define the vertex labeling $f : V(F) \to [0, 3m-1]$ such that $f(x_1) = m$, $f(y_1) = 0$, $f(z_1) = 3m/2$ and

$$f(w) = \begin{cases} i-1 & \text{if } w = x_{2i-1} \text{ and } i \in [2, m/2], \\ m/2 - 1 + i & \text{if } w = x_{2i} \text{ and } i \in [1, m/2], \\ 2m - 1 + i & \text{if } w = y_i \text{ and } i \in [2, m], \\ m - 1 + i & \text{if } w = z_{2i-1} \text{ and } i \in [2, m/2], \\ 3m/2 + i & \text{if } w = z_{2i} \text{ and } i \in [1, m/2]. \end{cases}$$

It remains to observe that

$$\{f(v) : v \in V(F)\} = [0, 3m-1]$$

and

$$\{|f(u) - f(v)| : uv \in E(F)\} = [m/2, m/2 + |E(F)| - 1].$$

Thus, $\beta(mP_3) \leq 3m - 1$ when $m \equiv 0 \pmod 4$. ∎

We turn our attention to the (strong) beta-number of forests whose components are all stars.

Theorem 4.7. *For every positive integer n,*

$$\beta(4S_n) = \beta_s(4S_n) = 4n + 3.$$

Proof. In light of Lemma 1.1, it suffices to show that $\beta_s(4S_n) \leq 4n+3$ for every positive integer n. To do this, define the forest $F \cong 4S_n$ with

$$V(F) = \{x_i : i \in [1,4]\} \cup \left\{y_i^j : i \in [1,4] \text{ and } j \in [1,n]\right\}$$

and

$$E(F) = \left\{x_i y_i^j : i \in [1,4] \text{ and } j \in [1,n]\right\},$$

and consider the cases according to the parity of the integer n.

First, let $n = 2k-1$, where k is a positive integer, and define the vertex labeling $f : V(F) \to [0, 8k-1]$ such that $f(x_1) = 0$, $f(x_2) = 8k-3$, $f(x_3) = 8k-1$, $f(x_4) = 8k-2$ and

$$f(y_1^j) = \begin{cases} k & \text{if } j = 1, \\ 2k & \text{if } j = 2, \\ 3k-3+j & \text{if } j \in [3,k], \\ 3k+j & \text{if } j \in [k+1, 2k-1], \end{cases}$$

$$f(y_2^j) = \begin{cases} j & \text{if } j \in [1,k-1], \\ 4k-2 & \text{if } j = k, \\ 6k-3+j & \text{if } j \in [k+1, 2k-1], \end{cases}$$

$$f(y_3^j) = \begin{cases} 2k+j & \text{if } j \in [1,k-1], \\ 4k-1 & \text{if } j = k, \\ 4k-1+j & \text{if } j \in [k+1, 2k-1], \end{cases}$$

$$f(y_4^j) = \begin{cases} k+j & \text{if } j \in [1,k-1], \\ 4k & \text{if } j = k, \\ 5k-2+j & \text{if } j \in [k+1, 2k-1]. \end{cases}$$

Notice that

$$\begin{aligned}
\left\{f\left(x_i\right) : i \in [1,4]\right\} &= \{0\} \cup [8k-3, 8k-1]\,,\\
\left\{f\left(y_1^j\right) : j \in [1,2k-1]\right\} &= \{k, 2k\} \cup [3k, 4k-3] \cup [4k+1, 5k-1]\,,\\
\left\{f\left(y_2^j\right) : j \in [1,2k-1]\right\} &= [1, k-1] \cup \{4k-2\} \cup [7k-2, 8k-4]\,,\\
\left\{f\left(y_3^j\right) : j \in [1,2k-1]\right\} &= [2k+1, 3k-1] \cup \{4k-1\} \cup [5k, 6k-2]\,,\\
\left\{f\left(y_4^j\right) : j \in [1,2k-1]\right\} &= [k+1, 2k-1] \cup \{4k\} \cup [6k-1, 7k-3]\,.
\end{aligned}$$

This implies that

$$\{f(v) : v \in V(F)\} = [0, 8k-1]\,,$$

which means that f is a bijective function. Notice also that

$$\begin{aligned}
\left\{\left|f(x_1) - f\left(y_1^j\right)\right| : j \in [1,2k-1]\right\} &= \{k, 2k\} \cup [3k, 4k-3] \cup [4k+1, 5k-1]\,,\\
\left\{\left|f(x_2) - f\left(y_2^j\right)\right| : j \in [1,2k-1]\right\} &= [1, k-1] \cup \{4k-1\} \cup [7k-2, 8k-4]\,,\\
\left\{\left|f(x_3) - f\left(y_3^j\right)\right| : j \in [1,2k-1]\right\} &= [2k+1, 3k-1] \cup \{4k\} \cup [5k, 6k-2]\,,\\
\left\{\left|f(x_4) - f\left(y_4^j\right)\right| : j \in [1,2k-1]\right\} &= [k+1, 2k-1] \cup \{4k-2\}\\
&\quad \cup [6k-1, 7k-3]\,.
\end{aligned}$$

It is now immediate that

$$\{|f(u) - f(v)| : uv \in E(F)\} = [1, |E(F)|]\,.$$

Consequently, $\beta_s(F) \le 8k-1$ for every positive integer k.

