GENERALIZED HAMMING GRAPHS:
SOME NEW RESULTS

AMARI BEDRANE

AND

BERRACHEDI ABDELHAFID

LIFORCE Laboratory
RO Department, USTHB, Algiers, Algeria

e-mail: bedrane.amari@gmail.com

Abstract

A projection of a vertex $x$ of a graph $G$ over a subset $S$ of vertices is a vertex of $S$ at minimal distance from $x$. The study of projections over quasi-intervals gives rise to a new characterization of quasi-median graphs.

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1. Introduction

All graphs considered in this paper are finite, undirected, without loops or multiple edges. We denote by $d(u, v)$ the length of a shortest $(u, v)$-path in the graph $G$. The interval $I(u, v)$ is the set of vertices of $G$ lying on shortest $(u, v)$-paths: $I(u, v) = \{x : d(u, x) + d(x, v) = d(u, v)\}$. The quasi-interval $I^*(u, v)$ is the set of vertices $x$ such that any shortest $(u, x)$-path and shortest $(x, v)$-path have only $x$ as common vertex. That is, $I^*(u, v) = \{x : I(u, x) \cap I(x, v) = \{x\}\}$. This notion was introduced by Nebeský [10]. The projection (introduced by Berrachedi [4]) of a vertex $x$ of a graph $G$ over a subset $S$ of vertices, is a subset of vertices of $S$ which are at minimal distance from $x$. It is denoted by $P(x, S)$. A graph $G$ is Hilbertian if $|P(x, I(u, v))| = 1$, for all $u, v, x \in G$. A graph $G$ is quasi-Hilbertian if, for all $u, v$ and $x$ in $G$, $|P(x, I^*(u, v))| = 1$. Quasi-median graphs have been introduced by Mulder [9] as a natural generalization of median graphs, in fact,
median graphs are just the bipartite quasi-median graphs. Many researchers are interested in studying this class of graphs. Among prominent examples of median graphs let us mention hypercubes, trees and grids. Berrachedi [4] proved that a graph $G$ is median if and only if $G$ is Hilbertian. From the fact that a quasi-interval is an enlarged interval and in median graphs a quasi-interval is also an interval, then another generalization of Hilbertian graphs is to consider graphs which are quasi-Hilbertian. In this paper, our aim is to show that the class of quasi-median graphs is the same as the class of quasi-Hilbertian graphs.

2. Preliminaries

In this section, we recall some classical definitions and notation following that of [7, 9]. Then we give a mini-review of some interesting results on median graphs, and results obtained analogously for quasi-median graphs. A connected subgraph $H$ of a graph $G$ is called convex if for any two vertices $u$ and $v$ from $H$ all shortest $(u, v)$-paths are contained in $H$. The convex closure of a subgraph $H$ of $G$ is defined as the smallest convex subgraph of $G$ which contains $H$. The Cartesian product $G \square H$ of two graphs $G$ and $H$ is the graph with vertex set $V(G) \times V(H)$ and $(a, x)(b, y) \in E(G \square H)$ whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. A clique in $G$ is a set of vertices $K \subseteq V(G)$ in which any two distinct vertices are adjacent. If $K$ is a clique and $K = V(G)$, then $G$ is the complete graph $K_n$, where $n$ is the number of vertices of $G$. The graph $K_4 - e$ is the complete graph on four vertices minus an edge. $K_{n,m}$ is the complete bipartite graph, where $n$ and $m$ are the number of vertices of the first and the second part of the partition. For $u \in V(G)$, $N(u)$ is the set of vertices adjacent to the vertex $u$. A Cartesian product of complete graphs is called a Hamming graph, a Cartesian power of the $K_2$ is called a hypercube. A graph $G$ satisfies the triangle property if for any vertices $u, x, y \in V(G)$, where $d(u, x) = d(u, y) = k$ such that $xy \in E(G)$, there exists a common neighbour $v$ of $x$ and $y$ with $d(u, v) = k - 1$. A graph $G$ satisfies the quadrangle property if for any $u, x, y, z \in V(G)$ such that $d(u, x) = d(u, y) = d(u, z) - 1$ and $d(x, y) = 2$, with $z$ a common neighbour of $x$ and $y$, there exists a common neighbour $v$ of $x$ and $y$ such that $d(u, v) = d(u, x) - 1$. A graph which fulfils the quadrangle property and the triangle property is called a weakly modular graph.

2.1. Median graphs

A vertex $x$ is a median of the triple of vertices $u, v$ and $w$ if

1. $d(u, x) + d(x, v) = d(u, v)$;
2. $d(v, x) + d(x, w) = d(v, w)$;
3. \( d(w, x) + d(x, u) = d(w, u) \).

A graph \( G \) is a median graph if any three vertices \( u, v \) and \( w \) in \( G \) have a unique median. Mulder gave the following characterization of median graphs using the procedure of convex expansions, see [9] for the necessary details.

**Theorem 1** (Mulder [9]). A graph \( G \) is a median graph if and only if \( G \) can be obtained from \( K_1 \) by a sequence of convex expansions.

**Theorem 2** (Mulder [8]). A graph \( G \) is a hypercube if and only if \( G \) is a regular median graph.

A retraction \( f \) from a graph \( G \) to a subgraph \( H \) is a mapping \( f \) of the vertex set \( V(G) \) of \( G \) onto the vertex set \( V(H) \) of \( H \) such that for every edge \( uv \) in \( G \) the image \( f(u)f(v) \) is an edge in \( H \), and \( f(w) = w \) for all vertices \( w \) of \( H \). Using retraction, Bandelt [2] characterized hypercubes as median graphs.

**Theorem 3** (Bandelt [2]). The retracts of hypercubes are precisely the median graphs.

Berrachedi in [4] introduced the class of Hilbertian graphs, using projections over intervals, he showed the following.

**Theorem 4** (Berrachedi [4]). Let \( G \) be a graph. Then \( G \) is Hilbertian if and only if \( G \) is a median graph.

Other characterizations of median graphs using projections over intervals and convex sets are given by Berrachedi and Mollard in [5].

### 2.2. Quasi-median graphs

A triple of vertices \((x, y, z)\) is a quasi-median of \((u, v, w)\) if we have:

1. \( d(u, x) + d(x, y) + d(y, v) = d(u, v) \);
   \( d(v, y) + d(y, z) + d(z, w) = d(v, w) \);
   \( d(w, z) + d(z, x) + d(x, u) = d(w, u) \).
2. \( d(x, y) = d(y, z) = d(z, x) = k \).
3. \( k \) is minimal under the two above conditions.

Mulder [9] defines a quasi-median graph \( G \) as follows.

(i) Each ordered triple of vertices of \( G \) has a unique quasi-median;
(ii) \( G \) does not admit \( K_4 - e \) as induced subgraph;
(iii) Each induced \( C_6 \) in \( G \) has \( K_3 \Box K_3 \) or \( Q_3 \) as convex closure.

He characterized the quasi-median graphs with the quasi-median expansion procedure.
Theorem 5 (Mulder [9]). A graph $G$ is quasi-median if and only if $G$ can be obtained from $K_1$ by a sequence of quasi-median expansions.

Theorem 6 (Mulder [9]). A graph $G$ is a Hamming graph if and only if $G$ is a regular quasi-median graph.

Theorem 7 (Wilkeit [11]). The retracts of Hamming graphs are precisely the quasi-median graphs.

Chung et al. [6], characterized quasi-median graphs as weakly modular graphs without $K_4-e$ or $K_2,3$ as induced subgraph.

Theorem 8 (Chung et al. [6]). A graph $G$ is quasi-median if and only if $G$ is weakly modular and does not contain $K_4-e$ or $K_2,3$ as an induced subgraph.

More characterizations of quasi-median graphs can be found in [1, 3, 6, 9, 11].