Next, let $n = 2k$, where k is a positive integer, and define the vertex labeling $f : V(F) \to [0, 8k+3]$ such that $f(x_1) = 0$, $f(x_2) = 8k+1$, $f(x_3) = 8k+2$, $f(x_4) = 8k+3$ and

$$\begin{aligned}
f\left(y_1^j\right) &= \begin{cases} k+1 & \text{if } j = 1,\\ 2k+2 & \text{if } j = 2,\\ 3k+j & \text{if } j \in [3, 2k]\,, \end{cases}\\
f\left(y_2^j\right) &= \begin{cases} j & \text{if } j \in [1, k]\,,\\ 6k+j & \text{if } j \in [k+1, 2k]\,, \end{cases}\\
f\left(y_3^j\right) &= \begin{cases} k+1+j & \text{if } j \in [1, k]\,,\\ 5k+j & \text{if } j \in [k+1, 2k]\,, \end{cases}\\
f\left(y_4^j\right) &= \begin{cases} 2k+2+j & \text{if } j \in [1, k]\,,\\ 4k+j & \text{if } j \in [k+1, 2k]\,. \end{cases}
\end{aligned}$$

Notice that

$$\begin{aligned}
\left\{f\left(x_i\right) : i \in [1,4]\right\} &= \{0\} \cup [8k+1, 8k+3]\,,\\
\left\{f\left(y_1^j\right) : j \in [1,2k]\right\} &= \{k+1, 2k+2\} \cup [3k+3, 5k]\,,\\
\left\{f\left(y_2^j\right) : j \in [1,2k]\right\} &= [1,k] \cup [7k+1, 8k]\,,\\
\left\{f\left(y_3^j\right) : j \in [1,2k]\right\} &= [k+2, 2k+1] \cup [6k+1, 7k]\,,\\
\left\{f\left(y_4^j\right) : j \in [1,2k]\right\} &= [2k+3, 3k+2] \cup [5k+1, 6k]\,.
\end{aligned}$$

This implies that

$$\{f(v) : v \in V(F)\} = [0, 8k+3]\,,$$

which means that f is a bijective function. Notice also that

$$\begin{aligned}
\left\{\left|f(x_1) - f\left(y_1^j\right)\right| : j \in [1,2k]\right\} &= \{k+1, 2k+2\} \cup [3k+3, 5k]\,,\\
\left\{\left|f(x_2) - f\left(y_2^j\right)\right| : j \in [1,2k]\right\} &= [1,k] \cup [7k+1, 8k]\,,\\
\left\{\left|f(x_3) - f\left(y_3^j\right)\right| : j \in [1,2k]\right\} &= [k+2, 2k+1] \cup [6k+1, 7k]\,,\\
\left\{\left|f(x_4) - f\left(y_4^j\right)\right| : j \in [1,2k]\right\} &= [2k+3, 3k+2] \cup [5k+1, 6k]\,.
\end{aligned}$$

It is now immediate that

$$\{|f(u) - f(v)| : uv \in E(F)\} = [1, |E(F)|]\,.$$

Consequently, $\beta_s(F) \leq 8k+3$ for every positive integer k.

Therefore, it follows from the above cases that $\beta_s(F) \leq 4n+3$ for every positive integer n, completing the proof. ■

Now, we obtain the following result from Corollary 2.4 and Theorem 4.7.

Corollary 4.8. *For every two positive integers m and n such that $m \equiv 4 \pmod 8$,*

$$\beta(mS_n) = mn + m - 1.$$

Every star S_n is clearly graceful. This together with Corollary 2.3 implies that $\beta(mS_n) = mn + m - 1$ when m is odd and $n \geq 1$. It is also known from Lemma 1.2 that $\beta(S_m \cup S_n) = m + n + 1$ when mn is even. This implies that $\beta(2S_n) = 2n + 1$ when n is even. It follows from this and Corollary 2.4 that $\beta(mS_n) = mn + m - 1$ when $m \equiv 2 \pmod 4$ and n is even. Combining all these, we have the following result.

Corollary 4.9. *For every two positive integers* m *and* n *such that* m *is odd, or* $m \equiv 2 \pmod 4$ *and* n *is even,*

$$\beta(mS_n) = mn + m - 1.$$

5. Conclusions

Ichishima *et al.* [9] have provided a constructive proof that the strong beta-number of forests is finite. This proof gives a crude upper bound for $\beta_s(F)$ when F is a forest. However, we believe that the actual value of $\beta_s(F)$ is always smaller than the one provided by the proof. The previous explorations on the (strong) beta-numbers of forests in [9] and the results included in this paper lead us to propose the following two conjectures. We first state the weaker of the two conjectures.

Conjecture 5.1. *If* F *is a forest of order* p*, then* $\beta(F)$ *is either* $p-1$ *or* p*.*

We actually believe that more is true.

Conjecture 5.2. *If* F *is a forest of order* p*, then* $\beta_s(F)$ *is either* $p-1$ *or* p*.*

Of course, if Conjecture 5.2 is true, so is Conjecture 5.1 by Lemma 1.1. Indeed, the truth of Conjecture 5.2 implies the truth of the following conjecture by Lemma 1.1.

Conjecture 5.3. *If* F *is a forest of order* p*, then* $\operatorname{grac}(F)$ *is either* $p-1$ *or* p*.*

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Received 27 July 2016
Revised 30 January 2017
Accepted 30 January 2017