3. Quasi-Hilbertian Graphs

In this section we shall prove that quasi-Hilbertian graphs are precisely quasi-median graphs. Chung et al. [6], established a relation between the quasi-median graphs and weakly modular graphs. We use their relation and some properties of quasi-Hilbertian graphs to prove that quasi-Hilbertian graphs are precisely quasi-median graphs.

Theorem 9 (the main result). A graph $G$ is a quasi-median graph if and only if $G$ is a quasi-Hilbertian graph.

This Theorem will be proved using a series of Lemmas that follow.

Lemma 10. A quasi-median graph is quasi-Hilbertian.

Proof. Let $u,v,w$ be three vertices of a quasi-median graph $G$. We assume that $P(u,P(v,w))$ contains at least two vertices $x$ and $x'$. We take the triple $(x,v,w)$. As known in [9], there exists a unique vertex $y$ in $I(x,v) \cap I(v,w) = I(v,y)$. Also, with the triple $(x,y,w)$ we get $I(x,w) \cap I(y,w) = I(w,z)$. In the same way, starting by the triple $(x',v,w)$, we find $I(x',v) \cap I(v,w) = I(v,y')$ and $I(x',w) \cap I(y',w) = I(w,z')$. Thus, $(x,y,z)$ and $(x',y',z')$ are two quasi-median of $(u,v,w)$ in $G$, which is a contradiction.

Lemma 11. A quasi-Hilbertian graph is $K_{2,3}$-free.

Proof. Let $u,v,w,x$ and $y$ be five vertices that induce a $K_{2,3}$ in the quasi-Hilbertian graph $G$. Let $v,w$ and $u$ be the vertices of degree 2. Consider the quasi-interval $I'(v,w)$. Since $I(v,u) \cap I(u,w) \supseteq \{u,x,y\}$, $u \notin I'(v,w)$. The vertices $v,w,x$ and $y$ are in $I'(v,w)$. As $d(u,x) = d(u,y) = 1$, $P(u,I'(v,w)) \supseteq \{x,y\}$. This contradicts the fact that $G$ is a quasi-Hilbertian graph.
Lemma 12. A quasi-Hilbertian graph is $K_4 - e$-free.

Proof. Let $u, v, w$ and $z$ be four vertices that induce a $K_4 - e$ in the quasi-Hilbertian graph $G$. Let $u$ and $w$ be the vertices of degree 2. Consider the quasi-interval $I^*(v, w)$. The vertices $v, w$ and $z$ are in $I^*(v, w)$, but $u \notin I^*(v, w)$. As $d(u, v) = d(u, z) = 1$, $P(u, I^*(v, w)) \supseteq \{v, z\}$. This contradicts the fact that $G$ is a quasi-Hilbertian graph.

Lemma 13. In a quasi-Hilbertian graph $G$, for all $vw \in E(G)$ and for all $x \in I^*(v, w) \setminus \{v, w\}$, we have $d(v, x) = d(w, x) = 1$.

Proof. By contrary. Let $vw$ be an edge in a quasi-Hilbertian graph $G$ and $x \in I^*(v, w) \setminus \{v, w\}$. Let us consider the two possible cases.

Case 1. $d(v, x) \neq d(w, x)$. We assume without loss of generality that $d(v, x) < d(w, x)$, then $d(v, x) + 1 \leq d(w, x)$, which implies that $I(v, x) \subset I(w, x)$. Thus $I(v, x) \cap I(w, x) = I(v, x)$, this is a contradiction with $x \in I^*(v, w) \setminus \{v, w\}$.

Case 2. $d(v, x) = d(w, x) > 1$. We suppose that $d(v, x)$ is minimal. Let $x_1$ be a vertex in $I(v, x) \cap N(v)$. As $I(x_1, v) \cap I(v, x) = \{v, x_1\}$, $v \notin I^*(x_1, x)$. $I(x, w) \cap I(w, x_1) \neq \{w\}$, otherwise $P(v, I^*(x, x_1)) \supseteq \{w, x_1\}$. Necessarily, there exists $x_2 \in I(x, w) \cap I(w, x_1) \setminus \{w\}$ and $d(x_1, x_2) = 1$. If $v \in N(x_2)$ and $w \notin N(x_1)$, then $K_4 - e$ will be an induced subgraph. The same result holds if $v \notin N(x_2)$ and $w \in N(x_1)$. If $v \in N(x_2)$ and $w \in N(x_1)$, then $P(v, I^*(x, x_1)) \supseteq \{x_1, x_2\}$ and $P(w, I^*(x, x_1)) \supseteq \{x_1, x_2\}$. Thus $d(v, x_2) = d(w, x_1) = 2$. From the minimality of $d(v, x)$, we have $d(x, x_1) = d(x, x_2) = 1$, so that $P(x_1, I^*(v, w)) \supseteq \{v, x\}$. Contradiction with the fact that $G$ is a quasi-Hilbertian graph. Consequently, we have $d(v, x) = d(w, x) = 1$, for all $x \in I^*(v, w) \setminus \{v, w\}$ with $vw \in E(G)$.

Lemma 14. For every two adjacent vertices $v$ and $w$ of a quasi-Hilbertian graph $G$, the quasi-interval $I^*(v, w)$ induces a complete subgraph.

Proof. Let $I^*(v, w)$ be the quasi-interval such that $d(v, w) = 1$, and $x, y \in I^*(v, w)$ such that $x \neq y$. From Lemma 13, we have

\[
\begin{align*}
\begin{cases}
d(v, x) = d(w, x) = d(v, w) = 1, \\
d(v, y) = d(w, y) = d(v, w) = 1.
\end{cases}
\end{align*}
\]

If $x = v$ or $x = w$, then $d(x, y) = 1$. The same result hold if $y = v$ or $y = w$. Else, if $d(x, y) \neq 1$, then the vertices $v, w, x$ and $y$ induce a forbidden $K_4 - e$.

Lemma 15. A quasi-Hilbertian graph satisfies the triangle property.
Proof. Consider three vertices $u, v$ and $w$ of a quasi-Hilbertian graph such that $d(u, v) = d(u, w) = k$ and $d(w, v) = 1$. If $k = 1$, we have the triangle property. Suppose that $k \geq 2$. Since $I^*(w, v)$ induce a complete subgraph, $u$ is not in $I^*(w, v)$. So, there exists $x$ in $I(w, u) \cap I(u, v) \setminus \{u\}$ such that $x \in I^*(w, v)$. Hence, $d(x, v) = d(w, x) = 1$ and $d(u, x) = k - 1$.

Lemma 16. A quasi-Hilbertian graph satisfies the quadrangle property.

Proof. Let $u, v, w$ and $z$ be four vertices in a quasi-Hilbertian graph such that $d(u, v) = d(u, z) = d(u, w) - 1 = k$, $d(z, v) = 2$, and $w \in I(v, z)$.

Consider the quasi-interval $I^*(u, z)$. If $k = 1$, we have the quadrangle property. Suppose that $k \geq 2$. $I(u, v) \cap I(v, z) \neq \{v\}$, otherwise $P(w, I^*(u, z)) \supseteq \{z, v\}$. Necessarily, there exists $x \in I(z, v) \cap I(v, u) \setminus \{v\}$, then $d(z, x) = d(v, x) = 1$ and $d(u, x) = k - 1$.

Proof of Theorem 9. From Lemma 10, a quasi-median graph is quasi-Hilbertian. As a quasi-Hilbertian graph is weakly modular (Lemmas 15 and 16), and does not contain $K_{2,3}$ or $K_4 - e$ as an induced subgraph, it is a quasi-median graph (Theorem 8).

Theorems 9 and 6 give a new characterization of Hamming graphs.

Theorem 17. A graph $G$ is a Hamming graph if and only if

\[
\begin{align*}
&G \text{ is regular}, \\
&\text{for all } u, v, w \in G \text{ we have } |P(w, I^*(u, v))| = 1.
\end{align*}
\]

References


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